A Lyapunov-Schmidt method for transition layers in reaction-diffusion systems[1]

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Abstract. We study reaction-diffusion systems of propagator-controller type in the one-dimensional unit interval. When propagator diffuses slowly, we establish the existence of transition layer equilibria by using singular perturbation expansions and a Lyapunov-Schmidt reduction method. Our approach to the existence also enables us to simultaneously obtain a stability criterion for the layer equilibria.

1. Introduction and main results

1.1. Reaction-diffusion system. We study the following one-dimensional system of reaction-diffusion equations

$$(1.1) \qquad \begin{cases} u_t = \varepsilon^2 u_{xx} + f(u,v) \\ v_t = D v_{xx} + g(u,v) \end{cases} \quad (x \in (0,1), t > 0),$$

under the homogeneous Neumann boundary conditions

$$(1.2) \qquad u_x(0,t) = u_x(1,t) = 0 = v_x(0,t) = v_x(1,t).$$

Systems like (1.1) have been employed in many fields [4, 5, 9, 12, 15, 21] to study pattern formation phenomena from a mathematical viewpoint.

When $\varepsilon > 0$ is small and the ODE $u_t = f(u,v)$ is bistable for each v fixed in some interval, we expect that solutions of (1.1)–(1.2) will develop transition layers. To see this, let us consider a specific example of reaction kinetics (f,g);

$$(1.3) \qquad \begin{cases} f(u,v) = (1-u^2)(u - h^0(v)) \quad \text{with } h^0(v) = q \tanh(v - \gamma), \\ g(u,v) = 2u - v, \end{cases}$$

with $|q| \le 1$. Formally setting $\varepsilon = 0$ in (1.1), we obtain

$$(1.4) \qquad \begin{cases} u_t = (1-u^2)(u - h^0(v)) \\ v_t = D v_{xx} + 2u - v \end{cases} \quad (x \in (0,1), t > 0).$$

Notice that $u = \pm 1$ (to be called $h^{\pm}(v)$ later) are asymptotically stable equilibria for the ordinary differential equation $u_t = (1 - u^2)(u - h^0(v))$ with each v being fixed. This suggests that solutions $(u(t, x), v(t, x))$ of (1.4) will generically behave as $\lim_{t \to \infty} u(t, x) = \pm 1$ for most $x \in [0, 1]$. For large t, $u(t, x)$ is expected to exhibit a sharp transition behavior, from $u(t, \cdot) \approx -1$ to $u(t, \cdot) \approx +1$, in a neighborhood of some points $x^* \in [0, 1]$.

Our objective in this paper is to show that, for a general class of functions f, g, there are equilibria of (1.1), (1.2) which exhibit transition layers for $0 < \varepsilon \ll 1$. The equilibria of (1.1) and (1.2) satisfy the equations

(1.5)
$$\begin{cases} 0 = \varepsilon^2 u_{xx} + f(u, v), \quad 0 = Dv_{xx} + g(u, v) \quad (x \in (0, 1)), \\ u_x(x) = 0 = v_x(x) \quad \text{at } x = 0, 1, \end{cases}$$

for $0 < \varepsilon \ll 1$. Stability properties of these solutions also will be determined. Our approach is based on a *singular perturbation method*, namely, we first examine *reduced solutions* of (1.5) for "$\varepsilon = 0$", and then construct solutions for small $\varepsilon > 0$ by a perturbation argument.

Let us outline the construction of transition layer equilibria with a single transition, by using the nonlinearity (f, g) of (1.3). We first consider a *reduced problem*

(1.6)
$$\begin{cases} 0 = (1 - u^2)(u - h^0(v)) & x \in (0, 1) \\ 0 = Dv_{xx} + 2u - v \\ u_x(x) = 0 = v_x(x) \quad \text{at } x = 0, 1. \end{cases}$$

As a solution of the first equation in (1.6), we choose

$$u = U^*(x) := \begin{cases} -1 & x \in [0, x^*] \\ +1 & x \in (x^*, 1] \end{cases},$$

where $x^* \in (0, 1)$ is the location of transition layer, which is to be determined.

This solution has a jump discontinuity at $x = x^*$. To obtain a smooth solution for small $\varepsilon > 0$, we will need to have a sharp transition layer near x^*. To accomplish this, we introduce a stretched variable $z = (x - x^*)/\varepsilon$ near the transition point, and rewrite the first equation of (1.5) in terms of z;

$$0 = u_{zz} + (1 - u^2)(u - q \tanh(v - \gamma)).$$

We now seek a solution satisfying $\lim_{z \to \pm \infty} u(z) = \pm 1$. Such a solution exists only if $v = \gamma$ when $|q| \leq 1$, $q \neq 0$. Therefore, at the transition point, we should require that $v(x^*) = \gamma$.

Using the above definition of U^* in the second equation of (1.6) and using the boundary conditions for v, it is natural to consider the following problem:

$$\begin{cases} 0 = Dv_{xx} - 2 - v & x \in (0, x^*), \\ 0 = Dv_{xx} + 2 - v & x \in (x^*, 1), \\ v_x(0) = 0 = v_x(1), \quad v(x^*) = \gamma. \end{cases}$$

For each $x^* \in (0, 1)$, there is the following solution

$$(1.7) \qquad v = V^*(x) := \begin{cases} (\gamma + 2) \frac{\cosh(x/\sqrt{D})}{\cosh(x^*/\sqrt{D})} - 2 & 0 \le x \le x^* \\ (\gamma - 2) \frac{\cosh((x-1)/\sqrt{D})}{\cosh((x^*-1)/\sqrt{D})} + 2 & x^* \le x \le 1 \end{cases}.$$

It is evident that $V^* \in C^0([0,1])$. If we impose the additional condition $V^* \in C^1([0,1])$ (C^1-matching condition), then $x^* \in (0,1)$ is uniquely given by

$$(1.8) \qquad x^* := \frac{\sqrt{D}}{2} \log\left(\frac{1}{2} \sqrt{\frac{\gamma^2}{4}(e^{2/\sqrt{D}} - 1)^2 + 4e^{2/\sqrt{D}}} - \frac{\gamma}{4}(e^{2/\sqrt{D}} - 1) \right),$$

which follows immediately from (1.7). For this choice of x^*, V^* also belongs to $C^1([0,1]) \cap C^\infty([0,1] \setminus \{x^*\})$.

The main result in this paper, when applied to the specific example (1.3), is the following one.

MAIN RESULT FOR (1.3). *There exist an $\varepsilon_* > 0$ and a family of solutions $(u^\varepsilon, v^\varepsilon)$ of (1.5) for $\varepsilon \in (0, \varepsilon_*]$ with the following properties.*
(i) $\lim_{\varepsilon \to 0} v^\varepsilon(x) = V^*(x)$ *uniformly on* $[0,1]$, *where* V^* *is defined by* (1.7).
(ii) *For each* $\delta > 0$, $\delta < \min\{x^*, 1 - x^*\}$

$$\lim_{\varepsilon \to 0} u^\varepsilon(x) = \begin{cases} -1 \\ +1 \end{cases} \quad uniformly \ on \quad \begin{cases} 0 \le x \le x^* - \delta \\ x^* + \delta \le x \le 1 \end{cases},$$

where x^* is given in (1.8).
(iii) *The solution $(u^\varepsilon, v^\varepsilon)$ is asymptotically stable if $0 < q \le 1$, and unstable if $-1 \le q < 0$.*

Results of this type have appeared in a series of articles [3, 11, 8, 14] between 1976 and 1987. One purpose of our article is to show that the method of Lyapunov-Schmidt used in [6] (see also [22]) for a scalar equation can be extended to apply to systems of equations. Furthermore, it will be shown elsewhere (cf. [16, 17]) that this method can be adapted to deal with internal layers for a multi-space-dimensional version of (1.1). Recently, Lin [10] developed a geometric-dynamical system appoach, combined with asymptotic expansions, to successfully discuss one-space-dimensional and multi-component systems including (1.1).

The method presented in this article also can be extended to apply to the evolution of internal layers for the parabolic system (1.1), (1.2). This problem

was discussed in [2, 1] when f, g have some specific types of monotonicity. With the extension of our method to parabolic systems, it is possible to have more general f, g which include the specific example (1.3). A preliminary version of such an extension is presented in [18, 19, 20].

1.2. Transition layer equilibria. We now state conditions and results in general terms. Throughout the paper, we let the nonlinear functions f and g satisfy the condtions listed below.

(A1) The function f is smooth on $\mathbf{R}^2$, and the ordinary differential equation $\dot{u} = f(u, v)$ is bistable in u for each fixed $v \in (\underline{v}, \bar{v}) =: I_0$. Namely, $f(u, v) = 0$ has exactly three zeros $u = h^\pm(v), h^0(v)$ for each $v \in I_0$ safisfying

$$h^-(v) < h^0(v) < h^+(v), \qquad f_u(h^\pm(v), v) < 0.$$

(A2) If, for $v \in I_0$, we define the function $J(v)$ by

$$J(v) := \int_{h^-(v)}^{h^+(v)} f(s, v)ds,$$

then there exists a $v^* \in I_0$ such that

$$J(v^*) = 0 \qquad \text{and} \qquad J'(v^*) \neq 0.$$

(A3) The function g is smooth on $\mathbf{R}^2$. If we define $g^\pm(v) := g(h^\pm(v), v)$ for $v \in I_0$, then we have

$$g^-(v) < 0 < g^+(v), \qquad g_v^\pm(v) := \frac{d}{dv}g^\pm(v) < 0 \qquad \text{for } v \in I_0.$$

(A4) The inequalities

$$f_u(u, v) + g_v(u, v) < 0 \qquad \text{at } u = h^\pm(v)$$

hold for $v \in I_0$.

Under these conditions, our main result is stated as follows.

THEOREM 1.1. *If the conditions* (A1), (A2), (A3) *and* (A4) *are satisfied, then*

(i) *there exist a constant $D_0 > 0$, determined solely by (f, g), a constant $x^* = x^*(D) \in (0, 1)$ and a family of C^1-functions $V^{*,D}$ defined for $D \geq D_0$, satisfying*

$$0 = DV_{xx}^{*,D} + \begin{cases} g(h^-(V^{*,D}), V^{*,D}) \\ g(h^+(V^{*,D}), V^{*,D}) \end{cases} \quad for \quad \begin{cases} 0 < x < x^* \\ x^* < x < 1 \end{cases}$$

and

$$V_x^{*,D}(0) = 0 = V_x^{*,D}(1), \qquad V^{*,D}(x^*) = v^*.$$

(ii)   *there exist $\varepsilon_* > 0$ and a family of solutions $(u^{\varepsilon,D}(x), v^{\varepsilon,D}(x))$ of (1.5),*
       defined for $(\varepsilon, D) \in (0, \varepsilon_*] \times [D_0, \infty)$, satisfying
       (a) $\lim\limits_{\varepsilon \to 0} v^{\varepsilon,D}(x) = V^{*,D}(x)$ *uniformly on* $[0, 1]$;
       (b) *for each* $\delta > 0$,

$$\lim_{\varepsilon \to 0} u^{\varepsilon,D}(x) = \begin{cases} h^-(V^{*,D}(x)) \\ h^+(V^{*,D}(x)) \end{cases} \quad \textit{uniformly on} \quad \begin{cases} 0 \le x \le x^* - \delta \\ x^* + \delta \le x \le 1. \end{cases}$$

(iii) *the solution $(u^{\varepsilon,D}(x), v^{\varepsilon,D}(x))$ is exponentially asymptotically stable if*
       $J'(v^*) < 0$, *and it is unstable if* $J'(v^*) > 0$.

In [11] the existence part of Theorem 1.1 was first proved by modifying a method developed in [3]. This was later improved in [8]. In these articles, the authors construct two boundary layer solutions and glue them smoothly across the transition point x^* (C^1-matching). In our approach, we do this matching while we construct approximate solutions. The stability result, Theorem 1.1 (iii), was proved much later in [14]. With our method, the stability result comes simultaneously with the existence result.

Theorem 1.1 is slightly more general than those of [11, 8, 14]. In our result, a situation $g_v(u, v) \ge 0$ at $u = h^\pm(v)$ is *allowed* to the extent that (A4) is satisfied, while $g_v(u, v) \le 0$ at $u = h^\pm(v)$ was *required* in [11, 8, 14]. For example, our theory applies to the situation where $f(u, v) = u - u^3 - v$ and $g(u, v) = u + \gamma v$, as long as $\gamma < 1/2$. This improvement comes from our derivation and treatment of SLEP (cf. §§ 5.2, 5.3).

In Theorem 1.1 (i), the function $V^{*,D}$ is similar to the one given in (1.7). Statement (ii)(b) clearly shows that our solutions exhibit a sharp transition layer at $x = x^*$. It is of interest to observe from Theorem 1.1 (iii) that the stability properties of the transition layer solutions are determined by a single quantity $J'(v^*)$, if $J'(v^*) \ne 0$. When $J'(v^*) = 0$, Theorem 1.1 is no longer valid. However, even if $J'(v^*) = 0$, our method of proof extends to establish a result similar to Theorem 1.1, if we assume an additional condition $J''(v^*) \ne 0$. Such problems will be treated elsewhere.

The proof of Theorem 1.1 consists of three steps;
- construction of approximate solutions with high degree of accuracy (§ 2.2, § 4);
- spectral analysis of linearization around the approximate solutions (§ 2.3, § 5);
- a perturbation argument: Lyapunov-Schmidt Reduction (§ 2.4).

2.   Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by using several propositions. The proofs of some of these propositions are postponed to §§ 4 and 5.

2.1.   Preliminaries.   For each $v^0 \in I_0 = (\underline{v}, \bar{v})$, we consider the reduced problem,

(2.1) $\begin{cases} 0 = DV_{xx} + g^-(V) & 0 < x < x^0, \quad V_x(0) = 0, \quad V(x^0) = v^0 \\ 0 = DV_{xx} + g^+(V) & x^0 < x < 1, \quad V_x(1) = 0, \quad V(x^0) = v^0 \\ V(\cdot) \in C^1([0,1]), \end{cases}$

where $x^0 \in (0,1)$ is a quantity to be determined so that the last condition $V(\cdot) \in C^1([0,1])$, called a C^1-*matching condition*, is fulfilled.

In multi-dimensional spaces, the problem (2.1) corresponds to a free-boundary problem (in which x^0 is replaced by a hypersurface). It is not so easy to find a solution of such problems. In the one-dimensional case, it has an easy solution.

Proposition 2.1.   *If conditions* (A1) *and* (A3) *are satisfied, then for each* $v^0 \in I_0$*, there exists a* $D_0 > 0$*, which depends only on* (f, g) *and* v^0*, so that the problem* (2.1) *has a unique solution pair* $(V^{v^0, D}(x), x^0(D, v^0))$ *for* $D \geq D_0$*, satisfying*

$$0 < x^0(D, v^0) < 1, \qquad V_x^{v^0, D}(x) > 0, \qquad x \in (0,1).$$

Proof.   In (2.1), rescaling $x \in [0,1]$ by $x - x^0 = \sqrt{D}y$ and defining $v(y) := V(x)$, we obtain the equivalent problem,

$$0 = v_{yy} + g^-(v), \quad -\frac{x^0}{\sqrt{D}} < y < 0, \qquad v_y\left(-\frac{x^0}{\sqrt{D}}\right) = 0, \quad v(0) = v^0,$$

$$0 = v_{yy} + g^+(v), \quad 0 < y < \frac{1 - x^0}{\sqrt{D}}, \qquad v_y\left(\frac{1 - x^0}{\sqrt{D}}\right) = 0, \quad v(0) = v^0.$$

We use the shooting method to find the desired C^1-matched solution. For $\alpha \in [\underline{v}, v^0]$, $\beta \in [v^0, \bar{v}]$, there are solutions of the problems

$$\begin{cases} 0 = v_{yy}^- + g^-(v^-), & -l^- < y < 0, \\ v^-(-l^-) = \alpha, \quad v_y^-(-l^-) = 0, \quad v^-(0) = v^0, \end{cases}$$

$$\begin{cases} 0 = v_{yy}^+ + g^+(v^+), & 0 < y < l^+, \\ v^+(l^+) = \beta, \quad v_y^+(l^+) = 0, \quad v^+(0) = v^0, \end{cases}$$

where $l^- > 0$ and $l^+ > 0$ are given by

$$l^-(\alpha) := \sqrt{\frac{v^0 - \alpha}{2}} \int_0^1 \frac{dt}{\sqrt{\int_t^0 g^-((v^0 - \alpha)s + \alpha)ds}},$$

$$l^+(\beta) := \sqrt{\frac{\beta - v^0}{2}} \int_0^1 \frac{dt}{\sqrt{\int_t^1 g^+((\beta - v^0)s + v^0)ds}}.$$

It is easy to verify

$$(2.2) \qquad \frac{dl^-(\alpha)}{d\alpha} < 0, \qquad \frac{dl^+(\beta)}{d\beta} > 0,$$

by using $g_v^\pm < 0$ (cf. (A3)). Rescaling back to the original x-variable will yield a solution of (2.1) if α and β are chosen so that

$$l^-(\alpha) + l^+(\beta) = \frac{1}{\sqrt{D}}.$$

Moreover, the third condition in (2.1) (C^1-matching condition) requires that $v_y^-(0) = v_y^+(0)$, in addition to the condition $v^-(0) = v^0 = v^+(0)$. Let us define $p^-(\alpha) > 0$ and $p^+(\beta) > 0$ by

$$p^-(\alpha) := v_y^-(0) = \sqrt{2\int_{v^0}^\alpha g^-(s)ds},$$

$$p^+(\beta) := v_y^+(0) = \sqrt{2\int_{v^0}^\beta g^+(s)ds},$$

and $\bar{p}$ by

$$\bar{p} := \min\left\{ \max_{\underline{v} \le \alpha \le v^0} p^-(\alpha), \max_{v^0 \le \beta \le \bar{v}} p^+(\beta) \right\}.$$

In order to fulfill $v_y^-(0) = v_y^+(0)$, it is necessary to have

$$0 \le p^-(\alpha) \le \bar{p}, \qquad 0 \le p^+(\beta) \le \bar{p}$$

satisfied. We immediately find that

$$(2.3) \qquad \frac{dp^-(\alpha)}{d\alpha} < 0, \qquad \frac{dp^+(\beta)}{d\beta} > 0,$$

by using $g^-(v) < 0 < g^+(v)$ for $v \in [\underline{v}, \bar{v}]$ (cf. (A3)). Therefore, we can express α and β as a function of $p \in [0, \bar{p}]$; $\alpha = \alpha(p)$, $\beta = \beta(p)$. By using (2.2) and (2.3), we find that

$$l(p) := l^-(\alpha(p)) + l^+(\beta(p))$$

is monotone increasing in $p \in [0, \bar{p}]$ and satisfies $l(0) = 0$. Therefore, if $D_0 := l(\bar{p})^{-2}$, then, for each $D \in [D_0, \infty)$, there exists a unique $p(D) \in [0, \bar{p}]$ so that

$$l(p) = \frac{1}{\sqrt{D}}.$$

As remarked earlier, scaling back to the original $x \in [0,1]$, we obtain the desired solution of (2.1) as follows:

$$(2.4) \qquad x^0(D, v^0) := \sqrt{D} l^-(\alpha(p(D)))$$

$$(2.5) \qquad V^{v^0, D}(x) := \begin{cases} v^-\left(\frac{x}{\sqrt{D}} - l^-(\alpha(p(D)))\right) & 0 \le x \le x^0(D, v^0), \\ v^+\left(\frac{x}{\sqrt{D}} - l^-(\alpha(p(D)))\right) & x^0(D, v^0) \le x \le 1. \end{cases} \qquad \square$$

We now *linearize* (2.1) at $(V, v^0) = (V^{v^0, D}, v^0)$.

