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Asymptotic behavior of solutions of a class of second order quasilinear ordinary differential equations

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Abstract. We study asymptotic behavior of solutions of a class of second order quasilinear ordinary differential equations. All solutions are classified into six types by means of their asymptotic behavior. Necessary and/or sufficient conditions are given for such equations to possess a solution of each of the six types.

0. Introduction

In this paper we consider second order quasilinear ordinary differential equations of the form

$$(|y'|^{\alpha-1}y')' = p(t)|y|^{\beta-1}y, \tag{A}$$

where we always assume that

$$\begin{cases} \alpha, \beta > 0 \text{ are constants;} \\ p \in C[t_0, \infty), \text{ and } p(t) > 0 \text{ on } [t_0, \infty), t_0 > 0. \end{cases}$$

For simplicity we often employ the notation

$$x^{\gamma *} = |x|^{\gamma-1}x = |x|^{\gamma} \operatorname{sgn} x, \qquad x \in \mathbf{R}, \gamma > 0,$$

in terms of which (A) can be expressed as

$$((y')^{\alpha *})' = p(t)y^{\beta *}.$$

By a solution of (A) on an interval $J \subset [t_0, \infty)$ we mean a function $y : J \to \mathbf{R}$ which is of class C^1 together with $|y'|^{\alpha-1}y'$, and satisfies (A) at every point of J.

We here call equation (A) *super-homogeneous* or *sub-homogeneous* according as $\alpha < \beta$ or $\alpha > \beta$. If $\alpha = \beta$, (A) is often called *half-linear*. In this paper our attention is mainly paid to the super-homogeneous and sub-homogeneous cases, and the half-linear case is almost excluded from our consideration.

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The purpose of the paper is to investigate the structure of the set of solutions of (A). If a solution y of (A) exists on an interval of the form $[t_1, \infty), t_1 \geq t_0$, and is eventually nontrivial, then it is called *proper*. A nontrivial solution which is not proper is called *singular*. Further, a singular solution y is classified into two types. One type consists of solutions existing on finite intervals of the form $[t_1, t_2)$, $t_1 < t_2 < \infty$. The other type consists of solutions y such that they exist on infinite intervals of the form $[t_1, \infty)$, $t_1 = t_1(y)$, and $y \not\equiv 0$ on $[t_1, \infty)$, but $y \equiv 0$ near $+\infty$. Proper solutions are classified according to their asymptotic behavior as $t \to \infty$, and it will be shown below that each proper solution satisfies one of four different features. Thus every solution of (A) is classified into six types. In this paper we present necessary and/or sufficient conditions for the existence of solutions of each of the six types, and clarify the structure of the set of solutions of (A).

If $\alpha = 1$, then equation (A) reduces to the Emden-Fowler equation

$$\text{(B)} \qquad y'' = p(t)|y|^{\beta-1}y.$$

In this case, asymptotic theory for solutions was discussed in detail, for example, by Kiguradze [3], Taliaferro [6], and Wong [7] for the superlinear case $\beta > 1$, and by Chanturiya [2] for the sublinear case $0 < \beta < 1$. The results in the present paper give an extension of the results for the Emden-Fowler equation (B).

Some parts of the results in this paper can be obtained also from those of Mirzov [5] which is concerned with the first order Emden-Fowler system

$$\text{(C)} \qquad \begin{cases} u_1' = p_1(t)|u_2|^{\lambda_1-1}u_2, \\ u_2' = p_2(t)|u_1|^{\lambda_2-1}u_1. \end{cases}$$

For the above system (C), it is assumed that $\lambda_i > 0$, $p_i \in C[t_0, \infty)$ and $p_i(t) > 0$, $t \geq t_0$, $i = 1, 2$. It is clear that, for a solution y of (A), $(u_1, u_2) = (y, (y')^{\alpha*})$ is a solution of (C) with $\lambda_1 = 1/\alpha$, $\lambda_2 = \beta$, $p_1 \equiv 1$ and $p_2 \equiv p$. Conversely, let (u_1, u_2) be a solution of (C). If $\int^{\infty} p_1(t)dt = \infty$ (which is essentially assumed in [5]), then $v(\tau) = u_1(t)$ with $\tau = \int_{t_0}^{t} p_1(s)ds$ satisfies the equation

$$(\hat{\text{C}}) \qquad (|\dot{v}|^{\delta-1}\dot{v})^{\cdot} = q(\tau)|v|^{\lambda_2-1}v,$$

where $\delta = 1/\lambda_1$, $q(\tau) = p_2(t)/p_1(t)$ and $\cdot = d/d\tau$. Note that the t-interval $[t_0, \infty)$ for (C) corresponds to the τ-interval $[0, \infty)$ for $(\hat{\text{C}})$. Therefore equation $(\hat{\text{C}})$ is of the form (A). In this sense, equation (A) and system (C) are the same.

The results for (A) may be obtained through the results for (C). But it is certain that direct consideration for (A) is easier to handle and gives a definite theory. Thus in this paper we discuss equation (A) directly. As stated above, some of our results can be obtained also from those of Mirzov [5], but the proofs of the corresponding results in [5] are different from those in the present paper.

The organization of the paper is as follows. In Section 1 we give a classification of all (local) solutions of (A). It will be found that all solutions of (A) are classified into six types in our context. In Sections 2 and 3, we state the main results for the super-homogeneous case and for the sub-homogeneous case, respectively. The proofs of all theorems in Sections 2 and 3 are given in Sections 6 and 7, respectively. In Section 4 we give basic results mainly concerning local properties of solutions of (A). Section 5 is devoted to constructing nonnegative solutions y of (A) with the particular property that $y(t) \geq 0$, $y'(t) \leq 0$ near ∞. Such solutions are called nonnegative nonincreasing solutions, and play important roles to prove our main results in Sections 6 and 7.

1. The classification of all solutions of (A)

To classify all solutions of (A) we need the following simple lemma.

Lemma 1.1. *Let y be a local solution of* (A) *near $t = T \geq t_0$, and $[T, \omega)$, $\omega \leq \infty$, be its right maximal interval of existence. Then we have either $y(t) \geq 0$ near ω, or $y(t) \leq 0$ near ω. That is, y does not change strictly its sign infinitely many times as $t \uparrow \omega$.*

The classification of all (local) solutions of (A) are given on the basis of Lemma 1.1. Since the proof is easy, we leave it to the reader.

Proposition 1.2. *Each local solution $y \not\equiv 0$ of* (A) *falls into exactly one of the following six types:*

(i) *(singular solution of the first kind; type* (S_1)*) there exists $t_1 \geq t_0$ such that*

$$y \not\equiv 0 \quad \textit{for } t \leq t_1, \qquad \textit{and} \qquad y \equiv 0 \quad \textit{for } t \geq t_1;$$

(ii) *(decaying solution; type* (D)*) y can be continued to ∞, and satisfies $y(t)y'(t) < 0$ for all large t, and*

$$\lim_{t\to\infty} y(t) = 0;$$

(iii) *(asymptotically constant solution; type* (AC)*) y can be continued to ∞, and satisfies $y(t)y'(t) < 0$ for all large t, and*

$$\lim_{t\to\infty} y(t) \in \mathbf{R}\backslash\{0\};$$

(iv) (*asymptotically linear solution; type* (AL)) *y can be continued to* ∞, *and satisfies* $y(t)y'(t) > 0$ *for all large* t, *and*

$$\lim_{t\to\infty} \frac{y(t)}{t} \in \mathbf{R}\backslash\{0\};$$

(v) (*asymptotically superlinear solution; type* (AS)) *y can be continued to* ∞, *and satisfies* $y(t)y'(t) > 0$ *for all large* t, *and*

$$\lim_{t\to\infty} \frac{y(t)}{t} = \pm\infty;$$

(vi) (*singular solution of the second kind; type* (S_2)) *y has the finite escape time; that is, there exists* $t_1 > t_0$ *such that*

$$\lim_{t\to t_1-0} y(t) = \pm\infty.$$

2. Main results for the super-homogeneous equation

Below we list our main results for the case of $\alpha < \beta$. Throughout this section we assume that $\alpha < \beta$.

THEOREM 2.1. *Equation* (A) *has no solutions of type* (S_1).

THEOREM 2.2. *Equation* (A) *has a solution of type* (D) *if and only if*

$$\int^{\infty}\left(\int_t^{\infty} p(s)ds\right)^{1/\alpha} dt = \infty. \tag{2.1}$$

THEOREM 2.3. *Equation* (A) *has a solution of type* (AC) *if and only if*

$$\int^{\infty}\left(\int_t^{\infty} p(s)ds\right)^{1/\alpha} dt < \infty. \tag{2.2}$$

THEOREM 2.4. *Equation* (A) *has a solution of type* (AL) *if and only if*

$$\int^{\infty} t^{\beta} p(t)dt < \infty. \tag{2.3}$$

THEOREM 2.5. *Equation* (A) *has a solution of type* (AS) *if* (2.3) *holds.*

THEOREM 2.6. *Equation* (A) *does not have solutions of type* (AS) *if*

$$\liminf_{t\to\infty} t^{1+\beta} p(t) > 0.$$

THEOREM 2.7. *Equation* (A) *does not have solutions of type* (AS) *if there are constants* $\rho > 0$ *and* $\sigma \in (0,1)$ *satisfying*

$$\liminf_{t\to\infty} t^{\rho} \int_t^{\infty} s^{\beta\sigma+\sigma-\rho-1}[p(s)]^{\sigma} ds > 0 \tag{2.4}$$

and

$$\begin{cases} \beta\sigma + \sigma - \rho - 1 \geq 0, \\ 1 - \sigma - \alpha\sigma - \alpha\rho \geq 0. \end{cases} \tag{2.5}$$

Remark 2.8. The set of all pairs $(\rho, \sigma) \in (0, \infty) \times (0, 1)$ satisfying inequalities (2.5) is not empty. In fact, the pair $(\rho, \sigma) = ((\beta - \alpha)/(\alpha\beta + 2\alpha + 1), (\alpha + 1)/(\alpha\beta + 2\alpha + 1))$ belongs to it.

