HIROSHIMA MATH. J.
32 (2002), 145–154

On Poincaré four-complexes with free fundamental groups

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(Received June 8, 1999)
(Revised October 23, 2001)

ABSTRACT. Let X be a closed connected and oriented PL manifold whose fundamental group π_1 is a free group of rank p. Let Λ be the integral group ring of π_1. Then $H_2(X, \Lambda)$ is Λ-free (see [6]). We show that there is a bijective correspondence between homotopy equivalence classes of 4-dimensional Poincaré complexes Y with $Y^{(3)} \cong X^{(3)}$ and invertible hermitian matrices of type k over Λ, where k is the rank of $H_2(X, \Lambda)$.

1. Introduction.

By a Poincaré 4-complex we understand a 4-dimensional CW-complex Y with a fundamental class $[Y] \in H_4(Y, \mathbf{Z}) \cong \mathbf{Z}$ inducing isomorphisms

$$\bigcap [Y] : H^q(Y, \Lambda) \to H_{4-q}(Y, \Lambda),$$

where $\Lambda = \mathbf{Z}[\pi_1(Y)]$ is the integral group ring of the fundamental group. Since we will discuss only fundamental groups which are freely generated, e.g., by p generators, there will be no Whitehead torsion. Therefore they will be finite Poincaré complexes in the sense of [8]. We can also assume that Y is obtained from the 3-skeleton $Y^{(3)}$ of Y by attaching only one 4-cell (see [8], p. 30), i.e., $Y = Y^{(3)} \cup_\varphi D^4$, where $\varphi : \mathbf{S}^3 \to Y^{(3)}$ is the attaching map. The following result of T. Matumoto and A. Katanaga will be crucial for our discussion (see [6], Proposition 2).

PROPOSITION 1.1. *Let X be a closed PL four-manifold with $\pi_1 = *^p\mathbf{Z}$. Then $X^{(3)}$ is homotopy equivalent to $\bigvee^p(\mathbf{S}^1 \vee \mathbf{S}^3) \vee (\bigvee^k \mathbf{S}^2)$.*

Let us suppose X to be a PL 4-manifold, hence $\pi_2(X) = H_2(X, \Lambda)$ is Λ-free of rank k. Recall the Whitehead exact sequence ([9])

$$0 \to \Gamma(\pi_2) \to \pi_3(X^{(3)}) \to H_3(X^{(3)}, \Lambda) \to 0,$$

where $\Gamma(\pi_2)$ is the (quadratic) Γ-functor applied to the abelian group $\pi_2(X^{(3)}) = H_2(X, \Lambda)$. Note that $H_3(X^{(3)}, \Lambda)$ is Λ-free of rank p. There are canonical

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2000 *Mathematics Subject Classification.* 57N65, 57R67, 57Q10.

Key words and phrases. Poincaré complexes, Free fundamental groups, Homotopy type, Hermitian forms.

maps $\Gamma(\pi_2) \to \pi_2 \otimes_{\mathbf{Z}} \pi_2$ and $\pi_2 \otimes_{\mathbf{Z}} \pi_2 \to \Gamma(\pi_2)$ such that their composition is multiplication by 2 on $\Gamma(\pi_2)$. This implies in our case that

$$\Gamma(\pi_2) \subset \pi_2 \otimes_{\mathbf{Z}} \pi_2.$$

In fact, $\Gamma(\pi_2)$ can be considered as the subset of symmetric tensors in $\pi_2 \otimes_{\mathbf{Z}} \pi_2$. Moreover, $\Gamma(\pi_2)$ is a Λ-submodule of $\pi_2 \otimes_{\mathbf{Z}} \pi_2$, and the induced homomorphism

$$\Gamma(\pi_2) \otimes_{\Lambda} \mathbf{Z} \to \pi_2 \otimes_{\Lambda} \pi_2$$

is injective. (Here as elsewhere in this paper, we have, if necessary, to shift from right- to left-Λ-module structures using the canonical anti-automorphism of Λ.) The purpose of this paper is to study 4-complexes Y obtained from $X^{(3)}$ by attaching one 4-cell with attaching map $\psi : \mathbf{S}^3 \to X^{(3)}$, i.e., $Y = X^{(3)} \cup_{\psi} D^4$. More precisely, we want to study conditions on ψ to obtain a Poincaré 4-complex Y. Our basic result is a correspondence between homotopy equivalence classes of PD-duality spaces Y and invertible hermitian matrices over Λ of type k (see Theorem 3.3 below).

