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Some acyclic relations in the lambda algebra

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Dedicated to the Memory of Professor Masahiro Sugawara

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ABSTRACT. We consider the relations $\omega\gamma = 0 \in \Lambda$, and show that if $\omega\alpha = 0$ then $\alpha = \gamma\beta$ for some β. These relations give the acyclic chain complex $\Lambda \xrightarrow{\gamma} \Lambda \xrightarrow{\omega} \Lambda$. We consider various cases, e.g. $\omega = \lambda_n$ and $\gamma = \lambda_{2n+1}$. Especially, we consider the case $\omega = w_n = d\lambda_n$ for $n = 2^{e+r} + 2^e - 1$, where $\gamma = (h_{e+r})^r$.

1. Introduction

Consider the stable homotopy groups of the sphere $\pi_*(S^0)$ localized at prime 2. We have the 2-local Adams spectral sequence converging to $\pi_*(S^0)$ with E_2-term $\mathrm{Ext}_A^{s,t}(\mathbf{Z}/2, \mathbf{Z}/2) = H^{s,t}(\Lambda)$ by [2]. Moreover, Λ contains a subcomplex $\Lambda(n)$ whose cohomology is the E_2-term of the unstable Adams spectral sequence converging to the 2-component of the unstable homotopy groups of S^n. There are corresponding p-local versions of Λ algebra that we will not consider.

The lambda algebra Λ (at the prime $p = 2$) is a bigraded $\mathbf{Z}/2$-algebra with generators $\lambda_n \in \Lambda^{1,n+1}$ $(n \geq 0)$ and relations

$$(1) \qquad \lambda_i \lambda_{2i+1+n} = \sum_{j \geq 0} \binom{n-1-j}{j} \lambda_{i+n-j} \lambda_{2i+1+j} \qquad (i, n \geq 0)$$

with differential

$$(2) \qquad d\lambda_n = \sum_{j \geq 1} \binom{n-j}{j} \lambda_{n-j} \lambda_{j-1} \qquad (n \geq 0).$$

We refer to [9] for these relations and [2, 5] for that d is a well-defined endomorphism of Λ. For a sequence $I = (n_1, n_2, \dots, n_s)$ of non-negative integers, a monomial $\lambda_I = \lambda_{n_1} \lambda_{n_2} \dots \lambda_{n_s}$ is said to be admissible if $2n_i \geq n_{i+1}$ for $1 \leq i \leq s-1$. The admissible monomials form an additive basis of Λ by [2, 5]. $\Lambda(n) \subset \Lambda$ is the subcomplex spanned by the admissible monomials with

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$n_1 < n$ (cf. [2, 9]). By [9], there is a unique differential algebra endomorphism $\theta : \Lambda \to \Lambda$ with $\theta(\lambda_n) = \lambda_{2n+1}$. This θ is usually called Sq^0. See [6] for a recent treatment of the lambda algebra.

The Adem relation $\lambda_m \lambda_{2m+1} = 0$ gives a chain complex of right Λ modules, using left-multiplication by λ_m and λ_{2m+1}. This complex, and an unstable analogue, are acyclic:

Theorem 1.1. *The following chain complexes are acyclic:*

$$\Lambda \xrightarrow{\lambda_{2n+1}\smile} \Lambda \xrightarrow{\lambda_n\smile} \Lambda,$$

$$\Lambda(p+2n+3) \xrightarrow{\lambda_{2n+1}\smile} \Lambda(p+1) \xrightarrow{\lambda_n\smile} \Lambda(p-n), \qquad \text{for } p \geq 2n+1.$$

For $p < 2n+1$, *the composite* $\Lambda(p+1) \xrightarrow{E} \Lambda \xrightarrow{\lambda_n\smile} \Lambda$ *is injective.*

The unstable maps above are defined in Lemma 2.2. The unstable Λ composition formulas in § 2 (of Wang, Mahowald and Singer) are crucial to our proofs. Furthermore, in Theorem 1.3 below, we prove the following chain complex of right Λ modules is acyclic:

$$\Lambda \xrightarrow{(\lambda_3)^2\smile} \Lambda \xrightarrow{(\lambda_1,\lambda_0)\smile} \Lambda \oplus \Lambda.$$

This implies that the following chain complex, defined by Proposition 2.3, is acyclic:

$$\Lambda(15) \xrightarrow{(h_2)^2\smile} \Lambda(7) \xrightarrow{w_4\smile} \Lambda(4),$$

where $h_i = \lambda_{2^i-1}$, $w_n = d\lambda_n$ (cf. Theorem 1.5). The unstable maps above are well-defined by Singer's result (Proposition 2.3 below which extends Wang's earlier result), which we use heavily. We have many other, more complicated, acyclic chain complexes, e.g. (cf. 1.4 and 1.6):

$$\Lambda \xrightarrow{(h_{i+2})^2\smile} \Lambda \xrightarrow{(h_{i+1},h_i)\smile} \Lambda \oplus \Lambda$$

$$\Lambda(2^{i+4}-1) \xrightarrow{(h_{i+2})^2\smile} \Lambda(2^{i+3}-1) \xrightarrow{w_{2^{i+2}+2^i-1}\smile} \Lambda(2^{i+2}+2^i-1)$$

$$\Lambda \xrightarrow{\lambda_5\lambda_3\smile} \Lambda \xrightarrow{(\lambda_2,\lambda_0)\smile} \Lambda \oplus \Lambda$$

$$\Lambda(20) \xrightarrow{\lambda_5\lambda_3\smile} \Lambda(10) \xrightarrow{w_6\smile} \Lambda(6)$$

Now we collect some acyclic chain complexes systematically. For integers $n_1 > \cdots > n_r \geq 0$, we denote

$$\gamma(n_1,\ldots,n_r) = \theta(\lambda_{n_1})\ldots\theta^r(\lambda_{n_r}).$$

Theorem 1.2. *If* $\lambda_{n_i}\gamma(n_1,\ldots,n_r) = 0$ *for* $1 \leq i \leq r$, *then the following chain complexes are acyclic:*

$$\Lambda \xrightarrow{\gamma(n_1,\dots,n_r)\smile} \Lambda \xrightarrow{(\lambda_{n_1},\dots,\lambda_{n_r})\smile} \bigoplus_{i=1}^{r} \Lambda,$$

$$\Lambda(p+1+t_r) \xrightarrow{\gamma(n_1,\dots,n_r)\smile} \Lambda(p+1) \xrightarrow{(\lambda_{n_1},\dots,\lambda_{n_r})\smile} \bigoplus_{i=1}^{r} \Lambda(p-n_i),$$

for $p \geq 2n_1 + 1$, *where* $t_r = \sum_{i=1}^{r} 2^i(n_i+1)$.