Theorem 2.9. *Equation* (A) *has solutions of type* (S_2).

Example 2.10. Let $\alpha < \beta$. Consider equation (A) with $p(t) = t^{\sigma}$:

$$(|y'|^{\alpha-1}y')' = t^{\sigma}|y|^{\beta-1}y, \qquad t \geq 1, \qquad \sigma \in \mathbf{R}. \tag{2.6}$$

For this equation we have the following results:

(i) (2.6) has a solution of type (D) if and only if $\sigma \geq -\alpha - 1$ (Theorem 2.2);

(ii) (2.6) has a solution of type (AC) if and only if $\sigma < -\alpha - 1$ (Theorem 2.3);

(iii) (2.6) has a solution of type (AL) if and only if $\sigma < -\beta - 1$ (Theorem 2.4);

(iv) (2.6) has a solution of type (AS) if and only if $\sigma < -\beta - 1$ (Theorems 2.5 and 2.6).

Remark 2.11. Theorems 2.6 and 2.7 have the same conclusion that there are not solutions of type (AS). Generally, Theorem 2.6 is easier to apply than Theorem 2.7. However, Theorem 2.7 is still valid for the case that p is nonnegative. For example, it is found by this extended version of Theorem 2.7 that the equation

$$(|y'|^{\alpha-1}y')' = (1 + \sin t)|y|^{\beta-1}y, \qquad t \geq 1,$$

does not have solutions of type (AS). Theorem 2.6 can not be applied to this equation.

Remark 2.12. Theorems 2.3, 2.4 and 2.6 can be obtained also from the results by Mirzov [5] in which the Emden-Fowler system (C) is considered. Theorems 2.1 and 2.9 are also given in [1].

3. Main results for the sub-homogeneous equation

Below we list our main results for the case of $\alpha > \beta$. Throughout this section we assume that $\alpha > \beta$.

Theorem 3.1. *Equation* (A) *has solutions of type* (S_1).

THEOREM 3.2. *Equation* (A) *has a solution of type* (D) *if*

$$\int^{\infty}\left(\int_{t}^{\infty} p(s)ds\right)^{1/\alpha} dt < \infty. \tag{3.1}$$

THEOREM 3.3. *Equation* (A) *does not have solutions of type* (D) *if*

$$\liminf_{t\to\infty} t^{1+\alpha}p(t) > 0. \tag{3.2}$$

THEOREM 3.4. *Equation* (A) *does not have solutions of type* (D) *if there are constants* $\rho > 0$ *and* $\sigma \in (0,1)$ *satisfying*

$$\liminf_{t\to\infty} t^{\rho\alpha}\int_{t}^{\infty} s^{\sigma+\alpha\sigma-\alpha\rho-1}[p(s)]^{\sigma}ds > 0 \tag{3.3}$$

and

$$\begin{cases} \beta\sigma + \sigma + \rho - 1 \le 0, \\ 1 - \sigma - \alpha\sigma + \alpha\rho \le 0. \end{cases} \tag{3.4}$$

REMARK 3.5. The set of all pairs $(\rho,\sigma) \in (0,\infty)\times(0,1)$ satisfying inequalities (3.4) is not empty. In fact, the pair $(\rho,\sigma) = ((\alpha-\beta)/(\alpha\beta+2\alpha+1), (\alpha+1)/(\alpha\beta+2\alpha+1))$ belongs to it.

THEOREM 3.6. *Equation* (A) *has a solution of type* (AC) *if and only if* (3.1) *holds.*

THEOREM 3.7. *Equation* (A) *has a solution of type* (AL) *if and only if*

$$\int^{\infty} t^{\beta}p(t)dt < \infty.$$

THEOREM 3.8. *Equation* (A) *has a solution of type* (AS) *if and only if*

$$\int^{\infty} t^{\beta}p(t)dt = \infty. \tag{3.5}$$

THEOREM 3.9. *Equation* (A) *has no solutions of type* (S_2).

EXAMPLE 3.10. Let $\alpha > \beta$, and consider equation (2.6) again. We have the following results:

(i) (2.6) has a solution of type (D) if and only if $\sigma < -\alpha - 1$ (Theorems 3.2 and 3.3);

(ii) (2.6) has a solution of type (AC) if and only if $\sigma < -\alpha - 1$ (Theorem 3.6);

(iii) (2.6) has a solution of type (AL) if and only if $\sigma < -\beta - 1$ (Theorem 3.7);

(iv) (2.6) has a solution of type (AS) if and only if $\sigma \geq -\beta - 1$ (Theorem 3.8).

REMARK 3.11. Theorems 3.3, 3.6 and 3.7 can be obtained from the results in [5]. In [1] several results related to Theorems 2.4 and 3.7 are obtained, and Theorems 3.1 and 3.9 are also given.

4. Auxiliary lemmas

In the section we collect auxiliary lemmas, which are mainly concerned with local solutions of (A).

A comparison lemma of the following type is useful, and will be used in many places.

LEMMA 4.1. *Suppose that* $p_1, p_2 \in C[a,b]$ *and* $0 < p_1(t) \leq p_2(t)$ *on* $[a,b]$. *Let* $y_i, i = 1,2,$ *be solutions on* $[a,b]$ *of the equations*

$$(|y_i'|^{\alpha-1}y_i')' = p_i(t)|y_i|^{\beta-1}y_i, \qquad i = 1,2,$$

respectively. If $y_1(a) \leq y_2(a)$ *and* $y_1'(a) < y_2'(a)$, *then*

$$y_1(t) < y_2(t) \qquad \text{and} \qquad y_1'(t) < y_2'(t) \qquad \text{on } (a,b].$$

PROOF. We have

$$[y_i'(t)]^{\alpha*} = [y_i'(a)]^{\alpha*} + \int_a^t p_i(s)[y_i(s)]^{\beta*}ds, \qquad a \leq t \leq b, \ \ i = 1,2; \tag{4.1}$$

$$y_i(t) = y_i(a) + \int_a^t \left([y_i'(a)]^{\alpha*} + \int_a^s p_i(r)[y_i(r)]^{\beta*}dr\right)^{1/\alpha*} ds, \qquad a \leq t \leq b, \ \ i = 1,2. \tag{4.2}$$

By the hypotheses we have $y_1(t) < y_2(t)$ in some right neighborhood of a. If $y_1(t) \geq y_2(t)$ for some point in $(a,b]$, we can find a point $c \in (a,b]$ satisfying

$$y_1(t) < y_2(t) \ \text{ for } \ a < t < c; \qquad \text{and} \qquad y_1(c) = y_2(c).$$

But, this yields a contradiction, because

$$\begin{aligned} 0 = y_1(c) - y_2(c) &= y_1(a) - y_2(a) \\ &+ \int_a^c \left[\left([y_1'(a)]^{\alpha*} + \int_a^t p_1(s)[y_1(s)]^{\beta*}ds\right)^{1/\alpha*}\right. \\ &\qquad \left. - \left([y_2'(a)]^{\alpha*} + \int_a^t p_2(s)[y_2(s)]^{\beta*}ds\right)^{1/\alpha*}\right] dt < 0. \end{aligned}$$

Hence we see that $y_1(t) < y_2(t)$ on $(a,b]$. Returning to (4.1), we find that $y_1'(t) < y_2'(t)$ on $[a,b]$. The proof is complete.

The uniqueness of local solutions with non-zero initial data is essentially proved in [4]. That is, for given T $(\geq t_0), y_0$, and y_1, equation (A) has a unique local solution y satisfying $y(T) = y_0$, $y'(T) = y_1$ provided $|y_0| + |y_1| \neq 0$. The uniqueness of the trivial solution can be concluded for the case $\alpha \leq \beta$:

LEMMA 4.2. *Let $\alpha \leq \beta$ and $T \geq t_0$. If y is a (local) solution of* (A) *satisfying $y(T) = y'(T) = 0$, then $y \equiv 0$ on $[t_0, \infty)$.*

PROOF. Assume the contrary. We may suppose that $y \not\equiv 0$ on $[T, \infty)$. Then, we can find t_1, t_2 $(T \leq t_1 < t_2)$ satisfying $|y(t_1)| + |y'(t_1)| = 0$ and $|y(t)| + |y'(t)| > 0$ on $(t_1, t_2]$. Integrating (A), we obtain

$$y'(t) = \left(\int_{t_1}^{t} p(s)[y(s)]^{\beta*} ds\right)^{1/\alpha*},$$

$$y(t) = \int_{t_1}^{t} \left(\int_{t_1}^{s} p(r)[y(r)]^{\beta*} dr\right)^{1/\alpha*} ds, \qquad t_1 \leq t \leq t_2.$$

We therefore have

$$|y'(t)| \leq \left(\int_{t_1}^{t} p(s)\{|y(s)| + |y'(s)|\}^{\beta} ds\right)^{1/\alpha}, \tag{4.3}$$

$$|y(t)| \leq \int_{t_1}^{t} \left(\int_{t_1}^{s} p(r)\{|y(r)| + |y'(r)|\}^{\beta} dr\right)^{1/\alpha} ds, \qquad t_1 \leq t \leq t_2. \tag{4.4}$$

