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Generalized solutions of nonlinear diffusion equations

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Abstract. We investigate generalized solutions of nonlinear diffusion equations in the sense of Colombeau and extend the results of Biagioni and Oberguggenberger [2] to more general equations. We prove the existence and uniqueness of a generalized solution which is shown to be consistent with the classical solution.

1. Introduction

Biagioni and Oberguggenberger [2] have studied generalized solutions of the Cauchy problem

$$\begin{cases} \tilde{u}_t + \tilde{u}\tilde{u}_x = \tilde{\mu}\tilde{u}_{xx}, & x \in \mathbf{R},\ t > 0, \\ \tilde{u}|_{t=0} = \tilde{u}_0, & x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where $\tilde{\mu}$ is a generalized constant belonging to the algebra $\mathscr{G}_{s,g}$ of generalized functions, which is a modified version of the algebra introduced by Colombeau [3, 4] to deal with the multiplication of distributions. This algebra contains the space of bounded distributions $\mathscr{D}'_{L^\infty}$ so that initial data with strong singularities can be considered in this setting. They formulated the classical Cauchy problem

$$\begin{cases} u_t + uu_x = 0, & x \in \mathbf{R},\ t > 0, \\ u|_{t=0} = u_0, & x \in \mathbf{R} \end{cases} \tag{1.2}$$

as

$$\begin{cases} \tilde{u}_t + \tilde{u}\tilde{u}_x \approx 0, & x \in \mathbf{R},\ t > 0, \\ \tilde{u}|_{t=0} = \tilde{u}_0, & x \in \mathbf{R} \end{cases} \tag{1.3}$$

in the present setting, where "$\approx$" denotes the association relation on $\mathscr{G}_{s,g}$, and proved that the generalized solution of (1.3) is obtained as the generalized solution of (1.1) with $\tilde{\mu} \approx 0$. Furthermore, they proved the existence and uniqueness of a generalized solution of (1.1) and showed that, if the initial data

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belong to $L^{\infty}(\mathbf{R})$ and $\tilde{\mu} \approx 0$, the generalized solution is associated with the weak entropy solution of (1.2).

In this paper we study generalized solutions of the Cauchy problem

$$\begin{cases} \tilde{u}_t + f(x,t,\tilde{u})\tilde{u}_x + g(x,t,\tilde{u})\tilde{u} = \tilde{\mu}\tilde{u}_{xx}, & x \in \mathbf{R},\ t > 0, \\ \tilde{u}|_{t=0} = \tilde{u}_0, & x \in \mathbf{R}, \end{cases} \tag{1.4}$$

where $\tilde{\mu}$ is a generalized constant. As in [2] we formulate the classical Cauchy problem

$$\begin{cases} u_t + f(x,t,u)u_x + g(x,t,u)u = 0, & x \in \mathbf{R},\ t > 0, \\ u|_{t=0} = u_0, & x \in \mathbf{R} \end{cases} \tag{1.5}$$

as

$$\begin{cases} \tilde{u}_t + f(x,t,\tilde{u})\tilde{u}_x + g(x,t,\tilde{u})\tilde{u} \approx 0, & x \in \mathbf{R},\ t > 0, \\ \tilde{u}|_{t=0} = \tilde{u}_0, & x \in \mathbf{R} \end{cases} \tag{1.6}$$

in the present setting.

In this paper, we extend results in [2] to (1.4) with general functions f and g. We first recall the definition of the Colombeau algebra $\mathscr{G}_{s,g}$ in Section 2. In Sections 3 and 4, we prove results concerning existence and uniqueness of a generalized solution of (1.4) (Theorems 3.1 and 4.4). In Section 5, we study the distribution with which the generalized solution of (1.4) is associated, when the initial data belong to $L^{\infty}(\mathbf{R})$. It turns out that if a coefficient $\tilde{\mu}$ is associated with zero, the generalized solution of (1.4) is associated with the weak solution of (1.5) which satisfies the entropy condition (Theorem 5.3) and, if $\tilde{\mu}$ is a positive number, the generalized solution is associated with the unique classical solution (Theorem 5.5).

2. The algebra of generalized functions

We briefly recall the definition of a modified version of the Colombeau algebra of generalized functions [3, 4].

NOTATIONS 2.1. Let Ω be a nonempty open subset of $\mathbf{R}^d$. We set

$$\mathscr{E}_s[\Omega] = \{R : (0,\infty) \times \Omega \to \mathbf{R} \text{ such that } R(\varepsilon, x) \text{ is of class } C^{\infty} \text{ in } x \in \Omega \text{ for each } \varepsilon > 0\};$$

$$\mathscr{E}_s[\overline{\Omega}] = \{R : (0,\infty) \times \overline{\Omega} \to \mathbf{R} \text{ such that } R|_{(0,\infty)\times\Omega} \in \mathscr{E}_s[\Omega] \text{ and that the map } x \in \Omega \mapsto R(\varepsilon, x) \in \mathbf{R} \text{ and all its derivatives can be continuously extended to } \overline{\Omega}, \text{ for each } \varepsilon > 0\};$$

$$\begin{aligned}\mathscr{E}_{M,s,g}[\overline{\Omega}] = \{R \in \mathscr{E}_s[\overline{\Omega}] &\text{ such that for all } \alpha \in \mathbf{N}^d,\\ &\text{there exist } N \in \mathbf{N}, c > 0 \text{ and } \eta > 0 \text{ such that}\\ &\sup_{x \in \overline{\Omega}} |D_x^\alpha R(\varepsilon, x)| < c\varepsilon^{-N} \text{ for all } 0 < \varepsilon < \eta\};\end{aligned}$$

$$\begin{aligned}\mathscr{N}_{s,g}[\overline{\Omega}] = \{R \in \mathscr{E}_s[\overline{\Omega}] &\text{ such that for all } \alpha \in \mathbf{N}^d \text{ and } q \in \mathbf{N},\\ &\text{there exist } c > 0 \text{ and } \eta > 0 \text{ such that}\\ &\sup_{x \in \overline{\Omega}} |D_x^\alpha R(\varepsilon, x)| < c\varepsilon^{q} \text{ for all } 0 < \varepsilon < \eta\}.\end{aligned}$$

The algebra of generalized functions $\mathscr{G}_{s,g}(\overline{\Omega})$ is defined by

$$\mathscr{G}_{s,g}(\overline{\Omega}) = \mathscr{E}_{M,s,g}[\overline{\Omega}]/\mathscr{N}_{s,g}[\overline{\Omega}].$$

We denote by $R_{\tilde{u}}(\varepsilon, x)$ a representative of a generalized function $\tilde{u} \in \mathscr{G}_{s,g}(\overline{\Omega})$. Then for any $\alpha \in \mathbf{N}^d$, we can define a generalized function $D_x^\alpha \tilde{u}$ to be the class of $\{D_x^\alpha R_{\tilde{u}}(\varepsilon, x)\}_{\varepsilon > 0}$. Also, for any generalized function $\tilde{u} \in \mathscr{G}_{s,g}(\mathbf{R} \times [0, T])$, we can define $\tilde{u}|_{t=0}$ to be the class of $\{R_{\tilde{u}}(\varepsilon, x, 0)\}_{\varepsilon > 0}$.

Concerning nonlinear functions f of elements of the algebra $\mathscr{G}_{s,g}(\mathbf{R}^d)$, we define the following notion.