PROPOSITION 2.2. *Assume that conditions* (A1) *and* (A3) *are satisfied.*
(i) *The boundary value problems*

$$(2.6) \qquad (-) \begin{cases} 0 = D\Psi_{xx}^- + g_v^-(V^{v^0, D}(x))\Psi^-, & 0 < x < x^0, \\ \Psi_x^-(0) = 0, \quad \Psi^-(x^0) = 1, \end{cases}$$

$$(+) \begin{cases} 0 = D\Psi_{xx}^+ + g_v^+(V^{v^0, D}(x))\Psi^+, & x^0 < x < 1, \\ \Psi_x^+(1) = 0, \quad \Psi^+(x^0) = 1, \end{cases}$$

have unique solutions satisfying

$$\Psi^-(x) > 0 \quad (0 \le x \le x^0), \qquad \Psi^+(x) > 0 \quad (x^0 \le x \le 1),$$
$$\Psi_x^-(x) > 0 \quad (0 < x \le x^0), \qquad \Psi_x^+(x) < 0 \quad (x^0 \le x < 1).$$

(ii) *If we define a constant* $\pi_0(v^0)$ *by*

$$(2.7) \qquad \pi_0(v^0) := \Psi_x^-(x^0) - \Psi_x^+(x^0) > 0,$$

then we have

$$(2.8) \quad 0 < V_x^{v^0, D}(x^0) < \frac{[g]_-^+(v^0)}{D\pi_0(v^0)} \qquad (\text{where } [g]_-^+(v^0) := g^+(v^0) - g^-(v^0) > 0).$$

PROOF. Since $g_v^\pm(v) < 0$ from (A3), the problems (2.6)($\pm$) have unique solutions.

The solution Ψ^- satisfies $\Psi_x^-(x^0) > 0$. If not, $\Psi_x^-(x^0) \le 0$, then

$$D\Psi_{xx}^-(x^0) = -g_v^-(V^{v^0, D}(x^0))\Psi^-(x^0) = -g_v^-(V^{v^0, D}(x^0)) > 0$$

implies $\Psi_x^-(x) < 0$ for x near x^0 ($x < x^0$), and hence this is true for all $x \in [0, x^0]$. This makes it impossible to fulfill the boundary condition $\Psi_x^-(0) = 0$. Hence $\Psi_x^-(x^0) > 0$. The same reasoning shows that for any $x \in (0, x^0]$, $\Psi^-(x) > 0$ implies $\Psi_x^-(x) > 0$.

Now, we will show that $\Psi^-(x) > 0$ for all $x \in [0, x^0]$. If not, there exists $x_1 \in [0, x^0)$ such that $\Psi^-(x_1) = 0$, $\Psi^-(x) > 0$ for all $x \in (x_1, x^0]$ and

$\Psi_x^-(x_1) \geq 0$. If $\Psi_x^-(x_1) = 0$, then $\Psi^-(x) \equiv 0$ follows from the uniqueness of solutions for the initial value problem. Therefore, we have $\Psi_x^-(x_1) > 0$, and hence

$$\Psi^-(x) < 0 \qquad \text{and} \qquad D\Psi_{xx}^-(x) = -g_v^-(V^{v^0,D}(x))\Psi^-(x) < 0$$

for x near x_1 with $x < x^1$. This yields $\Psi_x^-(x) > 0$ for x near x_1 $(x \leq x_1)$, and hence for all $x \in [0, x_1]$. This contradicts the boundary condition $\Psi_x^-(0) = 0$. We have thus established the statements in (i) for Ψ^-.

Similar arguments apply to Ψ^+. This completes the proof of Proposition 2.2 (i).

To prove the statement (ii), let us define $\hat{v}(x)$ by

$$\hat{v}(x) := \frac{V_x^{v^0,D}(x)}{V_x^{v^0,D}(x^0)}.$$

This function satisfies the differential equations in (2.6) on $0 < x < x^0$ and $x^0 < x < 1$, together with the boundary conditions

$$\hat{v}(x^0) = 1, \qquad \hat{v}(0) = 0 = \hat{v}(1).$$

If $w(x) := \Psi^-(x) - \hat{v}(x)$ for $x \in [0, x^0]$, then w satisfies $(2.6)(-)$ and $w(0) > 0$, $w(x^0) = 0$. Since $g_v^- < 0$, the maximum principle implies $w(x) > 0$ for $x \in [0, x^0)$, and hence $w_x(x^0) < 0$. Therefore, we have

$$(2.9) \qquad\qquad 0 < \Psi_x^-(x^0) < \hat{v}_x(x^0 - 0).$$

Similar arguments apply to Ψ^+, giving rise to

$$(2.10) \qquad\qquad 0 > \Psi_x^+(x^0) > \hat{v}_x(x^0 + 0).$$

From (2.9) and (2.10), we obtain

$$(2.11) \qquad 0 < \Psi_x^-(x^0) - \Psi_x^+(x^0) < \hat{v}_x(x^0 - 0) - \hat{v}_x(x^0 + 0)$$

$$= \frac{1}{V_x^{v^0,D}(x^0)}[V_{xx}^{v^0,D}(x^0 - 0) - V_{xx}^{v^0,D}(x^0 + 0)]$$

$$= \frac{1}{V_x^{v^0,D}(x^0)}\frac{1}{D}[g^+(v^0) - g^-(v^0)],$$

which is equivalent to (2.8). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For our discussion below, the solution of (2.1) with $v^0 = v^*$, where $v^* \in I_0$ is as in (A2), is of particular importance. We denote this solution by

$$x^* := x^0(D, v^*)$$

$$V^{*,D}(x) := V^{v^*,D}(x)$$

We also introduce simplified expressions $\pi_0 := \pi_0(v^*)$ and $[g]^+_- := [g]^+_-(v^*)$. The relations (2.7) and (2.8) are expressed as

$$\text{(2.12)} \qquad\qquad \pi_0 = \Psi^-_x(x^*) - \Psi^+_x(x^*) > 0,$$

$$\text{(2.13)} \qquad\qquad 0 < V^{*,D}_x(x^*) < \frac{[g]^+_-}{D\pi_0}.$$

The pair of functions $(V^{*,D}(x), U^{*,D}(x))$, where $U^{*,D}$ is defined by

$$U^{*,D}(x) := \begin{cases} h^-(V^{*,D}(x)) & 0 \le x \le x^* \\ h^+(V^{*,D}(x)) & x^* < x \le 1 \end{cases},$$

is a building block to construct transition layer solutions of (1.5).

For each $v \in I_0 := (\underline{v}, \bar{v})$, let us consider the determination of eigenpair $(Q(z,v), c(v))$ of the problem

$$\text{(2.14)} \qquad \begin{cases} Q_{zz} + c(v)Q_z + f(Q,v) = 0 & z \in \mathbf{R}, \\ Q(\pm\infty; v) = h^\pm(v), \quad Q(0;v) = h^0(v). \end{cases}$$

PROPOSITION 2.3. *Under the condition* (A1), *the problem* (2.14) *has a unique solution pair* $(Q(z;v), c(v))$ *for each* $v \in I_0$ *with the following properties.*

(i) *The function* $Q(z; v)$ *(respectively,* $Q_z(z;v)$*) approaches the limit* $h^\pm(v)$ *(respectively, zero) at an exponential rate as* $z \to \pm\infty$, *and* $Q_z(z;v) > 0$ *for* $z \in \mathbf{R}$.

(ii) *If we define* $m(v) := \int_{\mathbf{R}} Q_z(z;v)^2 dz > 0$ *for* $v \in I_0$, *then the function* $c(v)$ *is explicitly given by*

$$c(v) = -\frac{J(v)}{m(v)}.$$

(iii) *If, in addition,* (A2) *is satisfied, then we have*

$$c'(v^*) = -\frac{J'(v^*)}{m(v^*)}.$$

PROOF. For each fixed $v \in I_0$, we write the differential equation in (2.14) as a first order system;

$$\text{(2.15)} \qquad \begin{cases} Q_z = P \\ P_z = -f(Q,v) - cP \end{cases},$$

where $c \in \mathbf{R}$ is a free parameter. In the Q-P phase plane for (2.15), one recognizes immediately that $(Q, P) = (h^\pm(v), v)$ are hyperbolic equilibria for all $c \in \mathbf{R}$. Let us consider the unstable manifold of $(h^-(v), v)$, expressed as the graph of a function $P = P^-(Q, c, v) \ge 0$. We also consider the stable manifold of $(h^+(v), 0)$, expressed as $P = P^+(Q, c, v) \ge 0$. We need to find $c \in \mathbf{R}$ so that

(2.16) $P^-(h^0(v), c, v) = P^+(h^0(v), c, v).$

One can then verify that $P^-(h^0(v), c, v)$ is monotone decreasing in c and that $P^+(h^0(v), c, v)$ is monotone increasing in c;

$$\begin{cases} \lim_{c \to +\infty} P^-(h^0(v), c, v) = 0, & \lim_{c \to -\infty} P^-(h^0(v), c, v) = +\infty, \\ \lim_{c \to +\infty} P^+(h^0(v), c, v) = +\infty, & \lim_{c \to -\infty} P^+(h^0(v), c, v) = 0. \end{cases}$$

This ensures that there exists a unique value $c = c(v)$ so that (2.16) holds. Now, for this value of c, our solution $Q(z; v)$ is the function representing the heteroclinic orbit connecting $(h^-(v), 0)$ at $z = -\infty$ to $(h^+(v), 0)$ at $z = +\infty$. We have thus completed the proof of statement (i).

To prove statement (ii), multiply the differential equation in (2.14) by Q_z and integrate over $\mathbf{R}$. It then follows that

$$m(v)c(v) = -\int_{-\infty}^{\infty} f(Q(z), v)Q_z(z)dz = -J(v).$$

To prove (iii), we simply differentiate the relation in (ii) with respect to v at $v = v^*$ and use the fact $J(v^*) = 0$ if (A2) is satisfied.   □

2.2. Approximate solutions.

We state the existence of approximate solutions which solve (1.5) with an arbitrarily high degree of accuracy. The degree of the accuracy is measured by ε^α for $\alpha > 0$.

PROPOSITION 2.4. *Assume that the conditions* (A1), (A2) *and* (A3) *are satisfied. For each integer $k \geq 0$, there exists a family of smooth (C^∞) functions $(u_k^{\varepsilon, D}(x), v_k^{\varepsilon, D}(x))$, defined for $\varepsilon > 0$, $D \geq D_0$, that satisfies;*

(i) $\lim_{\varepsilon \to 0} v_k^{\varepsilon, D}(x) = V^{*, D}(x)$ *uniformly on* $[0, 1]$;

(ii) *for each $\delta > 0$, $\delta < \min\{x^*, 1 - x^*\}$,*

$$\lim_{\varepsilon \to 0} u_k^{\varepsilon, D}(x) = \begin{cases} h^-(V^{*, D}(x)) \\ h^+(V^{*, D}(x)) \end{cases} \quad \textit{uniformly on} \quad \begin{cases} 0 \leq x \leq x^* - \delta; \\ x^* + \delta \leq x \leq 1 \end{cases};$$

(iii) *for each $\beta \in (0, 1]$ there exists a constant $C_{k, \beta} > 0$, independent of small $\varepsilon > 0$, such that*

$$\|\mathscr{R}_i^\varepsilon\|_{L^\infty(0, 1)} \leq C_{k, \beta}\varepsilon^{k+1-\beta} \quad \textit{as } \varepsilon \to 0 \qquad (i = 1, 2),$$

where

$$\mathscr{R}_1^\varepsilon := \varepsilon^2(u_k^{\varepsilon, D})_{xx} + f(u_k^{\varepsilon, D}, v_k^{\varepsilon, D}),$$

$$\mathscr{R}_2^\varepsilon := D(v_k^{\varepsilon, D})_{xx} + g(u_k^{\varepsilon, D}, v_k^{\varepsilon, D}).$$

This will be proved in §4.

REMARK 2.1. *In the statement of Proposition* 2.4 (iii), *we could choose* $\beta = 0$, *if we were only content with the approximation being* C^0-matched *at* $x = x^*$. *By making the approximate solutions smooth across* $x = x^*$, *we slightly lose the degree of approximation. Compare Remark* 4.1 *at the end of* §4.

In order to prove Theorem 1.1, we will need Proposition 2.4 only for $k = 2$. The reason why we present the proposition for general $k \in \mathbf{N}$ is that we do not know in advance how accurate our approximations should be for successfully establishing the existence of true solutions nearby. The degree of accuracy of approximation has to be determined accroding to the magnitude of linear part. In fact, the reason why it suffices to use Proposition 2.4 with $k = 2$ for the proof of Theorem 1.1 comes from the fact that small eigenvalues of the linearization of (1.5) around our approximation behaves like $O(\varepsilon)$ as $\varepsilon \to 0$. The latter fact will be established in the next subsection.

2.3. Spectral analysis. The proof of Theorem 1.1 (iii) will follow from a spectral analysis of the linearization of (1.1) around the transition layer solutions $(u^{\varepsilon,D}, v^{\varepsilon,D})$. This involves an analysis of the eigenvalue problem

$$(2.17) \qquad \lambda\Phi = \mathscr{L}^{\varepsilon}\Phi, \qquad \Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \qquad \mathscr{L}^{\varepsilon} := \begin{pmatrix} L^{\varepsilon} & f_v^{\varepsilon} \\ g_u^{\varepsilon} & M^{\varepsilon} \end{pmatrix}$$

where $f_v^{\varepsilon} := f_v^{\varepsilon}(u^{\varepsilon,D}, v^{\varepsilon,D})$, $g_u^{\varepsilon} := g_u^{\varepsilon}(u^{\varepsilon,D}, v^{\varepsilon,D})$,

$$L^{\varepsilon} := \varepsilon^2 \frac{d^2}{dx^2} + f_u^{\varepsilon}(u^{\varepsilon,D}, v^{\varepsilon,D}), \qquad M^{\varepsilon} := D\frac{d^2}{dx^2} + g_v^{\varepsilon}(u^{\varepsilon,D}, v^{\varepsilon,D})$$

and the Neumann boundary conditions are imposed.

However, the desired family of solutions $(u^{\varepsilon,D}(x), v^{\varepsilon,D}(x))$ is not yet available. Therefore, we linearize (1.1) around the approximate solutions $(u_k^{\varepsilon,D}(x), v_k^{\varepsilon,D}(x))$ given in Proposition 2.4, and consider an eigenvalue problem

$$(2.18) \qquad\qquad\qquad \lambda\Phi = \mathscr{L}_k^{\varepsilon}\Phi$$

with Neumann boundary conditions, where $\mathscr{L}_k^{\varepsilon}$ has the same form as $\mathscr{L}^{\varepsilon}$, except that $(u^{\varepsilon,D}(x), v^{\varepsilon,D}(x))$ is replaced by the approximation $(u_k^{\varepsilon,D}(x), v_k^{\varepsilon,D}(x))$ of order $k \geq 0$. We then show that the information obtained for (2.18) can be used to prove Theorem 1.1 (iii), as well as Theorem 1.1 (ii).

PROPOSITION 2.5. *Assume that the conditions* (A1), (A2), (A3) *and* (A4) *are satisfied. Let* k, *the order of approximation in Proposition* 2.4, *satisfy* $k \geq 1$.
  (i)   *There exists* $\lambda_* > 0$ *so that there is only one eigenvalue (called a critical eigenvalue) of* (2.18) *in* $\mathbf{C}_{\lambda_*} := \{\lambda \in \mathbf{C}; \operatorname{Re} \lambda > -\lambda_*\}$.
  (ii) *The critical eigenvalue* $\lambda_0^{\varepsilon} \in \mathbf{C}_{\lambda_*}$ *of* (2.18) *is real, simple and has the following behavior as* $\varepsilon \to 0$:

$$\lambda_0^\varepsilon = \varepsilon\hat{\lambda}_0 + o(\varepsilon), \qquad \hat{\lambda}_0 := c'(v^*)\left[V_x^{*,D}(x^*) - \frac{[g]_-^+}{D\pi_0}\right].$$

(iii) *An L^2-normalized eigenfunction Φ_0^ε of $\mathscr{L}_k^\varepsilon$ associated with λ_0^ε enjoys the following property:*

$$\|\Phi_0^\varepsilon\|_{L^1(0,1)} = O(\sqrt{\varepsilon}), \qquad \|\Phi_0^\varepsilon\|_{L^\infty(0,1)} = O\left(\frac{1}{\sqrt{\varepsilon}}\right).$$

(iv) *Let $\hat{\mathscr{L}}_k^\varepsilon$ be the L^2-adjoint of $\mathscr{L}_k^\varepsilon$. An L^2-normalized eigenfunction $\hat{\Phi}_0^\varepsilon$ of $\hat{\mathscr{L}}_k^\varepsilon$ associated with λ_0^ε can be chosen so that the following conditions are fulfilled.*

$$\langle\hat{\Phi}_0^\varepsilon, \Phi_0^\varepsilon\rangle = 1, \qquad \|\hat{\Phi}_0^\varepsilon\|_{L^1(0,1)} = O(\sqrt{\varepsilon}), \qquad \|\hat{\Phi}_0^\varepsilon\|_{L^\infty(0,1)} = O\left(\frac{1}{\sqrt{\varepsilon}}\right),$$

where $\langle\cdot,\cdot\rangle$ stands for the L^2-inner product.

The proof of Proposition 2.5 is given in §5.
We now introduce function spaces.

$$\mathbf{X} := [H_N^2(0,1)]^2, \qquad \mathbf{Y} := [L^2(0,1)]^2,$$

where

$$H_N^2(0,1) := \{u \in H^2(0,1) \mid u_x(0) = 0 = u_x(1)\}.$$

By using Proposition 2.5, we decompose $\mathbf{Y}$ as

(2.19) $$\mathbf{Y} = [\Phi_0^\varepsilon] \oplus \mathbf{N}, \qquad \mathbf{N} := [\hat{\Phi}_0^\varepsilon]^\perp = \text{range}(\mathscr{L}_k^\varepsilon - \lambda_0^\varepsilon),$$

and $\mathbf{X}$ as

(2.20) $$\mathbf{X} = [\Phi_0^\varepsilon]^2 \oplus \mathbf{M}, \qquad \mathbf{M} := \mathbf{X} \cap \mathbf{N}.$$

Then $\mathscr{L}_k^\varepsilon : \mathbf{M} \to \mathbf{N}$ is not only an isomorphism, but also satisfies the following property.

PROPOSITION 2.6. *Let $k \geq 1$. There exists a $C > 0$, independent of $0 < \varepsilon \ll 1$, such that*

$$\|\mathbf{p}\|_{L^\infty(0,1)} \leq C\|\mathscr{L}_k^\varepsilon\mathbf{p}\|_{L^\infty(0,1)} \quad \forall\mathbf{p} \in \mathbf{M}, \qquad 0 < \forall\varepsilon \ll 1.$$

This will be proved in §5.

2.4. Lyapunov-Schmidt Reduction. In this subsection, we choose the order of approximation $k \geq 2$ and look for solutions of (1.5) in the following form.

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathscr{W}_k^{\varepsilon,D} + \mathbf{p} \qquad \text{with } \mathscr{W}_k^{\varepsilon,D} := \begin{pmatrix} u_k^{\varepsilon,D} \\ v_k^{\varepsilon,D} \end{pmatrix} \text{ and } \mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.$$

In terms of $\mathbf{p}$, (1.5) is equivalent to

$$(2.21) \qquad \mathscr{L}_k^\varepsilon \mathbf{p} + \mathscr{F}^\varepsilon(\mathbf{p}) = \mathscr{R}^\varepsilon,$$

where

$$\mathscr{F}^\varepsilon(\mathbf{p}) := \begin{bmatrix} f(\mathscr{W}_k^{\varepsilon,D} + \mathbf{p}) - f(\mathscr{W}_k^{\varepsilon,D}) - f_u(\mathscr{W}_k^{\varepsilon,D})p_1 - f_v(\mathscr{W}_k^{\varepsilon,D})p_2 \\ g(\mathscr{W}_k^{\varepsilon,D} + \mathbf{p}) - g(\mathscr{W}_k^{\varepsilon,D}) - g_u(\mathscr{W}_k^{\varepsilon,D})p_1 - g_v(\mathscr{W}_k^{\varepsilon,D})p_2 \end{bmatrix} = O(|\mathbf{p}|^2)$$

and

$$\mathscr{R}^\varepsilon := \begin{pmatrix} \mathscr{R}_1^\varepsilon \\ \mathscr{R}_2^\varepsilon \end{pmatrix}$$

with $\mathscr{R}_j^\varepsilon$, $j = 1, 2$, being defined in Proposition 2.4 (iii).