Put $w(t) = \max_{t_1 \leq \xi \leq t}(|y(\xi)| + |y'(\xi)|)$. We see that $w(t_1) = 0, w(t) > 0$ on $(t_1, t_2]$ and w is nondecreasing. From (4.3) and (4.4) we can get

$$|y'(t)| \leq [w(t)]^{\beta/\alpha} \left(\int_{t_1}^{t} p(s) ds\right)^{1/\alpha},$$

$$|y(t)| \leq [w(t)]^{\beta/\alpha} \int_{t_1}^{t} \left(\int_{t_1}^{s} p(r) dr\right)^{1/\alpha} ds, \qquad t_1 \leq t \leq t_2.$$

Let $t_1 \leq \tau \leq t \leq t_2$. Then from this observation we see that

$$|y'(\tau)| + |y(\tau)| \leq [w(\tau)]^{\beta/\alpha} G(\tau) \leq [w(t)]^{\beta/\alpha} G(t),$$

where

$$G(v) = \left(\int_{t_1}^{v} p(s) ds\right)^{1/\alpha} + \int_{t_1}^{v} \left(\int_{t_1}^{s} p(r) dr\right)^{1/\alpha} ds.$$

Consequently, we have

$$w(t) \le [w(t)]^{\beta/\alpha} G(t), \qquad t_1 \le t \le t_2. \tag{4.5}$$

If $\alpha = \beta$, from (4.5) we have $1 \le G(t)$, $t_1 < t \le t_2$. This is a contradiction because $G(t_1) = 0$. If $\alpha < \beta$, from (4.5) we have $[w(t)]^{-(\beta-\alpha)/\alpha} \le G(t)$, $t_1 < t \le t_2$. This is also a contradiction because $G(t_1 + 0) = w(t_1 + 0) = 0$. The proof is complete.

Next we discuss continuability of local solutions of (A).

Lemma 4.3. *Let $\alpha \ge \beta$. Then, all local solutions of* (A) *can be continued to ∞ and t_0; that is, all solutions of* (A) *exist on the whole interval* $[t_0, \infty)$.

Proof. Let y be a local solution of (A) in a neighborhood of $T \ge t_0$. Suppose the contrary that the right maximal interval of existence of y is of the form $[T, \omega), \omega < \infty$. Then, it is easily seen that $y(\omega - 0) = \pm\infty$. Integrating (A) twice, we have

$$y(t) = c_0 + \int_T^t \left(c_1^{\alpha *} + \int_T^s p(r)[y(r)]^{\beta *} dr \right)^{1/\alpha *} ds, \qquad T \le t < \omega,$$

where $c_0 = y(T)$ and $c_1 = y'(T)$. Accordingly,

$$|y(t)| \le |c_0| + \int_T^t \left(|c_1|^{\alpha} + \int_T^s p(r)|y(r)|^{\beta} dr \right)^{1/\alpha} ds, \qquad T \le t < \omega.$$

Put $z(t) = \max_{T \le \xi \le t} |y(\xi)|$. Then,

$$|y(t)| \le |c_0| + \int_T^t \left(|c_1|^{\alpha} + [z(s)]^{\beta} \int_T^s p(r) dr \right)^{1/\alpha} ds, \qquad T \le t < \omega.$$

Put moreover $u(t) = \max\{|c_1|^{\alpha/\beta}, z(t)\}$. Then, as in the proof of Lemma 4.2, we have

$$z(t) \le |c_0| + \int_T^t H(s)[u(s)]^{\beta/\alpha} ds, \qquad T \le t < \omega, \tag{4.6}$$

where $H(t) = (1 + \int_T^t p(s) ds)^{1/\alpha}$. Since $y(\omega - 0) = \pm\infty$, there is a $\overline{T} \in (T, \omega)$ such that $z(t) \ge |c_1|^{\alpha/\beta}$ on $[\overline{T}, \omega)$. Therefore it follows from (4.6) that

$$u(t) \le |c_0| + \int_T^t H(s)[u(s)]^{\beta/\alpha} ds, \qquad \overline{T} \le t < \omega. \tag{4.7}$$

Let $\alpha = \beta$. Then, using Gronwall's inequality, we see that $u(\omega - 0) < \infty$, which is a contradiction. Next let $\alpha > \beta$. Then, (4.7) implies that

$$u(t) \le |c_0| + [u(t)]^{\beta/\alpha} \int_T^t H(s) ds, \qquad \overline{T} \le t < \omega.$$

Since $\beta/\alpha < 1$, we have $u(\omega - 0) < \infty$. This is a contradiction too. Hence y can be continued to ∞. The continuability to the left end point t_0 is verified in a similar way. The proof is complete.

The following lemma establishes more than is stated in Theorem 2.9. Accordingly the proof of Theorem 2.9 will be omitted.

LEMMA 4.4. *Let $\alpha < \beta$, and $T \geq t_0$ and $c > 0$ be given. Then, there exists an $M = M(T, c) > 0$ such that the right maximal interval of existence of each solution y of* (A) *satisfying $y(T) \geq c$ and $y'(T) \geq M$ is a finite interval $[T, \hat{T})$, $\hat{T} = \hat{T}_y < \infty$, and* $\lim_{t\to\hat{T}-0} y(t) = \infty$.

PROOF. Let $t_1 > T$ be fixed, and put $\min_{T\leq t\leq t_1} p(t) = m > 0$. There is an $M > 0$ satisfying

$$\int_c^{\infty} \left(M^{\alpha+1} + \frac{m(\alpha+1)}{\alpha(\beta+1)} (v^{\beta+1} - c^{\beta+1}) \right)^{-1/(\alpha+1)} dv < t_1 - T.$$

We first claim that the solution z of (A) with the initial condition $z(T) = c$, $z'(T) = M$ does not exist on $[T, t_1]$; that is, z blows up at some $\hat{T} \in (T, t_1]$. To see this suppose the contrary that z exists at least on $[T, t_1]$. By the definition of m, we have

$$((z')^{\alpha})' = p(t) z^{\beta} \geq m z^{\beta}, \qquad T \leq t \leq t_1.$$

This inequality is equivalent to

$$\frac{\alpha}{\alpha+1} ((z')^{\alpha+1})' \geq \frac{m}{\beta+1} (z^{\beta+1})', \qquad T \leq t \leq t_1.$$

An integration yields

$$\frac{\alpha}{\alpha+1} ([z'(t)]^{\alpha+1} - M^{\alpha+1}) \geq \frac{m}{\beta+1} ([z(t)]^{\beta+1} - c^{\beta+1}), \qquad T \leq t \leq t_1,$$

and hence

$$z'(t) \left[M^{\alpha+1} + \frac{m(\alpha+1)}{\alpha(\beta+1)} ([z(t)]^{\beta+1} - c^{\beta+1}) \right]^{-1/(\alpha+1)} \geq 1, \qquad T \leq t \leq t_1.$$

Finally, we integrate the both sides on $[T, t_1]$ to obtain

$$\int_c^{z(t_1)} \left(M^{\alpha+1} + \frac{m(\alpha+1)}{\alpha(\beta+1)} (w^{\beta+1} - c^{\beta+1}) \right)^{-1/(\alpha+1)} dw \geq t_1 - T,$$

which is a contradiction to the choice of M. Hence z must blow up at some $\hat{T} \in (T, t_1]$: $\lim_{t\to\hat{T}-0} z(t) = \infty$.

If $y(T) \ge c$ and $y'(T) \ge M$, then Lemma 4.1 implies that $y(t) \ge z(t)$ on the common interval of existence of y and z, and therefore y blows up at some point before t_1. The proof is complete.

5. Nonnegative nonincreasing solutions

The main objective of the section is to prove the following theorem:

Theorem 5.1. *For each* $y_0 > 0$ *the problem*

$$\begin{cases} (|y'|^{\alpha-1}y')' = p(t)|y|^{\beta-1}y, \\ y(t_0) = y_0 \end{cases}$$

has exactly one solution $\tilde{y}$ *such that* $\tilde{y}$ *is defined on* $[t_0, \infty)$ *and satisfies*

$$\tilde{y}(t) \ge 0, \tilde{y}'(t) \le 0 \qquad \text{for } t \ge t_0. \tag{5.1}$$

Furthermore, if y *is a solution on* $[t_0, \infty)$ *of* (A) *satisfying* $y(t_0) = y_0$ *and* $y'(t_0) > \tilde{y}'(t_0)$ *[resp.* $y'(t_0) < \tilde{y}'(t_0)$*], then*

$$\lim_{t\to\infty} y(t) = \infty \,[\text{resp. } \lim_{t\to\infty} y(t) = -\infty]. \tag{5.2}$$

Remark 5.2. (i) In the case $\alpha \le \beta$, employing Lemma 4.2, we can strengthen (5.1) to the property that

$$\tilde{y}(t) > 0, \tilde{y}'(t) < 0 \qquad \text{for } t \ge t_0.$$

(ii) In the case $\alpha \ge \beta$, all local solutions of (A) can be continued to the whole interval $[t_0, \infty)$ (Lemma 4.3). Hence in this case property (5.2) always holds for all solutions y with $y(t_0) = y_0$ and $y'(t_0) > \tilde{y}'(t_0)$ [resp. $y'(t_0) < \tilde{y}'(t_0)$].

The property of nonnegative nonincreasing solutions $\tilde{y}$ described in Theorem 5.1 will play important roles through the paper. This section is entirely devoted to proving Theorem 5.1. To this end we prepare several lemmas.

Lemma 5.3. *Let* $A, B \in \mathbf{R}$, *and* $f \in C([a,b] \times \mathbf{R})$ *be a bounded function. Then, the two-point boundary value problem*

$$\begin{cases} (|y'|^{\alpha-1}y')' = f(t,y) \quad \text{on } [a,b], \\ y(a) = A, y(b) = B, \end{cases} \tag{5.3}$$

has a solution.