2. Homological properties of Y.

Let X be an oriented closed PL-manifold of dimension 4. We may assume that $X^{(3)}$ is homotopy equivalent to $\bigvee^p(\mathbf{S}^1 \vee \mathbf{S}^3) \vee (\bigvee^k \mathbf{S}^2)$. Let

$$\varphi : \mathbf{S}^3 \to \bigvee^p(\mathbf{S}^1 \vee \mathbf{S}^3) \vee (\bigvee^k \mathbf{S}^2)$$

be the attaching map of the 4-cell and let φ_0 be the composition

$$\mathbf{S}^3 \xrightarrow{\varphi} \bigvee^p(\mathbf{S}^1 \vee \mathbf{S}^3) \vee (\bigvee^k \mathbf{S}^2) \xrightarrow{c} \bigvee^p(\mathbf{S}^1 \vee \mathbf{S}^3)$$

with the collapsing map $c : \bigvee^p(\mathbf{S}^1 \vee \mathbf{S}^3) \vee (\bigvee^k \mathbf{S}^2) \to \bigvee^p(\mathbf{S}^1 \vee \mathbf{S}^3)$.

LEMMA 2.1. *The space $Q = \bigvee^p(\mathbf{S}^1 \vee \mathbf{S}^3) \cup_{\varphi_0} D^4$ is a Poincaré complex of dimension* 4. *The canonical map $f : X \to Q$ is of degree* 1.

PROOF. The collapsing map c induces a map $f : X \to Q$ such that the following diagram commutes

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_4(Q,\mathbf{Z}) & \longrightarrow & H_4(Q,Q^{(3)},\mathbf{Z}) & \longrightarrow & H_3(Q^{(3)},\mathbf{Z}) \\
 & & f_* \uparrow & & f_* \uparrow \cong & & \cong \uparrow \\
0 & \longrightarrow & H_4(X,\mathbf{Z}) & \longrightarrow & H_4(X,X^{(3)},\mathbf{Z}) & \longrightarrow & H_3(X^{(3)},\mathbf{Z}).
\end{array}$$

In particular, $f_*: H_4(X,\mathbf{Z}) \xrightarrow{\cong} H_4(Q,\mathbf{Z})$. Let $[Q] = f_*[X]$, then

$$\begin{array}{ccc} H^q(Q,\Lambda) & \xrightarrow{\bigcap[Q]} & H_{4-q}(Q,\Lambda) \\ {\scriptstyle f^*}\big\downarrow & & \big\uparrow{\scriptstyle f_*} \\ H^q(X,\Lambda) & \xrightarrow[\cong]{\bigcap[X]} & H_{4-q}(X,\Lambda) \end{array}$$

commutes. For $q=0$ we observe that $H^0(Q,\Lambda) = H^0(X,\Lambda) = 0$ since $H_4(X,\Lambda) = 0$. Hence Poincaré duality is trivial in this case. Moreover for $q = 1,3$ it follows that $f_*: H_{4-q}(X,\Lambda) \xrightarrow{\cong} H_{4-q}(Q,\Lambda)$ and $H^q(Q,\Lambda) \xrightarrow{\cong} H^q(X,\Lambda)$, hence Poincaré duality follows from the previous diagram. It follows from the diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^3(Q,\Lambda) & \longrightarrow & H^3(Q^{(3)},\Lambda) & \longrightarrow & H^4(Q,Q^{(3)},\Lambda) & \longrightarrow & H^4(Q,\Lambda) & \longrightarrow 0 \\ & \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle f^*} & & {\scriptstyle\cong}\big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle f^*} & \\ 0 \longrightarrow & H^3(X,\Lambda) & \longrightarrow & H^3(X^{(3)},\Lambda) & \longrightarrow & H^4(X,X^{(3)},\Lambda) & \longrightarrow & H^4(X,\Lambda) & \longrightarrow 0 \end{array}$$

that $f^*: H^4(Q,\Lambda) \xrightarrow{\cong} H^4(X,\Lambda)$ hence $\bigcap[Q]: H^4(Q,\Lambda) \xrightarrow{\cong} H_0(Q,\Lambda)$. Finally for $q = 2$, we see easily that $H^2(Q,\Lambda) = H_2(Q,\Lambda) = 0$. $\square$