Definition 2.2. We say that a function $f \in C^\infty(\mathbf{R}^d)$ is *slowly increasing at infinity* if for every $\alpha \in \mathbf{N}^d$ there exist $c > 0$ and $r \in \mathbf{N}$ such that, for all $x \in \mathbf{R}^d$,

$$|D^\alpha f(x)| \leq c(1 + |x|)^r.$$

We denote by $\mathscr{O}_M(\mathbf{R}^d)$ the space of slowly increasing functions at infinity.

If $f \in \mathscr{O}_M(\mathbf{R}^p)$ and $\tilde{u}_i \in \mathscr{G}_{s,g}(\mathbf{R}^d)$ for $i = 1, \ldots, p$, we can define a generalized function $f(\tilde{u}_1, \ldots, \tilde{u}_p) \in \mathscr{G}_{s,g}(\mathbf{R}^d)$ to be the class of $\{f(R_{\tilde{u}_1}, \ldots, R_{\tilde{u}_p})\}_{\varepsilon > 0}$. For details see [1, 3, 4].

Remark 2.3. The algebra $\mathscr{G}_{s,g}(\mathbf{R}^d)$ contains the space of bounded distributions $\mathscr{D}'_{L^\infty}(\mathbf{R}^d)$. Let f be an element of $\mathscr{D}'_{L^\infty}(\mathbf{R}^d)$. Since $R(\varepsilon, x) = f * \rho_\varepsilon(x)$ is a representative of f, we obtain $\mathscr{D}'_{L^\infty}(\mathbf{R}^d) \subset \mathscr{G}_{s,g}(\mathbf{R}^d)$, where ρ is a fixed element of $\mathscr{S}(\mathbf{R}^d)$ satisfying $\int \rho(x)dx = 1$, $\int x^\alpha \rho(x)dx = 0$, for all $\alpha \in \mathbf{N}^d$, $|\alpha| \geq 1$, and

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^d} \rho\left(\frac{x}{\varepsilon}\right).$$

Definition 2.4. A generalized function $\tilde{u} \in \mathscr{G}_{s,g}(\overline{\Omega})$ is said to be *associated with a distribution* $w \in \mathscr{D}'(\Omega)$ if it has a representative $R_{\tilde{u}} \in \mathscr{E}_{M,s,g}[\overline{\Omega}]$ such that

$$R_{\tilde{u}}(\varepsilon, x) \to w \qquad \text{in } \mathscr{D}'(\Omega) \text{ as } \varepsilon \to 0.$$

We denote by $\tilde{u} \approx w$ if $\tilde{u}$ is associated with w.

In other words, a generalized function $\tilde{u} \in \mathscr{G}_{s,g}(\overline{\Omega})$ is associated with a distribution w if $\tilde{u}$ behaves like w on the level of information of distribution theory. Actually, if $f \in \mathscr{D}'_{L^\infty}$, the class of $\{f * \rho_\varepsilon\}_{\varepsilon>0}$ is associated with f.

Definition 2.5. A generalized function $\tilde{\mu} \in \mathscr{G}_{s,g}(\overline{\Omega})$ is called a *generalized constant* if it has a representative which is constant for each $\varepsilon > 0$. We call $\tilde{\mu}$ a *generalized positive number* if it has a representative $R_{\tilde{\mu}}(\varepsilon)$ such that there exist $N \in \mathbf{N}$ and $\eta > 0$ such that $\varepsilon^N \le R_{\tilde{\mu}}(\varepsilon) \le \varepsilon^{-N}$ for $0 < \varepsilon < \eta$.

Definition 2.6. We say that $\tilde{u} \in \mathscr{G}_{s,g}(\overline{\Omega})$ is of *bounded type* if it has a representative $R_{\tilde{u}} \in \mathscr{E}_{M,s,g}[\overline{\Omega}]$ such that there exist $c > 0$ and $\eta > 0$ such that

$$\sup_{x \in \overline{\Omega}} |R_{\tilde{u}}(\varepsilon, x)| < c \qquad \text{for } 0 < \varepsilon < \eta.$$

We note that $u_0 \in L^\infty(\mathbf{R})$, viewed as an element of $\mathscr{G}_{s,g}(\mathbf{R})$, is of bounded type.

3. Existence theorem

Theorem 3.1. *Assume that $\tilde{\mu}$ is a generalized positive number and that $f, g \in C^\infty(\mathbf{R}^3)$ satisfy the following conditions: for every $\alpha \in \mathbf{N}^3$, there exist $c > 0$ and $r \in \mathbf{N}$ such that, for all $(x, t, u) \in \mathbf{R}^3$,*

$$|D^\alpha f(x, t, u)| \le c(1 + |u|)^r,$$

$$|D^\alpha g(x, t, u)| \le c(1 + |u|)^r,$$

and $g \ge 0$. Then for each $T > 0$ there exists a solution $\tilde{u} \in \mathscr{G}_{s,g}(\mathbf{R} \times [0, T])$ of (1.4).

In order to prove Theorem 3.1, we first prove Lemmas 3.3 and 3.4. Let us recall a next well-known result.

Lemma 3.2 ([7], Section 28, Theorems 7 and 8). *Let $0 < \alpha \le 1$ and $0 < \beta \le 1$, and let k and ℓ be two nonnegative integers such that $k + \alpha > \ell + \beta$. Furthermore, let an open subset D of $\mathbf{R}^d$ be bounded and convex or bounded with a C^∞-boundary. Then the identity map*

$$I : C^{k+\alpha}(D) \to C^{\ell+\beta}(D)$$

is compact.

Lemma 3.3. *Let g be a function on $\mathbf{R} \times (0, T)$, where $0 < T \le \infty$, such that for any $0 < t < T$,*

$$\sup_{x\in\mathbf{R}} |g(x,t)| \le Mt^{-\beta}$$

for some $M > 0$ *depending on* T *and for some* $0 \le \beta < 1$. *Then*

$$v(x,t) = -K \underset{x,t}{*} g = -\int_0^t \int_{-\infty}^{\infty} K(x-\xi, t-\eta) g(\xi,\eta) d\xi d\eta,$$

$$K(x,t) = \frac{1}{2\sqrt{\pi\mu t}} \exp\left(-\frac{x^2}{4\mu t}\right)$$

is a distributional solution of the problem

$$\begin{cases} v_t + g = \mu v_{xx}, & x \in \mathbf{R},\ 0 < t < T, \\ v|_{t=0} = 0, & x \in \mathbf{R}, \end{cases}$$

where $\mu > 0$. *Furthermore,* v *and* v_x *are Hölder continuous with respect to* x *and* t, *and for* $0 < \alpha \le 1$, *the sup norm* $\|\cdot\|$ *and Hölder norm* $|\cdot|_\alpha$ *on the domain* $\mathbf{R} \times [t,s]$ *satisfy*

$$\begin{cases} \|v\| \le \dfrac{s^{1-\beta}}{1-\beta} M, \\[2ex] \|v_x\| \le \dfrac{\mu^{-1/2}}{\sqrt{\pi}} B(1-\beta, 1/2) \max\{t^{1/2-\beta}, s^{1/2-\beta}\} M, \end{cases} \tag{3.1}$$

