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Extensions and the irreducibilities of the induced characters of cyclic p-groups

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ABSTRACT. Let ϕ be a faithful irreducible character of the cyclic group C_n of order p^n, where p is an odd prime. We study the p-group G containing C_n such that the induced character ϕ^G is also irreducible. The purpose of this paper is to determine the subgroups $N_G(N_G(C_n))$ and $N_G(N_G(N_G(C_n)))$ of G in the case when $[N_G(C_n); C_n] = p$.

1. Introduction

Let G be a finite group. We denote by $\mathrm{Irr}(G)$ the set of complex irreducible characters of G and by $\mathrm{FIrr}(G)$ $(\subset \mathrm{Irr}(G))$ the set of faithful irreducible characters of G.

Let p be a prime. For a non-negative integer n, we denote by C_n the cyclic group of order p^n. A finite group G is called an M-group, if every $\phi \in \mathrm{Irr}(G)$ is induced from a linear character of a subgroup of G.

It is well-known that every nilpotent group is an M-group. Hence, when G is a p-group, for any $\chi \in \mathrm{Irr}(G)$, there exists a subgroup H of G and a linear character ϕ of H such that $\phi^G = \chi$. If we set $N = \mathrm{Ker}\,\phi$, then $N \lhd H$ and ϕ is a faithful irreducible character of $H/N \cong C_n$, for some non-negative integer n. In this paper, we will consider the case when $N = 1$, that is, ϕ is a faithful linear character of $H \cong C_n$.

We consider the following:

PROBLEM 1. *Let p be an odd prime, and ϕ be a faithful irreducible character of C_n. Determine the p-group G such that $C_n \subset G$ and the induced character ϕ^G is also irreducible.*

Since all the faithful irreducible characters of C_n are algebraically conjugate to each other, the irreducibility of ϕ^G $(\phi \in \mathrm{FIrr}(C_n))$ is independent of the choice of ϕ, and depends only on n.

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This problem has been solved in each of the following cases:

(1) $C_n \lhd G$ ([2]),

(2) G has a subgroup H containing C_n such that $C_n \lhd H$ and $[G : H] = p$ ([6]).

On the other hand, when $p = 2$, Yamada and Iida [4] proved the following interesting result:

Let $\mathbf{Q}$ denote the rational field. Let G be a 2-group and χ a complex irreducible character of G. Then there exist subgroups $H \rhd N$ in G and a complex irreducible character ϕ of H such that $\chi = \phi^G$, $\mathbf{Q}(\chi) = \mathbf{Q}(\phi)$, $N = \operatorname{Ker} \phi$ and

$$H/N \cong Q_n \ (n \geq 2), \quad \text{or} \quad D_n \ (n \geq 2), \quad \text{or} \quad SD_n \ (n \geq 3), \quad \text{or} \quad C_n \ (n \geq 0).$$

Here, Q_n, D_n and SD_n denote the generalized quaternion group, the dihedral group of order 2^{n+1} $(n \geq 2)$ and the semidihedral group of order 2^{n+1} $(n \geq 3)$, respectively, and $\mathbf{Q}(\chi) = \mathbf{Q}(\chi(g), g \in G)$.

They considered the following:

PROBLEM 2. *Let ϕ be a faithful irreducible character of H, where $H = Q_n$ or D_n or SD_n. Determine the 2-group G such that $H \subset G$ and the induced character ϕ^G is also irreducible.*

Yamada and Iida [3] solved this problem in the case when $[G; H] = 2$ or 4 and we have solved it when $[G; H] = 8$ ([5]) for all $H = Q_n$ or D_n or SD_n.

Moreover, we have recently solved Problem 2 completely ([7]). In [7], we showed that

$$G = N_G(H) \qquad \text{or} \qquad N_G(N_G(H)),$$

for all $H = Q_n$ or D_n or SD_n, if G satisfies the conditions of Problem 2. Here, as usual, $N_G(H)$ and $N_G(N_G(H))$ are the normalizers of H and $N_G(H)$ in G, respectively. This means that, if we define subgroups of G by

$$M_1 = N_G(H), \qquad \text{and} \qquad M_{i+1} = N_G(M_i), \qquad \text{for } i \geq 1,$$

then

$$H \subset M_1 \subset M_2 = M_3 = M_4 = \cdots = G,$$

for all $H = Q_n$ or D_n or SD_n.

In this paper, we consider Problem 1. We also define subgroups of G by

$$N_1 = N_G(C_n), \qquad \text{and} \qquad N_{i+1} = N_G(N_i), \qquad \text{for } i \geq 1.$$

The purpose of this paper is to determine the groups $N_2 = N_G(N_G(C_n))$ and $N_3 = N_G(N_G(N_G(C_n)))$ in the case where $[N_G(C_n); C_n] = p$ and $[G; C_n] \geq p^3$. As a consequence of the results, we will see that

$$C_n \subsetneqq N_1 \subsetneqq N_2 \subsetneqq N_3,$$

in this case.