In the case $n_i = 2^{e+r-i} - 1$ $(e \geq 0, 1 \leq i \leq r)$, we have $\gamma(n_1,\dots,n_r) = (h_{e+r})^r$ and the assumption of Theorem above is satisfied.

Theorem 1.3. *The following chain complexes are acyclic:*

$$\Lambda \xrightarrow{(h_{e+r})^r\smile} \Lambda \xrightarrow{(h_{e+r-1},\dots,h_e)\smile} \bigoplus_{i=1}^{r} \Lambda,$$

$$\Lambda(p+1+r2^{e+r}) \xrightarrow{(h_{e+r})^r\smile} \Lambda(p+1) \xrightarrow{(h_{e+r-1},\dots,h_e)\smile} \bigoplus_{i=1}^{r} \Lambda(p-2^{e+r-i}+1),$$

for $p \geq 2^{e+r} - 1$.

In the case $n_i = 2^{e+r-i+1} - 2^e - 1$ $(e \geq 0, 1 \leq i \leq r)$, we have

$$\gamma(n_1,\dots,n_r) = \lambda_{2^{e+r+1}-2^{e+1}-1} \dots \lambda_{2^{e+r+1}-2^{e+i}-1} \dots \lambda_{2^{e+r+1}-2^{e+r}-1}.$$

We denote this element by $k_{e,r}$. By Lemma 3.5, $dk_{e,r} = 0$ and the assumption of Theorem 1.2 is satisfied.

Theorem 1.4. *The following chain complexes are acyclic:*

$$\Lambda \xrightarrow{k_{e,r}\smile} \Lambda \xrightarrow{(\lambda_{n_1},\dots,\lambda_{n_r})\smile} \bigoplus_{i=1}^{r} \Lambda,$$

for $n_i = 2^{e+r-i+1} - 2^e - 1$ $(e \geq 0, 1 \leq i \leq r)$,

$$\Lambda(p+1+t_r) \xrightarrow{k_{e,r}\smile} \Lambda(p+1) \xrightarrow{(\lambda_{n_1},\dots,\lambda_{n_r})\smile} \bigoplus_{i=1}^{r} \Lambda(p-n_i),$$

for $p \geq 2^{e+r+1} - 2^{e+1} - 1$, *where* $t_r = (r-1)2^{e+r+1} + 2^{e+1}$.

Using these acyclic chain complexes, we get the main theorems in this paper.

Theorem 1.5. *For* $n = 2^{e+r} + 2^e - 1$ $(e \geq 0, r \geq 1)$, *the following is an acyclic chain complex.*

$$\Lambda(2n+1+u) \xrightarrow{(h_{e+r})^r\smile} \Lambda(2n+1-2^{e+r-1}) \xrightarrow{w_n\smile} \Lambda(n-2^e+1),$$

where $u = (r-1)2^{e+r} + 2^{e+r-1}$.

Theorem 1.6. *For* $n = 2^{e+r+1} - 2^e - 1$ $(e \geq 0, r \geq 1)$, *the following is an acyclic chain complex.*

$$\Lambda(2n+1+u) \xrightarrow{k_{e,r}\smile} \Lambda(2n+1-2^{e+r}+2^e) \xrightarrow{w_n\smile} \Lambda(n-2^e+1),$$

where $u = (r-2)2^{e+r+1} + 2^{e+r} + 2^{e+1} + 2^e$.

Note that if $r = 1$ then Theorems 1.5 and 1.6 gives the same complexes, because

$$2^{e+1} + 2^e - 1 = 2^{e+1+1} - 2^e - 1, \qquad h_{e+1} = k_{e,1}, \qquad 2^{e+1-1} = 2^{e+1} - 2^e.$$

Since we can calculate $w_n = \sum_j \binom{n-j}{j} \lambda_{n-j}\lambda_{j-1}$ for $n = 2^{e+r} \pm 2^e - 1$ explicitly, we can conclude Theorems 1.5–6. In fact, if $w_n\alpha = 0$ and α is low-dimensional, then we get $\lambda_{j-1}\alpha = 0$ for each j with $\binom{n-j}{j} = 1$, and we can apply Theorems 1.2–3. In the case $n \neq 2^{e+r} \pm 2^e - 1$, we can calculate w_n partially, and get only a "partial acyclicity" result, which is too technical to state in this paper.

Before closing the introduction we compare with the possible acyclic relations in the Steenrod algebra $\mathscr{A}$ (cf. [5]). The sequence of left $\mathscr{A}$-modules

$$\mathscr{A} \xrightarrow{Sq^{2n-1}} \mathscr{A} \xrightarrow{Sq^n} \mathscr{A}$$

is exact for $n = 1$ and 2 (as is well-known from $\mathscr{A}$-module resolutions of the spectra KZ and bo), but not exact for any odd $n > 1$, as Sq^1 is in the homology. The sequence of right $\mathscr{A}$-modules

$$\mathscr{A} \xrightarrow{Sq^n} \mathscr{A} \xrightarrow{Sq^{2n-1}} \mathscr{A}$$

is not exact for $n = 3$, because $Sq^4Sq^2Sq^1$ is in the homology:

$$Sq^5Sq^4Sq^2Sq^1 = Sq^7Sq^2Sq^2Sq^1 = Sq^7Sq^3Sq^1Sq^1 = 0,$$

but $Sq^3Sq^4 = Sq^7$, and $Sq^3Sq^3Sq^1 = Sq^5Sq^1Sq^1 = 0$.