According to the decompositions in (2.19) and (2.20), we further look for solutions of (2.21) in the form $\mathbf{p} = p\Phi_0^\varepsilon + \mathbf{w}$, where $p \in \mathbf{R}$ and $\mathbf{w} \in \mathbf{M}$. Equation (2.21) becomes

$$(2.22) \qquad \begin{cases} \lambda_0^\varepsilon p + \langle \hat{\boldsymbol{\Phi}}_0^\varepsilon, \mathscr{F}^\varepsilon(p\Phi_0^\varepsilon + \mathbf{w}) \rangle = \langle \hat{\boldsymbol{\Phi}}_0^\varepsilon, \mathscr{R}^\varepsilon \rangle, \\ \mathscr{L}_k^\varepsilon \mathbf{w} + (I - \mathscr{E})\mathscr{F}^\varepsilon(p\Phi_0^\varepsilon + \mathbf{w}) = (I - \mathscr{E})\mathscr{R}^\varepsilon, \end{cases}$$

where $\mathscr{E} : \mathbf{Y} \to [\Phi_0^\varepsilon]$ is the projection defined by

$$\mathscr{E}\mathbf{p} := \langle \hat{\boldsymbol{\Phi}}_0^\varepsilon, \mathbf{p} \rangle \Phi_0^\varepsilon.$$

By using Proposition 2.4 (iii) with $\beta = 1/2$ and the estimates on $\hat{\boldsymbol{\Phi}}_0^\varepsilon$ from Proposition 2.5 (iv), we have

$$\|\mathscr{E}\mathscr{R}^\varepsilon\|_{L^\infty} = |\langle \hat{\boldsymbol{\Phi}}_0^\varepsilon, \mathscr{R}^\varepsilon \rangle| \cdot \|\Phi_0^\varepsilon\|_{L^\infty} \leq \|\mathscr{R}^\varepsilon\|_{L^\infty} |\langle \hat{\boldsymbol{\Phi}}_0^\varepsilon, 1 \rangle| O(1/\sqrt{\varepsilon})$$

$$= O(\varepsilon^{k+1/2}) O(\sqrt{\varepsilon}) O(1/\sqrt{\varepsilon}) = O(\varepsilon^{k+1/2}),$$

$$\|(I - \mathscr{E})\mathscr{R}^\varepsilon\|_{L^\infty} = O(\varepsilon^{k+1/2}), \qquad |\langle \hat{\boldsymbol{\Phi}}_0^\varepsilon, \mathscr{R}^\varepsilon \rangle| = O(\varepsilon^{k+1}).$$

It also follows, from $\|\Phi_0^\varepsilon\|_{L^\infty} = O(1/\sqrt{\varepsilon})$, that

$$\|\mathscr{F}^\varepsilon(p\Phi_0^\varepsilon + \mathbf{w})\|_{L^\infty} = O\left(\left(\frac{|p|}{\sqrt{\varepsilon}} + \|\mathbf{w}\|_{L^\infty} \right)^2 \right),$$

which suggests us to introduce $\hat{p}$ via $p = \sqrt{\varepsilon}\hat{p}$. From (2.22), $(\hat{p}, \mathbf{w})$ satisfy

$$(2.23) \qquad \begin{cases} \lambda_0^\varepsilon \hat{p} + \varepsilon^{-1/2} \langle \hat{\boldsymbol{\Phi}}_0^\varepsilon, \mathscr{F}^\varepsilon(\sqrt{\varepsilon}\hat{p}\Phi_0^\varepsilon + \mathbf{w}) \rangle = \varepsilon^{-1/2} \langle \hat{\boldsymbol{\Phi}}_0^\varepsilon, \mathscr{R}^\varepsilon \rangle, \\ \mathscr{L}_k^\varepsilon \mathbf{w} + (I - \mathscr{E})\mathscr{F}^\varepsilon(\sqrt{\varepsilon}\hat{p}\Phi_0^\varepsilon + \mathbf{w}) = (I - \mathscr{E})\mathscr{R}^\varepsilon. \end{cases}$$

We then have

$$\|(I - \mathscr{E})\mathscr{F}^\varepsilon(\sqrt{\varepsilon}\hat{p}\Phi_0^\varepsilon + \mathbf{w})\|_{L^\infty} = O((|\hat{p}| + \|\mathbf{w}\|_{L^\infty})^2).$$

Applying the implicit function theorem to the second equation in (2.23), and using Proposition 2.6, we obtain

(2.24) $\mathbf{w} = \mathbf{w}(\hat{p}, \varepsilon)$ where $\|\mathbf{w}(\hat{p}, \varepsilon)\|_{L^\infty} = O(|\hat{p}|^2 + \varepsilon^{k+1/2})$.

Substituting this into the first equation in (2.23), we finally arrive at:

(2.25) $\lambda_0^\varepsilon \hat{p} + B_2(\hat{p}, \varepsilon) = B_0(\varepsilon)$

where

$$B_2(\hat{p}, \varepsilon) := \frac{1}{\sqrt{\varepsilon}} \langle \hat{\Phi}_0^\varepsilon, \mathscr{F}^\varepsilon(\sqrt{\varepsilon}\hat{p}\Phi_0^\varepsilon + \mathbf{w}) \rangle$$

$$B_0(\varepsilon) := \frac{1}{\sqrt{\varepsilon}} \langle \hat{\Phi}_0^\varepsilon, \mathscr{R}^\varepsilon \rangle.$$

It is now evident that

$$|B_2(\hat{p}, \varepsilon)| = O(|\hat{p}|^2 + \varepsilon^{2k+1}), \qquad |B_0(\varepsilon)| = O(\varepsilon^{k+1/2}).$$

From Proposition 2.5 (ii), $\lambda_0^\varepsilon = \varepsilon\hat{\lambda}_0 + o(\varepsilon)$ with $\hat{\lambda}_0 \neq 0$. By scaling $\hat{p} = \varepsilon\rho$, (2.25) reduces to

(2.26) $(\hat{\lambda}_0 + o(1))\rho + \dfrac{1}{\varepsilon^2} B_2(\varepsilon\rho, \varepsilon) = \dfrac{1}{\varepsilon^2} B_0(\varepsilon)$.

If we choose $k \geq 2$, then we have

$$\frac{1}{\varepsilon^2} B_2(\varepsilon\rho, \varepsilon) = O(|\rho|^2 + \varepsilon^{2(k-1)+1}), \qquad \frac{1}{\varepsilon^2} B_0(\varepsilon) = O(\varepsilon^{k-3/2}) \quad \text{as } \varepsilon \to 0.$$

For $k \geq 2$, applying the Implicit Function Theorem to (2.26), we obtain a constant $\varepsilon_* > 0$ and a unique family of solutions $\rho = \rho^\varepsilon$ defined for $0 \leq \varepsilon \leq \varepsilon_*$. This immediately gives rise to a family of solutions of (2.25)

$$\hat{p}^\varepsilon = \varepsilon\rho^\varepsilon = \varepsilon O(\varepsilon^{k-3/2}) = O(\varepsilon^{k-1/2}).$$

By using (2.24), we have the estimate

$$\|\mathbf{w}(\hat{p}, \varepsilon)\|_{L^\infty} = O(\varepsilon^{k+1/2}).$$

Therefore, we obtain a unique family of solutions $\mathbf{p}^\varepsilon$ to (2.21) with

(2.27) $\|\mathbf{p}^\varepsilon\|_{L^\infty} = O(\varepsilon^{k-1/2})$.

This, together with Proposition 2.4, completes the proof of Theorem 1.1 (i) and (ii).

The estimate (2.27) also implies that the critical eigenvalue of $\mathscr{L}_k^\varepsilon$ and

that of $\mathscr{L}^{\varepsilon}$ are $O(\varepsilon^{k-1/2})$ away from each other. Since we have chosen $k \geq 2$, by making $\varepsilon_* > 0$ smaller if necessary, $O(\varepsilon^{k-1/2}) \ll \lambda_0^{\varepsilon}$ holds for $\varepsilon \in (0, \varepsilon_*]$. Therefore the statements (i), (ii), (ii) and (iv) of Proposition 2.5 are valid for $\mathscr{L}^{\varepsilon}$. Because of (2.13) and Proposition 2.3 (iii), the sign of $\hat{\lambda}_0$ in Proposition 2.5 (ii) is the same as that of $J'(v^*)$. This implies Theorem 1.1 (iii).

We emphasize again that to prove Theorem 1.1 we need Proposition 2.4 only for $k = 2$. However, as a merit of Proposition 2.4 for general $k \in \mathbf{N}$, the proof of Theorem 1.1 above gives the following.

THEOREM 2.1.   *For each $k \geq 2$ there exists a constant $C_k > 0$, independent of small $\varepsilon > 0$, so that the solution $(u^{\varepsilon, D}, v^{\varepsilon, D})$ in Theorem 1.1 is approximated by the pair of functions $(u_k^{\varepsilon, D}, v_k^{\varepsilon, D})$ in Proposition 2.4 as follows.*

$$\left\| \begin{pmatrix} u^{\varepsilon, D} \\ v^{\varepsilon, D} \end{pmatrix} - \begin{pmatrix} u_k^{\varepsilon, D} \\ v_k^{\varepsilon, D} \end{pmatrix} \right\|_{L^{\infty}(0, 1)} \leq C_k \varepsilon^{k-1/2}.$$

3. Solvability theory

In this section, we establish some technical results to be used in §4.

Let us denote by L^0 the linear operator defined by

$$(3.1) \qquad\qquad L^0 u(z) := u_{zz} + f_u(Q(z; v^*), v^*) u(z).$$

In the sequel, we simply write $Q(z; v^*)$ as $u^0(z) := Q(z; v^*)$. We also employ a constant $d_0 > 0$, $d_0 < \sqrt{f_u(h^{\pm}(v^*), v^*)}$, which measures the exponential decay rate of $u^0(z) - h^{\pm}(v^*)$ and $u_z^0(z)$ as $z \to \pm\infty$:

$$|u^0(z) - h^{\pm}(v^*)| = O(e^{-d_0|z|}), \qquad u_z^0(z) = O(e^{-d_0|z|}) \quad \text{as } |z| \to \infty.$$

LEMMA 3.1.   *Consider the linear inhomogeneous equations*

$$(3.2) \qquad\qquad 0 = L^0 u + p(z), \qquad z \in \mathbf{R},$$

$$(3.2^{\pm}) \qquad\qquad 0 = L^0 u^{\pm} + p^{\pm}(z), \qquad \pm z \in (0, \infty).$$

If p and $p^{\pm}$ are of polynomial order in z; that is,

$$p(z) = O(|z|^k), \qquad p_z(z) = O(|z|^{k-1}), \qquad p_{zz}(z) = O(|z|^{k-2}) \quad \text{as } |z| \to \infty,$$

$$p^{\pm}(z) = O(|z|^k), \qquad p_z^{\pm}(z) = O(|z|^{k-1}), \qquad p_{zz}^{\pm}(z) = O(|z|^{k-2}) \quad \text{as } z \to \pm\infty,$$

and satisfy

$$|p(z) - p^{\pm}(z)| = O(e^{-d_0|z|}) \qquad \text{as } z \to \pm\infty,$$

then, the following statements hold:

(i)   *The problem (3.2) has a solution satisfying*

(3.3) $$|u(z) - u^{\pm}(z)| = O(e^{-d_0|z|}) \qquad as \ z \to \pm\infty$$

   if and only if

(3.4) $$\int_{\mathbf{R}} u_z^0(z)p(z)dz = 0.$$

 (ii)   *When (3.4) holds, solutions of (3.2), satisfying (3.3), are expressed for an arbitrary constant $a \in \mathbf{R}$ as*

(3.5) $$u(z) = au_z^0(z) + \bar{u}(z),$$

   where $\bar{u}$ is given by

(3.6) $$\bar{u}(z) := -u_z^0(z)\int_0^z \frac{dz'}{(u_z^0(z'))^2}\int_{\pm\infty}^{z'} u_z^0(z'')p(z'')dz''.$$

(iii)   *When the condition (3.4) is satisfied, the solution $u(z)$ of (3.2) satisfies*

(3.7) $$\int_{\mathbf{R}} u_z^0(z)u_z(z)dz = \frac{1}{2}\int_{\mathbf{R}} zu_z^0(z)p(z)dz,$$

   and

(3.8) $$\left|\frac{u(z)}{p(z)} - \left(-\frac{1}{f_u(h^{\pm}(v^*), v^*)}\right)\right|, \ \left|\frac{u_z(z)}{p_z(z)} - \left(-\frac{1}{f_u(h^{\pm}(v^*), v^*)}\right)\right|,$$

$$\left|\frac{u_{zz}(z)}{p_{zz}(z)} - \left(-\frac{1}{f_u(h^{\pm}(v^*), v^*)}\right)\right| = O(e^{-d_0|z|}) \qquad as \ z \to \pm\infty.$$

PROOF.   (i)–(ii) The difference $\phi^{\pm}(z) := u(z) - u^{\pm}(z)$ satisfies

$$0 = L^0\phi^{\pm}(z) + \bar{p}^{\pm}(z), \qquad \pm z \in (0, \infty),$$

where $\bar{p}^{\pm}(z) := p(z) - p^{\pm}(z)$.   The variation of constants formula gives

$$\phi^{\pm}(z) = \phi^{\pm}(0)\frac{u_z^0(z)}{u_z^0(0)} - u_z^0(z)\int_0^z \frac{dz'}{(u_z^0(z'))^2}\int_{\pm\infty}^{z'} u_z^0(z'')\bar{p}^{\pm}(z'')dz'', \qquad \pm z \in (0, \infty).$$

Since $\bar{p}^{\pm}(z) = O(e^{-d_0|z|})$ as $z \to \pm\infty$, we can readily verify $\phi^{\pm}(z) = O(e^{-d_0|z|})$. On the other hand, representations similar to (3.5)–(3.6) hold for $u^{\pm}(z)$. Upon subtraction, we obtain

$$u(z) = u(0)\frac{u_z^0(z)}{u_z^0(0)} - u_z^0(z)\int_0^z \frac{dz'}{(u_z^0(z'))^2}\int_{\pm\infty}^{z'} u_z^0(z'')p(z'')dz'', \qquad \pm z \in (0, \infty).$$

To ensure the smoothness of this u across $z = 0$, we impose the matching condition

$$\lim_{z \nearrow 0} u_z(z) = \lim_{z \searrow 0} u_z(z),$$

which immediately gives rise to (3.4).

(iii) Since $p(z)$ is of polynomial growth in z satisfying (3.4) and $u_z^0(z) = O(e^{-d_0|z|})$, we have, by l'Hospital's rule,

$$\lim_{Z \to \infty} Z \int_{-Z}^{Z} u_z^0(z)p(z)dz = -\lim_{Z \to \infty} Z^2[u_z^0(Z)p(Z) + u_z^0(-Z)p(-Z)] = 0.$$

By using the representation (3.5)–(3.6), integrating by parts and exchanging orders of integration, we have

$$\begin{aligned}
\int_{\mathbf{R}} u_z^0(z)u_z(z)dz &= -\frac{1}{2} \int_{\mathbf{R}} dz \int_{-\infty}^{z} u_z^0(z')p(z')dz' \\
&= -\frac{1}{2} \lim_{Z \to \infty} \int_{-Z}^{Z} dz \int_{-Z}^{z} u_z^0(z')p(z')dz' \\
&= -\frac{1}{2} \lim_{Z \to \infty} \int_{-Z}^{Z} (Z - z')u_z^0(z')p(z')dz' \\
&= -\frac{1}{2} \lim_{Z \to \infty} Z \int_{-Z}^{Z} u_z^0(z')p(z')dz' + \frac{1}{2} \int_{\mathbf{R}} z'u_z^0(z')p(z')dz' \\
&= \frac{1}{2} \int_{\mathbf{R}} zu_z^0(z)p(z)dz,
\end{aligned}$$

which proves (3.7). To prove the first of (3.8), we apply l'Hospital's rule repeatedly. To prove the second and third of (3.8), we apply the same rule to the equation for u_z and u_{zz}, respectively. $\square$

LEMMA 3.2. *Consider the linear inhomogeneous equations*

$$(3.9) \qquad\qquad 0 = v_{zz} + q(z), \qquad z \in \mathbf{R},$$

$$(3.9^{\pm}) \qquad\qquad 0 = v_{zz}^{\pm} + q^{\pm}(z), \qquad \pm z \in (0, \infty).$$

If q and $q^{\pm}$ are of polynomial order in z; that is,

$$q(z) = O(|z|^k) \quad as \ |z| \to \infty, \qquad q^{\pm}(z) = O(|z|^k) \quad as \ z \to \pm\infty,$$

and satisfy

$$|q(z) - q^{\pm}(z)| = O(e^{-d_0|z|}) \qquad as \ z \to \pm\infty,$$

then the following statements hold true.

(i) *The problem (3.9) has a solution satisfying*

$$(3.10) \qquad\qquad |v(z) - v^{\pm}(z)| = O(e^{-d_0|z|}) \qquad as \ z \to \pm\infty$$

if and only if

$$(3.11) \qquad v(0) = v^-(0) - \int_{-\infty}^0 dz \int_{-\infty}^z (q(z') - q^-(z'))dz'$$

$$= v^+(0) - \int_{+\infty}^0 dz \int_{+\infty}^z (q(z') - q^+(z'))dz',$$

$$(3.12) \qquad v_z^-(0) - v_z^+(0) = \int_{-\infty}^0 (q(z) - q^-(z))dz - \int_{+\infty}^0 (q(z) - q^+(z))dz.$$

(ii)   *The difference $v(z) - v^\pm(z)$ depends only on $q(z) - q^\pm(z)$.*

PROOF.   The difference $v - v^\pm$ is given by

$$v(z) - v^\pm(z) = B^\pm + zA^\pm - \int_{\pm\infty}^z dz' \int_{\pm\infty}^{z'} (q(z'') - q^\pm(z''))dz'',$$

for some constants $A^\pm$ and $B^\pm$.   In this expression, the third term on the right hand side behaves like $O(e^{-d_0|z|})$ as $z \to \pm\infty$.   As a consequence, if $A^\pm = 0 = B^\pm$, the conditions in (3.10) are fulfilled.   This proves the first part of the lemma.   The second part immediately follows. $\square$

4.   Approximate solutions

In this section, we will prove Proposition 2.4 by using asymptotic expansions.   Let us denote by $\Omega^\pm$ the subintervals $\Omega^- := (0, x^*)$, $\Omega^+ := (x^*, 1)$.   Here and in the sequel, we use x^* in place of $x^*(D)$ to simplify notation.

4.1.   Outer Expansion.   We will determine $(V^{\pm,j}, U^{\pm,j})$ $(j = 0, 1, \ldots)$ in formal ε-power series

$$(4.1) \qquad u^{\pm,\varepsilon}(x) \sim \sum_{j\geq 0} \varepsilon^j U^{\pm,j}(x), \qquad v^{\pm,\varepsilon}(x) \sim \sum_{j\geq 0} \varepsilon^j V^{\pm,j}(x), \qquad x \in \overline{\Omega}^\pm$$

in such a way that they asymptotically satisfy (1.5) on the respective domain $\Omega^\pm$.   By substituting (4.1) into (1.5) and equating coefficients of like powers of ε in the equation, one obtains two sets of equations, one coming from u-component and the other from v-component.