Proof. Let $K > 0$ be a constant such that

$$|f(t,y)| \le K \qquad \text{for } (t,y) \in [a,b] \times \mathbf{R}.$$

We first claim that with each $y \in C[a,b]$ we can associate a unique constant $c(y)$ satisfying

$$\int_a^b \left(c(y) + \int_a^s f(r, y(r)) dr \right)^{1/\alpha *} ds = B - A. \tag{5.4}$$

Further, this $c(y)$ satisfies

$$-K(b-a) + \left(\frac{B-A}{b-a} \right)^{\alpha *} \le c(y) \le K(b-a) + \left(\frac{B-A}{b-a} \right)^{\alpha *}. \tag{5.5}$$

To see this let $y \in C[a,b]$ be fixed, and consider the function

$$I(\lambda) = \int_a^b \left(\lambda + \int_a^s f(r, y(r)) dr \right)^{1/\alpha *} ds, \qquad \lambda \in \mathbf{R}.$$

If $\lambda < -K(b-a) + \left(\dfrac{B-A}{b-a} \right)^{\alpha *}$, then $I(\lambda) < B - A$. Similarly, if $\lambda > K(b-a) + \left(\dfrac{B-A}{b-a} \right)^{\alpha *}$, then $I(\lambda) > B - A$. Since I is a strictly increasing continuous function, there is a unique constant $c(y)$ satisfying $I(c(y)) = B - A$, namely (5.4). Then, (5.5) is clearly satisfied.

By (5.5) we see that there is a constant $M = M(a, b, A, B, K) > 0$ satisfying $|c(y)| \le M$ for all $y \in C[a,b]$. Choose $L > 0$ so large that

$$|A| \le L \qquad \text{and} \qquad (M + K(b-a))^{1/\alpha}(b-a) \le L.$$

Now, we define the set $Y \subset C[a,b]$ and the mapping $\mathscr{F} : Y \to C[a,b]$ by

$$Y = \{ y \in C[a,b] : |y(t)| \le 2L \ \text{for} \ t \in [a,b] \}$$

and

$$\mathscr{F}y(t) = A + \int_a^t \left(c(y) + \int_a^s f(r, y(r)) dr \right)^{1/\alpha *} ds, \qquad a \le t \le b,$$

respectively. Then the boundary value problem (5.3) is equivalent to finding a fixed element of $\mathscr{F}$. We show that $\mathscr{F}$ has a fixed element in Y via the Schauder fixed point theorem.

Let $y \in Y$. then,

$$\begin{aligned}
|\mathscr{F}y(t)| &\le |A| + \int_a^t \left(|c(y)| + \int_a^s |f(r, y(r))| dr \right)^{1/\alpha} ds \\
&\le |A| + \int_a^t (M + K(s-a))^{1/\alpha} ds \\
&\le |A| + (M + K(b-a))^{1/\alpha}(b-a) \\
&\le L + L = 2L, \qquad a \le t \le b.
\end{aligned}$$

Hence, $\mathscr{F}$ maps Y into itself.

Next, to see the continuity of $\mathscr{F}$, assume that $\{y_n\}$ be a sequence converging to $y \in Y$ uniformly on $[a,b]$. We must prove that $\{\mathscr{F}y_n\}$ converges to $\mathscr{F}y$ uniformly on $[a,b]$. As a first step, we show that $\lim_{n\to\infty} c(y_n) = c(y)$. Assume that this is not the case. Then because of the boundedness of $\{c(y_n)\}$ (see (5.5)), there is a subsequence $\{c(y_{n_i})\}$ satisfying $c(y_{n_i}) \to \tilde{c} \neq c(y)$ for some finite value $\tilde{c}$. Noting the relation

$$\int_a^b \left(c(y_n) + \int_a^s f(r, y_n(r))dr \right)^{1/\alpha*} ds = B - A,$$

we have

$$\begin{aligned} B - A &= \lim_{n_i\to\infty} \int_a^b \left(c(y_{n_i}) + \int_a^s f(r, y_{n_i}(r))dr \right)^{1/\alpha*} ds \\ &= \int_a^b \left(\tilde{c} + \int_a^s f(r, y(r))dr \right)^{1/\alpha*} ds. \end{aligned}$$

This contradicts the uniqueness of the number $c(y)$. Hence $\lim_{n\to\infty} c(y_n) = c(y)$. Then we find similarly that $\lim_{n\to\infty} \mathscr{F}y_n(t) = \mathscr{F}y(t)$ uniformly on $[a,b]$.

It will be easily seen that the sets $\mathscr{F}Y = \{\mathscr{F}y : y \in Y\}$ and $\{(\mathscr{F}y)' : y \in Y\}$ are uniformly bounded on $[a,b]$. Thus, $\overline{\mathscr{F}Y}$ is compact in $C[a,b]$.

From the above observations we see that $\mathscr{F}$ has a fixed element y in Y. That this fixed element is a solution of boundary value problem (5.3) is easily proved. The proof is complete.

Lemma 5.4. *Let $t_1 > t_0$ and $y_0 > 0$. Then, the two-point boundary value problem*

$$\begin{cases} (|y'|^{\alpha-1}y')' = p(t)|y|^{\beta-1}y \quad \text{on } [t_0, t_1], \\ y(t_0) = y_0, \, y(t_1) = 0, \end{cases} \tag{5.6}$$

has a solution y such that $y(t) \geq 0$ and $y'(t) \leq 0$ on $[t_0, t_1]$.

Proof. Define the bounded function $f \in C([t_0, t_1] \times \mathbf{R})$ by

$$f(t, y) = \begin{cases} p(t)y_0^\beta & \text{for } t_0 \leq t \leq t_1, \; y \geq y_0; \\ p(t)y^\beta & \text{for } t_0 \leq t \leq t_1, \; 0 \leq y \leq y_0; \\ 0 & \text{for } t_0 \leq t \leq t_1, \; y \leq 0. \end{cases}$$

By Lemma 5.3, the BVP

$$\begin{cases} (|y'|^{\alpha-1}y')' = f(t, y) \quad \text{on } [t_0, t_1], \\ y(t_0) = y_0, \, y(t_1) = 0, \end{cases}$$

has a solution y.

We show that y satisfies $y(t) \geq 0$ on $[t_0, t_1]$. If this is not the case, we can find an interval $[\tau_0, \tau_1] \subset [t_0, t_1]$ such that

$$y(t) < 0 \text{ on } (\tau_0, \tau_1), \qquad \text{and} \qquad y(\tau_0) = y(\tau_1) = 0.$$

The definition of f implies that y satisfies the equation $(|y'|^{\alpha-1}y')' = 0$ on $[\tau_0, \tau_1]$. Hence $y(t)$ is a linear function on $[\tau_0, \tau_1]$. Obviously this is a contradiction. We see therefore that $y(t) \geq 0$ on $[t_0, t_1]$.

Since $y'(t_1) \leq 0$ and $(|y'|^{\alpha-1}y')' \geq 0$ on $[t_0, t_1]$ by the definition of f, we find that $y'(t) \leq 0$ on $[t_0, t_1]$. Hence $y(t) \leq y_0$, which implies that y is a desired solution of problem (5.6). The proof is complete.

PROOF OF THEOREM 5.1. The uniqueness of $\tilde{y}$ satisfying the properties mentioned here is easily established as in the proof of Lemma 4.1. Therefore we prove only the existence of such a $\tilde{y}$.

By Lemma 5.4, for each $n \in \mathbf{N}$, we have a solution $y = y_n$ of the BVP

$$\begin{cases} (|y'|^{\alpha-1}y')' = p(t)|y|^{\beta-1}y \quad \text{on } [t_0, t_0 + n], \\ y(t_0) = y_0, y(t_0 + n) = 0, \end{cases}$$

satisfying

$$y_n(t) \geq 0 \qquad \text{and} \qquad y_n'(t) \leq 0 \qquad \text{for } t_0 \leq t \leq t_0 + n.$$

Let us extend each y_n over the interval $[t_0, \infty)$ by defining $y_n \equiv 0$ for $t \geq t_0 + n$. Below we show that $\{y_n\}$ contains a subsequence converging to a desired solution of (A).

As a first step, we prove that

$$y_1'(t_0) \leq y_2'(t_0) \leq \cdots \leq y_n'(t_0) \leq 0. \tag{5.7}$$

In fact, if this is not the case, then $y_i'(t_0) > y_{i+1}'(t_0)$ for some i. Since $y_i(t_0) = y_{i+1}(t_0)$, Lemma 4.1 implies that $y_i(t) > y_{i+1}(t)$ on $[t_0, t_0 + i]$. Putting $t = t_0 + i$, we have $0 = y_i(t_0 + i) > y_{i+1}(t_0 + i) \geq 0$, a contradiction. Accordingly, (5.7) holds, and so $\lim_{n\to\infty} y_n'(t_0) = l \in (-\infty, 0]$ exists. Since

$$0 \leq y_n(t) \leq y_0 \qquad \text{on } [t_0, t_0 + n] \text{ for any } n \in \mathbf{N},$$

$\{y_n\}$ is uniformly bounded on each compact subinterval of $[t_0, \infty)$. Noting that $y_n'(t)$ is nondecreasing and nonpositive on $[t_0, t_0 + n]$, we have

$$y_1'(t_0) \leq y_n'(t_0) \leq y_n'(t) \leq 0 \qquad \text{on } [t_0, t_0 + n], \ n \in \mathbf{N}.$$

Hence $\{y_n\}$ is equicontinuous on each compact subinterval of $[t_0, \infty)$. From these considerations we find that there is a subsequence $\{y_{n_i}\} \subset \{y_n\}$ and a function $\tilde{y} \in C[t_0, \infty)$ satisfying $\lim_{n_i\to\infty} y_{n_i}(t) = \tilde{y}(t)$ uniformly on each compact subinterval of $[t_0, \infty)$. Finally, we will show that this $\tilde{y}$ is a desired solution of (A).