Adams and Margolis [1] proved there are exact sequences of right $\mathscr{A}$-modules

$$\mathscr{A} \xrightarrow{P_t^s} \mathscr{A} \xrightarrow{P_t^s} \mathscr{A}$$

for $0 \leq s < t$, where $P_t^s \in \mathscr{A}$ is the Milnor-basis dual of $\xi_t^{2^s}$, but their proof are quite different from ours.

I conjecture that the sequences of left $\mathscr{A}$-modules

$$\mathscr{A} \xrightarrow{Sq^{2^{n+1}-1}} \mathscr{A} \xrightarrow{Sq^{2^n}} \mathscr{A}$$

are exact. I wish to thank Mark Mahowald for verifying the case $n = 2$ of my conjecture, and pointing out that a proof follows from his paper with Gorbounov [4]. I wish to thank the referee for many useful comments, and explained how Singer's results streamline my proofs.

2. The lambda algebra EHP sequences

By [8, Lemma 2.6], if $a = \sum a_i 2^i$, $b = \sum b_i 2^i$ $(0 \le a_i, b_i < 2)$, then

$$\binom{b}{a} \equiv \prod \binom{b_i}{a_i} \pmod 2. \tag{3}$$

By this formula,

$$\binom{n}{m} \equiv \binom{2n+1}{2m+1} \equiv \binom{2n+1}{2m} \equiv \binom{2n}{2m}, \binom{2n}{2m+1} \equiv 0. \tag{4}$$

Consider a map $\theta : \mathbf{Z} \to \mathbf{Z}$ by taking $\theta(n) = 2n+1 = 2(n+1)-1$. Then $\theta^e(n) = 2^e(n+1) - 1 = n2^e + 2^e - 1$ and $\binom{\theta(n)-2j}{2j} \equiv \binom{n-j}{j}$, $\binom{\theta(n)-2j-1}{2j+1} \equiv 0$.

For $n \ge 0$, let $F(n) = \left\{ j : \binom{n-j}{j} = 1, 0 \le j \le \frac{n}{2} \right\}$. It is well-known that $h_r = \lambda_{2^r-1}$ is a cycle for $r \ge 0$. This is equivalent to $F(2^r - 1) = \{0\}$ by Equation (3). By Equations (3) and (4), we have $F(2^r) = \{0\} \amalg \{2^a : 0 \le a < r\}$, $F(2^r - 2) = \{2^a - 1 : 0 \le a < r\}$ and

$$F(\theta^e(2^r)) = \{0\} \amalg \{2^{e+a} : 0 \le a < r\} \tag{5}$$

$$F(\theta^e(2^r - 2)) = \{2^{e+a} - 2^e : 0 \le a < r\}. \tag{6}$$

They are used to get acyclic chain complexes for w_n, where $n = \theta^e(b)$ for $b = 2^r, 2^r - 2$.

By [9], there is a unique differential algebra endomorphism $\theta : \Lambda \to \Lambda$, $\Lambda^{s,t}(n) \to \Lambda^{s,2t}(2n)$ with $\theta(\lambda_i) = \lambda_{2i+1}$. This θ is usually called Sq^0, and it commutes with Adem relations.

Lemma 2.1 ([9, Proposition 1.7.3]). (i) *θ is injective.*
(ii) *If $d(\theta(x)) = 0$ then $d(x) = 0$.*

Now we explain the lambda algebra EHP sequence. We refer to [6] for recent proofs.

Lemma 2.2 ([3, Lemma 3.5]). *$\lambda_m \Lambda(n+m+1) \subset \Lambda(n)$ for $m < n$.*

In [3], this is proved by a double induction argument and it is similar to the proof of the dual result [9, Proposition 1.8.1]:

$$\Lambda^{s,t}(n)\lambda_k \subset \Lambda(n) \qquad \text{for } k < n+t.$$

By Lemma 2.2 (or Wang's dual) and induction on s, we have the following proposition which is due to Singer.

Proposition 2.3 ([7, Proposition 5.1]). *$\Lambda^{s,t}(n)\Lambda(n+t) \subset \Lambda(n)$.*

This proposition and $d\lambda_n \in \Lambda^{2,n+1}(n)$ give Wang's result:

Lemma 2.4 ([9, Proposition 1.8.3]). *$(d\lambda_n)x \in \Lambda(n)$ for $x \in \Lambda(2n+1)$.*

Following Wang [9], we see that this lemma implies the result of [2]:

Proposition 2.5 ([9, Proposition 1.8.4]). *$\Lambda(n)$ is a subcomplex of the chain complex Λ, i.e. $d\Lambda(n) \subset \Lambda(n)$, $d : \Lambda^{s,t}(n) \to \Lambda^{s+1,t}(n)$.*

Now we define a map (Hopf invariant)

$$H : \Lambda^{s,t}(n+1) \to \Lambda^{s-1,t-n-1}(2n+1)$$

by $H(\lambda_n\lambda_I) = \lambda_I$, $H(\lambda_i\lambda_I) = 0$ for the admissible sequences $(n,I),(i,I)$ with $i<n$. Lemma 2.4 also implies the following.