The equations coming from the u-component are given by

$$(4.2) \qquad \begin{cases} \text{(i)} & 0 = f(U^{\pm,0}, V^{\pm,0}), \\ \text{(ii)} & 0 = f_u^{\pm,0} U^{\pm,1} + f_v^{\pm,0} V^{\pm,1}, \\ \text{(iii)} & 0 = f_u^{\pm,0} U^{\pm,j} + f_v^{\pm,0} V^{\pm,j} + f_{\pm,j} \quad (j \geq 2), \end{cases}$$

where $f_u^{\pm,0} := f_u(U^{\pm,0}, V^{\pm,0})$, $f_v^{\pm,0} := f_v(U^{\pm,0}, V^{\pm,0})$ and $f_{\pm,j}$ $(j \geq 2)$ are given by

$$f_{\pm,j} := U_{xx}^{\pm,j-2} - [f_u^{\pm,0} U^{\pm,j} + f_v^{\pm,0} V^{\pm,j}]$$

$$+ \frac{1}{j!} \frac{d^j}{d\varepsilon^j} \left[f\left(\sum_{k\geq 0} \varepsilon^k U^{\pm,k}, \sum_{k\geq 0} \varepsilon^k V^{\pm,k} \right) \right] \Bigg|_{\varepsilon=0}.$$

Note that $f_{\pm,j}$ depends only on $(U^{\pm,k}, V^{\pm,k})$ with $0 \leq k \leq j-1$.

In accordance with the condition (A1), as a solution of (4.2)-(i), we choose $U^{\pm,0} = h^{\pm}(V^{\pm,0})$. For $j \geq 1$, $U^{\pm,j}$ is given by

$$U^{\pm,1} = h_v^{\pm}(V^{\pm,0}) V^{\pm,1},$$

$$U^{\pm,j} = h_v^{\pm}(V^{\pm,0}) V^{\pm,j} - (f_u^{\pm,0})^{-1} f_{\pm,j} \qquad (j \geq 2).$$

Therefore $U^{\pm,j}$ $(j \geq 0)$ is determined by $V^{\pm,k}$ $(0 \leq k \leq j)$.

The equations coming from the v-component give rise to a series of boundary value problems:

$$(4.3) \quad \begin{cases} 0 = V_{xx}^{\pm,0} + \frac{1}{D} g^{\pm}(V^{\pm,0}), & x \in \Omega^{\pm}, \\ V_x^{-,0}(0) = 0 = V_x^{+,0}(1), & V^{-,0}(x^*) = v^* = V^{+,0}(x^*), \end{cases}$$

$$(4.4) \quad \begin{cases} 0 = V_{xx}^{\pm,1} + \frac{1}{D} g_v^{\pm}(V^{\pm,0}(x)) V^{\pm,1}, & x \in \Omega^{\pm}, \\ V_x^{-,1}(0) = 0 = V_x^{+,1}(1), & V^{-,1}(x^*) = b^{-,1}, \quad V^{+,1}(x^*) = b^{+,1}, \end{cases}$$

and for $j \geq 2$,

$$(4.5) \quad \begin{cases} 0 = V_{xx}^{\pm,j} + \frac{1}{D} g_v^{\pm}(V^{\pm,0}(x)) V^{\pm,j} + \frac{1}{D} g_{\pm,j}, & r \in (0, R), \\ V_x^{-,j}(0) = 0 = V_x^{+,j}(1), & V^{-,j}(x^*) = b^{-,j}, \quad V^{+,j}(x^*) = b^{+,j}. \end{cases}$$

In the above,

$$g^{\pm}(v) := g(h^{\pm}(v), v), \qquad g_v^{\pm}(v) := \frac{d}{dv} g^{\pm}(v)$$

and

$$g_{\pm,j} := \frac{1}{j!} \frac{d^j}{d\varepsilon^j} \left[g\left(\sum_{k\geq 0} \varepsilon^k U^{\pm,k}, \sum_{k\geq 0} \varepsilon^k V^{\pm,k} \right) \right] \Bigg|_{\varepsilon=0} - g_v^{\pm}(V^{\pm,0}) V^{\pm,j}.$$

Note that $g_{\pm,j}$ depends only on $V^{\pm,k}$ $(0 \leq k \leq j-1)$ and hence one can determine $V^{\pm,j}$ successively, starting from $j = 0$. The boundary values $b^{\pm,j}$ at $x = x^*$ are to be determined later, when we impose C^1-matching conditions at $x = x^*$ (cf. §4.3).

As a solution of (4.3), we choose the function $V^{*,D}$ given in Proposition 2.1, namely,

$$V^{-,0}(x) := V^{*,D}(x) \quad \text{for } x \in [0, x^*], \qquad V^{+,0}(x) := V^{*,D}(x) \quad \text{for } x \in [x^*, 1].$$

Throughout the remainder of this paper, we denote this function simply by $V^0(x)$ (note that $V^0 \in C^1[0,1]$, but its second derivative has a jump at $x = x^*$). We also understand that $V^{-,j}(x)$ is defined for $x \in [0, x^*]$ and $V^{+,j}(x)$ for $x \in [x^*, 1]$.

Since we have from (A3) that $g_v^\pm(V^0(x)) < 0$ for $x \in [0, 1]$, (4.4) and (4.5) have a unique solution for arbitrarily given boundary data $b^{\pm,j}$. From now on, we denote these unique solutions by $V^{\pm,j}(x)$, keeping in mind that they depend upon $b^{\pm,k}$ $(0 \le k \le j)$. In this way, $U^{\pm,j}$ and $V^{\pm,j}$ in (4.1) have been determined. We refer to this as the outer expansion of the desired solution.

It is now easy to see that the outer expansion satisfies the following estimates.

PROPOSITION 4.1.   *For each $k \ge 0$, the outer approximation satisfies*

$$\left| \varepsilon^2 \left(\sum_{j=0}^{k} \varepsilon^j U^{\pm,j}(x) \right)_{xx} + f\left(\sum_{j=0}^{k} \varepsilon^j U^{\pm,j}(x), \sum_{j=0}^{k} \varepsilon^j V^{\pm,j}(x) \right) \right| \le C_k \varepsilon^{k+1},$$

$$\left| D \left(\sum_{j=0}^{k} \varepsilon^j V^{\pm,j}(x) \right)_{xx} + g\left(\sum_{j=0}^{k} \varepsilon^j U^{\pm,j}(x), \sum_{j=0}^{k} \varepsilon^j V^{\pm,j}(x) \right) \right| \le C_k \varepsilon^{k+1}$$

uniformly on $\Omega^\pm$, where $C_k > 0$ is a constant independent of $\varepsilon \in (0, \varepsilon_0]$ for some $\varepsilon_0 > 0$.

4.2.   Inner Expansion.   In order to discuss the sharp transition behavior in u near x^*, let us introduce a stretched variable $z = (x - x^*)/\varepsilon$. In terms of the new variable z, the differential equations in (1.5) are recast as follows:

$$(4.6) \qquad \begin{cases} 0 = u_{zz}^\varepsilon + f(u^\varepsilon, v^\varepsilon), \\ 0 = v_{zz}^\varepsilon + \varepsilon^2 \frac{1}{D} g(u^\varepsilon, v^\varepsilon), \end{cases} \quad z \in \mathbf{R}.$$

Note that we consider the equations in (4.6) for $z \in \mathbf{R}$ and impose the boundary conditions

$$(4.7) \qquad |u^\varepsilon(z) - \tilde{U}^{\pm,\varepsilon}(z)| = O(e^{-d_0|z|}) \qquad \text{as } z \to \pm\infty,$$

$$(4.8) \qquad |v^\varepsilon(z) - \tilde{V}^{\pm,\varepsilon}(z)| = O(e^{-d_0|z|}) \qquad \text{as } z \to \pm\infty$$

for some constant $d_0 > 0$, $d_0 < \sqrt{f_u(h^\pm(v^*), v^*)}$, where

$$\tilde{U}^{\pm,\varepsilon}(z) := \sum_{j \ge 0} \varepsilon^j \tilde{U}^{\pm,j}(z) := \sum_{j \ge 0} \varepsilon^j U^{\pm,j}(x^* + \varepsilon z),$$

$$\tilde{V}^{\pm,\varepsilon}(z) := \sum_{j \ge 0} \varepsilon^j \tilde{V}^{\pm,j}(z) := \sum_{j \ge 0} \varepsilon^j V^{\pm,j}(x^* + \varepsilon z)$$

are an expression of the outer expansion (4.1) in terms of the stretched z-coordinate system. Let us define our inner expansion by

$$(4.9) \qquad u^\varepsilon(z) := \sum_{j \geq 0} \varepsilon^j u^j(z), \qquad v^\varepsilon(z) := \sum_{j \geq 0} \varepsilon^j v^j(z).$$

Notice that our outer expansion is constructed so that it satisfies (1.5) and that (4.6) is an expression of (1.5) in the z-coordinate system. Therefore, we immediately obtain the following.

> PROPOSITION 4.2. (i) *The pairs of functions $(\tilde{U}^{-,\varepsilon}, \tilde{V}^{-,\varepsilon})$ and $(\tilde{U}^{+,\varepsilon}, \tilde{V}^{+,\varepsilon})$ satisfy the equations in (4.6), respectively, on $(-\infty, 0)$ and $(0, \infty)$.*
>
> (ii) *For $j = 0, 1, \dots$, the boundary conditions (4.7) and (4.8) are equivalent to*

$$(4.10) \qquad |u^j(z) - \tilde{U}^{\pm,j}(z)| = O(e^{-d_0|z|}) \qquad \text{as } z \to \pm\infty,$$

$$(4.11) \qquad |v^j(z) - \tilde{V}^{\pm,j}(z)| = O(e^{-d_0|z|}) \qquad \text{as } z \to \pm\infty.$$

Substituting (4.9) into (4.6) and equatin coefficients of powers of ε, we obtain equations for (u^j, v^j), which are valid on **R**. They are given for $j = 0, 1, 2$ and $j \geq 3$ by

$$(4.12) \qquad \begin{cases} 0 = v^0_{zz}, \\ 0 = u^0_{zz} + f(u^0, v^0), \end{cases}$$

$$(4.13) \qquad \begin{cases} 0 = v^1_{zz}, \\ 0 = u^1_{zz} + f_u(u^0, v^0)u^1 + f_v(u^0, v^0)v^1, \end{cases}$$

$$(4.14) \qquad \begin{cases} 0 = v^2_{zz} + \frac{1}{D} g(u^0, v^0), \\ 0 = u^2_{zz} + f_u(u^0, v^0)u^2 + f_2(z), \end{cases}$$

$$(4.15) \qquad \begin{cases} 0 = v^j_{zz} + \frac{1}{D} g_j(z), \\ 0 = u^j_{zz} + f_u(u^0, v^0)u^j + f_j(z). \end{cases}$$

In these equations, g_j $(j \geq 2)$ and f_j $(j \geq 1)$ are *lower order* terms defined by

$$(4.16) \quad g_j(z) := \frac{1}{(j-2)!} \frac{d^{j-2}}{d\varepsilon^{j-2}} g\left(\sum_{k \geq 0} \varepsilon^k u^k(z), \sum_{k \geq 0} \varepsilon^k v^k(z) \right)\bigg|_{\varepsilon=0},$$

$$(4.17) \quad f_j(z) := \frac{1}{j!} \frac{d^j}{d\varepsilon^j} f\left(\sum_{k \geq 0} \varepsilon^k u^k(z), \sum_{k \geq 0} \varepsilon^k v^k(z) \right)\bigg|_{\varepsilon=0} - f_u(u^0, v^0)u^j(z).$$

We also define $\tilde{g}_{\pm,j}(z)$ $(j \geq 0)$ and $\tilde{f}_{\pm,j}(z)$ $(j \geq 1)$, respectively, by the same formulae as (4.16) and (4.17) with $(\tilde{U}^{\pm,j}, \tilde{V}^{\pm,j})$ replacing (u^j, v^j). We then find from Proposition 4.2 (i) that

PROPOSITION 4.3. $(\tilde{U}^{\pm,j}, \tilde{V}^{\pm,j})$ $(j \geq 1)$ *satisfies* (4.13), (4.14) *and* (4.15) *with* $(\tilde{g}_{\pm,j}(z), \tilde{f}_{\pm,j}(z))$ *replacing* $(g_j(z), f_j(z))$ *on* $(-\infty, 0)$ *and* $(0, \infty)$, *respectively.*

Let us find the solutions of (4.12)–(4.15) satisfying the conditions (4.10)–(4.11).

From the definition, $\tilde{V}^{\pm,0}(z) \equiv v^*$. Therefore, the equation $v_{zz}^0 = 0$ in (4.12) and the condition (4.11) with $j = 0$ imply that $v^0(z) \equiv v^*$. Then, the second equation in (4.12) and (2.14) give rise to

(4.18) $u^0(z) = Q(z + a_0; v^*)$,   where $a_0 \in \mathbf{R}$ is an arbitrary constant.

The parameter a_0 is to be determined so that the second equation in (4.13) has a solution satisfying (4.10) with $j = 1$.

PROPOSITION 4.4.   (i) *The problem* (4.13) *has a solution pair* (u^1, v^1) *satisfying* (4.10)–(4.11) *with* $j = 1$, *if and only if* $b^{-,1} = b^1 = b^{+,1}$ *for some* $b^1 \in \mathbf{R}$ *and*

(4.19) $v^1(0) = b^1$,

(4.20) $V_x^0(x^*)J'(v^*)a_0 - J'(v^*)b^1 = \displaystyle\int_{\mathbf{R}} zf_v(Q(z), v^*)Q_z(z)dz =: C_0.$

Moreover, $v^1(z)$ *and* $u^1(z)$ *are explicitly expressed as*

$$v^1(z) = b^1 + V_x^*(x^*)z, \qquad u^1(z) = a_1u_z^0(z) + \bar{u}^1(z),$$

where $\bar{u}^1$ *is a solution of* (4.13) *satisfying* (4.10), *as well as* $\bar{u}^1(0) = 0$.

(ii)   *The problem* (4.14) *and* (4.15) *have a solution pair* (u^j, v^j) $(j \geq 2)$ *satisfying* (4.10)–(4.11), *if and only if the following conditions are satisfied for some* $b^j, a_j \in \mathbf{R}$:

(4.21) $v^j(0) = b^j = b^{\pm,j} + A_{j-2}^{\pm}(a_0, \ldots, a_{j-2}, b^1, \ldots, b^{j-2})$,

(4.22) $-\dfrac{[g]_-^+}{D}a_{j-2} + \pi_0 b^{j-1} = B_{j-2}(a_0, \ldots, a_{j-3}, b^1, \ldots, b^{j-2})$,

(4.23) $V_x^*(x^*)J'(v^*)a_{j-1} - J'(v^*)b^j = C_{j-1}(a_0, \ldots, a_{j-2}, b^1, \ldots, b^{j-1})$,

where $A_{j-2}^{\pm}$, B_{j-2}, C_{j-1} *are smooth functions of the variables indicated, and* π_0 *is given in* (2.12). *We also used expressions* $a_{-1} = 0$ *and* $b^0 = v^*$.

PROOF.   (i) From the definition, we have $\tilde{V}^{\pm,1}(z) = b^{\pm,1} + V_x^*(x^*)z$, while (4.13) implies that $v^1(z) = v^1(0) + v_z^1(0)z$. Then the condition (4.11) with $j = 1$ immediately yields $b^{\pm,1} = v(0) = b^1$ for some $b^1 \in \mathbf{R}$ and $v^1(z) = b^1 + V_x^*(x^*)z$. Since $b^{\pm,1} = b^1$, Proposition 2.2 implies

(4.24) $V^{\pm,1}(x) = b^1 \Psi^{\pm}(x)$.

In order to deal with the second equation of (4.13), let us apply Lemma 3.1 to $(u, u^{\pm}) = (u^1, \tilde{U}^{\pm,1})$ with

$$p(z) = f_v(u^0(z), v^*)v^1(z), \qquad p^{\pm}(z) = f_v(h^{\pm}(v^*), v^*)\tilde{V}^{\pm,1}(z).$$

The solvability condition (3.4) now reads as:

$$0 = \int_{\mathbf{R}} f_v(Q(z+a_0), v^*)(b^1 + V_x^*(x^*)z)Q_z(z+a_0)dz$$

$$= \int_{\mathbf{R}} f_v(Q(z), v^*)(b^1 + V_x^*(x^*)(z - a_0))Q_z(z)dz$$

$$= J'(v^*)b^1 - V_x^*(x^*)J'(v^*)a_0 + \int_{\mathbf{R}} zf_v(Q(z), v^*)Q_z(z)dz,$$

which is the same as (4.20). Now (3.5) gives the expression of u^1.

(ii) Let us first exhibit the proof for $j = 2$.

We apply Lemma 3.2 to $(v, v^{\pm}) = (v^2, \tilde{V}^{\pm,2})$ with

$$q(z) = \frac{1}{D}g(u^0(z), v^*), \qquad q^{\pm}(z) = \frac{1}{D}g(h^{\pm}(v^*), v^*).$$

From the definition, $\tilde{V}^{\pm,2}(z) = b^{\pm,2} + V_x^{\pm,1}(x^*)z + O(z^2)$. If we introduce $v^2(0) = b^2$, then (3.11) gives (4.21) $(j = 2)$ with

$$A_0^{\pm}(a_0) = -\frac{1}{D}\int_{\pm\infty}^{a_0} dz \int_{\pm\infty}^{z} [g(Q(z'), v^*) - g(h^{\pm}(v^*)v^*)]dz'.$$

Since (4.24) implies $V_x^{-,1}(x^*) - V_x^{+,1}(x^*) = \pi_0 b^1$, (3.12) gives rise to (4.22) $(j = 2)$ with

$$B_0 = \frac{1}{D}\int_{-\infty}^{0} [g(Q(z), v^*) - g(h^{-}(v^*), v^*)]dz$$

$$- \frac{1}{D}\int_{\infty}^{0} [g(Q(z), v^*) - g(h^{+}(v^*), v^*)]dz.$$

We now apply Lemma 3.1 to $(u, u^{\pm}) = (u^2, \tilde{U}^{\pm,2})$. The solvability condition (3.4) now reads $0 = \int_{\mathbf{R}} f_2(z)u_z^0(z)dz$, where f_2, by definition, is as follows.

$$f_2(z) = f_v(\#)v^2 + \frac{1}{2}f_{uu}(\#)(u^1)^2 + f_{uv}(\#)u^1v^1 + \frac{1}{2}f_{vv}(\#)(v^1)^2,$$

in which a short hand expression $(\#) = (u^0(z), v^*)$ is used. Note that

$$v^2(z) = b^2 + \text{terms depending only on } a_0,$$

and that $u^1(z) = a_1 u_z^0(z) + \bar{u}^1(z)$ with $\bar{u}^1$ depending only on a_0 and b^1. Therefore, the solvalibity condition now reads

$$0 = \int_{\mathbf{R}} f_2(z) u_z^0(z) dz$$

$$= b^2 \int_{\mathbf{R}} f_v(\#) u_z^0(z) dz + \frac{(a_1)^2}{2} \int_{\mathbf{R}} f_{uu}(\#)(u_z^0(z))^2 u_z^0(z) dz$$

$$+ a_1 \int_{\mathbf{R}} (f_{uu}(\#) \bar{u}^1 u_z^0 + f_{uv}(\#) v^1 u_z^0) u_z^0 \, dz - C_1(a_0, b^1).$$

We easily find that the coefficient of b^2 is equal to $J'(v^*)$. Integrating by parts, we also find that the coefficient of $(a_1)^2$ is 0 and that of a_1 is equal to $-V_x^*(x^*) J'(v^*)$. We do not give an explicit formula for C_1. We therefore established (4.23) with $j = 2$.