Throughout this paper, **Z** and **N** denote the set of rational integers and the natural numbers, respectively. We will frequently use the word "respectively" so it is abbreviated to "resp.".

2. Statements of the results

For the rest of this paper, we assume that p is an odd prime. First, we introduce the following groups:

(i) $G(n,m) = \langle a, b_m \rangle$ with $a^{p^n} = b_m^{p^m} = 1$, $b_m a b_m^{-1} = a^{1+p^{n-m}}$, $(m \le n-1)$.

(ii) $G(n,m,1) = \langle a, b_m, v \rangle$ $(\triangleright G(n,m) = \langle a, b_m \rangle)$ with $a^{p^n} = b_m^{p^m} = 1$, $b_m a b_m^{-1} = a^{1+p^{n-m}}$, $vav^{-1} = a^{1+p^{n-m-1}} b_m^{p^{m-1}}$, $v^p = b_m$, $v b_m v^{-1} = b_m$ $(2m \le n-1)$.

(iii) $G(n,1,1,1) = \langle a, b_1, v, x \rangle$ $(\triangleright G(n,1,1) = \langle a, b_1, v \rangle)$ with $a^{p^n} = b_1^p = 1$, $b_1 a b_1^{-1} = a^{1+p^{n-1}}$, $vav^{-1} = a^{1+p^{n-2}} b_1$, $v^p = b_1$, $v b_1 v^{-1} = b_1$, $xax^{-1} = a^{1+p^{n-3}} v$, $x^p = v$, $xvx^{-1} = v$, $x b_1 x^{-1} = b_1$ $(7 \le n)$.

We can see that $G(n,m,1)$ (resp. $G(n,1,1,1)$) is an extension group of $G(n,m)$ (resp. $G(n,1,1)$) by using Proposition 1 below:

Proposition 1. *Let N be a finite group such that $G \triangleright N$ and $G/N = \langle uN \rangle$ is a cyclic group of order m. Then $u^m = c \in N$. If we put $\sigma(x) = uxu^{-1}$, $x \in N$, then $\sigma \in Aut(N)$ and* (i) *$\sigma^m(x) = cxc^{-1}$, $(x \in N)$* (ii) *$\sigma(c) = c$.*

Conversely, if $\sigma \in Aut(N)$ and $c \in N$ satisfy (i) *and* (ii)*, then there exists one and only one extension group G of N such that $G/N = \langle uN \rangle$ is a cyclic group of order m and $\sigma(x) = vxv^{-1}$ $(x \in N)$ and $v^m = c$.*

Proof. For instance, see [8, III, §7].

Theorem 0.1 (Iida [2]). *Let G be a p-group which contains C_n as a normal subgroup of index p^m. Let $\phi \in \mathrm{FIrr}(C_n)$. Suppose that $\phi^G \in \mathrm{Irr}(G)$. Then $m \le n-1$, and $G \cong G(n,m)$.*

In particular, when $C_n \subset G$ and $[G : C_n] = p$, C_n is always a normal subgroup of G. Hence we have:

Corollary 0.1. *Let $\phi \in \mathrm{FIrr}(C_n)$. Suppose that a group G containing C_n satisfies $[G : C_n] = p$ and $\phi^G \in \mathrm{Irr}(G)$. Then $G \cong G(n,1)$.*

Theorem 0.2 ([6]). *Let G be a p-group which contains C_n, and let $\phi \in \mathrm{FIrr}(C_n)$. Suppose that $[G : C_n] = p^{m+1}$, $\phi^G \in \mathrm{Irr}(G)$, and $n-3 \ge 2m$. Further, suppose that there exists a subgroup H of G such that $H \triangleright C_n$ and $[G : H] = p$. Then*

(1) *$G \cong G(n, m+1)$ if C_n is a normal subgroup of G.*

(2) *$G \cong G(n,m,1)$ if C_n is not a normal subgroup of G.*

Corollary 0.2. *Let G be a p-group which contains C_n and let $\phi \in \mathrm{FIrr}(C_n)$. Suppose that $[G : C_n] = p^2$, $\phi^G \in \mathrm{Irr}(G)$ and $n \geq 5$. Then*

(1) *$G \cong G(n, 2)$ if C_n is a normal subgroup of G.*

(2) *$G \cong G(n, 1, 1)$ if C_n is not a normal subgroup of G.*

Our main theorem is the following:

Theorem. *Let p be an odd prime. Let G be a p-group which contains $C_n = \langle a \rangle$. We assume that $[G : C_n] \geq p^3$. Define the subgroups of G by*

$$N_1 = N_G(C_n), \qquad \text{and} \qquad N_{i+1} = N_G(N_i), \qquad \text{for } i = 1, 2.$$

Let $\phi \in \mathrm{FIrr}(C_n)$ and $7 \leq n$. Suppose that $\phi^G \in \mathrm{Irr}(G)$, and $[N_1 : C_n] = p$. Then

(1) $N_2/N_1 \cong C_1$ and $N_2 \cong G(n, 1, 1)$,

(2) $N_3/N_2 \cong C_1$ and $N_3 \cong G(n, 1, 1, 1)$.