Proposition 2.6. *$H : \Lambda(n+1) \to \Lambda(2n+1)$ is a chain map.*

Corollary 2.7 ([9, Theorem 1.8.5]). *If $d\alpha = 0$ then $dH(\alpha) = 0$.*

We define unstable composition product $\alpha \smile \beta = \alpha\beta \in \Lambda(n)$ for $\alpha \in \Lambda^{s,t}(n)$, $\beta \in \Lambda(n+t)$. Then we can define a chain map (Whitehead product) $P : \Lambda^{s,t}(2n+1) \to \Lambda^{s+2,t+n+1}(n)$ by $P(\alpha) = w_n \smile \alpha$, where $w_n = d\lambda_n \in \Lambda^{2,n+1}(n)$. Moreover, we have a chain map (suspension) $E : \Lambda^{s,t}(n) \to \Lambda^{s,t}(n+1)$ which is inclusion.

Then we have short exact sequences

$$0 \to \Lambda^{s,t}(n) \xrightarrow{E} \Lambda^{s,t}(n+1) \xrightarrow{H} \Lambda^{s-1,t-n-1}(2n+1) \to 0.$$

Proposition 2.8 ([7, Proposition 5.3]).

$$EH(\alpha \smile \beta) = EH(\alpha) \smile \beta + \theta(\alpha) \smile EH(\beta) \in \Lambda(2n+2)$$

for $\alpha \in \Lambda^{s,t}(n+1)$, $\beta \in \Lambda(n+t+1)$.

Since E is injective, $E(\alpha \smile \beta) = E\alpha \smile E\beta$ and $\theta(E\alpha) = E^2\theta(\alpha)$. We have two special cases and the second case is [7, Proposition 5.2]:

Corollary 2.9. *Let $\alpha \in \Lambda^{s,t}(n)$, $\beta \in \Lambda(n+t+1)$. Then*

$$H(E(\alpha) \smile \beta) = E\theta(\alpha) \smile H(\beta) \in \Lambda(2n+1).$$

Also, if $\alpha \in \Lambda^{s,t}(n+1)$, $\beta \in \Lambda(n+t)$, then

$$H(\alpha \smile E(\beta)) = H(\alpha) \smile \beta \in \Lambda(2n+1).$$

Singer gave proofs of Propositions 2.3, 2.6 and 2.8 in the preprint version of his paper [7], but unfortunately omitted them from the published version.

Proposition 2.8 is proved by generalizing the proof of [3, Lemma 3.1] which is the case of $\alpha = d(\lambda_{2n})$ and $\beta \in \Lambda(4n+1)$. This is essentially Singer's preprint proof. We prove Proposition 2.3 by double induction. Note that our proof does not use Lemma 2.2.

Proof of Proposition 2.3. We shall show that

$$\Lambda^{s_1,t}(n)\Lambda^{s_2,*}(n+t) \subset \Lambda(n)$$

by double induction on $s = s_1 + s_2$ and n. Consider $\alpha = \lambda_m x$, for $m < n$ and $x \in \Lambda^{s_1-1,t-m-1}(2m+1)$, and $\beta \in \Lambda^{s_2,*}(n+t)$. Since $m < n$, $x \in \Lambda^{s_1-1,t-m-1}(n+m+1)$, and so we have $\gamma = x\beta \in \Lambda^{s-1,*}(n+m+1)$ by induction on s. We shall show that $\lambda_m\gamma \in \Lambda(n)$.

If $m = n-1$ then this is trivial since $\Lambda(2n) = \Lambda(2n-1) + \lambda_{2n-1}\Lambda(4n-1)$.

If $m < n-1$ then we take the admissible form $\gamma = \lambda_{n+m}x + y$ with $x \in \Lambda(2n+2m+1)$ and $y \in \Lambda(n+m)$. By induction on n, $\lambda_m y \in \Lambda(n-1)$. We have an Adem relation $\lambda_m\lambda_{n+m} = \lambda_{n-1}\lambda_{2m+1} + z$ with $z \in \Lambda^{2,2m+n+2}(n-1)$. By induction on s, $\lambda_{2m+1}x \in \Lambda(2n-1)$ and $zx \in \Lambda(n-1)$. Thus $\lambda_m\gamma \in \Lambda(n)$. □

The case $s = 1$ for the first part of Corollary 2.9 is proved by a similar argument, and induction proves the case $s > 1$. The second part of Corollary 2.9 follows easily by Proposition 2.3. Proposition 2.8 requires in addition some tricky cancellation, which we leave to the reader, since we do not use Proposition 2.8, but only Corollary 2.9.

3. Some relations on the lambda algebra

Consider elements $\alpha, \alpha_i \in \Lambda$. We define

$$\alpha \smile : \Lambda \to \Lambda \qquad \text{and} \qquad (\alpha_1, \dots, \alpha_r) \smile : \Lambda \to \bigoplus_{i=1}^r \Lambda$$

by taking $\alpha \smile (x) = \alpha x$, $(\alpha_1, \dots, \alpha_r) \smile (x) = (\alpha_1 x, \dots, \alpha_r x)$.

If $\alpha\beta = 0$ then we have a chain complex

$$\Lambda \xrightarrow{\beta\smile} \Lambda \xrightarrow{\alpha\smile} \Lambda. \tag{7}$$

If $\alpha_i\beta = 0$ for $1 \le i \le r$ then we have a chain complex

$$\Lambda \xrightarrow{\quad\beta\smile\quad} \Lambda \xrightarrow{(\alpha_1,\dots,\alpha_r)\smile} \bigoplus_{i=1}^r \Lambda. \tag{8}$$

For $\alpha \in \Lambda^{s,t}(n)$ and $m \le n+t$, we define the map

$$\alpha \smile : \Lambda(m) \to \Lambda(n)$$

by Proposition 2.3. Sometimes we will suspend alpha without mentioning it to give a larger n, but this is clear from context. For instance, in Theorem 1.3, we use $(h_{e+r})^r \smile : \Lambda(p+1+r2^{e+r}) \to \Lambda(p+1)$, where $(h_{e+r})^r \in \Lambda^{r, r2^{e+r}}(2^{e+r})$, and $2^{e+r} \le p+1$. So we suspended to think of $(h_{e+r})^r \in \Lambda^{r, r2^{e+r}}(p+1)$.