The proof is similar for $j \geq 3$. $\qquad\qquad\qquad\qquad\qquad\qquad$ □

COROLLARY 4.1. *For each $k \geq 0$, there exists a constant $C_k > 0$ such that the following estimates hold for $z \in \mathbf{R}$.*

(i) $$|v^k(z)| \leq C_k(1 + |z|^k), \qquad |u^k(z)| \leq C_k(1 + |z|^k),$$

and

(ii) $$\left| \left(\sum_{j=0}^{k} \varepsilon^j u^j(z) \right)_{zz} + f\left(\sum_{j=0}^{k} \varepsilon^j u^j(z), \sum_{j=0}^{k} \varepsilon^j v^j(z) \right) \right| \leq \varepsilon^{k+1} C_k(1 + |z|^{k+1}),$$

(iii) $$\left| D\left(\sum_{j=0}^{k+2} \varepsilon^j v^j(z) \right)_{zz} + \varepsilon^2 g\left(\sum_{j=0}^{k} \varepsilon^j u^j(z), \sum_{j=0}^{k} \varepsilon^j v^j(z) \right) \right|$$

$$\leq \varepsilon^{k+3} C_k(1 + |z|^{k+1} + \varepsilon|z|^{k+2} + \varepsilon^2 |z|^{k+3}).$$

Moreover, the conditions (4.10) and (4.11) are valid for derivatives;

(iv) $$|u_z^j(z) - \tilde{U}_z^{\pm,j}(z)| = O(e^{-d_0|z|}) = |u_{zz}^j(z) - \tilde{U}_{zz}^{\pm,j}(z)| \qquad as \ z \to \pm\infty,$$

(v) $$|v_z^j(z) - \tilde{V}_z^{\pm,j}(z)| = O(e^{-d_0|z|}) = |v_{zz}^j(z) - \tilde{V}_{zz}^{\pm,j}(z)| \qquad as \ z \to \pm\infty$$

for $j \geq 0$.

PROOF. From the definition, $\tilde{V}^{\pm,j}(z)$ and $\tilde{U}^{\pm,j}$ are polynomials in z of order j for each $j \geq 0$. Note that $u^j(z)$ and $v^j(z)$ satisfy (4.10) and (4.11), respectively. Therefore, the estimates (i) on v^k and u^k immediately follow.

To prove the second part, notice that u^j and v^j are chosen, via (4.12)–(4.15), so that

$$\frac{\partial^i}{\partial \varepsilon^i}\left[\left(\sum_{j=0}^{k}\varepsilon^j u^j(z)\right)_{zz} + f\left(\sum_{j=0}^{k}\varepsilon^j u^j(z), \sum_{j=0}^{k}\varepsilon^j v^j(z)\right)\right]_{\varepsilon=0} \equiv 0$$

for $i = 0, \ldots, k$. Therefore, for some $0 \le \tau \le 1$, we have

$$\left(\sum_{j=0}^{k}\varepsilon^j u^j(z)\right)_{zz} + f\left(\sum_{j=0}^{k}\varepsilon^j u^j(z), \sum_{j=0}^{k}\varepsilon^j v^j(z)\right)$$

$$= \frac{\varepsilon^{k+1}}{(k+1)!}\frac{\partial^{k+1}}{\partial \varepsilon^{k+1}}\left[f\left(\sum_{j=0}^{k}\varepsilon^j u^j(z), \sum_{j=0}^{k}\varepsilon^j v^j(z)\right)\right]_{\varepsilon=\tau\varepsilon}$$

$$= \varepsilon^{k+1}\sum_{(i,j)}^{k} c_{ij}(z,\varepsilon)u^i(z)v^j(z),$$

where $\sum_{(i,j)}^{k}$ stands for summation over integers $1 \le i, j \le k$ satisfying $i + j = k + 1$, and $c_{ij}(z,\varepsilon)$ are bounded functions. Therefore, using the estimates (i) on v^j and u^j, the estimate in (ii) follows.

Similar arguments apply to establish (iii).

The estimates (iv) and (v) easily follow from the proof of Lemmas 3.1 and 3.2. $\qquad\square$

4.3. C^1-matching conditions.

The equations (4.20), (4.22) and (4.23) are called C^1-*matching conditions*. We will show now that these conditions are satisfied by adequately choosing the parameters a_j ($j \ge 0$) and b^j ($j \ge 1$). In this process, the following *non-degeneracy condition*

$$(4.25) \qquad \det\begin{pmatrix} V_x^{*,D}(x^*) & -1 \\ -[g]_-^+/D & \pi_0 \end{pmatrix} = \pi_0\left(V_x^{*,D}(x^*) - \frac{[g]_-^+}{D\pi_0}\right) \ne 0$$

plays an important role. The inequality in (4.25) follows from (2.12) and (2.13).

Let us first couple (4.20) and (4.22) with $j = 2$, which is equivalent to (note: $V^0 = V^{*,D}$)

$$\begin{pmatrix} V_x^{*,D}(x^*) & -1 \\ -[g]_-^+/D & \pi_0 \end{pmatrix}\begin{pmatrix} a_0 \\ b^1 \end{pmatrix} = \begin{pmatrix} C_0/J'(v^*) \\ B_0 \end{pmatrix}.$$

Thanks to (4.25), this has a unique solution.

For $j \ge 2$, we couple (4.23) and (4.22) (with j being replaced by $j + 1$). This gives rise to

$$\begin{pmatrix} V_x^{*,D}(x^*) & -1 \\ -[g]_-^+/D & \pi_0 \end{pmatrix}\begin{pmatrix} a_{j-1} \\ b^j \end{pmatrix} = \begin{pmatrix} C_{j-1}(a_0, \ldots, a_{j-2}, b^1, \ldots, b^{j-1})/J'(v^*) \\ B_{j-1}(a_0, \ldots, a_{j-2}, b^1, \ldots, b^{j-1}) \end{pmatrix}.$$

Note that the right hand side of the last equation involves only (a_i, b^i) with lower indices i. Thanks to (4.25), we obtain a unique solution (a_{j-1}, b^j). Therefore, one can inductively determine (a_j, b^{j+1}) for all $j \geq 0$. Then, from (4.21) one can also determine $b^{\pm,j}$ for all $j \geq 1$.

We are now ready to define the approximate solutions $(u_k^{\varepsilon,D}, v_k^{\varepsilon,D})$ stated in Proposition 2.4. Let us introduce short hand expressions:

$$(4.26) \quad \begin{cases} U_k^{\pm,\varepsilon}(x) := \sum_{j=0}^{k} \varepsilon^j U^{\pm,j}(x), & V_k^{\pm,\varepsilon}(x) := \sum_{j=0}^{k} \varepsilon^j V^{\pm,j}(x), \\ \tilde{U}_k^{\pm,\varepsilon}(z) := \sum_{j=0}^{k} \varepsilon^j \tilde{U}^{\pm,j}(z), & \tilde{V}_k^{\pm,\varepsilon}(z) := \sum_{j=0}^{k} \varepsilon^j \tilde{V}^{\pm,j}(z), \\ \tilde{u}_k^{\varepsilon}(z) := \sum_{j=0}^{k} \varepsilon^j u^j(z), & \tilde{v}_k^{\varepsilon}(z) := \sum_{j=0}^{k} \varepsilon^j v^j(z). \end{cases}$$

We choose a smooth cut-off fucntion $\theta(z)$ satisfying

$$0 \leq \theta(z) \leq 1, \quad z \in \mathbf{R}, \qquad \theta(z) \equiv 1, \quad |z| \leq 1, \qquad \theta(z) \equiv 0, \quad |z| \geq 2.$$

For $x \in \Omega^{\pm}$, the desired approximate solutions are given by

$$(4.27) \quad \begin{cases} u_k^{\varepsilon,D}(x) = U_k^{\pm,\varepsilon}(x) + \theta\left(\frac{-d_0(x-x^*)}{(k+1)\varepsilon \log \varepsilon}\right)\left[\tilde{u}_k^{\varepsilon}\left(\frac{x-x^*}{\varepsilon}\right) - U_k^{\pm,\varepsilon}(x)\right], \\ v_k^{\varepsilon,D}(x) = V_{k+2}^{\pm,\varepsilon}(x) + \theta\left(\frac{-d_0(x-x^*)}{(k+1)\varepsilon \log \varepsilon}\right)\left[\tilde{v}_{k+2}^{\varepsilon}\left(\frac{x-x^*}{\varepsilon}\right) - V_{k+2}^{\pm,\varepsilon}(x)\right]. \end{cases}$$

It remains to verify that these apporximate solutions satisfy the statements in Proposition 2.4.

4.4. Proof of Proposition 2.4.

One can easily verify that the following estimates are valid for $-x^*/\varepsilon \leq z \leq 0$ (with superscript "$-$") and for $0 \leq z \leq (1-x^*)/\varepsilon$ (with superscript "$+$").

$$(4.28) \quad \begin{cases} \left|\frac{d^i}{dz^i}(U_k^{\pm,\varepsilon}(x^* + \varepsilon z) - \tilde{U}_k^{\pm,\varepsilon}(z))\right| \leq C_k \varepsilon^{k+1}(1 + |z|^{k+1}), \\ \left|\frac{d^i}{dz^i}(V_k^{\pm,\varepsilon}(x^* + \varepsilon z) - \tilde{V}_k^{\pm,\varepsilon}(z))\right| \leq C_k \varepsilon^{k+1}(1 + |z|^{k+1}), \end{cases} \quad i = 0, 1, 2.$$

The smoothness of the functions defined in (4.27) is obvious.

We now introduce a short hand expression

$$(4.29) \qquad \rho(x, \varepsilon) := \frac{-d_0}{(k+1)\varepsilon \log \varepsilon}(x - x^*).$$

From the definition, it immediately follows that

$$\lim_{\varepsilon \to 0} |V_{k+2}^{\pm,\varepsilon}(x) - V^{*,D}(x)| = 0 \qquad \text{uniformly on } \overline{\Omega}^{\pm}.$$

Therefore, to prove Proposition 2.4 (i), it suffices to show that

$$(4.30) \qquad \lim_{\varepsilon \to 0} |v_k^{\varepsilon,D}(x) - V_{k+2}^{\pm,\varepsilon}(x)| = 0 \qquad \text{uniformly on } \overline{\Omega}^{\pm}.$$

For $|\rho(x, \varepsilon)| \geq 2$, the left hand side of (4.30) is identically zero. Hence, we deal

with the case $|\rho(x,\varepsilon)| \le 2$ which is the same thing as $|z| \le -2(k+1)\log\varepsilon/d_0$. In the sequel, we use the fact that $v^i(z) \equiv \tilde{V}^{\pm,i}(z)$ for $i = 0, 1$.

$$
\begin{aligned}
|v_k^{\varepsilon,D}(x) - V_{k+2}^{\pm,\varepsilon}(x)| &= \theta(\rho)\left|\tilde{v}_{k+2}^\varepsilon\left(\frac{x-x^*}{\varepsilon}\right) - V_{k+2}^{\pm,\varepsilon}(x)\right| \\
&\le \left|\sum_{j=2}^{k+2}\varepsilon^j\{v^j(z) - \tilde{V}^{\pm,j}(z)\}\right| + |\tilde{V}_{k+2}^{\pm,\varepsilon}(z) - V_{k+2}^{\pm,\varepsilon}(x^* + \varepsilon z)| \\
&=: I_1^\varepsilon(z) + I_2^\varepsilon(z).
\end{aligned}
$$

By using (4.11), Corollary 4.1 (i), (4.28) and $v^j(z) - \tilde{V}^{\pm,j}(z)$ is bounded uniformly in $\pm z \in [0, \infty)$, we obtain

$$
\begin{aligned}
I_1^\varepsilon(z) &\le \begin{cases} \varepsilon^2 C e^{-d_0|z|} \le C\varepsilon^{k+3} & (|z| \ge -(k+1)\log\varepsilon/d_0), \\ C\varepsilon^2 & (|z| \le -(k+1)\log\varepsilon/d_0), \end{cases} \\
I_2^\varepsilon(z) &\le C\varepsilon^{k+3}(-\log\varepsilon)^{k+3} \le C\varepsilon^{k+2},
\end{aligned}
$$

which establishes the validity of (4.30).

We now prove the statement (ii) of Proposition 2.4. For each $\delta > 0$ with $\delta < \min\{x^*, 1 - x^*\}$, the condition $|x - x^*| > \delta$ implies $|\rho(x,\varepsilon)| \ge 2$, if $\varepsilon > 0$ is small. Therefore, we obtain

$$
\lim_{\varepsilon\to 0}|u_k^{\varepsilon,D}(x) - h^\pm(V^{*,D}(x))| = \lim_{\varepsilon\to 0}\left|\sum_{j=1}^k\varepsilon^j U^{\pm,j}(x)\right| = 0
$$

uniformly for x with $|x - x^*| > \delta$. This proves Proposition 2.4 (ii).

Before we proceed, let us establish the following estimates.

$$
(4.31) \quad \begin{cases} |u_k^{\varepsilon,D}(x) - U_k^{\pm,\varepsilon}(x)| \le C\varepsilon^{k+1}(-\log\varepsilon)^{k+1} \\ |v_k^{\varepsilon,D}(x) - V_{k+2}^{\pm,\varepsilon}(x)| \le C\varepsilon^{k+3}(-\log\varepsilon)^{k+3} \end{cases}, \quad 1 \le |\rho(x,\varepsilon)| \le 2.
$$

The second line has been established in the above. Similarly, the first line is obtained as follows. We have

$$
\begin{aligned}
|u_k^{\varepsilon,D}(x) - U_k^{\pm,\varepsilon}(x)| &= \theta(\rho)\left|\tilde{u}_k^\varepsilon\left(\frac{x-x^*}{\varepsilon}\right) - U_k^{\pm,\varepsilon}(x)\right| \\
&\le \left|\sum_{j=0}^k\varepsilon^j\{u^j(z) - \tilde{U}^{\pm,j}(z)\}\right| + |\tilde{U}_k^{\pm,\varepsilon}(z) - U_k^{\pm,\varepsilon}(x^* + \varepsilon z)|.
\end{aligned}
$$

By using (4.10), Corollary 4.1 (i) and (4.28), this yields the first line of (4.31).

To prove Proposition 2.4 (iii), we only need to deal with the case $|\rho(x,\varepsilon)| \le 2$, thanks to Proposition 4.1.

When $|\rho(x,\varepsilon)| \le 1$, we have $u_k^{\varepsilon,D}(x) = \tilde{u}_k^\varepsilon((x-x^*)/\varepsilon)$ and $v_k^{\varepsilon,D}(x) = \tilde{v}_{k+2}^\varepsilon((x-x^*)/\varepsilon)$. Therefore, applying Corollary 4.1, we obtain for each $\beta \in (0,1]$

$$|\varepsilon^2(u_k^{\varepsilon,D}(x))_{xx} + f(u_k^{\varepsilon,D}(x), v_k^{\varepsilon,D}(x))|$$

$$\le \left| \left(\sum_{j=0}^{k} \varepsilon^j u^j(z) \right)_{zz} + f\left(\sum_{j=0}^{k} \varepsilon^j u^j(z), \sum_{j=0}^{k} \varepsilon^j v^j(z) \right) \right|$$

$$+ \left| f\left(\sum_{j=0}^{k} \varepsilon^j u^j(z), \sum_{j=0}^{k+2} \varepsilon^j v^j(z) \right) - f\left(\sum_{j=0}^{k} \varepsilon^j u^j(z), \sum_{j=0}^{k} \varepsilon^j v^j(z) \right) \right|$$

$$\le C\varepsilon^{k+1}(1 + |\log \varepsilon|^{k+1}) \le C_\beta \varepsilon^{k+1-\beta}.$$

We also obtain

$$|D(v_k^{\varepsilon,D}(x))_{xx} + g(u_k^{\varepsilon,D}(x), v_k^{\varepsilon,D}(x))|$$

$$\le \left| \frac{D}{\varepsilon^2} \left(\sum_{j=0}^{k} \varepsilon^j v^j(z) \right)_{zz} + g\left(\sum_{j=0}^{k} \varepsilon^j u^j(z), \sum_{j=0}^{k} \varepsilon^j v^j(z) \right) \right|$$

$$+ \left| g\left(\sum_{j=0}^{k} \varepsilon^j u^j(z), \sum_{j=0}^{k+2} \varepsilon^j v^j(z) \right) - g\left(\sum_{j=0}^{k} \varepsilon^j u^j(z), \sum_{j=0}^{k} \varepsilon^j v^j(z) \right) \right|$$

$$\le \frac{C}{\varepsilon^2} \varepsilon^{k+3}(1 + |\log \varepsilon|^{k+1}) \le C_\beta \varepsilon^{k+1-\beta}.$$

Next, we treat the case $1 \le |\rho(x,\varepsilon)| \le 2$. We introduce a short hand expression

$$l^\varepsilon := \frac{-d_0}{(k+1)\log \varepsilon}.$$

Note that $0 < l^\varepsilon \le 1$ for small $\varepsilon > 0$. By using Proposition 4.1, we have

$$|\varepsilon^2(u_k^{\varepsilon,D}(x))_{xx} + f(u_k^{\varepsilon,D}(x), v_k^{\varepsilon,D}(x))|$$

$$\le |\varepsilon^2(U_k^{\pm,\varepsilon}(x))_{xx} + f(U_k^{\pm,\varepsilon}(x), V_k^{\pm,\varepsilon}(x))|$$

$$+ |f(U_k^{\pm,\varepsilon}(x), V_k^{\pm,\varepsilon}(x)) - f(U_k^{\pm,\varepsilon}(x), V_{k+2}^{\pm,\varepsilon}(x))|$$

$$+ |\{\theta(l^\varepsilon z)[\tilde{u}_k(z) - U_k^{\pm,\varepsilon}(x^* + \varepsilon z)]\}_{zz}| \qquad (=: K_1^\varepsilon(z))$$

$$+ |f(u_k^{\varepsilon,D}(x), v_k^{\varepsilon,D}(x)) - f(U_k^{\pm,\varepsilon}(x), V_{k+2}^{\pm,\varepsilon}(x))| \qquad (=: K_2^\varepsilon(x))$$

$$\le C\varepsilon^{k+1} + K_1^\varepsilon(z) + K_2^\varepsilon(x).$$

By using Corollary 4.1 (iv) and (4.28), it follows that

$$K_1^\varepsilon(z) \leq |\theta''(l^\varepsilon z)|\{|\tilde{u}_k^\varepsilon(z) - \tilde{U}_k^{\pm,\varepsilon}(z)| + |\tilde{U}_k^{\pm,\varepsilon}(z) - U_k^{\pm,\varepsilon}(x^* + \varepsilon z)|\}$$

$$+ 2|\theta'(l^\varepsilon z)|\{|[\tilde{u}_k^\varepsilon(z) - \tilde{U}_k^{\pm,\varepsilon}(z)]_z| + |[\tilde{U}_k^{\pm,\varepsilon}(z) - U_k^{\pm,\varepsilon}(x^* + \varepsilon z)]_z|\}$$

$$+ \{|[\tilde{u}_k^\varepsilon(z) - \tilde{U}_k^{\pm,\varepsilon}(z)]_{zz}| + |[\tilde{U}_k^{\pm,\varepsilon}(z) - U_k^{\pm,\varepsilon}(x^* + \varepsilon z)]_{zz}|\}$$

$$\leq C\varepsilon^{k+1} + C\varepsilon^{k+1}|\log \varepsilon|^{k+1} \leq C_\beta \varepsilon^{k+1-\beta}.$$

By using (4.31), we also obtain $K_2^\varepsilon(x) \leq C_\beta \varepsilon^{k+1-\beta}$. Similar arguments apply to $v_k^{\varepsilon,D}(x)$. This completes the proof of Proposition 2.4.

REMARK 4.1. *If we were only to deal with boundary layers either on $\bar{\Omega}^-$ or on $\bar{\Omega}^+$, we could choose*

$$(4.32) \quad \begin{cases} u_k^{\varepsilon,D}(x) = U_k^{\pm,\varepsilon}(x) + \theta\left(\frac{x-x^*}{r_0}\right)\left[\tilde{u}_k^\varepsilon\left(\frac{x-x^*}{\varepsilon}\right) - \tilde{U}_k^{\pm,\varepsilon}\left(\frac{x-x^*}{\varepsilon}\right)\right], \\ v_k^{\varepsilon,D}(x) = V_{k+2}^{\pm,\varepsilon}(x) + \theta\left(\frac{x-x^*}{r_0}\right)\left[\tilde{v}_{k+2}^\varepsilon\left(\frac{x-x^*}{\varepsilon}\right) - \tilde{V}_{k+2}^{\pm,\varepsilon}\left(\frac{x-x^*}{\varepsilon}\right)\right], \end{cases}$$

as our approximation, where $r_0 > 0$ is a constant given by

$$r_0 := \frac{1}{4}\min\{x^*(D), 1 - x^*(D)\}.$$

Indeed, the functions in (4.32) are better approximations to the solution of (1.5) on Ω^- and Ω^+, in the sense that Proposition 2.4 (iii) holds with $\beta = 0$. Moreover, the entire proof of Proposition 2.4 for (4.32) is easier than that for (4.27). However, the functions in (4.32) have a fatal defect for our purpose here. Namely, they are not smooth across the interface $x = x^*$. They are C^0-matched at $x = x^*$, but not C^1-matched. The difference in the derivatives at $x = x^*$ is $O(\varepsilon^{k+1})$.