As explained in the introduction we shall study the space $Y = X^{(3)} \cup_\psi D^4$ and $[\psi] = [\varphi] + \theta$, with $\theta \in \Gamma(\pi_2)$. Note that $Q^{(3)} = \bigvee^p(\mathbf{S}^1 \vee \mathbf{S}^3)$, hence we have

$$\pi_3(Q^{(3)}) \underset{\cong}{\to} H_3(Q^{(3)},\Lambda).$$

It follows from the diagram

$$(*)\qquad \begin{array}{ccccccccc} & & \theta & & [\varphi] & & & & \\ & & {\scriptstyle\in}\big\downarrow & & {\scriptstyle\in}\big\downarrow & & & & \\ 0 & \longrightarrow & \Gamma(\pi_2) & \longrightarrow & \pi_3(X^{(3)}) & \longrightarrow & H_3(X^{(3)},\Lambda) & \longrightarrow & 0 \\ & & & & \big\downarrow & & {\scriptstyle\cong}\big\downarrow & & \\ & & & & \pi_3(Q^{(3)}) & \underset{\cong}{\longrightarrow} & H_3(Q^{(3)},\Lambda) & & \\ & & & & \big\uparrow{\scriptstyle\in} & & & & \\ & & & & [\varphi_0] & & & & \end{array}$$

that the composition ψ_0

$$\mathbf{S}^3 \xrightarrow{\psi} X^{(3)} \xrightarrow{c} \bigvee^p(\mathbf{S}^1 \vee \mathbf{S}^3)$$

is homotopic to φ_0, so this defines a map

$$Y \xrightarrow{g} Q.$$

Note that $g_* : H_4(Y, \mathbf{Z}) \xrightarrow{\cong} H_4(Q, \mathbf{Z})$ (as above) and let $[Y] \in H_4(Y, \mathbf{Z})$ be such that $g_*([Y]) = [Q]$.

Lemma 2.2. *Cap product with* $[Y]$ *induces isomorphisms* $H^q(Y, \Lambda) \to H_{4-q}(Y, \Lambda)$ *for all* $q \neq 2$.

Proof. The proof goes as in Lemma 2.1. □

On the other hand if $Y = X^{(3)} \cup_\psi D^4$ is a Poincaré duality space for $\psi : \mathbf{S}^3 \to X^{(3)}$, then we can construct Q as above and $g_* : \pi_3(X^{(3)}) = \pi_3(Y^{(3)}) \to \pi_3(Q^{(3)})$ is identified with f_*. So it follows from diagram (*) that $[\psi] - [\varphi] \in \Gamma(\pi_2)$. Hence this is also a necessary condition.

Remark 2.1. *From the universal coefficient spectral sequence we get*

$$H^2(Z, \Lambda) \cong \mathrm{Hom}_\Lambda(H_2(Z, \Lambda), \Lambda)$$

for all spaces Z *under consideration.*

3. Poincaré duality in the middle dimension.

As before let $\varphi : \mathbf{S}^3 \to X^{(3)}$ be the attaching map of a 4-cell of the 4-manifold X. Given an element $\theta \in \Gamma(\pi_2)$ we must study the effect of θ on the homomorphism

$$H^2(Y, \Lambda) \to H_2(Y, \Lambda),$$

for $Y = X^{(3)} \cup_\psi D^4$ with $[\psi] = [\varphi] + \theta$. To simplify notation we will write π_2, H_2, and H^2 for $\pi_2(X)$, $H_2(X, \Lambda)$, and $H^2(X, \Lambda)$, respectively. In particular, we have $\pi_2 = H_2$. Note that $H^2(Y, \Lambda) \cong H^2(X, \Lambda)$ and $H_2(Y, \Lambda) \cong H_2(X, \Lambda)$ in a canonical way, so we have particularly