Let $t \in [t_0, \infty)$ be fixed arbitrarily. For all sufficiently large n_i's, we have

$$y_{n_i}(t) = y_0 + \int_{t_0}^{t} \left([y'_{n_i}(t_0)]^{\alpha*} + \int_{t_0}^{s} p(r)[y_{n_i}(r)]^{\beta} dr \right)^{1/\alpha*} ds.$$

Letting $n_i \to \infty$, we obtain

$$\tilde{y}(t) = y_0 + \int_{t_0}^{t} \left(l^{\alpha*} + \int_{t_0}^{s} p(r)[\tilde{y}(r)]^{\beta} dr \right)^{1/\alpha*} ds.$$

Differentiating this formula, we see that $\tilde{y}$ solves (A) on $[t_0, \infty)$. That $\tilde{y}$ satisfies (5.1) is evident. The proof of Theorem 5.1 is complete.

Remark 5.6. Theorem 5.1 can be obtained from [5, Theorems 1.1 and 1.2] in which system (C) is discussed. However, our proof presented here is different from that in [5]. Related results are found in [1].

6. Proofs of main results for the super-homogeneous equation

Throughout this section we assume that $\alpha < \beta$.

Proof of Theorem 2.1. The theorem is an immediate consequence of the uniqueness of the trivial solution (Lemma 4.2).

Proof of Theorem 2.3. (*Necessity*) Let y be a positive solution of (A) on $[t_1, \infty)$ of type (AC). It is easy to see that $y'(t) \uparrow 0$ and $y(t) \downarrow y(\infty) > 0$ as $t \uparrow \infty$. Hence integrating (A) twice, we have

$$-y(\infty) + y(t_1) = \int_{t_1}^{\infty} \left(\int_{s}^{\infty} p(r)[y(r)]^{\beta} dr \right)^{1/\alpha} ds,$$

from which we find that

$$[y(\infty)]^{\beta/\alpha} \int_{t_1}^{\infty} \left(\int_{s}^{\infty} p(r) dr \right)^{1/\alpha} ds < \infty.$$

This is equivalent to (2.2).

(*Sufficiency*) Let (2.2) hold. Fix an $l > 0$, and choose $t_1 \geq t_0$ so that

$$\int_{t_1}^{\infty} \left(\int_{s}^{\infty} p(r) dr \right)^{1/\alpha} ds \leq \frac{(2l)^{(\alpha-\beta)/\alpha}}{2}.$$

Define the set $Y \subset C[t_1, \infty)$ and the mapping $\mathscr{F} : Y \to C[t_1, \infty)$ by

$$Y = \{y \in C[t_1, \infty) : l \leq y(t) \leq 2l, t \geq t_1\},$$

and

$$\mathscr{F}y(t) = l + \int_t^\infty \left(\int_s^\infty p(r)[y(r)]^\beta dr \right)^{1/\alpha} ds, \qquad t \geq t_1,$$

respectively. We below show via the Schauder-Tychonoff fixed point theorem that $\mathscr{F}$ has at least one fixed element in Y.

Firstly, let $y \in Y$. Then,

$$\begin{aligned} l \leq \mathscr{F}y(t) &\leq l + (2l)^{\beta/\alpha} \int_{t_1}^\infty \left(\int_s^\infty p(r)dr \right)^{1/\alpha} ds \\ &\leq l + l = 2l, \qquad t \geq t_1. \end{aligned}$$

Thus $\mathscr{F}y \in Y$, and hence $\mathscr{F}Y \subset Y$. Secondly, to see the continuity of $\mathscr{F}$ let $\{y_n\}$ be a sequence in Y converging to $y \in Y$ uniformly on each compact subinterval of $[t_1, \infty)$. Since $p \in L^1(t_1, \infty)$ and

$$0 \leq \int_s^\infty p(r)[y_n(r)]^\beta dr \leq (2l)^\beta \int_s^\infty p(r)dr \in L^{1/\alpha}(t_1, \infty), \qquad n \in \mathbf{N},$$

the Lebesgue dominated convergence theorem implies that $\mathscr{F}y_n \to \mathscr{F}y$ uniformly on each compact subinterval of $[t_1, \infty)$. Since for $y \in Y$

$$|(\mathscr{F}y)'(t)| \leq \left(\int_t^\infty p(s)[y(s)]^\beta ds \right)^{1/\alpha} \leq (2l)^{\beta/\alpha} \left(\int_{t_1}^\infty p(s)ds \right)^{1/\alpha}, \qquad t \geq t_1,$$

the set $\{(\mathscr{F}y)' : y \in Y\}$ is uniformly bounded on $[t_1, \infty)$. This implies that $\overline{\mathscr{F}Y}$ is compact.

From these observations we find that $\mathscr{F}$ has a fixed element y in $Y : \mathscr{F}y = y$. That this y is a solution of (A) of type (AC) is easily proved. The proof is complete.

PROOF OF THEOREM 2.2. (*Sufficiency*) Let $\tilde{y}$ be a solution of (A) satisfying $\tilde{y}(t) > 0, \tilde{y}'(t) < 0$ for $t \geq t_0$. The existence of such a solution is ensured by Theorem 5.1 (and (iii) of Remark 5.2). Obviously, $\tilde{y}$ is either of type (D) or type (AC). Theorem 2.3 shows that, under assumption (2.1), (A) does not possess solutions of type (AC). Hence, $\tilde{y}$ must be of type (D).

(*Necessity*) Let y be a positive solution of (A) for $t \geq t_1$ of type (D). Clearly y satisfies

$$y(t) = \int_t^\infty \left(\int_s^\infty p(r)[y(r)]^\beta dr \right)^{1/\alpha} ds, \qquad t \geq t_1.$$

To verify (2.1), suppose the contrary that (2.1) fails to hold. Then, noting that y is decreasing on $[t_1, \infty)$, we have

$$y(t) \le [y(t)]^{\beta/\alpha} \int_t^\infty \left(\int_s^\infty p(r)dr \right)^{1/\alpha} ds, \qquad t \ge t_1.$$

Accordingly,

$$[y(t)]^{1-(\beta/\alpha)} \le \int_t^\infty \left(\int_s^\infty p(r)dr \right)^{1/\alpha} ds, \qquad t \ge t_1.$$

The left hand side tends to ∞ as $t \to \infty$ because of $\alpha < \beta$; whereas the right hand side tends to 0 as $t \to \infty$. This contradiction verifies (2.1). The proof is complete.

Proof of Theorem 2.4. (*Necessity*) Let y be a positive solution of (A) near ∞ of type (AL). There is a constant $c > 0$ and $t_1 \ge t_0$ satisfying

$$y(t) \ge ct, \qquad t \ge t_1. \tag{6.1}$$

An integration of (A) on $[t_1, t], t \ge t_1$, yields

$$[y'(t)]^{\alpha *} - [y'(t_1)]^{\alpha *} = \int_{t_1}^t p(s)[y(s)]^\beta ds, \qquad t \ge t_1.$$

Since $\lim_{t\to\infty} y'(t) = y'(\infty) \in (0, \infty)$, this equality implies that

$$\int_{t_1}^\infty p(t)[y(t)]^\beta dt < \infty. \tag{6.2}$$

Combining (6.2) with (6.1), we find that (2.3) holds.

(*Sufficiency*) We fix $l > 0$ arbitrarily, and choose $t_1 \ge t_0$ large enough so that

$$\int_{t_1}^\infty t^\beta p(t)dt \le (2l)^{\alpha-\beta}(1 - 2^{-\alpha}).$$

Define the set $Y \subset C[t_1, \infty)$ and the mapping $\mathscr{F} : Y \to C[t_1, \infty)$ by

$$Y = \{ y \in C[t_1, \infty) : l(t - t_1) \le y(t) \le 2l(t - t_1) \text{ for } t \ge t_1\}$$

and

$$\mathscr{F}y(t) = \int_{t_1}^t \left((2l)^\alpha - \int_s^\infty p(r)[y(r)]^\beta dr \right)^{1/\alpha} ds, \qquad t \ge t_1,$$

respectively. As in the proof of the sufficiency part of Theorem 2.3, we can show that $\mathscr{F}$ has a fixed element $y \in Y$ by the Schauder-Tychonoff fixed point theorem:

$$y(t) = \int_{t_1}^{t} \left((2l)^{\alpha} - \int_{s}^{\infty} p(r)[y(r)]^{\beta} dr \right)^{1/\alpha} ds, \qquad t \geq t_1.$$

Differentiating this formula we see that y is a positive solution of (A) on $[t_1, \infty)$. L'Hospital's rule shows that $\lim_{t\to\infty} y(t)/t = 2l$. Thus y is a solution of (A) of type (AL). The proof is complete.

To prove Theorem 2.5 we prepare the following lemma, which gives a refinement of the "if" part of Theorem 2.4.

LEMMA 6.1. *Let* $y_0 > 0$. *If* (2.3) *holds, then there is a positive solution of* (A) *on* $[t_0, \infty)$ *of type* (AL) *satisfying* $y(t_0) = y_0$.