Proof of Theorem 1.1. By the Adem relation, $(\lambda_n \smile) \circ (\lambda_{2n+1} \smile) = 0$. Consider an element $\alpha \in \Lambda^{s,t}(p+1)$ with $\lambda_n \smile \alpha = 0$. For $p < 2n+1$, $\lambda_n \alpha$ is admissible, and so $\alpha = 0$.

For $p = 2n+1$, $\alpha = \lambda_{2n+1}x + y \in \Lambda(2n+2)$, where $x = H(\alpha) \in \Lambda(4n+1)$ and $y \in \Lambda(2n+1)$. So $\lambda_n \smile \alpha = \lambda_n y$, and so $y = 0$ by the case $p < 2n+1$ above. Thus $\alpha = \lambda_{2n+1} \smile H(\alpha)$.

For $p > 2n+1$, we have a commutative diagram by Corollary 2.9:

$$\begin{array}{ccccc} \Lambda(p+2n+3) & \xrightarrow{\lambda_{2n+1}\smile} & \Lambda(p+1) & \xrightarrow{\lambda_n\smile} & \Lambda(p-n) \\ \downarrow H & & \downarrow H & & \downarrow H \\ \Lambda(2p+4n+5) & \xrightarrow{\lambda_{4n+3}\smile} & \Lambda(2p+1) & \xrightarrow{\lambda_{2n+1}\smile} & \Lambda(2p-2n-1) \end{array}$$

Then $0 = H(\lambda_n \smile \alpha) = \lambda_{2n+1} \smile H(\alpha)$. By induction on s, $H(\alpha) = \lambda_{4n+3} \smile \gamma$ for some $\gamma \in \Lambda(2p+4n+5)$. Since H is surjective, we have an element $f \in \Lambda(p+2n+3)$ with $H(f) = \gamma$. Then $H(\lambda_{2n+1} \smile f) = \lambda_{4n+3} \smile H(f) = \lambda_{4n+3} \smile \gamma = H(\alpha)$. Hence $\alpha' = \alpha + \lambda_{2n+1} \smile f \in \Lambda(p+1)$ has $H(\alpha') = 0$, and so $\alpha' \in \Lambda(p)$ and $\lambda_n \smile \alpha' = 0$. By induction on p, $\alpha' = \lambda_{2n+1} \smile \beta'$ for some $\beta' \in \Lambda(p+2n+2)$. Thus $\alpha = \lambda_{2n+1} \smile \beta$ for $\beta = f + \beta' \in \Lambda(p+2n+3)$. □

Lemma 3.1. *For integers $n_1 > \cdots > n_r \ge 0$, if $s < r$ then a composite $\Lambda^{s,t}(p+1) \xrightarrow{E} \Lambda \xrightarrow{(\lambda_{n_1}, \dots, \lambda_{n_r})\smile} \bigoplus_{i=1}^r \Lambda$ is injective.*

Proof. Consider $\alpha \in \Lambda^{s,t}(p+1)$ with $\lambda_{n_i} \smile \alpha = 0$ $(1 \le i \le r)$. We prove this lemma by induction on r, s, p. For $r = 1$ or $p = 0$ or $s = 0$, this is trivial.

If $p < 2n_1 + 1$ then $\alpha = 0$ by $\lambda_{n_1} \smile \alpha = 0$ and Theorem 1.1. If $p = 2n_1 + 1$ then $\alpha = \lambda_p \smile H(\alpha)$ by the proof of Theorem 1.1 for the case $p = 2n+1$. Now $0 = H(\lambda_{n_i} \smile \alpha) = \theta(\lambda_{n_i}) \smile H(\alpha)$ for $2 \le i \le r$, and so $H(\alpha) = 0$ by induction on r and $\alpha = \lambda_p \smile H(\alpha) = 0$. If $p > 2n_1 + 1$ then $0 = H(\lambda_{n_i} \smile \alpha) = \theta(\lambda_{n_i}) \smile H(\alpha)$ for $1 \le i \le r$, and so $H(\alpha) = 0$ by induction on s, and $\alpha \in \Lambda(p)$. By induction on p, $\alpha = 0$. □

For integers $n_1 > \cdots > n_i > \cdots > n_r \ge 0$, we denote $t_r = \sum_{i=1}^r 2^i(n_i + 1)$ and

$$\gamma(n_1, \dots, n_r) = \theta(\lambda_{n_1}) \dots \theta^i(\lambda_{n_i}) \dots \theta^r(\lambda_{n_r}) \in \Lambda^{r, t_r}(2n_1 + 2). \tag{9}$$

The proof of Theorem 1.2 is very similar to the proof of Theorem 1.1, which is the case $r = 1$.

Proof of Theorem 1.2. By the assumption,

$$((\lambda_{n_1}, \dots, \lambda_{n_r}) \smile) \circ (\gamma(n_1, \dots, n_r) \smile) = 0.$$

Consider an element $\alpha \in \Lambda^{s,t}(p+1)$ with $\lambda_{n_i} \smile \alpha = 0$ for $1 \le i \le r$. If $r = 1$ then this is Theorem 1.1. If $s = 0$ then $\alpha = 0$, because the generator is the identity element $* \in \Lambda^{0,0}(p+1) = \mathbf{Z}/2$ where $*$ is the monomial of length 0. But $*$ is not in the kernel since $\lambda_n \smile * = \lambda_n \neq 0$.