$$\begin{cases} |v|_\alpha \le K(\mu^{-\alpha/2} + 1)M, \quad \text{with } \alpha < 1, \alpha \le 1-\beta, \\[2ex] |v_x|_\alpha \le K(\mu^{-(1+\alpha)/2} + \mu^{-1/2}) M t^{-\beta}, \quad \text{with } \alpha < \dfrac{1}{2}, \\[2ex] |v_x|_\alpha \le K(\mu^{-(1+\alpha)/2} + \mu^{-1/2}) M, \quad \text{with } \alpha < \dfrac{1}{2}, \alpha \le \dfrac{1}{2} - \beta, \text{ if } \beta < \dfrac{1}{2} \end{cases} \tag{3.2}$$

for some positive constant K, *where* B *is the beta function.*

Proof. Since the inequalities of (3.1) are clear, we only prove (3.2). Since

$$\exp\left(-\frac{(x-\xi)^2}{4\mu(t-\eta)}\right) - \exp\left(-\frac{(y-\xi)^2}{4\mu(t-\eta)}\right)$$
$$= \int_{|y-\xi|^\alpha}^{|x-\xi|^\alpha} \left(-\frac{2}{\alpha} \frac{\zeta^{2/\alpha-1}}{4\mu(t-\eta)}\right) \exp\left(-\frac{\zeta^{2/\alpha}}{4\mu(t-\eta)}\right) d\zeta,$$

using the change of variable and $\left||x-\xi|^\alpha - |y-\xi|^\alpha\right| \le |x-y|^\alpha$, $0 < \alpha \le 1$, we obtain that

$$|v(x,t)-v(y,t)| \le |x-y|^{\alpha} B(1-\beta, 1-\alpha/2) \cdot \frac{2^{3-\alpha}}{\sqrt{\pi}\alpha} \int_0^{\infty} \zeta^{2-\alpha} e^{-\zeta^2} d\zeta \cdot \mu^{-\alpha/2} M t^{1-\alpha/2-\beta}$$

from the hypothesis on g. Similarly, for $0 < t \le t_1 < t_2 \le s$,

$$\begin{aligned}
&|v(x,t_2)-v(x,t_1)| \\
&\le \left| \int_0^{t_1} \int_{-\infty}^{\infty} \int_{(t_1-\eta)^{\alpha}}^{(t_2-\eta)^{\alpha}} \frac{\partial}{\partial \tau} \left[\frac{1}{2\sqrt{\pi\mu\tau^{1/\alpha}}} \exp\left(-\frac{(x-\xi)^2}{4\mu\tau^{1/\alpha}} \right) \right] d\tau \cdot g(\xi,\eta) d\xi d\eta \right| \\
&\quad + \left| \int_{t_1}^{t_2} \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi\mu(t_2-\eta)}} \exp\left(-\frac{(x-\xi)^2}{4\mu(t_2-\eta)} \right) g(\xi,\eta) d\xi d\eta \right| \\
&\le (t_2-t_1)^{\alpha} B(1-\beta, 1-\alpha) \frac{1}{\sqrt{\pi}\alpha} \int_{-\infty}^{\infty} (1/2+\zeta^2) e^{-\zeta^2} d\zeta \cdot M t_1^{1-\alpha-\beta} \\
&\quad + \frac{1}{1-\beta} (t_2-t_1)^{1-\beta} M.
\end{aligned}$$

If we take α such that $\alpha \le 1-\beta$, then $1-\alpha/2-\beta > 1-\alpha-\beta \ge 0$. Therefore the first inequality of (3.2) holds. Similarly

$$\begin{aligned}
|v_x(x,t)-v_x(y,t)| &\le |x-y|^{\alpha} B(1-\beta, 1/2-\alpha/2) \frac{2^{2-\alpha}}{\sqrt{\pi}\alpha} \\
&\quad \cdot \int_0^{\infty} (\zeta^{1-\alpha} + 2\zeta^{3-\alpha}) e^{-\zeta^2} d\zeta \cdot \mu^{-(1+\alpha)/2} t^{1/2-\alpha/2-\beta} M \\
&\quad + |x-y|^{\alpha} B(1-\beta, 1/2-\alpha/2) \frac{2^{1-\alpha}}{\sqrt{\pi}\alpha} \\
&\quad \cdot \sup_{\zeta \in \mathbf{R}} (|\zeta|^{2-\alpha} e^{-\zeta^2}) \cdot \mu^{-(1+\alpha)/2} t^{1/2-\alpha/2-\beta} M,
\end{aligned}$$

and, for $0 < t \le t_1 < t_2 \le s$,

$$\begin{aligned}
&|v_x(x,t_2)-v_x(x,t_1)| \\
&\le (t_2-t_1)^{\alpha} B(1-\beta, 1/2-\alpha) \cdot \frac{1}{\sqrt{\pi}\alpha} \int_{-\infty}^{\infty} (3|\zeta|/2+|\zeta|^3) e^{-\zeta^2} d\zeta \cdot \mu^{-1/2} t_1^{1/2-\alpha-\beta} M \\
&\quad + \frac{1}{\sqrt{\pi}} \mu^{-1/2} \int_{t_1}^{t_2} (t_2-\eta)^{-1/2} \eta^{-\beta} d\eta \cdot M \qquad (3.3) \\
&\le (t_2-t_1)^{\alpha} B(1-\beta, 1/2-\alpha) \cdot \frac{1}{\sqrt{\pi}\alpha} \int_{-\infty}^{\infty} (3|\zeta|/2+|\zeta|^3) e^{-\zeta^2} d\zeta \cdot \mu^{-1/2} t_1^{1/2-\alpha-\beta} M \\
&\quad + (t_2-t_1)^{1/2} \frac{2}{\sqrt{\pi}} \mu^{-1/2} t^{-\beta} M.
\end{aligned}$$

If $\beta < 1/2$, (3.3) can also be estimated as follows:

$$\begin{aligned}
&\frac{1}{\sqrt{\pi}}\mu^{-1/2}\int_{t_1}^{t_2}(t_2-\eta)^{-1/2}\eta^{-\beta}d\eta\cdot M \\
&\quad\le \frac{M}{\sqrt{\pi}\mu^{1/2}}\left\{\int_{t_1}^{(t_1+t_2)/2}\left(\frac{t_2-t_1}{2}\right)^{-1/2}\eta^{-\beta}d\eta\right. \\
&\qquad\qquad \left.+\int_{(t_1+t_2)/2}^{t_2}(t_2-\eta)^{-1/2}\left(\frac{t_1+t_2}{2}\right)^{-\beta}d\eta\right\} \\
&\quad= \frac{M}{\sqrt{\pi}\mu^{1/2}}\left\{\left(\frac{t_2-t_1}{2}\right)^{-1/2}\frac{1}{1-\beta}\left[\left(\frac{t_1+t_2}{2}\right)^{1-\beta}-t_1^{1-\beta}\right]\right. \\
&\qquad\qquad \left.+2\left(\frac{t_2-t_1}{2}\right)^{1/2}\left(\frac{t_1+t_2}{2}\right)^{-\beta}\right\} \\
&\quad\le \frac{M}{\sqrt{\pi}\mu^{1/2}}\left(\frac{1}{1-\beta}+2\right)\left(\frac{t_2-t_1}{2}\right)^{1/2-\beta}.
\end{aligned}$$