PROOF. By Theorem 2.4, there is an (AL)-type positive solution z of (A) defined in some neighborhood of the infinity: $0 < \lim_{t\to\infty} z(t)/t = \lim_{t\to\infty} z'(t) < \infty$. Let $\tilde{y}$ be a positive solution of (A) on $[t_0, \infty)$ satisfying

$$\tilde{y}(t_0) = y_0 \text{ and } \tilde{y}(t) > 0, \tilde{y}'(t) < 0 \qquad \text{for } t \geq t_0.$$

Take a $t_1 > t_0$ such that $\tilde{y}(t) < z(t)$ and $\tilde{y}'(t) < z'(t)$ for $t \geq t_1$. By Lemma 4.1 if $\lambda > \tilde{y}'(t_0)$ is sufficiently close to $\tilde{y}'(t_0)$, then the solution y of (A) with $y(t_0) = y_0$ and $y'(t_0) = \lambda$ exists at least on $[t_0, t_1]$ and satisfies

$$\tilde{y}(t_1) < y(t_1) < z(t_1), \qquad \tilde{y}'(t_1) < y'(t_1) < z'(t_1).$$

Then Lemma 4.1 again implies that $\tilde{y}(t) < y(t) < z(t)$ as long as $y(t)$ exists. Since $\tilde{y}(t)$ and $z(t)$ exist on $[t_1, \infty)$, this means that $y(t)$ exists on $[t_1, \infty)$ and satisfies $\tilde{y}(t) < y(t) < z(t), t \geq t_1$. Then we have

$$\frac{\tilde{y}(t)}{t} < \frac{y(t)}{t} < \frac{z(t)}{t}, \qquad t \geq t_1.$$

Noting that $\tilde{y}$ is the unique solution of (A) satisfying $\lim_{t\to\infty} \tilde{y}(t)/t = 0$ and passing through the point (t_0, y_0), we have $\lim_{t\to\infty} y(t)/t \in (0, \infty)$. Therefore y is of type (AL). The proof is complete.

PROOF OF THEOREM 2.5. For $\lambda > 0$, we denote by y_λ the unique solution of (A) with the initial condition $y(t_0) = y_0$ and $y'(t_0) = \lambda$. The maximal interval of existence of y_λ may be finite or infinite. Define the set $S \subset (0, \infty)$ by

$$S = \{\lambda > 0 : y_\lambda \text{ exists on } [t_0, \infty), \text{ and is of type (AL)}\}.$$

We know by Lemma 6.1 that $S \neq \phi$, and by Lemma 4.4 that $\lambda \notin S$ for all sufficiently large $\lambda > 0$. Hence $\sup S = \bar{\lambda} \in (0, \infty)$ exists. For $\bar{\lambda}$ there are three possibilities:

(a) $\bar{\lambda} \in S$;
(b) $\bar{\lambda} \notin S$, and $y_{\bar{\lambda}}$ is of type (AS).
(c) $\bar{\lambda} \notin S$, and $y_{\bar{\lambda}}$ is of type (S_2);

To prove the theorem, we below show that case (b) occurs. For simplicity, we write $\bar{y}$ for $y_{\bar{\lambda}}$ below.

Suppose that case (a) occurs. Then, $\lim_{t\to\infty} \bar{y}'(t) = \bar{y}'(\infty) = l \in (0, \infty)$ and $\bar{y}'(t) < l, t \geq t_0$. By condition (2.3), we can find a $t_1 > t_0$ satisfying

$$\int_{t_1}^{\infty} p(s)(y_0 + 2ls)^{\beta} ds < (2l)^{\alpha} - l^{\alpha}.$$

Choose $\lambda > \bar{\lambda}$ close enough to $\bar{\lambda}$ so that y_λ exists at least on $[t_0, t_1]$ and $y_\lambda'(t_1) < l$. Then, for such a λ, y_λ can be extended to ∞, and satisfies $y_\lambda'(t) < 2l, t \geq t_1$. In fact, if this is not the case, there is $\bar{t} > t_1$ satisfying

$$y_\lambda'(t) < 2l \text{ on } [t_0, \bar{t}) \qquad \text{and} \qquad y_\lambda'(\bar{t}) = 2l.$$

It follows therefore that $y_\lambda(t) \leq y_0 + 2lt, t \in [t_0, \bar{t}]$. An integration of (A) (with $y = y_\lambda$) on $[t_1, \bar{t}]$ yields

$$\begin{aligned}
(2l)^{\alpha} = (y_\lambda'(\bar{t}))^{\alpha} &= (y_\lambda'(t_1))^{\alpha} + \int_{t_1}^{\bar{t}} p(s)[y_\lambda(s)]^{\beta} ds \\
&\leq l^{\alpha} + \int_{t_1}^{\bar{t}} p(s)(y_0 + 2ls)^{\beta} ds \\
&\leq l^{\alpha} + \int_{t_1}^{\infty} p(s)(y_0 + 2ls)^{\beta} ds < (2l)^{\alpha}.
\end{aligned}$$

This contradiction implies that y_λ exists on $[t_0, \infty)$ and satisfies $y_\lambda'(t) < 2l$, $t \geq t_0$. These observations show that $S \ni \lambda > \bar{\lambda}$, which contradicts the definition of $\bar{\lambda}$. Hence, case (a) does not occur.

Next, suppose that case (c) occurs. Let $T > t_0$ be the point such that $\bar{y}(T - 0) = \bar{y}'(T - 0) = \infty$. By Lemma 4.4, there is an $M > 0$ such that solutions y of (A) satisfying $y(T) \geq 1$, $y'(T) \geq M$ must blow up at some finite $\hat{T} = \hat{T}(y) \in (T, \infty) : y(\hat{T} - 0) = y'(\hat{T} - 0) = \infty$. For sufficiently small $\varepsilon > 0$ we have $\bar{y}(T - \varepsilon) > 1$, $\bar{y}'(T - \varepsilon) > M$. Thus, if $\lambda < \bar{\lambda}$ is sufficiently close to $\bar{\lambda}$, then y_λ can be continued at least to $T - \varepsilon$, and satisfies $y_\lambda(T - \varepsilon) > 1$, $y_\lambda'(T - \varepsilon) > M$. Then, even though y_λ can be continued to T, y_λ blows up at some finite point by the definition of M. This fact shows that such a λ $(< \bar{\lambda})$ does not belong to S, contradicting the definition of $\bar{\lambda}$, again.

Consequently, case (b) occurs, and hence the proof of Theorem 2.5 is complete.

To prove Theorem 2.6, we prepare the following simple lemma.

LEMMA 6.2. *Let* $k > 0$, *and* $\rho = \rho(k)$ *be the unique positive root of the equation* $\alpha\rho^{\alpha}(\rho - 1) = k$. *Then,*

(i) *the function* $z(t) = Ct^{\rho}$, C *being an arbitrary positive constant, solves the equation*

$$((z')^{\alpha})' = kt^{-\alpha-1}z^{\alpha}, \qquad t \geq t_0.$$

(ii) $\lim_{k\to\infty} \rho(k) = \infty$.

PROOF OF THEOREM 2.6. Let $c_1 > 0$ be a number satisfying

$$p(t) \geq c_1 t^{-1-\beta}, \qquad t \geq t_0. \tag{6.3}$$

The proof is done by contradiction. Suppose that there is a positive solution y of (A) of type (AS).

We can find a sufficiently large $M > 0$ such that the positive root $\rho = \rho_M$ of the equation $\alpha\rho^{\alpha}(\rho - 1) = c_1 M^{\beta-\alpha}$ satisfies

$$-1 - \beta + \frac{\beta - \alpha}{2}\rho \geq 0. \tag{6.4}$$

This is possible by (ii) of Lemma 6.2. Since $\lim_{t\to\infty} y(t)/t = \infty$, there is a $t_1 \geq t_0$ such that y exists on $[t_1, \infty)$, and

$$y'(t) > 0 \text{ and } y(t) \geq Mt \qquad \text{for } t \geq t_1.$$

Lemma 6.2-(i) asserts that, for arbitrary $C > 0, z(t) = z(t; C) = Ct^{\rho}$ solves the equation

$$((z')^{\alpha})' = c_1 M^{\beta-\alpha} t^{-\alpha-1} z^{\alpha}, \qquad t \geq t_1. \tag{6.5}$$

Now, we choose $C > 0$ small enough so that

$$0 < z^{(i)}(t_1) < y^{(i)}(t_1), \qquad i = 0, 1. \tag{6.6}$$

We rewrite (A) in the form

$$((y')^{\alpha})' = p(t)[y(t)]^{\beta-\alpha} y^{\alpha}.$$

Note that the coefficient function of this equation satisfies

$$p(t)[y(t)]^{\beta-\alpha} \geq c_1 t^{-1-\beta}(Mt)^{\beta-\alpha} = c_1 M^{\beta-\alpha} t^{-1-\alpha}, \qquad t \geq t_1.$$

Thus, in view of (6.5), (6.6), and Lemma 4.1, we see that

$$y(t) \geq z(t) \equiv Ct^{\rho}, \qquad t \geq t_1.$$

Accordingly, y satisfies

$$\begin{aligned} ((y')^{\alpha})' &= p(t)[y(t)]^{(\beta-\alpha)/2} y^{(\alpha+\beta)/2} \\ &\geq c_1 t^{-1-\beta}(Ct^{\rho})^{(\beta-\alpha)/2} y^{(\alpha+\beta)/2} \geq c_2 y^{(\alpha+\beta)/2}, \qquad t \geq t_1. \end{aligned}$$

Here $c_2 > 0$ is a constant, and the last inequality follows from (6.4). This inequality is equivalent to

$$\frac{\alpha}{\alpha+1}((y')^{\alpha+1})' \geq \frac{c_2}{\mu+1}(y^{\mu+1})', \qquad \mu = \frac{\alpha+\beta}{2} > \alpha, \qquad t \geq t_1.$$

An integration yields

$$\frac{\alpha}{\alpha+1}([y'(t)]^{\alpha+1} - [y'(t_1)]^{\alpha+1}) \geq \frac{c_2}{\mu+1}([y(t)]^{\mu+1} - [y(t_1)]^{\mu+1}), \qquad t \geq t_1.$$

Since $y(\infty) = \infty$, there are constants $c_3 > 0$ and $t_2 > t_1$ satisfying

$$y'(t) \geq c_3[y(t)]^{(\mu+1)/(\alpha+1)}, \qquad t \geq t_2.$$

Dividing the both sides by $[y(t)]^{(\mu+1)/(\alpha+1)}$, and integrating on $[t_2, t]$, we get

$$\frac{\mu-\alpha}{\alpha+1}[y(t_2)]^{-(\mu-\alpha)/(\alpha+1)} - \frac{\mu-\alpha}{\alpha+1}[y(t)]^{-(\mu-\alpha)/(\alpha+1)} \geq c_3(t-t_2), \qquad t \geq t_2.$$

Letting $t \to \infty$, we have a contradiction. The proof is complete.