$$\bigcap[Y] : H^2 \xrightarrow{\cong} H_2.$$

Hence we have to study the isomorphism

$$\bigcap[Y] - \bigcap[X] : H^2 \xrightarrow{\cong} H_2.$$

Let $\theta = \sum_i u_i \otimes v_i \in \pi_2 \otimes_{\mathbf{Z}} \pi_2$ be a symmetric tensor, i.e., $\theta \in \Gamma(\pi_2)$. The effect of θ on H^2 is given as the image of θ under the canonical homomorphisms

$$\pi_2 \otimes_{\mathbf{Z}} \pi_2 \to \pi_2 \otimes_\Lambda \pi_2 \xrightarrow{\cong} \mathrm{Hom}_\Lambda(H_2^*, H_2) \xrightarrow{\cong} \mathrm{Hom}_\Lambda(H^2, H_2)$$

where $H_2^* = \mathrm{Hom}_\Lambda(H_2, \Lambda)$. Because π_2 is Λ-free these are isomorphisms (except the first map). Recall also that $H^2 \xrightarrow{\cong} H_2^*$. For convenience we identify $\mathrm{Hom}_\Lambda(H^2, H_2)$ with $\mathrm{Hom}_\Lambda(H_2, H_2)$ via the PD-isomorphism

$$\bigcap [X] : H^2(X, \Lambda) \to H_2(X, \Lambda).$$

The following lemma can be easily checked:

Lemma 3.1. *Under the above composition map the element* $\theta = \sum_i u_i \otimes v_i \in \Gamma(\pi_2)$ *gives the following homomorphism*

$$H_2(X, \Lambda) \to H_2(X, \Lambda),$$

sending x *to* $\sum_i u_i \lambda_X(v_i, x)$. *Here* $\lambda_X : H_2(X, \Lambda) \times H_2(X, \Lambda) \to \Lambda$ *is the intersection form over the group ring.*

Proof. The only difficulty is to write down the correct Λ-module structures. If H_2 is considered as a Λ-right module, then H_2^* is in a natural way a Λ-left module. For $\xi \in H_2^*$ and $\lambda \in \Lambda$ we have $(\lambda\xi)(x) = \lambda\xi(x)$. Then H_2^* is a right module as follows: $\xi \cdot \lambda(x) = \bar{\lambda}\xi(x)$, where $^- : \Lambda \to \Lambda$ is the canonical anti-involution. Now $\mathrm{Hom}_\Lambda(H_2^*, H_2)$ are the Λ-right module homomorphisms. Let us for simplicity consider $u \otimes_\Lambda v \in \pi_2 \otimes_\Lambda \pi_2$. This defines in $\mathrm{Hom}_\Lambda(H_2^*, H_2)$ the element given by $\xi \to u\overline{\xi(v)}$. In $\mathrm{Hom}_\Lambda(H^2, H_2)$ the expression $\xi(v)$ becomes $\xi \cap v$. Going from $\mathrm{Hom}_\Lambda(H^2, H_2)$ to $\mathrm{Hom}_\Lambda(H_2, H_2)$ via $\mathrm{PD}_X : H^2(X, \Lambda) \to H_2(X, \Lambda)$ we have to start with $x \in H_2$, i.e.,

$$x \to u\overline{(\mathrm{PD}_X^{-1}(x) \cap v)} = u\overline{\lambda_X(x, v)} = u\lambda_X(v, x).$$

Recall for this that $\lambda_X(x, y) = (\mathrm{PD}_X^{-1}(x) \cup \mathrm{PD}_X^{-1}(y)) \cap [X] \in H_0(X, \Lambda \otimes_{\mathbf{Z}} \Lambda) \cong \Lambda \otimes_\Lambda \Lambda \cong \Lambda$. □

Corollary 3.2. *If we compose* $\bigcap [Y] : H^2 \to H_2$ *with the Poincaré duality inverse* $\mathrm{PD}_X^{-1} : H_2 \to H^2$ *we obtain*

$$\left(\bigcap [Y]\right) \circ \mathrm{PD}_X^{-1}(x) = x + \sum_i u_i \lambda_X(v_i, x).$$