For $p = 2n_1 + 1$, $\alpha = \theta(\lambda_{n_1}) \smile H(\alpha)$ by the proof of Theorem 1.1 for the case $p = 2n + 1$. Now $0 = H(\lambda_{n_i} \smile \alpha) = \theta(\lambda_{n_i}) \smile H(\alpha)$ for $2 \le i \le r$, and

$$\begin{aligned} 0 &= H(\lambda_{n_i} \smile \gamma(n_1, \dots, n_r)) \\ &= \theta(\lambda_{n_i}) \smile H(\gamma(n_1, \dots, n_r)) \\ &= \theta(\lambda_{n_i}) \smile \theta(\gamma(n_2, \dots, n_r)). \end{aligned}$$

By induction on r, $H(\alpha) = \theta(\gamma(n_2, \dots, n_r)) \smile \beta$, where $\beta \in \Lambda(2p + 1 + \sum_{i=2}^{r} 2^{i-1}(2n_i + 1 + 1)) = \Lambda(p + t_r)$. Then $\alpha = \gamma(n_1, \dots, n_r) \smile \beta$.

For $p > 2n_1 + 1$, we have a commutative diagram by Corollary 2.9:

$$\begin{array}{ccccc} \Lambda(p+1+t_r) & \xrightarrow{\gamma\smile} & \Lambda(p+1) & \xrightarrow{(\lambda_{n_1},\dots,\lambda_{n_r})\smile} & \bigoplus_{i=1}^{r} \Lambda(p-n_i) \\ \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H} \\ \Lambda(2p+1+2t_r) & \xrightarrow{\theta(\gamma)\smile} & \Lambda(2p+1) & \xrightarrow{(\theta(\lambda_{n_1}),\dots,\theta(\lambda_{n_r}))\smile} & \bigoplus_{i=1}^{r} \Lambda(2p-2n_i-1), \end{array}$$

where $\gamma = \gamma(n_1, \dots, n_r) \in \Lambda^{r,t_r}(2n_1 + 2) \subset \Lambda^{r,t_r}(p)$. Then $0 = H(\lambda_{n_i} \smile \alpha) = \theta(\lambda_{n_i}) \smile H(\alpha)$ for $1 \le i \le r$. By induction on s, $H(\alpha) = \theta(\gamma) \smile \beta'$ for some $\beta' \in \Lambda(2p + 1 + 2t_r)$. Since H is surjective, we have an element $f \in \Lambda(p + 1 + t_r)$ with $H(f) = \beta'$. Then $H(\gamma \smile f) = \theta(\gamma) \smile H(f) = \theta(\gamma) \smile \beta' = H(\alpha)$. Hence $\alpha' = \alpha + \gamma \smile f \in \Lambda(p+1)$ has $H(\alpha') = 0$. So $\alpha' \in \Lambda(p)$, and $\lambda_{n_i} \smile \alpha' = 0$ for $1 \le i \le r$ by the assumption. By induction on p, $\alpha' = \gamma \smile \beta''$ for some $\beta'' \in \Lambda(p + t_r)$. Thus $\alpha = \gamma \smile \beta$ for $\beta = f + \beta'' \in \Lambda(p + 1 + t_r)$. □

Lemma 3.2. *If* $\lambda_{n_i}\gamma(n_1, \dots, n_i) = 0$ *then* $\lambda_{n_i}\gamma(n_1, \dots, n_r) = 0$.

Two examples where the hypotheses of Theorem 1.2 are satisfied are given in Lemmas 3.3 and 3.5 below. By [9], $h_i(h_{i+r})^r = 0$ for $h_i = \lambda_{2^i - 1}$, and so we have the following.

Lemma 3.3. *Let* $n_i = 2^{e+r-i} - 1$ $(e \ge 0, 1 \le i \le r)$ *be integers. Then*

$$\gamma(n_1, \dots, n_r) = (\lambda_{2^{e+r}-1})^r = (h_{e+r})^r$$

and $\gamma(n_j, \dots, n_i) = (h_{e+r-j+1})^{i-j+1}$. *Moreover* $\lambda_{n_i}\gamma(n_j, \dots, n_i) = 0$ *for* $1 \le j \le i \le r$.

This lemma and Theorem 1.2 imply Theorem 1.3.

Our next example leads to Theorem 1.4, and the proof is similar to Wang's calculation $h_i(h_{i+r})^r = 0$, so let's recall Wang's proof. It suffices by θ to prove that $h_0 h_r^r = 0$. An Adem relation writes $h_0 h_r$ as a sum of terms $\lambda_{m_i} h_i$, and by induction, $h_i h_r^{r-1} = 0$.

Next we consider integers $n_a = 2^a - 2$. Then we shall show that $\gamma(n_b, \dots, n_a)$ satisfies the conditions in Theorem 1.2. We write $\beta(b,a) = \gamma(n_b, \dots, n_a)$ for $b \geq a$.

Lemma 3.4. (i) $\lambda_{n_x}\beta(a+r,a) = 0$ *for* $a+r \geq x \geq a$.
(ii) $d(\beta(r,1)) = 0$.

Proof. (i) Because $\beta(a+r,a) = \beta(a+r,x)\theta^{a+r-x+1}(\beta(x-1,a))$ for $x > a$, it suffices to prove that $\lambda_{n_a}\beta(a+r,a) = 0$.

For $r = 0$, this is the Adem relation. We assume $r > 0$ and induction on r. Then

$$\beta(a+r,a) = \theta(\lambda_{n_{a+r}})\theta(\beta(a+r-1,a)).$$

The Adem relations imply

$$\lambda_p \lambda_{\theta(p)+2^e n} = \sum_{k \in F(n-1)} \lambda_{p+2^e(n-k)} \lambda_{\theta(p)+2^e k}.$$

Now $F(2^r - 2) = \{2^b - 1 : 0 \leq b < r\}$ by (6), and

$$\theta(n_{a+r}) = \theta(n_a) + 2(n_{a+r} - n_a) = \theta(n_a) + 2^{a+1}(2^r - 1).$$

By substituting b for r, we have $\theta(n_a) + 2^{a+1}(2^b - 1) = \theta(n_{a+b})$. Hence

$$\lambda_{n_a}\theta(\lambda_{n_{a+r}}) = \sum_{b=0}^{r-1} \lambda_{m(a,r,b)}\theta(\lambda_{n_{a+b}})$$

for some $m(a,r,b)$ we are not concerned with. This implies

$$\lambda_{n_a}\beta(a+r,a) = \sum_{b=0}^{r-1} \lambda_{m(a,r,b)}\theta(\lambda_{n_{a+b}}\beta(a+r-1,a)) = 0$$

by induction on r.