Remark 1. Conversely, it is easy to see that the groups $G(n, 1, 1)$ and $G(n, 1, 1, 1)$ satisfy the condition (EX, C), which is defined in section 3 of this paper. Hence these groups satisfy the conditions of Problem 1.

Remark 2. By results of Iida ([2], see Theorem 0.1. in this paper), we can see that $N_1 \cong G(n, 1)$.

3. Some preleminary results

In this section, we state some results concerning the criterion of the irreducibilities of induced characters and others, which we need in section 4.

We denote by $\zeta = \zeta_{p^n}$ a primitive p^nth root of unity. It is known that, for $C_n = \langle a \rangle$, there are p^n irreducible characters ϕ_v $(1 \leq v \leq p^n)$ of C_n:

$$\phi_v(a^i) = \zeta^{vi}, \qquad (1 \leq i \leq p^n).$$

The irreducible character ϕ_v is faithful if and only if $(v, p) = 1$. It is well-known that

$$\mathrm{Aut}\langle a \rangle \cong (\mathbf{Z}/p^n\mathbf{Z})^* \cong C_* \times C_{n-1}$$

where $(\mathbf{Z}/p^n\mathbf{Z})^*$ is the unit group of the factor ring $\mathbf{Z}/p^n\mathbf{Z}$ and C_* is the cyclic group of order $p - 1$. Further, C_{n-1} is generated by the element $1 + p$ in $\mathbf{Z}/p^n\mathbf{Z}$.

First, we state the following result of Shoda (cf [1, p. 329]):

Proposition 2. *Let G be a group and H be a subgroup of G. Let ϕ be a linear character of H. Then the induced character ϕ^G of G is irreducible if* *and only if, for each* $x \in G - H = \{g \in G \mid g \notin H\}$*, there exists* $h \in xHx^{-1} \cap H$ *such that* $\phi(h) \neq \phi(xhx^{-1})$*.* (*Note that, when* ϕ *is faithful, the condition* $\phi(h) \neq \phi(xhx^{-1})$ *holds if and only if* $h \neq xhx^{-1}$).

Using this result, we have the following:

Proposition 3. *Let* $\langle a\rangle = C_n \subset G$*, and* ϕ *be a faithful irreducible character of* C_n*. Then the following conditions are equivalent:*

(1) ϕ^G *is irreducible,*

(2) *For each* $x \in G - C_n$*, there exists* $y \in \langle a\rangle \cap x\langle a\rangle x^{-1}$ *such that* $xyx^{-1} \neq y$*.*

Definition. When the condition (2) of Proposition 3 holds, we say that G satisfies (EX,C).

Let H be a group. For a normal subgroup N of H, and any $g, h \in H$, we write

$$g \equiv h \qquad (\mathrm{mod}\ N)$$

when $g^{-1}h \in N$. For an element $g \in H$, we denote by $|g|$ the order of g.

4. Proof of Theorem

Let $\phi \in \mathrm{FIrr}(C_n)$. Since $\phi^G = (\phi^{N_1})^G \in \mathrm{Irr}(G)$, we must have $\phi^{N_1} \in \mathrm{Irr}(N_1)$. Therefore, by Corollary 0.1, we can take an element $b_1 \in N_1 - C_n = \{g \in N_1 \mid g \notin C_n\}$ such that

$$N_1 = \langle a, b_1 \mid a^{p^n} = b_1^p = 1, b_1 a b_1^{-1} = a^{1+p^{n-1}}\rangle \cong G(n,1).$$

Proof of (1). Since G is a p-group and $[G : N_1] \geq p^2$, by our assumption, we have

$$N_1 \subsetneqq N_G(N_1) = N_2.$$

Take an element $v \in N_2 - N_1 = \{g \in N_2 \mid g \notin N_1\}$ such that $v^p \in N_1$. Denote by N_1^0 the subgroup of G generated by v and the elements of N_1. Then

$$[N_1^0 : N_1] = p \qquad \text{and} \qquad N_1^0 \rhd N_1.$$

Since C_n is not a normal subgroup of N_1^0, we have

$$N_1^0 = \langle a, b_1, v\rangle \cong G(n,1,1)$$

by Corollary 0.2. Hence we may assume that the elements a, b_1, and v satisfy the following relations:

$$a^{p^n} = b_1^p = 1, \qquad b_1 a b_1^{-1} = a^{1+p^{n-1}}, \qquad v a v^{-1} = a^{1+p^{n-2}} b_1,$$
$$v^p = b_1, \qquad v b_1 v^{-1} = b_1. \tag{I}$$

Hereafter, we write b instead of b_1 for the sake of simplicity.