Let us now choose a Λ-basis $a_1, \ldots, a_k$ of H_2. Then

$$\{a_i g \otimes a_j g' \mid i, j = 1, \ldots, k, g, g' \in \pi_1\}$$

is a $\mathbf{Z}$-basis of $\pi_2 \otimes_{\mathbf{Z}} \pi_2 = H_2 \otimes_{\mathbf{Z}} H_2$. Let $\theta = \sum_{i,j,g,g'} a_i g \otimes a_j g' n_{ji}(g,g')$ be a symmetric element, i.e., $\theta \in \Gamma(\pi_2)$, with $n_{ji}(g,g') \in \mathbf{Z}$. Then we must have $n_{ji}(g,g') = n_{ij}(g',g)$. Note that the sum which defines θ is finite. Let us write the element $\theta \otimes_\Lambda 1 \in \pi_2 \otimes_\Lambda \pi_2$ as $\sum_{i,j} a_i \otimes a_j \gamma_{ji}$ with $\gamma_{ji} = \sum_{g,g'} n_{ji}(g,g') \cdot g' g^{-1} \in \Lambda$. The above symmetry condition then implies $\gamma_{ji} = \overline{\gamma_{ij}}$, i.e., the matrix $\Gamma = (\gamma_{ji}) \in M(k,k;\Lambda)$ is hermitian: $\bar{\Gamma}^t = \Gamma$. Conversely, let be given any hermitian matrix $\Gamma = (\gamma_{ji})$ over Λ. Let us write the elements $\gamma_{ji} \in \Lambda$ as $\gamma_{ji} = \sum_{g \in \pi_1} n_{ji}(g) g$. Then $\gamma_{ji} = \bar{\gamma}_{ij}$ implies $n_{ji}(g) = n_{ij}(g^{-1})$. Let

$$\theta = \sum_{i,j,g} \{a_i \otimes a_j g n_{ji}(g) + a_j g \otimes a_i n_{ij}(g^{-1})\} + \sum_i a_i \otimes a_i n_{ii}(1).$$

The first sum is taken over all $i,j = 1,\dots,k$ and $g \in \pi_1' \cup \{1\}$, where π_1' is a subset of $\pi_1 \backslash \{1\}$ containing for any $g \in \pi_1$ either g or g^{-1}. Then $\theta \in \Gamma(\pi_2)$ and $\theta \otimes_\Lambda 1 = \sum_{i,j} a_i \otimes a_j \gamma_{ji}$. Applying $(\bigcap[Y]) \circ \mathrm{PD}_X^{-1}$ to the basis $a_1,\dots,a_k$ we get from Corollary 3.2

$$\begin{aligned}(\textstyle\bigcap[Y]) \circ \mathrm{PD}_X^{-1}(a_\ell) &= a_\ell + \sum_{i,j} a_i \lambda_X(a_j \gamma_{ji}, a_\ell) \\ &= a_\ell + \sum_{i,j} a_i \bar{\gamma}_{ji} \lambda_X(a_j, a_\ell) \\ &= a_\ell + \sum_{i,j} a_i \gamma_{ij} \lambda_X(a_j, a_\ell).\end{aligned}$$

If we denote by L_X the intersection matrix $(\lambda_X(a_j, a_\ell))$ we get the matrix

$$I_k + \Gamma L_X = \Sigma$$

associated to $(\bigcap[Y]) \circ \mathrm{PD}_X^{-1}$. Since L_X is invertible and hermitian, we can solve this equation for Γ. Hence, beginning with an invertible hermitian matrix $\Sigma L_X^{-1} = \Omega$ we obtain an hermitian Λ-matrix Γ from which we can construct $\theta \in \Gamma(\pi_2)$ such that $Y = X^{(3)} \cup_\psi D^4$ with $[\psi] = [\varphi] + \theta$ is a Poincaré duality complex. On the other hand it was shown in [2] that the isomorphic intersection forms λ_X and λ_Y determine homotopy equivalent Poincaré spaces X and Y. This means that if the cup product pairings

$$H^2 \otimes H^2 \to \Lambda$$

given by $[X]$ and $[Y]$ are the same then X and Y are homotopy equivalent. Note that the cup product with respect to $[Y]$ is defined by the composition

$$H^2 \otimes H^2 \longrightarrow H^4(Y, \Lambda \otimes_{\mathbf{Z}} \Lambda) \xrightarrow{\bigcap[Y]} H_0(Y, \Lambda \otimes_{\mathbf{Z}} \Lambda) \cong \Lambda \otimes_\Lambda \Lambda \cong \Lambda.$$