Thus the assertion follows. □

Under the assumptions that $\mu > 0$ and f, g satisfy the conditions given in Theorem 3.1, we consider the classical Cauchy problem

$$\begin{cases} u_t + f(x,t,u)u_x + g(x,t,u)u = \mu u_{xx}, & x\in\mathbf{R}, 0<t<T, \quad (3.4)\\ u|_{t=0} = u_0, & x\in\mathbf{R} \quad (3.5)\end{cases}$$

in $C(\mathbf{R}\times[0,T))$ if $u_0\in C_b(\mathbf{R})$, where C_b denotes the space of all bounded continuous functions. For the initial data $u_0\in L^\infty(\mathbf{R})$, we need to weaken the initial condition (3.5) as follows: For the dual space $C_0'(\mathbf{R})$ of the space of all continuous functions with compact support,

$$u|_{t=0} = u_0 \qquad \text{in } C_0'(\mathbf{R}), \tag{3.6}$$

which means that for any continuous function φ on $\mathbf{R}$ with compact support,

$$\lim_{t\to+0}\int_{-\infty}^{\infty}u(x,t)\varphi(x)dx = \int_{-\infty}^{\infty}u_0(x)\varphi(x)dx.$$

Then, we can easily see that for the solution u in the following lemma

$$\lim_{t\to0,\,x\to x_0}u(x,t) = u_0(x_0)$$

holds at every continuous point x_0 of $u_0(x)$.

Lemma 3.4. *Let $u_0 \in L^\infty(\mathbf{R})$. Then for each $T > 0$ there exists a bounded classical solution of* (3.4) *satisfying* (3.6). *Furthermore, it is unique.*

Proof. In order to prove the existence of a solution, it suffices to consider the integral equation

$$u = K \underset{x}{*} u_0 - K \underset{x,t}{*} \{f(x,t,u)u_x + g(x,t,u)u\},$$

where

$$K \underset{x}{*} u_0 = \int_{-\infty}^{\infty} K(x-\xi,t)u_0(\xi)d\xi.$$

Let M be a positive number such that

$$\sup\{|f(x,t,u)t^{-1/2}p + g(x,t,u)u| \mid x \in \mathbf{R}, |u| + |p| \\ \le (1 + (\pi\mu)^{-1/2})\|u_0\|_{L^\infty(\mathbf{R})} + 1\} \le Mt^{-1/2} \tag{3.7}$$

for any $t \in (0,T]$ and let $s \le T$ be a positive number such that

$$2\left(1 + \frac{B(1/2,1/2)}{2\sqrt{\pi\mu}}\right)s^{1/2}M \le 1. \tag{3.8}$$

Furthermore, let $\mathbf{B}$ be the Banach space of all continuous functions u on $(-n,n) \times (0,s]$ which are continuously differentiable with respect to x on the same domain with the finite norm $\|u\|_{\mathbf{B}}$ defined by

$$\|u\|_{\mathbf{B}} = \sup_{\substack{-n<x<n \\ 0<t\le s}} |u(x,t)| + \sup_{\substack{-n<x<n \\ 0<t\le s}} t^{1/2}|u_x(x,t)|.$$

Let E be the closed, bounded and convex set in $\mathbf{B}$ defined by

$$E = \{u \in \mathbf{B} \mid \|u\|_{\mathbf{B}} \le (1 + (\pi\mu)^{-1/2})\|u_0\|_{L^\infty(\mathbf{R})} + 1\}.$$

We define a map A from E into $\mathbf{B}$ by

$$Au = K \underset{x}{*} u_0 - K \underset{x,t}{*} F[u]_n,$$

where

$$F[u]_n = \begin{cases} f(x,t,u)u_x + g(x,t,u)u, & \text{for } (x,t) \in (-n,n) \times (0,s], \\ 0, & \text{otherwise.} \end{cases}$$

Then by Lemma 3.3 we have

$$\|Au\|_{\mathbf{B}} \le \|u_0\|_{L^\infty(\mathbf{R})} + 2s^{1/2}M + \frac{1}{\sqrt{\pi\mu}}\|u_0\|_{L^\infty(\mathbf{R})} + \frac{B(1/2,1/2)}{\sqrt{\pi\mu}}s^{1/2}M \\ \le (1 + (\pi\mu)^{-1/2})\|u_0\|_{L^\infty(\mathbf{R})} + 1,$$

which shows that A is a map of E into E. Since for $u, v \in E$, there exists $c > 0$ such that

$$\|Au - Av\|_{\mathbf{B}} \le c(s^{1/2} + s)\|u - v\|_{\mathbf{B}},$$

A is continuous. Furthermore, from Lemma 3.2 and Lemma 3.3 it follows that the set $\{-Au + K *_x u_0 \mid u \in E\}$ is compact in $\mathbf{B}$, and hence that the map A is compact. By Schauder's Fixed Point Theorem, there exists a fixed point u_n in E with $u_n = Au_n$, and obviously u_n is a distributional solution of (3.4) in $(-n, n) \times (0, s]$.

The function $v_n = K *_{x,t} F[u_n]_n$ is defined for all $x \in \mathbf{R}$, $0 \le t \le s$. Since all the estimates are independent of n in the discussion above, it follows from Lemma 3.3 that the sequence $\{v_n\}_{n=1}^{\infty}$ is bounded in the Hölder norm with exponent 1/2 on $\mathbf{R} \times [0, s]$. Therefore according to Lemma 3.2, there exists a uniformly convergent subsequence which converges to a Hölder continuous function v with exponent $0 < \alpha < 1/2$. Since also by Lemma 3.3, $\{\partial v_n/\partial x\}_{n=1}^{\infty}$ is bounded in a Hölder norm with exponent $0 < \alpha' < 1/2$ on $\mathbf{R} \times [t, s]$, for every $0 < t < s$, it follows that v is differentiable in x and $\partial v/\partial x$ is a Hölder continuous with exponent $0 < \alpha'' < \alpha'$ in $\mathbf{R} \times (0, s]$. Now we have

$$\left(\frac{\partial}{\partial t} - \mu \frac{\partial^2}{\partial x^2}\right)(-v_n) \to \left(\frac{\partial}{\partial t} - \mu \frac{\partial^2}{\partial x^2}\right)\left(-v + K \underset{x}{*} u_0\right) \qquad \text{as } n \to \infty$$

in the distributional sense on $\mathbf{R} \times (0, s]$, since $K *_x u_0$ is a solution of the heat equation. Furthermore, we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \mu \frac{\partial^2}{\partial x^2}\right)(-v_n) &= -F\left[-v_n + K \underset{x}{*} u_0\right]_n \\ &\to -f\left(x, t, -v + K \underset{x}{*} u_0\right) \frac{\partial}{\partial x}\left(-v + K \underset{x}{*} u_0\right) \\ &\quad - g\left(x, t, -v + K \underset{x}{*} u_0\right)\left(-v + K \underset{x}{*} u_0\right) \qquad \text{as } n \to \infty\end{aligned}$$

in the distributional sense. Hence, $u = -v + K *_x u_0$ is a distributional solution of (3.4) in $\mathbf{R} \times (0, s]$, which can be written as

$$u = K \underset{x}{*} u_0 - K \underset{x,t}{*} \{f(x, t, u)u_x + g(x, t, u)u\}, \tag{3.9}$$

by the above considerations. Furthermore, from the hypotheses on f and g and (3.9) it follows that u is a classical solution of (3.4) and satisfies (3.6). Taking $u(x, s)$ as the initial data and applying the above argument, we have a solution in $\mathbf{R} \times [s, s + s_1]$, where s_1 is determined by (3.7) and (3.8) from

$\|u(\cdot, s)\|_{L^\infty(\mathbf{R})}$. By the Maximum Principle, we have $\|u(\cdot, s+s_1)\|_{L^\infty(\mathbf{R})} \le \|u(\cdot, s)\|_{L^\infty(\mathbf{R})}$. Therefore, by repeating the above method k times we have a solution in $\mathbf{R} \times [s, s(k+1)]$. Consequently, we have a solution for each $T > 0$.