(ii) For $r > 1$, $\beta(r,1) = \theta(\lambda_{n_r})\theta(\beta(r-1,1))$, so it by induction, it suffices to show that $d(\lambda_{n_r})\beta(r-1,1) = 0$. Then

$$d(\lambda_{n_r}) = \sum_{0<k \in F(n_r)} \lambda_{n_r - k}\lambda_{k-1} = \sum_{b=1}^{r-1} \lambda_{n_r - 2^b + 1}\lambda_{n_b}$$

since $F(n_r) = \{2^b - 1 : 0 \leq b < r\}$ as above. Hence (i) implies $d(\lambda_{n_r})\beta(r-1,1) = 0$. □

We write $k_r = \beta(r,1)$ and $k_{e,r} = \lambda_I \in \Lambda^{r,t_r}(2^{e+r+1} - 2^{e+1})$ for $t_r = (r-1)2^{e+r+1} + 2^{e+1}$ and an admissible sequence

$$I = (2^{e+r+1} - 2^{e+1} - 1, \dots, 2^{e+r+1} - 2^{e+i} - 1, \dots, 2^{e+r+1} - 2^{e+r} - 1).$$

Then $k_{e,r} = \theta^e(k_r) = \gamma(\theta^e(n_r), \dots, \theta^e(n_1))$, and this lemma implies $\lambda_{n_a} k_r = \lambda_{n_a}\beta(r,1) = 0$ for $1 \le a \le r$ and $d(k_r) = 0$.

Lemma 3.5. *If* $n_a = 2^a - 2$ *then* $\theta^e(n_a) = 2^{a+e} - 2^e - 1$,

$$\gamma(\theta^e(n_r), \dots, \theta^e(n_1)) = \theta^e(k_r) = k_{e,r}$$

and $d(k_{e,r}) = d(\theta^e(k_r)) = 0$. *Moreover*

$$\lambda_{\theta^e(n_a)}\gamma(\theta^e(n_r), \dots, \theta^e(n_1)) = \theta^e(\lambda_{n_a}k_r) = 0$$

for $1 \le a \le r$.

This lemma and Theorem 1.2 imply Theorem 1.4.

4. Proofs of the main theorems

For integers $0 < j_1 < j_2 < \cdots < j_r \le \dfrac{2n+1}{3}$ and an element $w' \in \Lambda^{2,n+1}(n - j_r)$, we take an element

$$w = w' + \sum_{a \in \{j_1,\dots,j_r\}} \lambda_{n-a}\lambda_{a-1}.$$

We shall use the direct sum decomposition

$$\Lambda(n) = \Lambda(n - j_r) + \sum_{n - j_r \le f < n} \lambda_f \Lambda(2f + 1).$$

Suppose $\alpha \in \Lambda(x)$ for some $x \le 2n + 1$. We want $w \smile \alpha$ to be expressed in terms of this decomposition. That is,

$$w \smile \alpha = w' \smile \alpha + \sum_{a \in \{j_1,\dots,j_r\}} \lambda_{n-a}(\lambda_{a-1} \smile \alpha)$$

and we want $w' \smile \alpha \in \Lambda(n - j_r)$ and $\lambda_{a-1} \smile \alpha \in \Lambda(2(n-a) + 1)$.

Lemma 2.2 tells us that this last condition is achieved for $x \le 2(n-a) + 1 + (a-1) + 1 = 2n - a + 1$ since $a - 1 < 2(n-a) + 1$ by $3a \le 2n + 1$. Proposition 2.3 tells us that $w' \smile \alpha \in \Lambda(n - j_r)$ if $x \le n - j_r + n + 1 = 2n - j_r + 1$. We have now proved:

Lemma 4.1. *If* $w \smile \alpha = 0$ *for* $\alpha \in \Lambda(2n - j_r + 1)$ *then* $\lambda_{j_i - 1} \smile \alpha = 0$ *for* $1 \le i \le r$.

For any integer $n \geq 0$, let $F(n) = \{j_0 = 0, j_1, j_2, \dots, j_r, \dots\}$ with $j_0 = 0 < j_1 < j_2 < \cdots$. We notice that $\frac{2n+1}{3} > j_i$ since $2n + 1 - 3j_i \geq j_i + 1 > 0$ by $n - j_i \geq j_i$. Then $w_n = d\lambda_n = \sum_{i=1}^{r} \lambda_{n-j_i}\lambda_{j_i-1} + w' \in \Lambda^{n+1}(n - j_1 + 1)$, where $w' \in \Lambda(n - j_r)$. The lemma above and Theorem 1.2 imply the following.

Lemma 4.2. *If* $w_n\alpha = 0$ *for* $\alpha \in \Lambda(2n + 1 - j_r)$, *then* $\lambda_{j_i-1}\alpha = 0$ *for* $1 \leq i \leq r$.

Moreover, if $\lambda_{j_i-1}\gamma(j_r - 1, \dots, j_1 - 1) = 0$ *for* $1 \leq i \leq r$ *then* $\alpha = \gamma(j_r - 1, \dots, j_1 - 1)\beta$ *for some* $\beta \in \Lambda(2n + 1 - j_r + t_r)$, *where* $t_r = \sum_{i=1}^{r} 2^i j_{r-i+1}$.