If $\underline{\xi, \eta \in H^2}$ we have $(\xi \cup \eta) \cap [Y] = \xi \cap (\eta \cap [Y])$. Now $\eta \cap [Y] = \eta \cap [X] + \sum_{i,j} a_i \overline{\eta(a_j \gamma_{ji})}$ (considering $\eta \in H^2 \cong H_2^*$). If we calculate the products of the Hom_Λ-dual basis $a_1^*, \ldots, a_k^* \in H^2$ we obtain

$$(a_r^* \cup a_s^*) \cap [Y] = (a_r^* \cup a_s^*) \cap [X] + \gamma_{rs}.$$

We can therefore summarize to get the following

Theorem 3.3. *Let X^4 be a closed connected PL 4-manifold with $\pi_1(X) \cong *^p\mathbf{Z}$. Fixing a Λ-basis $\{a_1, \ldots, a_k\}$ of $H_2(X, \Lambda)$, there is a bijective correspondence between hermitian invertible matrices Ω of type k and homotopy equivalence classes of Poincaré duality 4-complexes Y with $Y^{(3)} \cong X^{(3)}$.*

4. A remark on special hermitian forms and their realization by 4-manifolds.

If $\pi_1 \cong \mathbf{Z}$ any non-singular hermitian form has a realization by a 4-manifold (see [4]). There are forms such that the resulting manifold is not homotopy equivalent to $(\mathbf{S}^1 \times \mathbf{S}^3) \# M'$ with M' simply-connected, because there are forms over $\Lambda = \mathbf{Z}[\mathbf{Z}]$ which are not extended from $\mathbf{Z}$ (see [5]). If the rank of the free group is greater than 1 the realization of a non-singular hermitian form as intersection form of a closed 4-manifold is a difficult problem because free non-abelian groups are supposed to be not "good" in the sense of surgery theory (see [4]). An analogous problem arises by trying to realize surgery obstructions in dimension 4 (see [9], p. 54). Here the relevant hermitian forms are "special hermitian forms" (G, λ, μ) (see [9], p. 47). Besides being based (which we can ignore since the Whitehead group of $\pi_1 = *^p\mathbf{Z}$ is zero), a special hermitian form can be considered as an even hermitian space (see [7], Ch. 1). An even hermitian form is an orthogonal complement of a hyperbolic space (see [9], Lemma 5.4, or [7], Corollary 3.5.4). A hyperbolic space can be realized by $M = \#^p(\mathbf{S}^1 \times \mathbf{S}^3) \# (\#^n(\mathbf{S}^2 \times \mathbf{S}^2))$ for some n. The collapsing map $c : M \to \#^p(\mathbf{S}^1 \times \mathbf{S}^3)$ is a normal map of degree 1 with associated surgery obstruction $\sigma(c) = 0 \in L_4(\pi_1)$, where $L_4(\pi_1)$ is the Wall group of π_1. Recall that $L_4(\pi_1) = \mathbf{Z}$ and the surgery obstruction is the signature of the special hermitian form in question (see [1]). Now let us realize the even hermitian form (G, λ) by a Poincaré duality space Y and suppose that $sign(Y) = 0$. We consider $H_2(Y, \Lambda)$ as an orthogonal summand in $H_2(M, \Lambda)$.