On the other hand, when we deal with boundary layers, it is better to employ the approximations in (4.32), as clearly described in [7].

5. Eigenvalue problems

In this section, we will prove Propositions 2.5 and 2.6. We stress that the proof below works for any $k \geq 1$, where $k \in \mathbf{N}$ is the order of approximation in Proposition 2.4.

5.1. Linearized operator of Allen-Cahn type. Let us define a linear operator L_k^ε by

$$(5.1) \quad \begin{cases} L_k^\varepsilon\varphi(x) := \varepsilon^2\varphi_{xx} + f_u^\varepsilon(x)\varphi(x), \quad \varphi_x(0) = 0 = \varphi_x(1), \\ \text{where } f_u^\varepsilon(x) := f_u(u_k^{\varepsilon,D}(x), v_k^{\varepsilon,D}(x)). \end{cases}$$

We denote by μ_j^ε $(j = 0, 1, \ldots)$ the eigenvalues of L_k^ε;

$$\sigma(L_k^\varepsilon) := \{\mu_j^\varepsilon\}_{j=0}^\infty, \qquad \mu_0^\varepsilon > \mu_1^\varepsilon > \cdots > \mu_j^\varepsilon \to -\infty \;\; (j \to \infty).$$

PROPOSITION 5.1. *There exists an $\varepsilon_0 > 0$ so that the following items are valid for $\varepsilon \in (0, \varepsilon_0]$.*

(i) *If $\phi_0^\varepsilon(x)$ is the L^2-normalized, positive, principal eigenfunction of L_k^ε corresponding to μ_0^ε, then*

(5.2) $$0 < \mu_0^\varepsilon \;\; \text{for } \varepsilon \in (0, \varepsilon_0], \qquad \mu_0^\varepsilon \to 0 \;\; \text{as } \varepsilon \to 0,$$

(5.3) $$\lim_{\varepsilon \to 0} \sqrt{\varepsilon}\phi_0^\varepsilon(x^* + \varepsilon z) = \frac{1}{\sqrt{m(v^*)}} u_z^0(z) \qquad \text{in } C_{\text{loc}}^2(\mathbf{R}),$$

where $m(v)$ is defined in Proposition 2.3 (ii).

(ii) *Let $[\phi_0^\varepsilon]^\perp$ be the orthogonal complement of ϕ_0^ε in $L^2(0,1)$. There exist constants $\mu_* > 0$ and $C_R > 0$ for each $R > 0$ such that*

(5.4) $$\|(L_k^\varepsilon - \mu)w\|_{L^\infty} \geq C_R\|w\|_{L^\infty}$$

holds for $w \in H^2(0,1) \cap [\phi_0^\varepsilon]^\perp$ and $\mu \in \mathbf{C}$, $\operatorname{Re}\mu > -\mu_*$, $|\mu| \leq R$.

(iii) *There exists a $\hat{\mu} > \mu_*$ such that*

(5.5) $$\mu_1^\varepsilon \leq -\hat{\mu} \qquad \text{and} \qquad \mu_0^\varepsilon = \varepsilon c'(v^*) V_x^{*,D}(x^*) + o(\varepsilon) \quad \text{as } \varepsilon \to 0$$

where $c'(v^*)$ is defined in Proposition 2.3 (iii).

(iv) *Let $P^\varepsilon : L^2(0,1) \to L^2(0,1)$ be the orthogonal projection operator onto $[\phi_0^\varepsilon]^\perp$. The following estimate holds true for $\mu \in \mathbf{C}$, $\operatorname{Re}\mu \geq -\mu_*$.*

$$\|(L_k^\varepsilon - \mu)^{-1} P^\varepsilon q\|_{L^2} \leq \frac{1}{|\mu - \underline{\mu}|}\|q\|_{L^2} \qquad \text{for } q \in L^2(0,1),$$

where $\underline{\mu}$ is a constant satisfying $\mu_* < \underline{\mu} < \hat{\mu}$, say, $\underline{\mu} = (\mu_* + \hat{\mu})/2$.

(v) *For $q \in L^\infty(0,1)$ and $\mu \in \mathbf{C}$, $\operatorname{Re}\mu \geq -\mu_*$,*

(5.6) $$\lim_{\varepsilon \to 0}[(L_k^\varepsilon - \mu)]^{-1} P^\varepsilon q = \frac{q(x)}{f_u^0(x) - \mu} \qquad \text{strongly in } L^2(0,1),$$

where $f_u^0(x) := f_u(h^\pm(V^{*,D}(x)), V^{*,D}(x))$ for $x \in \Omega^\pm$. Moreover, the convergence in (5.6) is uniform with respect to $\mu \in \{\mu \in \mathbf{C} \,|\, \operatorname{Re}\mu > -\mu_*\}$ and q on H^1-bounded sets.

Statement (ii) was also proved in [22] for $\mu = 0$. Our proof for (ii) below is very similar to that of [22].

PROOF. (i) By the variational characterization of the principal eigenvalue for L_k^ε, we readily find that $\mu_0^\varepsilon > 0$. It is also obvious that μ_0^ε is bounded above.

Let us express L_k^ε in terms of the stretched variable $z = (x - x^*)/\varepsilon$.

$$\begin{cases} \tilde{L}_k^\varepsilon p(z) := p_{zz}(z) + \tilde{f}_u^\varepsilon(z) p(z), \\ \qquad \text{where} \ \ \tilde{f}_u^\varepsilon(z) := f_u^\varepsilon(x^* + \varepsilon z). \end{cases}$$

There exists $b > 0$, independent of $\varepsilon \in (0, \varepsilon_0]$, so that

$$(5.7) \qquad \tilde{f}_u^\varepsilon(z) \leq -(d_0)^2 \qquad \text{for } |z| \geq b,$$

where $d_0 > 0$ is the constant appearing in (4.7) and (4.8). We choose a positive eigenfunction $p^\varepsilon(z)$ of $\tilde{L}_k^\varepsilon$ associated with μ_0^ε, normalized as $\max_z p(z) = 1$. By using comparison arguments and (5.7), we find that there exists a constant $C > 0$ so that

$$(5.8) \quad p^\varepsilon(z) \leq Ce^{d_0(b-|z|)}, \quad |p_z^\varepsilon(z)| \leq Ce^{d_0(b-|z|)}, \quad |p_{zz}^\varepsilon(z)| \leq Ce^{d_0(b-|z|)}, \quad |z| \geq b.$$

The equation for p^ε is

$$(5.9) \qquad p_{zz}^\varepsilon + \tilde{f}_u^\varepsilon(z) p^\varepsilon = \mu_0^\varepsilon p^\varepsilon.$$

For any sequence $\{\varepsilon_n\}$ with $\lim_{n\to\infty} \varepsilon_n = 0$, there exists a subsequence (which we still denote by ε_n) such that $\lim_{n\to\infty} \mu_0^{\varepsilon_n} = \mu_0^* \geq 0$. Since $|p^\varepsilon(z)| \leq 1$, by using (5.9), we find $|p_{zz}^{\varepsilon_n}|$ is bounded uniformly in $\varepsilon_n \in (0, \varepsilon_0]$. Therefore, there exists a subsequence (still denoted by ε_n) such that p^{ε_n} is convergent in $C_{\mathrm{loc}}^1(\mathbf{R})$ as $n \to \infty$. Differentiating (5.9), we obtain

$$(5.10) \qquad p_{zzz}^\varepsilon + \tilde{f}_u^\varepsilon(z) p_z^\varepsilon + (\tilde{f}_u^\varepsilon(z))_z p^\varepsilon = \mu_0^\varepsilon p_z^\varepsilon.$$

Since $(\tilde{f}_u^\varepsilon(z))_z$ is of polynomial order in z and $p^\varepsilon(z)$ decays at an exponential order (cf. (5.8)), (5.10) says that $|p_{zzz}^{\varepsilon_n}|$ is bounded uniformly in $\varepsilon > 0$. Therefore, there exists a subsequence (still denoted by ε_n) so that p^{ε_n} converges in $C_{\mathrm{loc}}^2(\mathbf{R})$ as $n \to \infty$. We consider (5.9) with $\varepsilon = \varepsilon_n$. Passing to the limit $n \to \infty$, we obtain

$$(5.11) \qquad p_{zz}^*(z) + f_u(u^0(z), v^*) p^*(z) = \mu_0^* p^*(z), \qquad z \in \mathbf{R},$$

where $p^* \in C^2(\mathbf{R})$ is bounded (since $\max|p^\varepsilon(z)| = 1 = \max|p^*(z)|$) and $p^*(z) \geq 0$. On the other hand, (5.11) has a bounded solution if and only if

$$\mu_0^* \int_{\mathbf{R}} p^*(z) u_z^0(z) dz = 0.$$

However, the integral $\int p^* u_z^0 \, dz > 0$, because $p^\varepsilon(z)$ attains its positive maximum for some z with $|z| \leq b$ and hence so does $p^*(z)$. This implies $\mu_0^* = 0$. Therefore $p^*(z) = k_*^{-1} u_z^0(z)$ where $k_* = \max u_z^0(z)$. Since the original sequence ε_n could be chosen arbitrarily, we conclude that

$$(5.12) \qquad \lim_{\varepsilon \to 0} \mu_0^\varepsilon = 0, \qquad \lim_{\varepsilon \to 0} p^\varepsilon(z) = k_*^{-1} u_z^0(z) \quad \text{in } C_{\mathrm{loc}}^2(\mathbf{R}).$$

Now expressing $\phi_0^\varepsilon(x^* + \varepsilon z) = a_\varepsilon p^\varepsilon(z)$, where $a_\varepsilon > 0$ is a constant which L^2-normalizes $\phi_0^\varepsilon(x)$, (5.2) and (5.3) follow from (5.12).

(ii) We prove this statement by means of contradiction

Let us recall the definition $f_u^0(x) := f_u(h^\pm(V^{*,D}(x)), V^{*,D}(x))$ for $x \in \Omega^\pm$. If (5.4) were to fail for any constant $C_R > 0$, we could find sequences $w^n \in H^2 \cap [\phi_0^\varepsilon]^\perp$, $\mu_n \in \mathbf{C}$ with $\mathrm{Re}\,\mu_n > \max_{x \in [0,1]}\{f_u^0(x)\}$, $|\mu_n| \leq R$ and ε_n such that $\max|w^n| = 1$, $\lim_{n\to\infty} \varepsilon_n = 0$ and $\|q_n\|_{L^\infty} \to 0$ as $n \to \infty$, where $q^n = (L_k^{\varepsilon_n} - \mu_n)w^n$. We express the last relation in terms of a stretched variable $z = (x - x_n)/\varepsilon$, where x_n is such that $|w^n(x_n)| = 1$. We may assume, without loss of generality, that x_n and μ_n converge to $x_\infty \in [0,1]$ and μ^0 with $\mathrm{Re}\,\mu^0 > \max_{x \in [0,1]}\{f_u^0(x)\}$, $|\mu^0| \leq R$.

We assume, for the moment, that $x_\infty \in (0,1)$. (The cases $x_\infty = 0, 1$ will be treated similarly.) The equation for $\tilde{w}^n(z) := \tilde{w}^n(x_n + \varepsilon_n z)$, with $\tilde{q}^n(z) := \tilde{q}^n(x_n + \varepsilon_n z)$, is

$$(5.13) \qquad \tilde{q}^n(z) = \tilde{w}_{zz}^n(z) + (f_u^{\varepsilon_n}(x_n + \varepsilon_n z) - \mu_n)\tilde{w}^n(z).$$

Applying to (5.13) arguments similar to the proof of (i), we find that $\tilde{w}^n$ (possibly, a subsequence) is convergent in $C_{loc}^1(\mathbf{R})$ to $\tilde{w}^*(z)$ as $n \to \infty$. Passing to the limit $n \to \infty$ in the weak version (H^1-formulation) of (5.13), and using regularity arguments, we obtain $\tilde{w}^* \in C^2(\mathbf{R})$ which is bounded, satisfies $\tilde{w}^*(0) = 1$, $\tilde{w}_z^*(0) = 0$ and

$$(5.14) \quad \begin{cases} \text{(a)} \quad \mu^0 \tilde{w}^* = \tilde{w}_{zz}^* + f_u^0(x_\infty)\tilde{w}^*, \quad \text{if } x_\infty \neq x^*, \\ \text{(b)} \quad \mu^0 \tilde{w}^* = L_{z_*}^0 \tilde{w}^* := \tilde{w}_{zz}^* + f_u(u^0(z + z_*), v^*)\tilde{w}^*, \quad \text{if } x_\infty = x^*. \end{cases}$$

In (5.14)(b), $z_* := \lim_{n\to\infty}(x_n - x^*)/\varepsilon_n \in \mathbf{R}$ (if $z_* = \pm\infty$, then $x_\infty \neq x^*$).

Recall from (A1) that $f_u^0(x) < 0$ for $x \in [0,1]$. In case (5.14)(a), if $\mathrm{Re}\,\mu^0 > \max_{x \in [0,1]}\{f_u^0(x)\}$, then the only bounded solution to the differential equation is identically equal to 0, and hence contradicting $|\tilde{w}^*(0)| = 1$. In case (5.14)(b), we recall that the principal eigenvalue of $L_{z_*}^0$ is 0 which is isolated in the specturm $\sigma(L_{z_*}^0)$ of $L_{z_*}^0$. Therefore, there exists a constant $\bar{\mu} > 0$ so that

$$(5.15) \qquad \sigma(L_{z_*}^0) \cap \{\mu \in \mathbf{C} \mid \mathrm{Re}\,\mu > -\bar{\mu}\} = \{0\}.$$

Then (5.14)(b) implies that $\tilde{w}^* \equiv 0$, if $0 \neq \mu^0$, $\mathrm{Re}\,\mu^0 > -\bar{\mu}$. This is a contradiction, since $\max|\tilde{w}^*| = 1$. If $\mu^0 = 0$, then $\tilde{w}^*$ must be a multiple of the principal eigenfunction $u_z^0(z + z_*)$ of $L_{z_*}^0$, namely, $\tilde{w}^*(z) = au_z^0(z + z_*)$ for some a. However, since we have chosen $\tilde{w}^n \in [\phi_0^\varepsilon]^\perp$, (5.3) implies

$$0 = \int_{\mathbf{R}} \tilde{w}^*(z)u_z^0(z + z_*)dz = a \int (u_z^0(z))^2 dz.$$

Therefore, $a = 0$ and $\tilde{w}^*(z) \equiv 0$, which is a contradiction.

When $x_\infty = 0$ or $x_\infty = 1$, we obtain (5.14)(a) with the differential equation being posed on half intervals $\{z > 0\}$ or $\{z < 0\}$. However, for μ^0 with Re $\mu^0 > \max_{x \in [0,1]}\{f_u^0(x)\}$, $|\mu^0| \le R$, these problems have no *bounded* solution satisfying $|\tilde{w}^*(0)| = 1$ and $\tilde{w}_z^*(0) = 0$, arriving at a contradiction.

By choosing $\mu_* > 0$ so that

$$\mu_* < \hat{\mu} := \min\left\{\bar{\mu}, -\max_{x \in [0,1]}\{f_u^0(x)\}\right\},$$

we complete the proof of (ii).

(iii) The first statement $\mu_1^\varepsilon \le -\hat{\mu}$ follows from part (ii), since eigenfunctions associated to μ_1^ε belong to $[\phi_0^\varepsilon]^\perp$. To prove the second statement, we use the following elementary result whose proof is omitted.

LEMMA 5.1. *Let L be a self-adjoint operator on a Hilbert space $\mathbf{H}$. Assume that L has an isolated eigenvalue $\tilde{\mu}$ of multiplicity one which is bounded away from the other part of the spectrum by a constant $\delta > 0$.*

If we can find $p^\varepsilon \in \mathbf{H}$, $|p^\varepsilon|_{\mathbf{H}} = 1$ and $\lambda^\varepsilon \in \mathbf{R}$ such that

$$|Lp^\varepsilon - \lambda^\varepsilon p^\varepsilon|_{\mathbf{H}} = O(\varepsilon^i)$$

for some $i \ge 1$, and

$$\operatorname{dist}(\lambda^\varepsilon, \sigma(L) - \{\tilde{\mu}\}) \ge \delta,$$

then the eigenpair $(\tilde{\mu}, \tilde{\phi})$ of L, with $|\tilde{\phi}|_{\mathbf{H}} = 1$ and $\langle p^\varepsilon, \tilde{\phi} \rangle_{\mathbf{H}} > 0$, is approximated as

$$|\tilde{\mu} - \lambda^\varepsilon| = O(\varepsilon^i), \qquad |\tilde{\phi} - p^\varepsilon|_{\mathbf{H}} = O(\varepsilon^i).$$

We now apply Lemma 5.1 to $L = L_k^\varepsilon$ and $\mathbf{H} = L^2(0,1)$ with $\tilde{\mu} = \mu_0^\varepsilon$ and $\delta = \hat{\mu}$. We will find an expansion

$$\lambda^\varepsilon = \varepsilon\lambda_1 + o(\varepsilon) \qquad \text{and} \qquad p^\varepsilon(x) = p^0(x) + \varepsilon p^1(x) + o(\varepsilon)$$

so that

$$\|L_k^\varepsilon p^\varepsilon - \lambda^\varepsilon p^\varepsilon\|_{L^2} = o(\varepsilon)$$

is valid. The normalization $\|p^\varepsilon\|_{L^2} = 1$ will be done afterwards. To find the coefficients λ_1, p^0 and p^1, we follow the same line of arguments as in §4 (and if fact, procedures here are less complicated). Let us write down the equation to deal with.

$$\varepsilon\lambda_1(p^0 + \varepsilon p^1) = \varepsilon^2(p^0 + \varepsilon p^1)_{xx} + f_u^\varepsilon(x)(p^0 + \varepsilon p^1) + o(\varepsilon).$$

We immediately find that outer solutions for this equation are identically equal to 0, reflecting the fact that $\phi_0^\varepsilon = O(e^{-d_0|x-x^*|/\varepsilon})$. To find inner solutions, let us rewrite the last equation in terms of the stretched variable z;

$$\varepsilon\lambda_1(\tilde{p}^0(z) + \varepsilon\tilde{p}^1(z)) = (\tilde{p}^0(z) + \varepsilon\tilde{p}^1(z))_{zz} + \tilde{f}_u^\varepsilon(z)(\tilde{p}^0(z) + \varepsilon\tilde{p}^1(z)) + o(\varepsilon).$$

The equation for $\tilde{p}^0$ reads: $0 = L^0 \tilde{p}^0$. This has a unique solution $\tilde{p}^0 = c_0 u_z^0$ for some constant $c_0 \neq 0$. The equation for $\tilde{p}^1$ reads

$$\lambda_1 c_0 u_z^0 = \tilde{p}_{zz}^1 + f_u(\#)\tilde{p}^1 + [f_{uu}(\#)u^1 + f_{uv}(\#)v^1]c_0 u_z^0,$$

where $(\#) = (u^0(z), v^*)$. Applying the solvability condition (3.4) and using $c_0 \neq 0$, we find $\lambda_1 = c'(v^*)V_x^{*,D}(x^*)$ and $\tilde{p}^1(z) = O(e^{-d_0|z|})$. Therefore our approximate solution p^ε is now defined, with c_ε being a normalizing constant, by

$$p^\varepsilon(x) := c_\varepsilon \theta(\rho(x,\varepsilon)) \left[\tilde{p}^0\left(\frac{x - x^*}{\varepsilon}\right) + \varepsilon\tilde{p}^1\left(\frac{x - x^*}{\varepsilon}\right)\right],$$

where ρ is as in (4.29) and θ is the cut-off function introduced at the end of §4.3. We can now verify that p^ε and $\lambda^\varepsilon = \varepsilon c'(v^*)V_x^{*,D}(x^*)$ satisfy the conditions in Lemma 5.1, establishing the second statement in (5.5).