Proof of Theorem 2.7. The proof is done by contradiction. Let y be a solution of (A) of type (AS). We suppose that y exists on $[t_1, \infty)$ and satisfies

$$y(t) \geq C_1 t, \ \ y'(t) \geq C_1, \ \ t \geq t_1 \qquad \text{for some } C_1 > 0. \tag{6.7}$$

Put $z = y(y')^{\alpha}$ $(> 0), t \geq t_1$. Then

$$\begin{aligned} z' &= (y')^{\alpha+1} + y\{(y')^{\alpha}\}' = (y')^{\alpha+1} + p(t)y^{\beta+1} \\ &= y(y')^{\alpha}\left(\frac{y'}{y} + p(t)\frac{y^{\beta}}{(y')^{\alpha}}\right) = z\left(\frac{y'}{y} + p(t)\frac{y^{\beta}}{(y')^{\alpha}}\right), \qquad t \geq t_1. \end{aligned}$$

Now, we employ the Young inequality of the form

$$X + Y \geq \sigma^{-\sigma}(1-\sigma)^{-(1-\sigma)}X^{1-\sigma}Y^{\sigma} \qquad \text{for } X, Y \geq 0 \text{ and } 0 < \sigma < 1 \tag{6.8}$$

in the last expression. It follows therefore that

$$z' \geq C_2 z(y')^{1-\sigma-\sigma\alpha}y^{\beta\sigma+\sigma-1}[p(t)]^{\sigma}, \qquad t \geq t_1,$$

where $C_2 = C_2(\sigma, \alpha, \beta) > 0$ is a constant. We rewrite this inequality as

$$z' \geq C_2 y^{\beta\sigma+\sigma-\rho-1}(y')^{1-\sigma-\sigma\alpha-\rho\alpha}[p(t)]^{\sigma}z^{1+\rho}, \qquad t \geq t_1.$$

Noting (6.7) and condition (2.5), we obtain

$$z' \geq C_3 t^{\beta\sigma+\sigma-\rho-1}[p(t)]^{\sigma}z^{1+\rho}, \qquad t \geq t_1,$$

where $C_3 = C_3(\alpha, \beta, \sigma, \rho, C_2, C_1) > 0$ is a constant. Dividing the both sides by $z^{1+\rho}$ and integrating on $[t, \infty)$, we have

$$\frac{1}{\rho}[z(t)]^{-\rho} \geq C_3 \int_t^\infty s^{\beta\sigma+\sigma-\rho-1}[p(s)]^\sigma ds, \qquad t \geq t_1,$$

because $\lim_{t\to\infty} z(t) = \infty$. Consequently, we have

$$\frac{1}{\rho}\left(\frac{t}{y(t)}\right)^\rho [y'(t)]^{-\alpha\rho} \geq C_3 t^\rho \int_t^\infty s^{\beta\sigma+\sigma-\rho-1}[p(s)]^\sigma ds, \qquad t \geq t_1.$$

Letting $t \to \infty$, we get a contradiction to assumption (2.4). This completes the proof.

As was mentioned in Section 4, the proof of Theorem 2.9 is omitted. In fact, a more general result is proved in Lemma 4.4.

7. Proofs of main results for the sub-homogeneous equation

Throughout this section we assume that $\alpha > \beta$.

PROOF OF THEOREM 3.1. Let t_1, t_2 be fixed so that $t_0 \leq t_1 < t_2$, and put

$$m = \min_{t_1 \leq t \leq t_2} p(t) > 0 \qquad \text{and} \qquad \rho = \frac{\alpha+1}{\alpha-\beta} > 0.$$

Then there are constants $L > 0$ and $c > 0$ satisfying

$$L^{\beta/\alpha} \int_{t_1}^{t_2} \left(\int_s^{t_2} p(r)dr\right)^{1/\alpha} ds \leq L,$$

$$\frac{c^{\beta/\alpha} m^{1/\alpha}}{(\rho\beta+1)^{1/\alpha}} \cdot \frac{\alpha}{\alpha+\rho\beta+1} \geq c,$$

and

$$c(t_2 - t)^\rho \leq L \qquad \text{on } [t_1, t_2].$$

Define the closed convex subset Y of $C[t_1, t_2]$ by

$$Y = \{y \in C[t_1, t_2] : c(t_2 - t)^\rho \leq y(t) \leq L \text{ on } [t_1, t_2]\}$$

and the mapping $\mathscr{F} : Y \to C[t_1, t_2]$ by

$$\mathscr{F}y(t) = \int_t^{t_2} \left(\int_s^{t_2} p(r)[y(r)]^\beta dr\right)^{1/\alpha} ds, \qquad t_1 \leq t \leq t_2,$$

respectively. We show that the hypotheses of the Schauder fixed point theorem is satisfied for Y and $\mathscr{F}$.

Let $y \in Y$. Then, obviously $\mathscr{F}y(t) \leq L$ on $[t_1, t_2]$. Moreover,

$$\begin{aligned}\mathscr{F}y(t) &\geq m^{1/\alpha} c^{\beta/\alpha} \int_t^{t_2} \left(\int_s^{t_2} (t_2 - r)^{\beta\rho} dr \right)^{1/\alpha} ds \\ &= \frac{c^{\beta/\alpha} m^{1/\alpha}}{(\rho\beta + 1)^{1/\alpha}} \cdot \frac{\alpha}{\alpha + \rho\beta + 1} (t_2 - t)^{(\alpha+\rho\beta+1)/\alpha} \geq c(t_2 - t)^{\rho}\end{aligned}$$

on $[t_1, t_2]$. Hence $\mathscr{F}Y \subset Y$. The continuity of $\mathscr{F}$ and the boundedness of the sets $\mathscr{F}Y$ and $\{(\mathscr{F}y)' : y \in Y\}$ can be easily established. Accordingly there is a $\hat{y} \in Y$ satisfying $\mathscr{F}\hat{y} = \hat{y}$. By differentiating, we find that $\hat{y}$ is a solution of (A) on $[t_1, t_2]$ and that

$$\hat{y}(t) > 0 \text{ on } [t_1, t_2), \qquad \hat{y}(t_2) = \hat{y}'(t_2) = 0.$$

Now, we put

$$y(t) = \begin{cases} \hat{y}(t) & \text{on } [t_1, t_2], \\ 0 & \text{on } [t_2, \infty). \end{cases}$$

It is easy to see that y is a solution of (A) on $[t_1, \infty)$, and is of type (S_1). The proof is complete.

Theorems 3.6 and 3.7 can be proved exactly as in the proof of Theorems 2.3 and 2.4, respectively. We therefore omit the proofs.

Proof of Theorem 3.2. By our assumption we can find a positive solution $y_n, n \in \mathbf{N}$, of (A) satisfying $y_n(\infty) = 1/n$. Since $\alpha > \beta$, we see by Lemma 4.3 that each y_n exists on the whole interval $[t_0, \infty)$. We show that the sequence $\{y_n\}$ has the limit function y, and it gives rise to a positive solution of (A) of type (D).

We first claim that

$$y_1(t) > y_2(t) > \cdots > y_n(t) > y_{n+1}(t) > \cdots > 0, \qquad t \geq t_0. \tag{7.1}$$

If this is not true, then

$$y_\nu(\bar{t}) = y_{\nu+1}(\bar{t}) \qquad \text{for some } \nu \in \mathbf{N} \text{ and } \bar{t} \in [t_0, \infty).$$

This means however that there are two nonnegative nonincreasing solutions of (A) passing through the point $(\bar{t}, y_\nu(\bar{t}))$. This contradicts to Theorem 5.1. We therefore have (7.1), and so $\lim_{n\to\infty} y_n(t) \equiv y(t)$ exists. Observe that y_n satisfies

$$y_n(t) = \frac{1}{n} + \int_t^\infty \left(\int_s^\infty p(r)[y_n(r)]^\beta dr \right)^{1/\alpha} ds, \qquad t \geq t_0.$$

Letting $n \to \infty$, we obtain via the dominated convergence theorem

$$y(t) = \int_t^\infty \left(\int_s^\infty p(r)[y(r)]^\beta dr \right)^{1/\alpha} ds, \qquad t \geq t_0.$$

We see that y is a nonnegative solution of (A) satisfying $y(\infty) = 0$. It remains to prove that $y(t) > 0$ for $t \geq t_0$.

Fix $T > t_0$ arbitrarily. The proof of Theorem 3.1 implies that there is a solution y_T of (A) satisfying

$$y_T(t) > 0 \text{ on } [t_0, T); \qquad y_T(t) \equiv 0 \text{ on } [T, \infty).$$

We claim that

$$y_n(t) > y_T(t) \text{ on } [t_0, T] \qquad \text{for all } n \in \mathbf{N}. \tag{7.2}$$

In fact, if this fails to hold, then

$$y_\nu(\bar{t}) = y_T(\bar{t}) \qquad \text{for some } \nu \in \mathbf{N} \text{ and } \bar{t} \in [t_0, T).$$

But this means, as before, that there are two nonnegative nonincreasing solution of (A) passing through the point $(\bar{t}, y_T(\bar{t}))$. This contradiction shows that (7.2) holds. Hence by letting $n \to \infty$ in (7.2), we have $y(t) \geq y_T(t) > 0$ on $[t_0, T)$. Since $T > t_0$ is arbitrary, we see that $y(t) > 0$ on $[t_0, \infty)$. The proof is complete.