Remark 1. More precisely, in [6], we have shown that there exist an integer $s_1, (s_1, p) = 1$ and $v \in N_1^0$, such that the elements $a_1 = a^{s_1}, b$, and v satisfy the same relations as (I). Since $\langle a \rangle = \langle a^{s_1} \rangle$, we can take a_1 instead of a, and hence, we may assume that $N_1^0 = \langle a, b, v \rangle$ and a, b and v satisfy the same relations as (I).

To prove the theorem, we need the following:

Lemma 1. *For any integers i, j, the following equalities hold.*
(i) $ab \equiv ba \pmod{\langle a^{p^{n-1}} \rangle}$.
(ii) $b a^p b^{-1} = a^p$.
(iii) $(a^i b^j)^p = a^{ip}$.
(iv) $v a^{p^2} v^{-1} = a^{p^2}$.
(v) $v^j a v^{-j} = a^{1+jp^{n-2}} b^j$.
(vi) $(a^i v^j)^p \equiv a^{pi} v^{pj} = a^{pi} b^j \pmod{\langle a^{p^{n-1}} \rangle}$.
(vii) $(a^i v^j)^{p^2} = a^{p^2 i}$.

Proof of Lemma 1. (i), (ii), (iii) and (iv) can be shown by direct calculations.

(v). Since $n \geq 7$, by our assumption, we have $v a^{p^{n-2}} v^{-1} = a^{p^{n-2}}$, by (iv). Hence, $v^j a v^{-j} = a^{1+jp^{n-2}} b^j$, for any $j \in \mathbf{Z}$.

(vi). By (i) and (v),

$$v^j a^i v^{-j} = (a^{1+jp^{n-2}} b^j)^i \equiv a^{i(1+jp^{n-2})} b^{ij} \qquad \pmod{\langle a^{p^{n-1}} \rangle}.$$

Using this relation repeatedly, we can get

$$\begin{aligned}(a^i v^j)^p &\equiv a^{pi} a^{ijp^{n-2}(1+2+\cdots+(p-1))} b^{ij(1+2+\cdots+(p-1))} v^{pj} \qquad \pmod{\langle a^{p^{n-1}} \rangle} \\ &= a^{pi} a^{ijp^{n-2}(p(p-1)/2)} b^{ij(p(p-1)/2)} v^{pj} \\ &\equiv a^{pi} v^{pj} = a^{pi} b^j \qquad \pmod{\langle a^{p^{n-1}} \rangle},\end{aligned}$$

since p is odd.

(vii) follows from (vi).

The assertion (1) follows from the following

Claim I. $N_1^0 = N_2$.

Proof of Claim I. Suppose that $N_1^0 \subsetneqq N_2$. Take an element $w \in N_2 - N_1^0 = \{g \in N_2 \mid g \notin N_1^0\}$ such that $w^p \in N_1^0$.

Write $waw^{-1} = a^{i_0}b^{j_0}$ for some $i_0, j_0 \in \mathbf{Z}$, $0 \leq i_0 \leq p^n - 1$, $0 \leq j_0 \leq p - 1$. Then

$$wa^p w^{-1} = (a^{i_0}b^{j_0})^p = a^{pi_0}$$

by Lemma 1 (iii). Therefore

$$w^{p^2} a^p w^{-p^2} = a^{pi_0^{p^2}}.$$

Since $w^{p^2} \in N_1$, we must have

$$i_0^{p^2} \equiv 1 \qquad (\mathrm{mod}\ p^{n-1}).$$

So,

$$i_0 \equiv 1 \qquad (\mathrm{mod}\ p^{n-3}).$$

Hence we can write as $i_0 = 1 + k_0 p^{n-3}$, and

$$waw^{-1} = a^{1+k_0p^{n-3}}b^{j_0}$$

for some integer k_0. Since $n - 3 \geq 4$, by our assumption, we have

$$va^{p^{n-3}}v^{-1} = a^{p^{n-3}},$$

by Lemma 1 (iv). Hence

$$v^{p-j_0}waw^{-1}v^{-p+j_0} = v^{p-j_0}a^{1+k_0p^{n-3}}b^{j_0}v^{-p+j_0} = a^{1+k_0p^{n-3}+(p-j_0)p^{n-2}},$$

by Lemma 1 (v). This means that $v^{p-j_0}w \in N_1$, which contradicts the hypothesis that $w \notin N_1^0$. Hence the proof of Claim I is completed.