Since f and g belong to $C^\infty(\mathbf{R}^3)$, the uniqueness is obtained by a similar way to the proof for Theorem 7 of Oleĭnik [6]. □

Proof of Theorem 3.1. In order to prove the existence of a generalized solution of (1.4), it suffices to prove that for any initial data $u_0^\varepsilon \in \mathscr{E}_{M,s,g}[\mathbf{R}]$ and any coefficient μ^ε satisfying $\varepsilon^N \le \mu^\varepsilon \le \varepsilon^{-N}$ for suitable N, (3.4) and (3.5) have a solution $u^\varepsilon \in \mathscr{E}_{M,s,g}[\mathbf{R} \times [0, T]]$ for each $T > 0$.

From Lemma 3.4 there exists a classical solution u^ε of (3.4) and (3.5) with the initial data u_0^ε in $\mathbf{R} \times [0, T]$, because u_0^ε is continuous. Then, by the Maximum Principle we have $\|u^\varepsilon\|_{L^\infty(\mathbf{R}\times[0,T])} \le \|u_0^\varepsilon\|_{L^\infty(\mathbf{R})}$. Furthermore from the boundedness of $\partial_x u_0^\varepsilon$ and Lemma 3.4 we obtain the boundedness of $\partial_x u^\varepsilon$. In fact, from the proof of Lemma 3.4, $K *_{x,t} \{f(x,t,u)u_x + g(x,t,u)u\}$ is Hölder continuous with exponent α for any $0 < \alpha < 1/2$. Since $K *_x u_0^\varepsilon$ is bounded, Lipschitz continuous in x and 1/2-Hölder continuous in t globally, it is α-Hölder continuous for any α $(0 < \alpha < 1/2)$. Therefore it follows that the solution u^ε of (3.4) and (3.5) is Hölder continuous with exponent $0 < \alpha < 1/2$. From the Hölder continuity of u^ε we obtain the boundedness of $\partial_x u^\varepsilon$, namely, there exist $N \in \mathbf{N}, c > 0$ and $\eta > 0$ such that

$$\sup_{(x,t)\in\mathbf{R}\times[0,T]} |\partial_x u^\varepsilon| \le c\varepsilon^{-N}$$

for each $0 < \varepsilon < \eta$. Similarly we can prove that all derivatives of u^ε are dominated by $c\varepsilon^{-N}$ with suitable c and N. Finally by taking $R_{\tilde{u}}(\varepsilon, x, t) = u^\varepsilon(x, t)$ as a representative of $\tilde{u}$, the assertion is obtained. □

4. Uniqueness theorem

Here, we will prove the uniqueness of a generalized solution of (1.4). We first prove Lemma 4.1 and Proposition 4.2.

Lemma 4.1. *For a smooth function f on $\mathbf{R}$, put*

$$G_n(f; u, v) = \begin{cases} (u-v)^{-n}\left(f(u) - \displaystyle\sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(v)(u-v)^k\right), & \text{if } u \ne v, \\ \dfrac{1}{n!} f^{(n)}(u), & \text{if } u = v \end{cases}$$

for each $n \in \mathbf{N}$. Then

$$\frac{\partial}{\partial u} G_n(f; u, v) = -nG_{n+1}(f; u, v) + G_n(f'; u, v), \tag{4.1}$$

$$\frac{\partial}{\partial v} G_n(f; u, v) = nG_{n+1}(f; u, v). \tag{4.2}$$

Furthermore, if $f \in \mathcal{O}_M(\mathbf{R})$, then we obtain that for any $n \in \mathbf{N}$ there exist $c > 0$ and $r \in \mathbf{N}$ such that

$$|G_n(f; u, v)| \leq c(1 + |u| + |v|)^r. \tag{4.3}$$

Proof. Differentiating G_n with respect to u, for $n \geq 2$ we have

$$\frac{\partial}{\partial u} G_n(f; u, v) = \begin{cases} (u - v)^{-1}(-nG_n(f; u, v) + G_{n-1}(f'; u, v)), & \text{if } u \neq v, \\ \dfrac{1}{(n+1)!} f^{(n+1)}(u), & \text{if } u = v. \end{cases}$$

For each $n \in \mathbf{N}$ we have

$$G_n(f; u, v) = (u - v)G_{n+1}(f; u, v) + \frac{1}{n!} f^{(n)}(v),$$

so that for each $n \in \mathbf{N}$

$$\frac{\partial}{\partial u} G_n(f; u, v) = -nG_{n+1}(f; u, v) + G_n(f'; u, v).$$

Furthermore, since

$$\frac{\partial}{\partial v}\left(\sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(v)(u - v)^k\right) = \frac{1}{(n-1)!} f^{(n)}(v)(u - v)^{n-1},$$

we have

$$\frac{\partial}{\partial v} G_n(f; u, v) = nG_{n+1}(f; u, v).$$

Also, since

$$G_n(f; u, v) = \frac{1}{n!} f^{(n)}(\theta u + (1 - \theta)v)$$

holds for some $0 \leq \theta \leq 1$, if $f \in \mathcal{O}_M(\mathbf{R})$, then for any $n \in \mathbf{N}$ there exist $c > 0$ and $r \in \mathbf{N}$ such that

$$|G_n(f; u, v)| \leq c(1 + |u| + |v|)^r.$$

Thus the assertion follows. □

Proposition 4.2. *Assume that f is a function as in Theorem* 3.1. *Let u and v be elements of $\mathscr{E}_{M,s,g}[\mathbf{R} \times [0,T]]$. Put*

$$a(f;\varepsilon,x,t) = G_1(f(x,t,\cdot);u(\varepsilon,x,t),v(\varepsilon,x,t))$$

for each $(x,t) \in \mathbf{R} \times [0,T]$. Then we obtain that $a \in \mathscr{E}_{M,s,g}[\mathbf{R} \times [0,T]]$.