(iv) This follows immediately from Proposition 5.1 (iii) and the eigenfunction expansion of $(L_k^\varepsilon - \mu)^{-1}$.

We refer the proof of (v) to the proof of Lemma 2.2 in [14].

This completes the proof of Proposition 5.1. □

COROLLARY 5.1. *For each* $p \in H^1(0,1)$, *we have*

$$\text{(i)} \qquad \lim_{\varepsilon \to 0}\left\langle p, \left(\frac{\phi_0^\varepsilon}{\sqrt{\varepsilon}}\right)g_u^\varepsilon \right\rangle = p(x_*)\frac{[g]_-^+}{\sqrt{m(v^*)}},$$

$$\text{(ii)} \qquad \lim_{\varepsilon \to 0}\left\langle p, \left(\frac{\phi_0^\varepsilon}{\sqrt{\varepsilon}}\right)f_v^\varepsilon \right\rangle = p(x_*)\frac{J'(v^*)}{\sqrt{m(v^*)}},$$

where $m(v)$ *is as defined in Proposition* 2.3 (ii).

PROOF. It suffices to prove (i) and (ii) for $p \in C^1[0,1]$. For $p \in C^1[0,1]$, by using (5.3), we have

$$\left\langle p, \left(\frac{\phi_0^\varepsilon}{\sqrt{\varepsilon}}\right)g_u^\varepsilon \right\rangle = \int_0^1 p(x)\phi_0^\varepsilon(x)g_u^\varepsilon \frac{dx}{\sqrt{\varepsilon}}$$

$$= \int_{-x^*/\varepsilon}^{(1-x^*)/\varepsilon} p(x^* + \varepsilon z)(\sqrt{\varepsilon}\phi_0^\varepsilon(x^* + \varepsilon z))g_u^\varepsilon(x^* + \varepsilon z)dz$$

$$\text{(as } \varepsilon \to 0) \quad \to \frac{p(x^*)}{\sqrt{m(v^*)}}\int_{-\infty}^{\infty} u_z^0(z)g_u(u_0(z), v^*)dz$$

$$= p(x^*)\frac{g(h^+(v^*), v^*) - g(h^-(v^*), v^*)}{\sqrt{m(v^*)}} = p(x^*)\frac{[g]_-^+}{\sqrt{m(v^*)}},$$

establishing (i). The proof of (ii) is similar. □

5.2. Proof of Proposition 2.5. Let us now deal with the eigenvalue of $\mathscr{L}_k^\varepsilon$ with the largest real part. We call this the principal eigenvalue of $\mathscr{L}_k^\varepsilon$ and denote its principal eigenfunction[2] by

$$\Phi_0^\varepsilon := \begin{pmatrix} \varphi_0^\varepsilon \\ \psi_0^\varepsilon \end{pmatrix}$$

Let $\lambda_* > 0$ be defined by

$$\lambda_* := \min\left\{\hat{\mu}, -\frac{1}{2}\max_{x \in [0,1]}(f_u^0(x) + g_v^0(x))\right\} > 0,$$

where $g_v^0(x) := g_v(h^\pm(V^0(x), V^0(x)))$ for $x \in \Omega^\pm$ and $\hat{\mu}$ is the same as in Proposition 5.1 (ii). The positivity of λ_* follows from $\hat{\mu} > 0$ and the condition in (A4). Let us define

$$\mathbf{C}_{\lambda_*} := \{\lambda \in \mathbf{C} \,|\, \mathrm{Re}\,\lambda \geq -\lambda_*\}.$$

We characterize the eigenvalues of $\mathscr{L}_k^\varepsilon$ contained in $\mathbf{C}_{\lambda_*}$ to establish Proposition 2.5. In the sequel, we always consider $\lambda \in \mathbf{C}_{\lambda_*}$.

By decomposing the first component of the eigenfunction of $\mathscr{L}_k^\varepsilon$ as $\varphi^\varepsilon = a\phi_0^\varepsilon + w^\varepsilon$, the eigenvalue problem (2.18) is recast as

$$a(\mu_0^\varepsilon - \lambda) = -\langle f_v^\varepsilon \psi, \phi_0^\varepsilon \rangle = -\langle \psi, f_v^\varepsilon \phi_0^\varepsilon \rangle$$

(5.16)
$$(L_k^\varepsilon - \lambda)w^\varepsilon = -P^\varepsilon(f_v^\varepsilon \psi)$$

$$-(M_k^\varepsilon - \lambda)\psi - g_u^\varepsilon w^\varepsilon = ag_u^\varepsilon \phi_0^\varepsilon$$

where $(\mu_0^\varepsilon, \phi_0^\varepsilon(x))$ is the principal eigenpair of L_k^ε, $a \in \mathbf{C}$, w^ε satisfies $\langle w^\varepsilon, \phi_0^\varepsilon \rangle = 0$ and P^ε is the orthogonal projection onto the orthogonal complement of ϕ_0^ε. Thanks to Proposition 5.1 (iv) and the fact that $\lambda \in \mathbf{C}_{\lambda_*}$, the second equation in (5.16) is solved in w^ε as

$$w^\varepsilon = -(L_k^\varepsilon - \lambda)^{-1} P^\varepsilon(f_v^\varepsilon \psi).$$

By using this relation, the third equation in (5.16) becomes

(5.17)
$$\mathscr{N}^\varepsilon(\lambda)\psi = ag_u^\varepsilon \phi_0^\varepsilon,$$

where

(5.18)
$$\mathscr{N}^\varepsilon(\lambda)\psi := -(M_k^\varepsilon - \lambda)\psi + g_u^\varepsilon(L_k^\varepsilon - \lambda)^{-1}P^\varepsilon(f_v^\varepsilon \psi).$$

The following summarizes the properties of $\mathscr{N}^\varepsilon(\lambda)$.

---

[2] In the sequel we use the symbol φ only for this first component of Φ, and distinguish it from another similar symbol ϕ.

PROPOSITION 5.2.   *There exists an $\varepsilon_0 > 0$ such that the following statements hold.*

(i)   *For $\lambda \in \mathbf{C}_{\lambda_*}$, the operator $\mathcal{N}^\varepsilon(\lambda)$ is invertible and*

$$[\mathcal{N}^\varepsilon(\lambda)]^{-1} : L^2(0,1) \to H^2(0,1)$$

is bounded uniformly in $(\varepsilon, \lambda) \in (0, \varepsilon_0] \times \mathbf{C}_{\lambda_*}$.

(ii)   *Denoting by $[H^1(0,1)]'$ the dual space of $H^1(0,1)$, the inverse operator $[\mathcal{N}^\varepsilon(\lambda)]^{-1}$ extends to $[H^1(0,1)]'$ and*

$$[\mathcal{N}^\varepsilon(\lambda)]^{-1} : [H^1(0,1)]' \to H^1(0,1)$$

is bounded uniformly in $(\varepsilon, \lambda) \in (0, \varepsilon_0] \times \mathbf{C}_{\lambda_*}$.

We prove this result later in §5.3.

We now resume the proof of Proposition 2.5.

If $a = 0$, then Proposition 5.2 (i) and (5.17) imply $\psi = 0$, which in turn implies $w^\varepsilon = 0$ and hence $\varphi = 0 = \psi$. Therefore, in order for $\lambda \in \mathbf{C}_{\lambda_*}$ to be an eigenvalue of $\mathscr{L}_k^\varepsilon$, $a \neq 0$ must be satisfied. Therefore, an eigenvalue $\lambda \in \mathbf{C}_{\lambda_*}$ has to satisfy

(5.19)
$$\begin{cases} \mu_0^\varepsilon - \lambda = -\langle \psi, f_v^\varepsilon \phi_0^\varepsilon \rangle \\ \mathcal{N}^\varepsilon(\lambda)\psi = g_u^\varepsilon \phi_0^\varepsilon. \end{cases}$$

LEMMA 5.2.   *Eigenvalues of $\mathscr{L}_k^\varepsilon$ in $\mathbf{C}_{\lambda_*}$ are bounded uniformly in $\varepsilon \in (0, \varepsilon_0]$ for some $\varepsilon_0 > 0$.*

PROOF.   From (5.19) and Proposition 5.2 (i), the ψ-component of the eigenfunction is non-zero. We normalize it as $\|\psi\|_{L^2} = 1$. Multiply the second equation in (5.19) by $\bar{\psi}$, the complex conjugate of ψ, and integrate over $[0,1]$. We separate the Re- and Im-parts and use $\|\psi\|_{L^2} = 1$ to obtain

$$\text{Re } \lambda = -D\|\psi_x\|_{L^2}^2 + \langle g_v^\varepsilon \psi, \bar{\psi} \rangle - \text{Re}\{\langle g_u^\varepsilon (L_k^\varepsilon - \lambda)^{-1} P^\varepsilon (f_v^\varepsilon \psi), \bar{\psi} \rangle\} + \text{Re}\langle g_u^\varepsilon \phi_0^\varepsilon, \bar{\psi} \rangle$$

$$< \langle g_v^\varepsilon \psi, \bar{\psi} \rangle - \text{Re}\{\langle g_u^\varepsilon (L_k^\varepsilon - \lambda)^{-1} P^\varepsilon (f_v^\varepsilon \psi), \bar{\psi} \rangle\} + \text{Re}\langle g_u^\varepsilon \phi_0^\varepsilon, \bar{\psi} \rangle,$$

$$\text{Im } \lambda = -\text{Im}\{\langle g_u^\varepsilon (L_k^\varepsilon - \lambda)^{-1} P^\varepsilon (f_v^\varepsilon \psi), \bar{\psi} \rangle\} + \text{Im}\langle g_u^\varepsilon \phi_0^\varepsilon, \bar{\psi} \rangle.$$

Since f_v^ε, g_u^ε and g_v^ε are L^∞-bounded, by using $\|\psi\|_{L^2} = 1$, $\text{Re } \lambda \geq -\lambda_*$ and Proposition 5.1 (iv), we immediately find that $|\text{Re } \lambda|$ and $|\text{Im } \lambda|$ are bounded. $\square$

Thanks to Corollary 5.1 (i) and Proposition 5.2 (ii), we deduce from the second equation in (5.19) that

$$\psi = \sqrt{\varepsilon}\hat{\psi}^\varepsilon \qquad \text{with } \hat{\psi}^\varepsilon = [\mathcal{N}^\varepsilon(\lambda)]^{-1}(g_u^\varepsilon \phi_0^\varepsilon / \sqrt{\varepsilon}) \in H^1(0,1),$$

where $\hat{\psi}^\varepsilon$ is bounded in H^1 uniformly with respect to $(\varepsilon, \lambda) \in (0, \varepsilon_0] \times \mathbf{C}_{\lambda_*}$. On the other hand, it also follows from Corollary 5.1 (ii) that

$$\langle \psi, f_v^\varepsilon \phi_0^\varepsilon \rangle = \varepsilon \langle \hat{\psi}, f_v^\varepsilon \hat{\phi}_0^\varepsilon \rangle = O(\varepsilon), \qquad \text{where } \hat{\phi}_0^\varepsilon(x) := \frac{1}{\sqrt{\varepsilon}} \phi_0^\varepsilon(x).$$

Putting these facts together and using the fact that $\mu_0^\varepsilon = O(\varepsilon)$ (cf. (5.5)), we find from (5.19) that $\lambda = O(\varepsilon)$, and hence we may set

$$\lambda = \varepsilon \hat{\lambda} \qquad \text{and} \qquad \mu_0^\varepsilon = \varepsilon \hat{\mu}_0^\varepsilon \qquad \text{with } \hat{\mu}_0^\varepsilon = c'(v^*) V_x^{*,D}(x^*) + o(1).$$

It is now easy to see that (5.19) is equivalent to

$$(5.20) \qquad\qquad \hat{\mu}_0^\varepsilon - \hat{\lambda} = -\langle [\mathcal{N}^\varepsilon(\varepsilon\hat{\lambda})]^{-1}(g_u^\varepsilon \hat{\phi}_0^\varepsilon), f_v^\varepsilon \hat{\phi}_0^\varepsilon \rangle.$$

This equation, called a SLEP-equation in [14] (SLEP = singular limit eigenvalue problem), determines the eigenvalues of $\mathcal{L}_k^\varepsilon$ in $\mathbf{C}_{\lambda_*}$. Let us now examine the right hand side of (5.20).

PROPOSITION 5.3. *If $\hat{\lambda}$ is a solution of the* SLEP-*equation* (5.20), *then it is real and the right hand side of* (5.20) *is characterized as follows.*

$$(5.21) \qquad -\langle [\mathcal{N}^\varepsilon(\varepsilon\hat{\lambda})]^{-1}(g_u^\varepsilon \hat{\phi}_0^\varepsilon), f_v^\varepsilon \hat{\phi}_0^\varepsilon \rangle = \frac{c'(v^*)[g]_-^+}{D\pi_0} + o(1) \in \mathbf{R} \qquad as \ \varepsilon \to 0,$$

where

$$\pi_0 := \Psi_x^-(x^*) - \Psi_x^+(x^*) \qquad (cf. \ (2.12) \ in \ \S 2).$$

We will prove this result in §5.3.

Recall from (5.5) that $\hat{\mu}_0^\varepsilon = c'(v^*) V_x^{*,D}(x^*) + o(1)$. By using this and (5.21), we find that (5.20) is equivalent to

$$\hat{\lambda}^\varepsilon = c'(v^*) \left[V_x^{*,D}(x^*) - \frac{[g]_-^+}{D\pi_0} \right] + o(1) \in \mathbf{R} \qquad \text{as } \varepsilon \to 0.$$

This completes the proof of Proposition 2.5 (i)(ii). The simplicity of the eigenvalue λ_0^ε is an implication of Proposition 2.5 (iii), which will be proved below.

In order to prove Proposition 2.5 (iii), we recall from the line of arguments above that the principal eigenfunction $\Phi_0^\varepsilon = \begin{pmatrix} \varphi_0^\varepsilon \\ \psi_0^\varepsilon \end{pmatrix}$ of $\mathcal{L}_k^\varepsilon$ is of the following form:

$$(5.22) \qquad \begin{cases} \varphi_0^\varepsilon(x) = \phi_0^\varepsilon(x) + w^\varepsilon(x), \\ \psi_0^\varepsilon(x) = \sqrt{\varepsilon} \hat{\psi}^\varepsilon(x) := \sqrt{\varepsilon} [\mathcal{N}^\varepsilon(\varepsilon\hat{\lambda}^\varepsilon)]^{-1}(g_u^\varepsilon \phi_0^\varepsilon / \sqrt{\varepsilon})(x), \\ w^\varepsilon(x) = -\sqrt{\varepsilon} (L_k^\varepsilon - \varepsilon\hat{\lambda}^\varepsilon)^{-1} P^\varepsilon(f_v^\varepsilon \hat{\psi}^\varepsilon)(x). \end{cases}$$

From Corollary 5.1 (i), we find that $g_u^\varepsilon \phi_0^\varepsilon / \sqrt{\varepsilon} \in [H^1(0,1)]'$ is bounded uniformly in $\varepsilon \in (0, \varepsilon_*]$. Therefore Proposition 5.2 (ii) implies $\hat{\psi}^\varepsilon \in H^1(0,1)$ is bounded uniformly in $\varepsilon > 0$. By using the Sobolev embedding $H^1(0,1) \subset L^\infty(0,1)$, we find that $\|\hat{\psi}^\varepsilon\|_{H^1}$ and $\|\hat{\psi}^\varepsilon\|_{L^\infty}$ are bounded uniformly in $\varepsilon > 0$. This immediately implies

$$(5.23) \quad \|\psi_0^\varepsilon\|_{H^1} = O(\sqrt{\varepsilon}), \qquad \|\psi_0^\varepsilon\|_{L^\infty} = O(\sqrt{\varepsilon}), \qquad \|\psi_0^\varepsilon\|_{L^1} = O(\sqrt{\varepsilon}).$$

On the other hand, (5.3) implies $\|\phi_0^\varepsilon\|_{L^\infty} = O(1/\sqrt{\varepsilon})$. Hence, (5.4) and Corollary 5.1 (ii) give rise to

$$\|(L_k^\varepsilon - \varepsilon\hat{\lambda}^\varepsilon)^{-1} P^\varepsilon(f_v^\varepsilon \hat{\psi}^\varepsilon)\|_{L^\infty} \leq C\|P^\varepsilon(f_v^\varepsilon \hat{\psi}^\varepsilon)\|_{L^\infty}$$

$$\leq C\|f_v^\varepsilon \hat{\psi}^\varepsilon\|_{L^\infty} + C\sqrt{\varepsilon}\left|\left\langle \hat{\psi}^\varepsilon, f_v^\varepsilon \frac{\phi_0^\varepsilon}{\sqrt{\varepsilon}} \right\rangle\right| \|\phi_0^\varepsilon\|_{L^\infty} = O(1) \qquad \text{as } \varepsilon \to 0,$$

where we used the definition of P^ε; $P^\varepsilon \psi := \psi - \langle \psi, \phi_0^\varepsilon \rangle \phi_0^\varepsilon$. From this estimate and the third line of (5.22), we obtain

$$(5.24) \quad \|w^\varepsilon\|_{L^\infty} = O(\sqrt{\varepsilon}), \qquad \|w^\varepsilon\|_{L^2} = O(\sqrt{\varepsilon}), \qquad \|w^\varepsilon\|_{L^1} = O(\sqrt{\varepsilon}).$$

Using $\|\phi_0^\varepsilon\|_{L^2} = 1$, (5.23) and (5.24), we find from (5.22) that

$$\left\| \begin{pmatrix} \varphi_0^\varepsilon \\ \psi_0^\varepsilon \end{pmatrix} \right\|_{L^2} = 1 + O(\sqrt{\varepsilon}).$$

Therefore, multiplying (5.22) by a normalization constant $1 + O(\sqrt{\varepsilon})$ (and still denoting the resulting function by Φ_0^ε), we find

$$\|\Phi_0^\varepsilon\|_{L^1} = O(\sqrt{\varepsilon}), \qquad \|\Phi_0^\varepsilon\|_{L^\infty} = O\left(\frac{1}{\sqrt{\varepsilon}}\right),$$

establishing Proposition 2.5 (iii).

(iv) The proof of this part is accomplished by applying the same line of reasoning as in the proof of statements (i), (ii) and (iii) to $\hat{\mathscr{L}}_k^\varepsilon$. The only change one needs to make is to exchange the roles of f_v^ε and g_u^ε. We then obtain an expression for the principal eigenfunction $\hat{\Phi}_0^\varepsilon := \begin{pmatrix} \varphi_0^{*\varepsilon} \\ \psi_0^{*\varepsilon} \end{pmatrix}$, similar to (5.22), as follows.

$$(5.25) \quad \begin{cases} \varphi_0^{*\varepsilon}(x) := \phi_0^\varepsilon(x) + w^{*\varepsilon}(x), \\ \psi_0^{*\varepsilon}(x) := \sqrt{\varepsilon}\hat{\psi}^{*\varepsilon}(x) := \sqrt{\varepsilon}[\mathscr{N}^{*\varepsilon}(\varepsilon\hat{\lambda}^\varepsilon)]^{-1}(f_v^\varepsilon \phi_0^\varepsilon/\sqrt{\varepsilon})(x), \\ w^{*\varepsilon}(x) := -\sqrt{\varepsilon}(L_k^\varepsilon - \varepsilon\hat{\lambda}^\varepsilon)^{-1} P^\varepsilon(g_u^\varepsilon \hat{\psi}^\varepsilon)(x), \end{cases}$$

where $\mathscr{N}^{*\varepsilon}$ is the operator defined by (5.18) with g_u^ε and f_v^ε being interchanged. We also note that $\psi_0^{*\varepsilon}$ and $w^{*\varepsilon}$ satisfy estimates in (5.23) and (5.24).