Proof of (2). Since G is a p-group and $[G : N_2] \geq p$, by our assumption, we have

$$N_2 \subsetneqq N_G(N_2) = N_3.$$

Take an element $y \in N_3 - N_2$ such that $y^p \in N_2$. Denote by N_2^0 the subgroup of G generated by y and the elements of N_2. Then

$$[N_2^0 : N_2] = p \qquad \text{and} \qquad N_2^0 \rhd N_2.$$

First, we show the following

Claim II. We can write as

$$yay^{-1} = a^{1+kp^{n-3}}v^j,$$

$$yby^{-1} = a^{p^{n-1}d}b,$$

$$yvy^{-1} = a^{p^{n-2}s}b^d v,$$

for some $k, j, d, s \in \mathbf{Z}$, such that $(k, p) = (j, p) = 1$.

PROOF OF CLAIM II. First, we consider the elements yby^{-1}. By Lemma 1 (vi), $|a^i v^j| \geq p^2$ when $(j,p)=1$. So, we must have $yby^{-1} = a^{d_0}b^{t_0}$ for some $d_0, t_0 \in \mathbf{Z}$. But

$$1 = yb^p y^{-1} = (a^{d_0}b^{t_0})^p = a^{d_0 p},$$

by Lemma 1 (iii). Therefore

$$d_0 \equiv 0 \qquad (\mathrm{mod}\ p^{n-1}).$$

Hence, we may write $d_0 = p^{n-1}d$ and

$$yby^{-1} = a^{p^{n-1}d}b^{t_0},$$

for some $d \in \mathbf{Z}$.

Next, consider the element yay^{-1}. Since $y \notin N_2$, we must have

$$yay^{-1} = a^i v^j,$$

for some $j \in \mathbf{Z}$, $(j,p)=1$. By Lemma 1 (vi), $ya^p y^{-1} = (a^i v^j)^p = a^{pi+mp^{n-1}}b^j$ for some $m \in \mathbf{Z}$. Since $yay^{-1} \notin \langle a \rangle$, $ya^p y^{-1} \notin \langle a \rangle$, and $ya^{p^2}y^{-1} = (a^{pi+mp^{n-1}}b^j)^p = a^{p^2 i} \in \langle a \rangle$, we must have

$$a^{p^2 i} \neq a^{p^2},$$

by the condition (EX,C). Therefore

$$i \notin \langle 1+p^{n-2} \rangle,$$

where $\langle 1+p^{n-2} \rangle$ is the subgroup of $(\mathbf{Z}/p^n\mathbf{Z})^*$ generated by $1+p^{n-2}$. But $y^p a^{p^2} y^{-p} = a^{p^2 i^p}$ and $y^p \in N_2 = G(n,1,1)$. Hence

$$i^p \in \langle 1+p^{n-2} \rangle.$$

Thus we may write as $i = 1+kp^{n-3}$ and

$$yay^{-1} = a^{1+kp^{n-3}}v^j,$$

for some integers k, j such that $(k,p)=(j,p)=1$.

Since $n-1 \geq 6$, by our assumption, we have,

$$ya^{p^{n-1}}y^{-1} = (a^{1+kp^{n-3}}v^j)^{p^{n-1}} = a^{p^{n-1}}.$$

by Lemma 1 (vii). Taking the conjugate of both sides of the equality, $bab^{-1} = a^{1+p^{n-1}}$, by y, we get

$$(a^{p^{n-1}d}b^{t_0})(a^{1+kp^{n-3}}v^j)(a^{p^{n-1}d}b^{t_0})^{-1} = (a^{1+kp^{n-3}}v^j)a^{p^{n-1}}.$$

Hence,

$$a^{(1+kp^{n-3})(1+t_0 p^{n-1})}v^j = a^{1+kp^{n-3}+p^{n-1}}v^j.$$

Therefore,

$$t_0 \equiv 1 \pmod{p},$$

and hence

$$yby^{-1} = a^{p^{n-1}d}b.$$

Finally, we consider the element yvy^{-1}. Write $yvy^{-1} = a^{s_0}v^{h_0}$. Then