Now let $V \subset H_2(M, \Lambda)$ be the orthogonal complement of $H_2(Y, \Lambda) \subset H_2(M, \Lambda)$. Then $sign(V) = 0$ since $sign(M) = sign(Y) = 0$. This means that V is stably a hyperbolic form. In other words, let us consider $N = M \# (\#^r(\mathbf{S}^2 \times \mathbf{S}^2))$ for a large enough r. Let $H \subset H_2(N, \Lambda)$ be the hyperbolic space defined by $\#^r(\mathbf{S}^2 \times \mathbf{S}^2)$. Then $V \oplus H$ is a sum of hyperbolic planes $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with respect to a symplectic base $(a_1, b_1, \dots, a_m, b_m)$ of $V \oplus H \subset H_2(N, \Lambda)$. In higher dimension, surgery on $a_1, \dots, a_m$ can be done to kill $V \oplus H$. In dimension 4 surgery can be done to kill $V \oplus H$ if the basis $(a_1, b_1, \dots, a_m, b_m)$ can be represented by a map

$$\varphi : \bigcup^m (\mathbf{S}^2 \vee \mathbf{S}^2) \to N$$

such that $\pi_1(\operatorname{Im} \varphi) \to \pi_1(N)$ is the zero-map (see [3]). Completing these surgeries on N we get a manifold W^4 with intersection form isomorphic to that of Y, hence W is homotopy equivalent to Y (see [2]). The above discussion gives also a "stable" result. Note that we can use the interior of the attached 4-cell of Y to form connected sums with manifolds. Then, if as before, λ_Y is even and $sign(Y) = 0$, then for some $r \geq 0$, $Y \# (\#^r(\mathbf{S}^2 \times \mathbf{S}^2))$ is homotopy equivalent to $\#^p(\mathbf{S}^1 \times \mathbf{S}^3) \# (\#^{r+k}(\mathbf{S}^2 \times \mathbf{S}^2))$, where k is the rank of $H_2(Y, \Lambda)$. This follows again from [2], since the intersection forms are isomorphic.

5. Non-extended hermitian forms.

Let (G, λ) be a hermitian form over the group ring $\Lambda = \mathbf{Z}[\pi_1]$. Note that $\mathbf{Z} \subset \Lambda$. Let $\varepsilon : \Lambda \to \mathbf{Z}$ be the augmentation map. Suppose $G \cong \bigoplus^k \Lambda$. The hermitian form λ is extended from $\mathbf{Z}$ if there is a symmetric bilinear form $b : (\bigoplus^k \mathbf{Z}) \times (\bigoplus^k \mathbf{Z}) \to \mathbf{Z}$ such that (G, λ) is isomorphic to $(G, \bar{b})$, where $\bar{b} : (\bigoplus^k \mathbf{Z}) \otimes_{\mathbf{Z}} \Lambda \times (\bigoplus^k \mathbf{Z}) \otimes_{\mathbf{Z}} \Lambda \to \Lambda$ is defined by $\bar{b}(x \otimes \alpha, y \otimes \beta) = \bar{\alpha} b(x, y) \beta$ for $x, y \in \bigoplus^k \mathbf{Z}$, $\alpha, \beta \in \Lambda$. This construction is a special case of the "change of ring"-construction (see [7]). One can apply this construction to the augmentation homomorphism $\varepsilon : \Lambda \to \mathbf{Z}$. With respect to an associated matrix $B = (b_{ij}) \in M(k, k; \Lambda)$ one gets the matrix $\varepsilon(B) = (\varepsilon(b_{ij})) \in M(k, k; \mathbf{Z})$. It becomes clear that if (G, λ) is extended, it must be extended from this $\mathbf{Z}$-bilinear form. A connected sum $\#^p(\mathbf{S}^1 \times \mathbf{S}^3) \# M'$, $\pi_1(M') = \{1\}$, has an intersection form over Λ which is extended from the $\mathbf{Z}$-intersection form of $H_2(M', \mathbf{Z})$. Even if there are many non-singular hermitian forms over Λ which are not extended from forms over $\mathbf{Z}$, concrete examples seem to be rare. Here we report an example of Quebbemann used in [5] to construct a 4-manifold X with $\pi_1(X) \cong \mathbf{Z}$ and $H_2(X, \Lambda) \cong \bigoplus^4 \Lambda$ which is not a connected sum $(\mathbf{S}^1 \times \mathbf{S}^3) \# M'$, $\pi_1(M') = \{1\}$. Since we will consider $\pi_1 = *^p \mathbf{Z}$, $p \geq 1$, the calculations are slightly different. Let $h \in \pi_1$ be an arbitrary element $\neq 1$. Denote $t = h + h^{-1}$ and let

$$L = \begin{pmatrix} 1+t+t^2 & t+t^2 & 1+t & t \\ t+t^2 & 1+t+t^2 & t & 1+t \\ 1+t & t & 2 & 0 \\ t & 1+t & 0 & 2 \end{pmatrix}.$$