Proof. Since all derivatives of u and v with respect to x and t belong to $\mathscr{E}_{M,s,g}[\mathbf{R} \times [0,T]]$, it suffices to prove that all derivatives of a with respect to u and v belong to $\mathscr{E}_{M,s,g}[\mathbf{R} \times [0,T]]$. Since u and v are elememts of $\mathscr{E}_{M,s,g}[\mathbf{R} \times [0,T]]$, (4.3) implies that for any $n \in \mathbf{N}$ there exist $N \in \mathbf{N}$, $c > 0$ and $\eta > 0$ such that

$$\sup_{(x,t) \in \mathbf{R} \times [0,T]} |G_n(f(x,t,\cdot);u(\varepsilon,x,t),v(\varepsilon,x,t))| \le c\varepsilon^{-N} \tag{4.4}$$

for all $0 < \varepsilon < \eta$. Consequently, from (4.1), (4.2) and (4.4) we obtain that all derivatives of a with respect to u and v belong to $\mathscr{E}_{M,s,g}[\mathbf{R} \times [0,T]]$. Thus the assertion follows. □

Lemma 4.3 ([2], Lemma 2.2). *Let u be a nonnegative, continuous function on $[0,\infty)$ and assume that*

$$u(t) \le a_1 + a_2 \int_0^t \frac{u(s)}{\sqrt{t-s}}\,ds \qquad \text{for } t \ge 0$$

with some constants $a_1, a_2 \ge 0$. Then

$$u(t) \le a_1(1 + 2a_2\sqrt{t})\exp(\pi a_2^2 t) \qquad \text{for any } t \ge 0.$$

Theorem 4.4. *Assume that a representative $R_{\tilde{\mu}}(\varepsilon)$ of a generalized positive number $\tilde{\mu}$ satisfies*

$$R_{\tilde{\mu}}(\varepsilon)\log\frac{1}{\varepsilon} \ge 1 \tag{4.5}$$

for any $0 < \varepsilon < \eta$ with $\eta > 0$. Then for each $T > 0$ the solution $\tilde{u} \in \mathscr{G}_{s,g}(\mathbf{R} \times [0,T])$ of bounded type of (1.4) *is unique.*

Proof. Let $\tilde{u}_1, \tilde{u}_2 \in \mathscr{G}_{s,g}(\mathbf{R} \times [0,T])$ be two solutions of (1.4) with representatives $R_{\tilde{u}_1}, R_{\tilde{u}_2}$ of bounded type, respectively. Then there exist $H \in \mathscr{N}_{s,g}[\mathbf{R} \times [0,T]]$ and $h \in \mathscr{N}_{s,g}[\mathbf{R}]$ such that

$$\begin{cases} (R_{\tilde{u}_1} - R_{\tilde{u}_2})_t = R_{\tilde{\mu}}(R_{\tilde{u}_1} - R_{\tilde{u}_2})_{xx} - (f(x,t,R_{\tilde{u}_1})(R_{\tilde{u}_1})_x - f(x,t,R_{\tilde{u}_2})(R_{\tilde{u}_2})_x) \\ \qquad\qquad - (g(x,t,R_{\tilde{u}_1})R_{\tilde{u}_1} - g(x,t,R_{\tilde{u}_2})R_{\tilde{u}_2}) + H, \\ R_{\tilde{u}_1} - R_{\tilde{u}_2}|_{t=0} = h. \end{cases}$$

By changing representatives suitably, we may assume that $h \equiv 0$. Let F be a function which satisfies

$$F_u(x,t,R_{\tilde{u}_i}) = f(x,t,R_{\tilde{u}_i})$$

for $i = 1, 2$ and set $R_{\tilde{u}} = R_{\tilde{u}_1} - R_{\tilde{u}_2}$. Then we have

$$\begin{cases} (R_{\tilde{u}})_t = R_{\tilde{\mu}}(R_{\tilde{u}})_{xx} - [F(x,t,R_{\tilde{u}_1}) - F(x,t,R_{\tilde{u}_2})]_x + F_x(x,t,R_{\tilde{u}_1}) - F_x(x,t,R_{\tilde{u}_2}) \\ \qquad - g(x,t,R_{\tilde{u}_1})R_{\tilde{u}} - [g(x,t,R_{\tilde{u}_1}) - g(x,t,R_{\tilde{u}_2})]R_{\tilde{u}_2} + H, \\ R_{\tilde{u}}|_{t=0} = 0. \end{cases}$$

From Proposition 4.2 it follows that

$$\begin{cases} (R_{\tilde{u}})_t = R_{\tilde{\mu}}(R_{\tilde{u}})_{xx} - [a(F;\varepsilon,x,t)R_{\tilde{u}}]_x + a(F_x;\varepsilon,x,t)R_{\tilde{u}} \\ \qquad - g(x,t,R_{\tilde{u}_1})R_{\tilde{u}} - a(g;\varepsilon,x,t)R_{\tilde{u}}R_{\tilde{u}_2} + H, \\ R_{\tilde{u}}|_{t=0} = 0. \end{cases}$$

Since $R_{\tilde{u}}$ satisfies

$$\begin{aligned} R_{\tilde{u}} = \int_0^t \int_{-\infty}^{\infty} & K(x-\xi, t-\eta) \cdot (-[a(F;\varepsilon,\xi,\eta)R_{\tilde{u}}]_x + a(F_x;\varepsilon,\xi,\eta)R_{\tilde{u}} \\ & - g(\xi,\eta,R_{\tilde{u}_1})R_{\tilde{u}} - a(g;\varepsilon,\xi,\eta)R_{\tilde{u}}R_{\tilde{u}_2} + H)\,d\xi\,d\eta, \end{aligned}$$

we have

$$\begin{aligned} |R_{\tilde{u}}(\varepsilon,x,t)| \le T & \sup_{(x,t)\in\mathbf{R}\times[0,T]} |H(\varepsilon,x,t)| \\ & + \sup_{(x,t)\in\mathbf{R}\times[0,T]} |a(F;\varepsilon,x,t)| \int_0^t \frac{1}{\sqrt{\pi R_{\tilde{\mu}}(\varepsilon)(t-\eta)}} \sup_{\xi\in\mathbf{R}} |R_{\tilde{u}}(\varepsilon,\xi,\eta)|\,d\eta \\ & + \Bigg(\sup_{(x,t)\in\mathbf{R}\times[0,T]} |a(F_x;\varepsilon,x,t)| + \sup_{(x,t)\in\mathbf{R}\times[0,T]} |g(x,t,R_{\tilde{u}_1})| \\ & \quad + \sup_{(x,t)\in\mathbf{R}\times[0,T]} |a(g;\varepsilon,x,t)R_{\tilde{u}_2}| \Bigg) \int_0^t \sup_{\xi\in\mathbf{R}} |R_{\tilde{u}}(\varepsilon,\xi,\eta)|\,d\eta. \end{aligned}$$

Furthermore, from $\sqrt{t}/\sqrt{t-\eta} \ge 1$ for any $0 < \eta < t$, it follows that

$$
\begin{aligned}
|R_{\tilde{u}}(\varepsilon,x,t)| \le {} & T \sup_{(x,t)\in\mathbf{R}\times[0,T]} |H(\varepsilon,x,t)| \\
& + \sup_{(x,t)\in\mathbf{R}\times[0,T]} |a(F;\varepsilon,x,t)| \int_0^t \frac{1}{\sqrt{\pi R_{\tilde{\mu}}(\varepsilon)(t-\eta)}} \sup_{\xi\in\mathbf{R}} |R_{\tilde{u}}(\varepsilon,\xi,\eta)|d\eta \\
& + \Bigg(\sup_{(x,t)\in\mathbf{R}\times[0,T]} |a(F_x;\varepsilon,x,t)| + \sup_{(x,t)\in\mathbf{R}\times[0,T]} |g(x,t,R_{\tilde{u}_1})| \\
& \quad + \sup_{(x,t)\in\mathbf{R}\times[0,T]} |a(g;\varepsilon,x,t)R_{\tilde{u}_2}| \Bigg) \int_0^t \frac{\sqrt{t}}{\sqrt{t-\eta}} \sup_{\xi\in\mathbf{R}} |R_{\tilde{u}}(\varepsilon,\xi,\eta)|d\eta \\
\le {} & T \sup_{(x,t)\in\mathbf{R}\times[0,T]} |H(\varepsilon,x,t)| \\
& + \frac{1}{\sqrt{\pi}} \Bigg(\frac{1}{\sqrt{R_{\tilde{\mu}}(\varepsilon)}} \sup_{(x,t)\in\mathbf{R}\times[0,T]} |a(F;\varepsilon,x,t)| \\
& \quad + \sqrt{\pi T} \sup_{(x,t)\in\mathbf{R}\times[0,T]} |a(F_x;\varepsilon,x,t)| + \sqrt{\pi T} \sup_{(x,t)\in\mathbf{R}\times[0,T]} |g(x,t,R_{\tilde{u}_1})| \\
& \quad + \sqrt{\pi T} \sup_{(x,t)\in\mathbf{R}\times[0,T]} |a(g;\varepsilon,x,t)R_{\tilde{u}_2}| \Bigg) \\
& \cdot \int_0^t \frac{1}{\sqrt{t-\eta}} \sup_{\xi\in\mathbf{R}} |R_{\tilde{u}}(\varepsilon,\xi,\eta)|d\eta.
\end{aligned}
$$