$$yby^{-1} = yv^p y^{-1} = (a^{s_0}v^{h_0})^p = a^{ps_0+ep^{n-1}}b^{h_0},$$

for some integer e, by Lemma 1 (vi). Therefore

$$a^{p^{n-1}d}b = a^{ps_0+ep^{n-1}}b^{h_0},$$

and, we have

$$h_0 \equiv 1 \pmod{p}, \qquad \text{and} \qquad s_0 \equiv 0 \pmod{p^{n-2}}.$$

Write

$$h_0 = 1 + pl \qquad \text{and} \qquad s_0 = p^{n-2}s,$$

for some $l, s \in \mathbf{Z}$. Then we have

$$yvy^{-1} = a^{p^{n-2}s}v^{1+pl} = a^{p^{n-2}s}b^l v.$$

Taking the conjugate of both sides of the equality, $vav^{-1} = a^{1+p^{n-2}}b$ by y, we get

$$(a^{p^{n-2}s}b^l v)(a^{1+kp^{n-3}}v^j)(a^{p^{n-2}s}b^l v)^{-1} = (a^{1+kp^{n-3}}v^j)a^{p^{n-2}}(a^{p^{n-1}d}b),$$

since $ya^{p^{n-2}}y^{-1} = a^{p^{n-2}}$. Hence, we have

$$a^{1+kp^{n-3}+p^{n-2}+p^{n-1}l}bv^j = a^{1+kp^{n-3}+p^{n-2}+p^{n-1}d}bv^j.$$

Therefore,

$$d \equiv l \pmod{p},$$

Thus the proof of Claim II is completed.

Now, we consider the element y^p $(\in N_2 = G(n,1,1) = \langle a, b, v \rangle)$. Write $y^p = a^{r_0}v^h$. Then

$$a^{r_0}v^h = y^p = yy^p y^{-1} = y(a^{r_0}v^h)y^{-1} = (a^{1+kp^{n-3}}v^j)^{r_0}(a^{p^{n-2}s}b^d v)^h.$$

Therefore

$$v^h \equiv v^{jr_0}v^h \qquad (\text{mod } N_1 = \langle a, b \rangle).$$

Since $(j, p) = 1$, we have $r_0 \equiv 0 \pmod{p}$. Hence we can write as $r_0 = pr$, and

$$y^p = a^{pr} v^h,$$

for some $r \in \mathbf{Z}$.

We show the following:

Claim III. There exists an integer ε such that $(a^{\varepsilon} y)^p = v^{pt_0 + h}$, for some integer t_0. Further, $(h, p) = 1$.

Proof of Claim III. It is easy to see that $\langle a^{p^{n-3}} \rangle$ is a normal subgroup of N_2^0. By Claim II, the following equalities hold:

$$yay^{-1} \equiv av^j \qquad (\mathrm{mod} \langle a^{p^{n-3}} \rangle),$$

$$yby^{-1} \equiv b \qquad (\mathrm{mod} \langle a^{p^{n-3}} \rangle),$$

$$yvy^{-1} \equiv b^d v \qquad (\mathrm{mod} \langle a^{p^{n-3}} \rangle).$$

Using these relations repeatedly, we can get

$$y^p a y^{-p} \equiv ab^{dj(1+2+\cdots+(p-1))} v^{pj} = ab^{dj(p(p-1)/2)} v^{pj} = av^{pj} = ab^j \qquad (\mathrm{mod} \langle a^{p^{n-3}} \rangle),$$

since p is odd. Hence we can write as

$$y^p a y^{-p} = a^{1+\beta p^{n-3}} b^j,$$

for some $\beta \in \mathbf{Z}$. On the other hand, since $y^p = a^{pr} v^h$, we have

$$y^p a y^{-p} = (a^{pr} v^h) a (a^{pr} v^h)^{-1} = a^{1+hp^{n-2}} b^h,$$

by Lemma 1 (v). Thus we have $b^j = b^h$, and, in particular, $(h, p) = 1$. Define the subgroup H of N_2^0 as

$$H = \langle a^{p^{n-3}} \rangle \times \langle b \rangle.$$

It is easy to see that $N_2^0 \rhd H$, and the following equalities hold:

$$va \equiv av \qquad (\mathrm{mod}\ H),$$

$$yay^{-1} \equiv av^j \qquad (\mathrm{mod}\ H),$$

$$yvy^{-1} \equiv v \qquad (\mathrm{mod}\ H).$$

Using these relations repeatedly, we can get

$$y^c a^m y^{-c} \equiv a^m v^{mcj} \qquad (\mathrm{mod}\ H),$$

for any $m, c \in \mathbf{Z}$. Using this equality, we have

$$(a^m y)^p \equiv a^{pm} v^{jm(1+2+\cdot+(p-1))} y^p = a^{pm} v^{jm(p(p-1)/2)} a^{pr} v^h = a^{p(m+r)} v^h \qquad (\mathrm{mod}\ H),$$

for any $m \in \mathbf{Z}$. Therefore we may write as

$$(a^m y)^p = a^{p(m+r)+\theta p^{n-3}} b^t v^h,$$

for some integers θ and t. Note that θ and t are not independent of the choice of m. If we set $y_1 = a^{-r} y$, then

$$y_1^p = a^{\theta_0 p^{n-3}} b^{t_0} v^h = a^{\theta_0 p^{n-3}} v^{pt_0+h},$$

for some integers θ_0 and t_0. Further, set $\varepsilon = -\theta_0 p^{n-4} - r$, and

$$y_2 = a^{\varepsilon} y = a^{-\theta_0 p^{n-4} - r} y = a^{-\theta_0 p^{n-4}} y_1.$$