As is indicated in [5], we have $\det(L) = 1$ and $\varepsilon(L)$ is equivalent to the standard form

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let e_1, e_2, e_3, e_4 be the standard basis of $\bigoplus^4 \Lambda$. Let λ be the form defined by L. Then $\lambda(e_1, e_1) = 1 + t + t^2$. The extended form of A is the standard form μ on $\bigoplus^4 \Lambda$. Note that $\mu(v, v) = \sum \bar{v}_i v_i$ if $v = (v_1, v_2, v_3, v_4) \in \bigoplus^4 \Lambda$. The proof of the non-extendibility consists in showing that there does not exist a vector $v \in \bigoplus^4 \Lambda$ such that $\mu(v, v) = 1 + t + t^2$. Let us write $v_i = \sum_g n_i(g) \in \Lambda$. Then $1 + t + t^2 = 3 + h + h^{-1} + h^2 + h^{-2} = \sum_{i=1}^4 \bar{v}_i v_i = \sum_{i,g,g'} n_i(g) n_i(g') g' g^{-1}$ would imply $3 = \sum_{i,g} n_i(g)^2$, hence $n_i(g) \neq 0$ only for three cases (i, g). If $n_i(g) \neq 0$, then $n_i(g) = \pm 1$. Let us suppose that $n_{i_1}(g_1) = \pm 1$, $n_{i_2}(g_2) = \pm 1$, $n_{i_3}(g_3) = \pm 1$. As in [5] we distinguish three cases:

1. case i_1, i_2, i_3 are distinct: then we have $v_{i_1} = \pm g_1$, $v_{i_2} = \pm g_2$, $v_{i_3} = \pm g_3$. It follows $\sum \bar{v}_i v_i = 3 \neq 3 + h + h^{-1} + h^2 + h^{-2}$.

2. case $i_1 = i_2 \neq i_3$: then we have $v_{i_1} = \varepsilon g_1 + \varepsilon' g_2$ with $\varepsilon, \varepsilon' \in \{\pm 1\}$ and $v_{i_3} = \pm g_3$. One obtains $\sum \bar{v}_i v_i = 3 + \varepsilon\varepsilon'(g_1 g_2^{-1} + g_2 g_1^{-1})$. Write $g = g_1 g_2^{-1}$, then we must have

$$\varepsilon\varepsilon'(g + g^{-1}) = 3 + h + h^{-1} + h^2 + h^{-2}$$

in Λ with $g, h \in \pi_1$. This cannot hold either.

3. case $i_1 = i_2 = i_3$: here we have $v_{i_1} = \varepsilon g_1 + \varepsilon' g_2 + \varepsilon'' g_3$ with $\varepsilon, \varepsilon', \varepsilon'' \in \{\pm 1\}$. Let us write $x = \varepsilon g_1$, $y = \varepsilon' g_2$, $z = \varepsilon'' g_3$. Then we obtain $\bar{v}_{i_1} v_{i_1} = \bar{x}x + \bar{y}y + \bar{z}z = 3 + xy^{-1} + xz^{-1} + yx^{-1} + yz^{-1} + zx^{-1} + zy^{-1}$. Putting $\alpha = xy^{-1}$, $\beta = xz^{-1}$, $\gamma = yz^{-1}$ we get the condition

$$\alpha + \alpha^{-1} + \beta + \beta^{-1} + \gamma + \gamma^{-1} = 3 + h + h^{-1} + h^2 + h^{-2}$$

in Λ with $\alpha, \beta, \gamma \in \pi_1$ (up to sign), which cannot hold.

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