Applying Lemma 4.3, we have

$$
\sup_{(x,t)\in\mathbf{R}\times[0,T]} |R_{\tilde{u}}(\varepsilon,x,t)| \le a_1(1+2a_2\sqrt{T}) \exp(\pi a_2^2 T),
$$

where

$$
\begin{aligned}
a_1 &= T \sup_{(x,t)\in\mathbf{R}\times[0,T]} |H(\varepsilon,x,t)|, \\
a_2 &= \frac{1}{\sqrt{\pi}} \Bigg(\frac{1}{\sqrt{R_{\tilde{\mu}}(\varepsilon)}} \sup_{(x,t)\in\mathbf{R}\times[0,T]} |a(F;\varepsilon,x,t)| \\
& \quad + \sqrt{\pi T} \sup_{(x,t)\in\mathbf{R}\times[0,T]} |a(F_x;\varepsilon,x,t)| + \sqrt{\pi T} \sup_{(x,t)\in\mathbf{R}\times[0,T]} |g(x,t,R_{\tilde{u}_1})| \\
& \quad + \sqrt{\pi T} \sup_{(x,t)\in\mathbf{R}\times[0,T]} |a(g;\varepsilon,x,t)R_{\tilde{u}_2}| \Bigg).
\end{aligned}
$$

By Lemma 4.1, Proposition 4.2 and the hypothesis that $\tilde{u}_i$ is of bounded type for $i = 1, 2$, it follows that a is uniformly bounded for sufficiently small $\varepsilon > 0$. Therefore, there exists $C > 0$ such that for sufficiently small $\varepsilon > 0$

$$a_2^2 \le \frac{C}{\pi}\left(\frac{1}{R_{\tilde{\mu}}(\varepsilon)} + \frac{1}{\sqrt{R_{\tilde{\mu}}(\varepsilon)}} + 1\right).$$

Consequently, we obtain that for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}\exp(\pi a_2^2 T) &\le \exp\left(CT\left(\frac{1}{R_{\tilde{\mu}}(\varepsilon)} + \frac{1}{\sqrt{R_{\tilde{\mu}}(\varepsilon)}} + 1\right)\right)\\ &\le \exp\left(CT\left(\log\frac{1}{\varepsilon} + \sqrt{\log\frac{1}{\varepsilon}} + 1\right)\right)\\ &\le \exp\left(CT\left(2\log\frac{1}{\varepsilon} + 1\right)\right) = \left(\frac{1}{\varepsilon}\right)^{2CT} e^{CT},\end{aligned}$$

which means that for all $q \in \mathbf{N}$, there exist $c > 0$ and $\eta > 0$ such that

$$\sup_{(x,t)\in\mathbf{R}\times[0,T]} |R_{\tilde{u}}(\varepsilon, x, t)| \le c\varepsilon^q$$

for each $0 < \varepsilon < \eta$. Similarly we can prove the same type of estimate for any derivative of $R_{\tilde{u}}$ with respect to x and t. Hence $R_{\tilde{u}}(\varepsilon, x, t) \in \mathcal{N}_{s,g}[\mathbf{R} \times [0, T]]$, that is, $\tilde{u}_1 - \tilde{u}_2 = 0$ in $\mathcal{G}_{s,g}(\mathbf{R} \times [0, T])$. Thus the assertion follows. □

5. Relationship to classical solutions

In [5], Lax introduced the following pseudo-norm, defined for locally integrable functions f on $\mathbf{R}$, but possibly infinite:

$$|f|_* = \sup_{y\in\mathbf{R}}\left|\int_0^y f(x)dx\right|.$$

We will investigate the relationship between generalized solutions and classical solutions by using this pseudo-norm. We first prove Lemma 5.1 and Proposition 5.2.

Lemma 5.1. *Let* $\mu > 0$, $u_0 \in L^\infty(\mathbf{R})$ *and* u *be the solution constructed in Lemma* 3.4 *of the problem*

$$\begin{cases} u_t + (F(x,t,u))_x = \mu u_{xx}, \\ u|_{t=0} = u_0 \quad \text{in } C_0'(\mathbf{R}), \end{cases}$$

where F *is an element of* $C^\infty(\mathbf{R}^3)$ *such that for any* $\alpha \in \mathbf{N}^3$ *there exist* $c > 0$ *and* $r \in \mathbf{N}$ *such that for all* $(x,t,u) \in \mathbf{R}^3$

$$|D^\alpha F(x,t,u)| \le c(1 + |u|)^r, \tag{5.1}$$

and for all $(x,t,u) \in \mathbf{R}^3$

$$u \cdot F_x(x,t,u) \geq 0. \tag{5.2}$$

Then $\int_0^x u(\xi,t)d\xi$ *converges to* $\int_0^x u_0(\xi)d\xi$ *as* t *tends to* 0.

PROOF. It suffices to prove the case where $x > 0$. Let $\varphi_\varepsilon(\xi)$ be a positive continuous function on $\mathbf{R}$ such that for each $0 < \varepsilon < x$

$$\varphi_\varepsilon(\xi) = \begin{cases} 1 & \text{for } \dfrac{\varepsilon}{2} \leq \xi \leq x - \dfrac{\varepsilon}{2}, \\ 0 & \text{for } \xi \leq 0,\ x \leq \xi, \end{cases}$$

and $\varphi_\varepsilon \leq 1$. Let $L = \sup_{(x,t) \in \mathbf{R}\times[0,T]} |u(x,t)|$. For fixed $x > 0$ we have

$$\left|\int_0^x u(\xi,t)d\xi - \int_0^x u(\xi,t)\varphi_\varepsilon(\xi)d\xi\right| = \left|\int_0^x u(\xi,t)(1-\varphi_\varepsilon(\xi))d\xi\right| \leq L\varepsilon.$$

Similarly,

$$\left|\int_0^x u_0(\xi)d\xi - \int_0^x u_0(\xi)\varphi_\varepsilon(\xi)d\xi\right| \leq L\varepsilon.$$

Furthermore from the hypothesis, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $0 < t < \delta$,

$$\left|\int_0^x u(\xi,t)\varphi_\varepsilon(\xi)d\xi - \int_0^x u_0(\xi)\varphi_\varepsilon(\xi)d\xi\right| \leq \varepsilon.$$