Since $n - 4 \geq 3$, by our assumption, we have

$$y_1 a^{p^{n-4}} y_1^{-1} = a^{p^{n-4}}.$$

Hence,

$$y_2^p = (a^{\varepsilon} y)^p = (a^{-\theta_0 p^{n-4}} y_1)^p = a^{-\theta_0 p^{n-3}} y_1^p = v^{pt_0+h}.$$

This completes the proof of Claim III.

Since $(pt_0 + h, p) = 1$, there exists $k' \in \mathbf{Z}$, such that $(pt_0 + h)k' \equiv 1 \pmod{p^2}$. Hence

$$y_2^{k'p} = v^{(pt_0+h)k'} = v.$$

Therefore

$$y_2 v y_2^{-1} = v, \qquad \text{and} \qquad y_2 b y_2^{-1} = b.$$

Further, we have

$$\begin{aligned}
y_2 a y_2^{-1} &= a^{-\theta_0 p^{n-4} - r} y a y^{-1} a^{\theta_0 p^{n-4} + r} \\
&= a^{-r} (a^{1+kp^{n-3}} v^j) a^r \\
&= a^{-r} a^{1+kp^{n-3}} (a^{1+jp^{n-2}} b^j)^r v^j \\
&\equiv a^{-r} a^{1+kp^{n-3}} a^r b^{jr} v^j \qquad (\mathrm{mod} \langle a^{p^{n-2}} \rangle) \\
&\equiv a^{1+kp^{n-3}} b^{jr} v^j \qquad (\mathrm{mod} \langle a^{p^{n-2}} \rangle) \\
&= a^{1+kp^{n-3}} v^{pjr+j}.
\end{aligned}$$

Therefore

$$y_2 a y_2^{-1} = a^{1+kp^{n-3}+\delta p^{n-2}} v^{pjr+j},$$

for some integer δ. If we set $k_1 = k + \delta p$, then $(k_1, p) = 1$, and

$$y_2 a y_2^{-1} = a^{1+k_1 p^{n-3}} v^{pjr+j}.$$

Since $n-3 \geq 4$, by our assumption, we have

$$y_2 a^{p^{n-3}} y_2^{-1} = a^{p^{n-3}}.$$

Thus,

$$y_2^p a y_2^{-p} = a^{1+k_1 p^{n-2}} v^{pj} = a^{1+k_1 p^{n-2}} b^j.$$

On the other hand, since $y_2^p = v^{pt_0+h}$, we have

$$y_2^p a y_2^{-p} = (v^{pt_0+h}) a (v^{pt_0+h})^{-1} = a^{1+(pt_0+h)p^{n-2}} b^{pt_0+h} = a^{1+(pt_0+h)p^{n-2}} b^h,$$

by Lemma 1 (v). Therefore we have

$$k_1 \equiv pt_0 + h \qquad (\mathrm{mod}\ p^2),$$

and

$$j \equiv h \qquad (\mathrm{mod}\ p).$$

Summarizing the results, we have

$$\begin{aligned} y_2 a y_2^{-1} &= a^{1+k_1 p^{n-3}} v^{jpr+j} = a^{1+k_1 p^{n-3}} b^{hr} v^j, \\ y_2^p &= v^{k_1}, \\ y_2 b y_2^{-1} &= b, \\ y_2 v y_2^{-1} &= v. \end{aligned}$$

There exists an integer l_1, such that

$$l_1 k_1 \equiv 1 \qquad (\mathrm{mod}\ p^3).$$

Since

$$k_1 \equiv h \equiv j \qquad (\mathrm{mod}\ p),$$

we have

$$l_1 k_1 \equiv l_1 j \equiv 1 \qquad (\mathrm{mod}\ p).$$

Hence, we may write as

$$l_1 j = 1 + ps_2,$$

for some $s_2 \in \mathbf{Z}$. Set $y_3 = y_2^{l_1}$. Then

$$\begin{aligned} y_3 a y_3^{-1} = y_2^{l_1} a y_2^{-l_1} &= a^{1+p^{n-3} k_1 l_1} v^{l_1(jpr+j)} \\ &= a^{1+p^{n-3}} v^{l_1 j(pr+1)} = a^{1+p^{n-3}} v^{(ps_2+1)(pr+1)} \\ &= a^{1+p^{n-3}} v^{p(r+s_2)+1} = a^{1+p^{n-3}} b^{r_1} v, \end{aligned}$$

where $r_1 = r + s_2$. Further

$$y_3^p = y_2^{pl_1} = v^{k_1 l_1} = v, \qquad y_3 v y_3^{-1} = v, \qquad \text{and} \qquad y_3 b y_3^{-1} = b.$$