This completes the proof of Proposition 2.5.

5.3. Proof of Propositions 5.2 and 5.3. We first deal with Proposition 5.2. We only need to prove (ii) since (i) follows immediately from (ii) and the elliptic regularity theory. The proof of (ii) depends on an application of the Lax-Milgram Theorem ([13], Theorem 5.21.2).

Let us consider a sesquilinear form $\mathscr{B}^{\varepsilon}_{\lambda}$ associated with $\mathscr{N}^{\varepsilon}(\lambda)$:

$$\mathscr{B}^{\varepsilon}_{\lambda} : H^1(0,1) \times H^1(0,1) \to \mathbf{C}$$

defined by

$$\mathscr{B}^{\varepsilon}_{\lambda}(\psi,\phi) := D\langle \psi_x, \bar{\phi}_x \rangle - \langle (g^{\varepsilon}_v - \lambda)\psi, \bar{\phi} \rangle + \langle g^{\varepsilon}_u (L^{\varepsilon}_k - \lambda)^{-1} P^{\varepsilon}(f^{\varepsilon}_v \psi), \bar{\phi} \rangle,$$

where $\bar{\phi}$ stands for the complex conjugate of ϕ. Our aim is to show that $\mathscr{B}^{\varepsilon}_{\lambda}$ is bounded and coercive for $(\varepsilon, \lambda) \in [0, \varepsilon_0] \times \mathbf{C}_{\lambda_*}$, where $\varepsilon_0 > 0$ is an appropriate constant. It is easy to see that $\mathscr{B}^{\varepsilon}_{\lambda}$ is bounded. Moreover, it depends continuously on $\varepsilon \in [0, \varepsilon_0]$. Therefore, to show the coercivity of $\mathscr{B}^{\varepsilon}_{\lambda}$ for $\lambda \in \mathbf{C}_{\lambda_*}$, we only need to do so for the limit $\mathscr{B}^0_{\lambda} := \lim_{\varepsilon \to 0} \mathscr{B}^{\varepsilon}_{\lambda}$.

By using Proposition 5.1 (v) (5.6), the operator $\mathscr{B}^0_{\lambda}$ is given by

$$(5.26) \qquad \mathscr{B}^0_{\lambda}(\psi,\phi) := D\langle \psi_x, \bar{\phi}_x \rangle + \left\langle \frac{f^0_v g^0_u - (f^0_u - \lambda)(g^0_v - \lambda)}{f^0_u - \lambda} \psi, \bar{\phi} \right\rangle.$$

It follows from this that

$$|\mathscr{B}^0_{\lambda}(\psi,\psi)| \geq D\|\psi_x\|^2_{L^2} + \int_0^1 \mathrm{Re}\left(\frac{\lambda^2 - \mathrm{tr}^0\,\lambda + \det^0}{\lambda - f^0_u} \right)|\psi|^2 dx,$$

where

$$\mathrm{tr}^0 := f^0_u + g^0_v < 0 \quad \text{(cf. (A4))} \qquad \text{and} \qquad \det^0 := f^0_u g^0_v - f^0_v g^0_u > 0 \quad \text{(cf. (A3))}.$$

By the choice of $\lambda_* > 0$ at the beginning of §5.2, there exists a constant $C_0 > 0$ such that for $x \in [0,1]$ and $\lambda \in \mathbf{C}_{\lambda_*}$, the inequality

$$\mathrm{Re}\left(\frac{\lambda^2 - \mathrm{tr}^0\,\lambda + \det^0}{\lambda - f^0_u} \right) \geq C_0$$

holds true. Therefore, we have

$$|\mathscr{B}^0_{\lambda}(\psi,\psi)| \geq \min\{D, C_0\}\|\psi\|^2_{H^1}.$$

This establishes, on account of the continuity of $\mathscr{B}^{\varepsilon}_{\lambda}$ in ε, the coercivity of $\mathscr{B}^{\varepsilon}_{\lambda}$ for $\lambda \in \mathbf{C}_{\lambda_*}$. Now the Lax-Milgram Theorem ([13], Theorem 5.21.2) proves Proposition 5.2 (ii) with

$$\|(\mathscr{N}^{\varepsilon}(\lambda))^{-1}\|_{[H^1]' \to H^1} \leq \frac{1}{2\min\{D, C_0\}} \qquad \text{for } (\varepsilon, \lambda) \in [0, \varepsilon_0] \times \mathbf{C}_{\lambda_*},$$

where $\varepsilon_0 > 0$ is an adequate constant. This concludes the proof of Proposition 5.2.

Next, we prove Proposition 5.3.

We first split $\mathcal{N}^\varepsilon(\varepsilon\hat{\lambda})$ and $\hat{\psi}^\varepsilon = [\mathcal{N}^\varepsilon(\varepsilon\hat{\lambda})]^{-1}g_u^\varepsilon\hat{\phi}_0^\varepsilon$ into real and imaginary parts.

$$\mathcal{N}^\varepsilon(\varepsilon\hat{\lambda}) = \mathcal{N}_R^\varepsilon + i\varepsilon(\text{Im } \hat{\lambda})\mathcal{N}_I^\varepsilon, \qquad \hat{\psi}^\varepsilon = \hat{\psi}_R^\varepsilon + i\hat{\psi}_I^\varepsilon,$$

where

$$\mathcal{N}_R^\varepsilon\psi := -(M_k^\varepsilon - \varepsilon\,\text{Re }\hat{\lambda})\psi + g_u^\varepsilon[\varepsilon^2|\text{Im }\hat{\lambda}|^2 + (L_k^\varepsilon - \varepsilon\,\text{Re }\hat{\lambda})^2]^{-1}P^\varepsilon(f_v^\varepsilon\psi)$$

$$\mathcal{N}_I^\varepsilon\psi := \psi + [\varepsilon^2|\text{Im }\hat{\lambda}|^2 + (L_k^\varepsilon - \varepsilon\,\text{Re }\hat{\lambda})^2]^{-1}P^\varepsilon(f_v^\varepsilon\psi)$$

are real operators and $\hat{\psi}_R^\varepsilon$, $\hat{\psi}_I^\varepsilon$ are real valued functions. Then, the relation $\mathcal{N}^\varepsilon(\varepsilon\hat{\lambda})\hat{\psi}^\varepsilon = g_u^\varepsilon\hat{\phi}_0^\varepsilon$ translates into

$$\mathcal{N}_R^\varepsilon\hat{\psi}_R^\varepsilon - \varepsilon(\text{Im }\hat{\lambda})\mathcal{N}_I^\varepsilon\hat{\psi}_I^\varepsilon = g_u^\varepsilon\hat{\phi}_0^\varepsilon, \qquad \mathcal{N}_R^\varepsilon\hat{\psi}_I^\varepsilon + \varepsilon(\text{Im }\hat{\lambda})\mathcal{N}_I^\varepsilon\hat{\psi}_R^\varepsilon = 0.$$

From the proof of Proposition 5.2 above, the conclusions of Proposition 5.2 are valid for $\mathcal{N}_R^\varepsilon$. Therefore, the relations above yield

$$\mathcal{N}_R^\varepsilon\hat{\psi}_R^\varepsilon + \varepsilon^2(\text{Im }\hat{\lambda})^2\mathcal{N}_I^\varepsilon(\mathcal{N}_R^\varepsilon)^{-1}\mathcal{N}_I^\varepsilon\hat{\psi}_R^\varepsilon = g_u^\varepsilon\hat{\phi}_0^\varepsilon.$$

Evidently, $\mathcal{N}_I^\varepsilon : L^2(0,1) \to L^2(0,1)$ is a bounded operator uniformly in small $\varepsilon > 0$. From Proposition 5.1 (ii), the same is true for $\mathcal{N}_I^\varepsilon : L^\infty(0,1) \to L^\infty(0,1)$. Therefore, we have

$$\hat{\psi}_R^\varepsilon = (\mathcal{N}_R^\varepsilon)^{-1}g_u^\varepsilon\hat{\phi}_0^\varepsilon + O(\varepsilon^2), \qquad \hat{\psi}_I^\varepsilon = -\varepsilon(\text{Im }\hat{\lambda})(\mathcal{N}_R^\varepsilon)^{-1}\mathcal{N}_I^\varepsilon\hat{\psi}_R$$

with $\hat{\psi}_R^\varepsilon$, $(\mathcal{N}_R^\varepsilon)^{-1}\mathcal{N}_I^\varepsilon\hat{\psi}_R \in H^1(0,1)$ uniformlay bounded for small $\varepsilon > 0$. Substituting these into (5.20), we have

$$\hat{\mu}_0^\varepsilon - \text{Re }\hat{\lambda} = -\langle\hat{\psi}^\varepsilon, f_v^\varepsilon\hat{\phi}_0^\varepsilon\rangle$$

$$-\text{Im }\hat{\lambda} = \varepsilon(\text{Im }\hat{\lambda})\langle(\mathcal{N}_R^\varepsilon)^{-1}\mathcal{N}_I^\varepsilon\hat{\psi}_R, f_v^\varepsilon\hat{\phi}_0^\varepsilon\rangle.$$

The last equation is equivalent to

$$(1 + \varepsilon\langle(\mathcal{N}_R^\varepsilon)^{-1}\mathcal{N}_I^\varepsilon\hat{\psi}_R, f_v^\varepsilon\hat{\phi}_0^\varepsilon\rangle)\,\text{Im }\hat{\lambda} = 0,$$

which implies $\text{Im }\hat{\lambda} = 0$ when $\varepsilon > 0$ is small. This establishes the first part of Proposition 5.3.

Let us now prove (5.21). If $\hat{\psi}^\varepsilon = [\mathcal{N}^\varepsilon(\varepsilon\hat{\lambda})]^{-1}(g_u^\varepsilon\hat{\phi}_0^\varepsilon)$, then $\hat{\psi}^* := \lim_{\varepsilon\to 0}\hat{\psi}^\varepsilon$ satisfies

$$\mathcal{B}_0^0(\hat{\psi}^*, p) = \lim_{\varepsilon\to 0}\int_0^1 g_u^\varepsilon(x)\frac{\phi_0^\varepsilon(x)}{\sqrt{\varepsilon}}p(x)dx = \frac{[g]_-^+}{\sqrt{m(v^*)}}p(x^*) \qquad \text{for all } p \in H^1(0,1).$$

This is equivalent, via (5.26) with $\lambda = 0$, to

$$(5.27) \qquad D \int_0^1 \hat{\psi}_x^*(x) p_x(x) dx - \int_0^1 g_v^*(x) \hat{\psi}^*(x) p(x) dx = \frac{[g]_-^+}{\sqrt{m(v^*)}} p(x^*)$$

for all $p \in H^1(0,1)$, where $g_v^*(x) := g_v^{\pm}(V^{*,D}(x))$ for $x \in \Omega^{\pm}$, and we used the fact $\det^0(x)/f_u^0(x) = g_v^*(x)$ (cf. (A3)). Taking as a test function

$$p(x) = \Psi^{\pm}(x) \qquad x \in \Omega^{\pm},$$

in (5.27), where $\Psi^{\pm}$ is the solution of (2.6) with $v^0 = v^*$, and integrating by parts, we find

$$(5.28) \quad \hat{\psi}^*(x^*) = \frac{[g]_-^+}{D\pi_0} \frac{1}{\sqrt{m(v^*)}}, \qquad \hat{\psi}^*(x) = \hat{\psi}^*(x^*) \Psi^{\pm}(x) \quad \text{for } x \in \Omega^{\pm}.$$

Therefore, using Corollary 5.1 (ii), the left hand side of (5.21) is

$$-\lim_{\varepsilon \to 0} \langle [\mathcal{N}^{\varepsilon}(\varepsilon\hat{\lambda})]^{-1}(g_u^{\varepsilon}\hat{\phi}_0^{\varepsilon}), f_v^{\varepsilon}\hat{\phi}_0^{\varepsilon} \rangle = -\hat{\psi}^*(x^*) \frac{J'(v^*)}{\sqrt{m(v^*)}} = \frac{c'(v^*)[g]_-^+}{D\pi_0},$$

establishing Proposition 5.3.

Let us also find an expression of $\lim_{\varepsilon \to 0} w^{\varepsilon}(x)/\sqrt{\varepsilon}$. From (5.22) and (5.6),

$$(5.29) \quad \hat{w}^*(x) := \lim_{\varepsilon \to 0} \frac{w^{\varepsilon}(x)}{\sqrt{\varepsilon}} = -\frac{f_v^0(x)}{f_u^0(x)} \hat{\psi}^*(x) = h_v^{\pm}(V^{*,D}(x))\hat{\psi}^*(x) \quad \text{for } x \in \Omega^{\pm}$$

although we do not use it in this paper.

5.4. Proof of Proposition 2.6.
This is proved by means of contradiction.

If the statement of the proposition were to fail, there would exist sequences $\{\varepsilon_j\}$ with $\lim_{j \to \infty} \varepsilon_j = 0$ and $\{\mathbf{p}^j = (p_1^j, p_2^j)\} \subset \mathbf{M}$ such that

$$|\mathbf{p}^j|_{L^{\infty}} = \max_{x \in [0,1]} (|p_1^j(x)| + |p_2^j(x)|) = 1, \qquad \lim_{j \to \infty} |\mathcal{L}_k^{\varepsilon_j}\mathbf{p}^j|_{L^{\infty}} = 0.$$

Let us consider the equation

$$(5.30) \qquad\qquad \mathbf{q}^j = \mathcal{L}_k^{\varepsilon_j}\mathbf{p}^j, \qquad \lim_{j \to \infty} \|\mathbf{q}^j\|_{L^{\infty}} = 0.$$

We denote as $\mathbf{q}^j = (q_1^j, q_2^j)$.

We will apply to (5.30) the same procedures as appeared in the proof of Proposition 2.5 with $\lambda = 0$. In the sequel, we use ε for ε_j and treat the limits $j \to \infty$ and $\varepsilon \to 0$ interchangeably.

Decomposing p_1^j as $p_1^j = a_{\varepsilon}\phi_0^{\varepsilon} + p_{1\perp}^j$, and inserting it into (5.30), we obtain an inhomogeneous version of (5.16). Following the same line of arguments as above, we find

$$\mu_0^\varepsilon a_\varepsilon + \langle \mathcal{N}^\varepsilon(0)^{-1} p_2^j, f_v^\varepsilon \phi_0^\varepsilon \rangle = \langle q_1^j, \phi_0^\varepsilon \rangle,$$

$$p_2^j = a_\varepsilon \mathcal{N}^\varepsilon(0)^{-1}(g_u^\varepsilon \phi_0^\varepsilon) + \mathcal{N}^\varepsilon(0)^{-1}[g_u^\varepsilon (L_k^\varepsilon)^{-1} P^\varepsilon q_1^j - q_2^j],$$

$$p_{1\perp}^j = -(L_k^\varepsilon)^{-1} P^\varepsilon (f_v^\varepsilon p_2^j - q_1^j).$$

Substituting the second equation into the first equation, we have

$$a_\varepsilon \{\mu_0^\varepsilon + \langle \mathcal{N}^\varepsilon(0)^{-1}(g_u^\varepsilon \phi_0^\varepsilon), f_v^\varepsilon \rangle\}$$
$$= \langle \mathcal{N}^\varepsilon(0)^{-1} q_2^j, f_v^\varepsilon \phi_0^\varepsilon \rangle - \langle \mathcal{N}^\varepsilon(0)^{-1} g_u^\varepsilon (L_k^\varepsilon)^{-1} P^\varepsilon q_1^j, f_v^\varepsilon \phi_0^\varepsilon \rangle + \langle q_1^j, \phi_0^\varepsilon \rangle,$$

in which the coefficient of a_ε is equal to $\lambda_0^\varepsilon + o(\varepsilon)$. Therefore, we obtain

$$(5.31) \qquad a_\varepsilon = \frac{\langle \mathcal{N}^\varepsilon(0)^{-1} q_2^j, f_v^\varepsilon \phi_0^\varepsilon \rangle - \langle \mathcal{N}^\varepsilon(0)^{-1} g_u^\varepsilon (L_k^\varepsilon)^{-1} P^\varepsilon q_1^j, f_v^\varepsilon \phi_0^\varepsilon \rangle + \langle q_1^j, \phi_0^\varepsilon \rangle}{\lambda_0^\varepsilon + o(\varepsilon)},$$

$$(5.32) \qquad p_2^j = a_\varepsilon \mathcal{N}^\varepsilon(0)^{-1}(g_u^\varepsilon \phi_0^\varepsilon) - \mathcal{N}^\varepsilon(0)^{-1} q_2^j + \mathcal{N}^\varepsilon(0)^{-1} g_u^\varepsilon (L_k^\varepsilon)^{-1} P^\varepsilon q_1^j$$

$$(5.33) \qquad p_{1\perp}^j = -(L_k^\varepsilon)^{-1} P^\varepsilon (f_v^\varepsilon p_2^j - q_1^j).$$

Since $\|\phi_0^\varepsilon\|_{L^1} = O(\sqrt{\varepsilon})$ and $\lambda_0^\varepsilon + o(\varepsilon) = \varepsilon(\hat{\lambda}_0 + o(1))$ as $\varepsilon \to 0$ with $\hat{\lambda}_0 \neq 0$ (cf. Proposition 2.5 (ii)), (5.31) implies

$$a_\varepsilon = \frac{O(\|q_1^j\|_{L^\infty} + \|q_2^j\|_{L^\infty})}{\sqrt{\varepsilon}}.$$

Using this, $\|q_i^j\|_{L^\infty} = o(1)$ as $\varepsilon \to 0$ for $i = 1, 2$ and that $\phi_0^\varepsilon / \sqrt{\varepsilon} \in [H^1(0,1)]'$ is bounded, we immediately find from (5.32) and Proposition 5.2 that

$$\|p_2^j\|_{H^1} = o(1) \qquad \text{as } \varepsilon \to 0,$$

and hence

$$(5.34) \qquad \|p_2^j\|_{L^\infty} = o(1), \qquad \text{as } \varepsilon \to 0.$$

Thanks to this and Proposition 5.1 (ii), (5.33) implies that

$$(5.35) \qquad \|p_{1\perp}^j\|_{L^\infty} = o(1) \qquad \text{as } \varepsilon \to 0.$$

Therefore, the normalization

$$1 = \max_{x \in [0,1]} (|p_1^j(x)| + |p_2^j(x)|) = |a_\varepsilon| \, \|\phi_0^\varepsilon\|_{L^\infty} + o(1),$$

$\|\phi_0^\varepsilon\|_{L^\infty} = O(1/\sqrt{\varepsilon})$ and (5.3) imply that

$$a_\varepsilon := \sqrt{\varepsilon}\hat{a}_\varepsilon \qquad \text{with} \quad \lim_{\varepsilon \to 0} \hat{a}_\varepsilon = \hat{a}_* := \frac{\sqrt{m(v^*)}}{\max_{z \in \mathbf{R}} u_z^0(z)} > 0.$$

Let us now show that this is a contradiction.   Note that $\mathbf{p}^j \in \mathbf{M} \subset [\hat{\boldsymbol{\Phi}}_0^{\varepsilon}]^{\perp}$. Therefore, by using (5.25), estimates similar to (5.23)–(5.24) for $\psi_0^{*\varepsilon}$ and $w^{*\varepsilon}$ and (5.34)–(5.35), we have

$$0 = \langle \hat{\boldsymbol{\Phi}}_0^{\varepsilon_j}, \mathbf{p}^j \rangle = \sqrt{\varepsilon_j}(\hat{a}_* \langle \phi_0^{\varepsilon_j}, \phi_0^{\varepsilon_j} \rangle + o(1)) = \sqrt{\varepsilon_j}(\hat{a}_* + o(1)) > 0$$

for j sufficiently large, arriving at a contradiction.   This concludes the proof of Proposition 2.6.

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