PROOF OF THEOREM 3.3. The proof is done by contradiction. Let y be a positive solution of (A) on $[t_1, \infty)$ of type (D). Multiplying (A) by $-y' > 0$ and using (3.2), we obtain

$$\alpha(-y')^\alpha y'' \geq C_1 t^{-1-\alpha} y^\beta(-y'), \qquad t \geq t_1,$$

that is,

$$((-y')^{\alpha+1})' \leq C_2 t^{-\alpha-1}(y^{\beta+1})', \qquad t \geq t_1, \tag{7.3}$$

where C_1 and C_2 are positive constants. We fix a $T \geq t_1$ arbitrarily, and consider inequality (7.3) only on the interval $[T, 2T]$ for a moment. An integration of (7.3) on $[t, 2T], T \leq t \leq 2T$, gives

$$\begin{aligned}[-y'(2T)]^{\alpha+1} - [-y'(t)]^{\alpha+1} &\leq C_2 \int_t^{2T} s^{-\alpha-1}([y(s)]^{\beta+1})' ds \\ &\leq C_2 T^{-\alpha-1}([y(2T)]^{\beta+1} - [y(t)]^{\beta+1}), \qquad T \leq t \leq 2T.\end{aligned}$$

Thus

$$[-y'(t)]^{\alpha+1} \geq C_2 T^{-\alpha-1}([y(t)]^{\beta+1} - [y(2T)]^{\beta+1}), \qquad T \leq t \leq 2T,$$

or equivalently,

$$-y'(t) \geq C_3 T^{-1}([y(t)]^{\beta+1} - [y(2T)]^{\beta+1})^{1/(\alpha+1)}, \qquad T \leq t \leq 2T,$$

where $C_3 = C_2^{1/(\alpha+1)}/2$. We therefore have

$$\frac{-y'(t)}{([y(t)]^{\beta+1} - [y(2T)]^{\beta+1})^{1/(\alpha+1)}} \geq C_3 T^{-1}, \qquad T \leq t \leq 2T.$$

Finally, integrating on $[T, 2T]$, we obtain

$$-\int_{y(T)}^{y(2T)} \frac{du}{(u^{\beta+1} - [y(2T)]^{\beta+1})^{1/(\alpha+1)}} \geq C_3, \qquad T \geq t_1,$$

that is,

$$[y(2T)]^{(\alpha-\beta)/(\alpha+1)} \int_1^{y(T)/y(2T)} (v^{\beta+1} - 1)^{-1/(\alpha+1)} dv \geq C_3, \qquad T \geq t_1. \tag{7.4}$$

Noting that

$$\int_1^x (v^{\beta+1} - 1)^{-1/(\alpha+1)} dv = O((x-1)^{(\alpha-\beta)/(\alpha+1)}) \qquad \text{as } x \to \infty,$$

and

$$\int_1^x (v^{\beta+1} - 1)^{-1/(\alpha+1)} dv = O((x-1)^{\alpha/(\alpha+1)}) \qquad \text{as } x \to 1+0,$$

we can find a constant $C > 0$ satisfying

$$\int_1^x (v^{\beta+1} - 1)^{-1/(\alpha+1)} dv \leq C(x-1)^{(\alpha-\beta)/(\alpha+1)}, \qquad x \geq 1.$$

Therefore (7.4) implies that

$$C[y(2T)]^{(\alpha-\beta)/(\alpha+1)} \left(\frac{y(T)}{y(2T)} - 1\right)^{(\alpha-\beta)/(\alpha+1)} \geq C_3, \qquad T \geq t_1,$$

from which we have

$$C[y(T) - y(2T)]^{(\alpha-\beta)/(\alpha+1)} \geq C_3, \qquad T \geq t_1.$$

Letting $T \to \infty$, we have a contradiction. The proof is complete.

PROOF OF THEOREM 3.4. The proof is done by contradiction. Let y be a solution of (A) of type (D), We notice first that

$$\lim_{t\to\infty} ty'(t) = 0. \tag{7.5}$$

In fact, since $y''(t) > 0$, we can compute as follows:

$$\begin{aligned} y(t) &= \int_t^\infty [-y'(s)]ds \geq \int_t^{2t} [-y'(s)]ds \\ &\geq [-y'(2t)] \int_t^{2t} ds = t[-y'(2t)] \geq 0 \qquad \text{for large } t. \end{aligned}$$

Therefore (7.5) holds.

We may suppose that for some $C_1 > 0$ and $t_1 \geq t_0$

$$0 < y(t) \leq C_1, \ \ 0 < -ty'(t) \leq C_1, \ \ t \geq t_1. \tag{7.6}$$

Put $z = y(-y')^\alpha \ (> 0), t \geq t_1$. Then

$$\begin{aligned} -z' &= (-y')^{\alpha+1} - y\{(-y')^\alpha\}' = (-y')^{\alpha+1} + p(t)y^{\beta+1} \\ &= y(-y')^\alpha\left(\frac{-y'}{y} + p(t)\frac{y^\beta}{(-y')^\alpha}\right) = z\left(\frac{-y'}{y} + p(t)\frac{y^\beta}{(-y')^\alpha}\right), \qquad t \geq t_1. \end{aligned}$$

Preceding as in the proof of Theorem 2.7, we obtain

$$-z' \geq C_2 y^{\beta\sigma+\sigma+\rho-1}(-y')^{1-\sigma-\sigma\alpha+\rho\alpha}[p(t)]^\sigma z^{1-\rho}, \qquad t \geq t_1,$$

where $C_2 > 0$ is a constant. We obtain from (7.6) and assumption (3.4)

$$-z' \geq C_3 t^{\sigma+\sigma\alpha-\rho\alpha-1}[p(t)]^\sigma z^{1-\rho}, \qquad t \geq t_1,$$

where $C_3 > 0$ is a constant. Dividing the both sides by $z^{1-\rho}$ and integrating on $[t.\infty)$, we have

$$\frac{1}{\rho}[z(t)]^\rho \geq C_3 \int_t^\infty s^{\sigma+\sigma\alpha-\rho\alpha-1}[p(s)]^\sigma ds, \qquad t \geq t_1,$$

that is,

$$\frac{1}{\rho}[y(t)]^\rho[-ty'(t)]^{\alpha\rho} \geq C_3 t^{\rho\alpha} \int_t^\infty s^{\sigma+\sigma\alpha-\rho\alpha-1}[p(s)]^\sigma ds, \qquad t \geq t_1.$$

Letting $t \to \infty$, we get a contradiction to assumption (3.3) by (7.5). The proof is complete.

Proof of Theorem 3.8. (*Sufficiency*) By Theorem 5.1 and (ii) of Remark 5.2, there is a positive solution y of (A) satisfying $y(\infty) = \infty$. This y is either of type (AL) or of type (AS). But, by Theorem 3.7, we see that y must be of type (AS).

(*Necessity*) Let y be a positive solution of (A) on $[t_1, \infty)$ of type (AS). To prove (3.5), we suppose the contrary that $\int^\infty t^\beta p(t)dt < \infty$. As in the proof of Lemma 4.3, we have

$$|y(t)| \le |c_0| + \int_{t_1}^t \left(|c_1|^\alpha + \int_{t_1}^s p(r)|y(r)|^\beta dr\right)^{1/\alpha} ds, \qquad t \ge t_1,$$

where $c_0 = y(t_1)$ and $c_1 = y'(t_1)$. Let $z(t) = \max_{t_1 \le \xi \le t} |y(\xi)|/\xi$. It follows that

$$\begin{aligned}\frac{|y(t)|}{t} &\le C_2 + \frac{1}{t}\int_{t_1}^t \left(|c_1|^\alpha + [z(s)]^\beta \int_{t_1}^s r^\beta p(r)dr\right)^{1/\alpha} ds \\ &\le C_2 + \left(|c_1|^\alpha + [z(t)]^\beta \int_{t_1}^t r^\beta p(r)dr\right)^{1/\alpha}, \qquad t \ge t_1,\end{aligned}$$

where $C_2 > 0$ is a constant. Put $w(t) = \max\{|c_1|^{\alpha/\beta}, z(t)\}$. We then have

$$z(t) \le C_2 + [w(t)]^{\beta/\alpha}\left(1 + \int_{t_1}^t r^\beta p(r)dr\right)^{1/\alpha}, \qquad t \ge t_1.$$

Since y is of type (AS), $|y(t)|/t$ is unbounded on $[t_1, \infty)$, and so is $z(t)$. Accordingly, there is a $t_2 \ge t_1$ satisfying $w(t) \equiv z(t)$ for $t \ge t_2$. Thus

$$\begin{aligned}w(t) &\le C_2 + [w(t)]^{\beta/\alpha}\left(1 + \int_{t_1}^t r^\beta p(r)dr\right)^{1/\alpha} \\ &\le C_2 + [w(t)]^{\beta/\alpha}\left(1 + \int_{t_1}^\infty r^\beta p(r)dr\right)^{1/\alpha}, \qquad t \ge t_2.\end{aligned}$$

since $\beta/\alpha < 1$, this implies the boundedness of w, which is a contradiction. Hence we must have (3.5). The proof is complete.

Theorem 3.9 is clear because all solutions of (A) with $\alpha > \beta$ exist on the whole interval $[t_0, \infty)$ (see Lemma 4.4).

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