Proof of Theorem 1.5. Let $n = 2^{e+r} + 2^e - 1 = \theta^e(2^r)$. Then $j_i = 2^{e+i-1}$ for $1 \leq i \leq r$ by Equation (5), and so

$$\gamma(j_r - 1, \dots, j_1 - 1) = (h_{e+r})^r \in \Lambda^{r,t_r}(2^{e+r}) \qquad \text{and} \qquad t_r = r2^{e+r}.$$

Hence $w_n(h_{e+r})^r = \sum_{i=1}^{r} \lambda_{n-j_i}\lambda_{j_i-1}(h_{e+r})^r = 0$ by Lemma 3.3.

If $w_n \smile \alpha = 0$ for $\alpha \in \Lambda(2n + 1 - 2^{e+r-1})$ then $\alpha = (h_{e+r})^r \smile \beta$ for some $\beta \in \Lambda(2n + 1 + (r-1)2^{e+r} + 2^{e+r-1})$ by Lemma 3.3 and 4.2. □

Proof of Theorem 1.6. Let $n = 2^{e+r+1} - 2^e - 1 = \theta^e(2^{r+1} - 2)$. Then $j_i = 2^{e+i} - 2^e$ for $1 \leq i \leq r$ by Equation (6), and so

$$\gamma(j_r - 1, \dots, j_1 - 1) = k_{e,r} \qquad \text{and} \qquad t_r = (r-1)2^{e+r+1} + 2^{e+1}.$$

Hence $w_n k_{e,r} = \sum_{i=1}^{r} \lambda_{n-j_i}\lambda_{j_i-1}k_{e,r} = 0$ by Lemma 3.5.

If $w_n \smile \alpha = 0$ for $\alpha \in \Lambda(2n + 1 - 2^{e+r} + 2^e)$ then $\alpha = k_{e,r} \smile \beta$ for some $\beta \in \Lambda(2n + 1 + (r-2)2^{e+r+1} + 2^{e+r} + 2^{e+1} + 2^e)$ by Lemma 3.5 and 4.2. □

For a general n, we do not get chain complexes. That is, our methods produce necessary but not sufficient conditions. If $w_n\alpha = 0$, we can conclude that $\alpha = \gamma\beta$ for some β, but it's not generally true that $w_n\gamma = 0$, and we have "partial acyclicity" result. Consider $n = 10, 12$:

By $F(10) = \{0, 1, 3, 4, 5\}$, $F(12) = \{0, 1, 2, 5, 6\}$,

$$w_{10} = \lambda_9\lambda_0 + \lambda_7\lambda_2 + \lambda_6\lambda_3 + \lambda_5\lambda_4 = w' + \lambda_9\lambda_0 + \lambda_7\lambda_2,$$

$$\gamma(4, 3, 2, 0) = \lambda_9\lambda_{15}\lambda_{23}\lambda_{15},$$

$$w_{12} = \lambda_{11}\lambda_0 + \lambda_{10}\lambda_1 + \lambda_7\lambda_4 + \lambda_6\lambda_5 = w'' + \lambda_{11}\lambda_0 + \lambda_{10}\lambda_1,$$

$$\gamma(5, 4, 1, 0) = \lambda_{11}\lambda_{19}(\lambda_{15})^2,$$

in which $w' \in \Lambda^{2,11}(7)$, $w'' \in \Lambda^{2,13}(10)$.

Now $\lambda_{j_i-1}\gamma(j_k-1,\dots,j_i-1)=0$ except for

$$\lambda_0\gamma(3,2,0)=\lambda_0\lambda_7\lambda_{11}\lambda_7=\lambda_4(\lambda_7)^3,$$

$$\lambda_0\gamma(4,3,2,0)=\lambda_0\lambda_9\lambda_{15}\lambda_{23}\lambda_{15}=\lambda_8\lambda_9(\lambda_{15})^3,$$

$$\lambda_1\gamma(4,1)=\lambda_1\lambda_9\lambda_7=(\lambda_5)^2\lambda_7,$$

$$\lambda_1\gamma(5,4,1)=\lambda_1\lambda_{11}\lambda_{19}\lambda_{15}=\lambda_9(\lambda_{11})^2\lambda_{15},$$

$$\lambda_0\gamma(5,4,1,0)=\lambda_0\lambda_{11}\lambda_{19}(\lambda_{15})^2=\lambda_8(\lambda_{11})^2(\lambda_{15})^2.$$

Moreover $\lambda_1\gamma(4,1,0)=(\lambda_5)^2(\lambda_7)^2$. Hence $\gamma(2,0)$ and $\gamma(1,0)$ satisfy the condition of Theorem 1.2, but the other $\gamma(j_r-1,\dots,j_1-1)$ don't satisfy this condition. So we apply Lemma 4.2 to $\gamma(2,0)=\lambda_5\lambda_3$ and $\gamma(1,0)=(h_2)^2$ as follows:

If $\alpha\in\Lambda(18)$ and $w_{10}\smile\alpha=0$ then $\alpha=\gamma(2,0)\beta$ for some $\beta\in\Lambda(28)$.
If $\alpha\in\Lambda(23)$ and $w_{12}\smile\alpha=0$ then $\alpha=\gamma(1,0)\beta$ for some $\beta\in\Lambda(31)$.
However, we don't have chain complexes

$$\Lambda(28)\xrightarrow{\gamma(2,0)\smile}\Lambda(18)\xrightarrow{w_{10}\smile}\Lambda(10),$$

$$\Lambda(31)\xrightarrow{\gamma(1,0)\smile}\Lambda(23)\xrightarrow{w_{12}\smile}\Lambda(12)$$

because

$$w_{10}\gamma(2,0)=w_{10}\lambda_5\lambda_3=\lambda_6\lambda_3\lambda_5\lambda_3+\lambda_5\lambda_4\lambda_5\lambda_3,$$

$$w_{12}\gamma(1,0)=w_{12}(\lambda_3)^2=\lambda_7\lambda_4(\lambda_3)^2+\lambda_6\lambda_5(\lambda_3)^2.$$

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