Consequently, we obtain that

$$\left|\int_0^x u(\xi,t)d\xi - \int_0^x u_0(\xi)d\xi\right| \leq \varepsilon(1+2L).$$

Thus the assertion follows. □

PROPOSITION 5.2. *Assume that* F *satisfies* (5.1) *and* (5.2). *Let* $\mu > 0$, $u_{0,i} \in L^\infty(\mathbf{R})$ *and* u_i *be the solution constructed in Lemma* 3.4 *of the problem*

$$\begin{cases} u_t + (F(x,t,u))_x = \mu u_{xx}, \\ u|_{t=0} = u_{0,i} \quad \text{in } C_0'(\mathbf{R}) \end{cases}$$

for $i = 1,2$ *and assume that* $|u_{0,1} - u_{0,2}|_*$ *is finite. Then for each* $T > 0$,

$$\sup_{0 \leq t \leq T} |u_1(\cdot,t) - u_2(\cdot,t)|_* \leq 2|u_{0,1} - u_{0,2}|_*.$$

PROOF. Let

$$U_i(x,t) = \int_0^x u_i(\xi,t)d\xi + h_i(t),$$

where for $i = 1, 2$,

$$h_i'(t) = \mu(u_i)_x(0,t) - F(0,t,u_i(0,t)), \qquad h_i(0) = 0.$$

Then U_i satisfies

$$\begin{cases} (U_i)_t + F(x,t,(U_i)_x) = \mu(U_i)_{xx}, \\ U_i|_{t=0} = \int_0^x u_{0,i}(\xi)d\xi. \end{cases}$$

Thus letting $W = U_1 - U_2$, we have

$$\begin{cases} W_t + F_u(x,t,u_3)W_x = \mu W_{xx}, \\ W|_{t=0} = \int_0^x [u_{0,1}(\xi) - u_{0,2}(\xi)]d\xi, \end{cases}$$

where u_3 denotes a value between u_1 and u_2. It follows from Lemma 5.1 that the function W is continuous up to $t = 0$. Hence by the Maximum Principle we obtain the inequality

$$\sup_{(x,t)\in\mathbf{R}\times[0,T]} |W(x,t)| \le \sup_{x\in\mathbf{R}} |W(x,0)|.$$

Since

$$|u_1(\cdot,t) - u_2(\cdot,t)|_* \le \sup_{x\in\mathbf{R}} |W(x,t)| + |W(0,t)|,$$

the assertion follows. □

Theorem 5.3. *Assume that F satisfies* (5.1), (5.2) *and that for all u and $(x,t)\in\mathbf{R}\times[0,T]$ $F_{uu}\ge 0$, and that there exist positive numbers τ and λ such that $F_{uu}\ge\lambda>0$ for bounded u and $0\le t\le\tau$. Let $u_0\in L^\infty(\mathbf{R})$ and u be the weak entropy solution of the problem*

$$\begin{cases} u_t + (F(x,t,u))_x = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{5.3}$$

Furthermore, let $\tilde\mu$ be as in Theorem 4.4 *and $\tilde\mu\approx 0$. Finally, let $\tilde v\in\mathscr{G}_{s,g}(\mathbf{R}\times[0,T])$ be the solution of the problem*

$$\begin{cases} \tilde v_t + (F(x,t,\tilde v))_x = \tilde\mu\tilde v_{xx}, \\ \tilde v|_{t=0} = \tilde u_0, \end{cases} \tag{5.4}$$

where $\tilde u_0 = u_0$ in $\mathscr{G}_{s,g}(\mathbf{R})$. Then we have $\tilde v\approx u$.

Lemma 5.4 ([2], Lemma 3.4). *Let $u_0 \in L^\infty(\mathbf{R})$ and ρ be as in Remark 2.3. Then it follows that $|u_0 - u_0 * \rho_\varepsilon|_*$ converges to 0 as ε tends to 0.*

Proof of Theorem 5.3. Let $R_{\tilde{v}}$ be a representative of $\tilde{v}$ satisfying

$$\begin{cases} (R_{\tilde{v}})_t + (F(x,t,R_{\tilde{v}}))_x = R_{\tilde{\mu}}(R_{\tilde{v}})_{xx}, \\ R_{\tilde{v}}|_{t=0} = R_{\tilde{u}_0} = u_0 * \rho_\varepsilon. \end{cases}$$

Applying Proposition 5.2 with $\mu = R_{\tilde{\mu}}(\varepsilon)$, where $R_{\tilde{\mu}}$ is a representative of $\tilde{\mu}$, $u_{0,1} = u_0$, $u_{0,2} = u_0 * \rho_\varepsilon$, $u_1 = u^\varepsilon$, $u_2 = R_{\tilde{v}}$ for fixed ε, we obtain that

$$\sup_{0 \le t \le T} |u^\varepsilon(\cdot, t) - R_{\tilde{v}}(\varepsilon, \cdot, t)|_* \le 2|u_0 - u_0 * \rho_\varepsilon|_*.$$

By Lemma 5.4 and the fact that u^ε converges to the weak entropy solution u in $\mathscr{D}'(\mathbf{R} \times (0,T))$ ([6], Theorem 8), the assertion follows. □

Theorem 5.5. *Let f and g be functions satisfying the conditions given in Theorem 3.1. Let μ be a fixed positive real number and u be the solution constructed in Lemma 3.4 of the problem*

$$\begin{cases} u_t + f(x,t,u)u_x + g(x,t,u)u = \mu u_{xx}, \\ u|_{t=0} = u_0 \quad \text{in } C_0'(\mathbf{R}). \end{cases} \tag{5.5}$$

Furthermore, let $\tilde{v} \in \mathscr{G}_{s,g}(\mathbf{R} \times [0,T])$ be the solution of the problem

$$\begin{cases} \tilde{v}_t + f(x,t,\tilde{v})\tilde{v}_x + g(x,t,\tilde{v})\tilde{v} = \mu \tilde{v}_{xx}, \\ \tilde{v}|_{t=0} = \tilde{u}_0, \end{cases}$$

where $\tilde{u}_0 = u_0$ in $\mathscr{G}_{s,g}(\mathbf{R})$, then we have $\tilde{v} \approx u$.

Proof. From the hypotheses on f and g, it follows that $R_{\tilde{v}}$ is uniformly bounded in $C^{2+\alpha, 1+\alpha}(K)$ for some $\alpha > 0$ and every compact subset K of $\mathbf{R} \times (0,T)$. Therefore, we can pass to the limit $\varepsilon \to 0$ along a subsequence and obtain a function $u(x,t)$ in the same space, which is a classical solution of (5.5) in $\mathbf{R} \times (0,T)$. From the uniqueness of a solution of (5.5) we obtain that $R_{\tilde{v}}$ converges to u, that is, $\tilde{v}$ is associated with u. □

Remark 5.6. Let $F(x,t,u) = u^2/2$. Then $F_u(x,t,u) = u$, $F_{uu}(x,t,u) = 1$ and $F_x(x,t,u) = 0$. Hence, it is seen that our results include the ones in Biagioni and Oberguggenberger [2].

Remark 5.7. It is well-known (Oleĭnik [6]) that problem (5.3) has the unique weak entropy solution under some condition on F. In Theorem 5.3 we need, in addition to the condition on F as in Oleĭnik [6], only the requirement that F is smooth and polynomially bounded, together with all derivatives, to obtain the result that the generalized solution of (5.4) is associated with the weak entropy solution of (5.3).

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