If we set $a_0 = a^{1-pr_1}$ $(=a^{1+(p^n-p)r_1})$, then we have

$$a_0^{p^n} = 1 \qquad \text{and} \qquad b a_0 b^{-1} = a_0^{1+p^{n-1}}.$$

Further, we have

$$\begin{aligned} v a_0 v^{-1} &= v a^{1-pr_1} v^{-1} = v a a^{-pr_1} v^{-1} = (a^{1+p^{n-2}} b)(a^{1+p^{n-2}} b)^{-pr_1} \\ &= (a^{1+p^{n-2}} b) a^{(1+p^{n-2})(-pr_1)} = a^{(1+p^{n-2})(1-pr_1)} b = a_0^{1+p^{n-2}} b, \end{aligned}$$

by using Lemma 1 (iii). We also have

$$\begin{aligned} y_3 a_0 y_3^{-1} &= y_3 a a^{-pr_1} y_3^{-1} \\ &= (a^{1+p^{n-3}} b^{r_1} v)(a^{1+p^{n-3}} b^{r_1} v)^{-pr_1} \\ &\equiv (a^{1+p^{n-3}} b^{r_1} v)(a^{p(1+p^{n-3})} v^p)^{-r_1} \qquad (\mathrm{mod} \langle a^{p^{n-1}} \rangle) \\ &= (a^{1+p^{n-3}} b^{r_1} v) a^{-pr_1(1+p^{n-3})} b^{-r_1}, \\ &\equiv a^{1+p^{n-3}} a^{-pr_1(1+p^{n-3})} v \qquad (\mathrm{mod} \langle a^{p^{n-1}} \rangle) \\ &= a^{(1+p^{n-3})(1-pr_1)} v = a_0^{1+p^{n-3}} v. \end{aligned}$$

Hence we may write as $y_3 a_0 y_3^{-1} = a_0^{1+p^{n-3}} a^{\gamma p^{n-1}} v$, for some integer γ. But $a_0^{\gamma p^{n-1}} = a^{\gamma p^{n-1}}$, so

$$y_3 a_0 y_3^{-1} = a_0^{1+p^{n-3}+\gamma p^{n-1}} v.$$

Finally, we set $y_4 = b^{-\gamma} y_3$. Then we have

$$y_4^p = y_3^p = v, \qquad y_4 v y_4^{-1} = v, \qquad \text{and} \qquad y_4 b y_4^{-1} = b.$$

Further

$$\begin{aligned} y_4 a_0 y_4^{-1} &= b^{-\gamma} y_3 a_0 y_3^{-1} b^{\gamma} = b^{-\gamma} (a_0^{1+p^{n-3}+\gamma p^{n-1}} v) b^{\gamma} \\ &= a_0^{(1+p^{n-3}+\gamma p^{n-1})(1+p^{n-1})^{-\gamma}} v = a_0^{1+p^{n-3}} v. \end{aligned}$$

Therefore, the relations of the elements $a_0, b, v,$ and y_4 are the same as that of $G(n,1,1,1)$. So, the group $N_2^0 = \langle a_0, b, v, y_4 \rangle$ is clearly isomorphic to $G(n,1,1,1)$.

We will complete the proof of (2), by showing the following:

Claim IV. $N_2^0 = N_3$.

PROOF OF CLAIM IV. Since the proof of this claim is similar to that of Claim I, we state only an outline of the proof. Suppose that $N_2^0 \subsetneqq N_3$. Take an element $g \in N_3 - N_2^0$ such that $g^p \in N_2^0$. Then we can write as

$$ga_0g^{-1} = a_0^{1+k_2p^{n-4}}v^{j_1}$$

for some integers k_2 and j_1, by the same way as in the proof of Claim I. Since $n-4 \geq 3$, by our assumption, we have

$$y_4a_0^{p^{n-4}}y_4^{-1} = a_0^{p^{n-4}}.$$

Thus

$$y_4^{p^2-j_1}ga_0g^{-1}y_4^{-p^2+j_1} = a_0^{1+k_2p^{n-4}+(p^2-j_1)p^{n-3}}.$$

This means that $y_4^{p^2-j_1}g \in N_1$, which contradicts the hypothesis that $g \in N_3 - N_2^0$. Hence the proof of Claim IV is completed.

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