HIROSHIMA MATH. J.
33 (2003), 27–30

Remarks on universal functions of $\mathcal{O}(\mathbf{C}^*)$

Makoto ABE, Teisuke JIN and Tadashi SHIMA

(Received August 9, 2001)
(Revised March 27, 2002)

ABSTRACT. Let $A(\mathbf{C}^*)$ be the family of all $\mathcal{O}(\mathbf{C}^*)$-convex compact sets of $\mathbf{C}^*$ and $B(\mathbf{C}^*)$ the family of all compact sets of $\mathbf{C}^*$ whose complements in $\mathbf{C}^*$ are connected. Then the family $B(\mathbf{C}^*)$ is the maximal subfamily of $A(\mathbf{C}^*)$ on which there exists a universal function of $\mathcal{O}(\mathbf{C}^*)$. We also prove the transcendence of the universal functions of $\mathcal{O}(\mathbf{C}^*)$ on $B(\mathbf{C}^*)$.

1. Introduction and preliminaries

Let X be a complex manifold. We denote by $\mathcal{O}(X)$ the set of all holomorphic functions on X. For any compact set K of X the set

$$\hat{K}_X := \left\{ z \in X \,|\, |f(z)| \le \max_{x \in K} |f(x)| \text{ for every } f \in \mathcal{O}(X) \right\}$$

is said to be the *holomorphically convex hull* of K in X. A compact set K of X is said to be $\mathcal{O}(X)$-*convex* if $\hat{K}_X = K$. According to Zappa [8] we denote by $A(X)$ the family of all $\mathcal{O}(X)$-convex compact sets of X.

Let G be a Stein group (see for example Grauert-Remmert [5, p. 136]) and $\mathcal{S}$ a subfamily of $A(G)$. A function $F \in \mathcal{O}(G)$ is said to be a *universal function* of $\mathcal{O}(G)$ on $\mathcal{S}$ if for every $f \in \mathcal{O}(G)$, $K \in \mathcal{S}$ and $\varepsilon > 0$ there exists an element $c \in G$ such that $\max_{x \in K} |F(c \cdot x) - f(x)| < \varepsilon$.

For the additive group $\mathbf{C}^n$, $n \ge 1$, there exists a universal function of $\mathcal{O}(\mathbf{C}^n)$ on $A(\mathbf{C}^n)$ by Birkhoff [4], Luh [6], Y. Abe [1] and Abe-Zappa [3]. For the multiplicative group $\mathbf{C}^* = GL(1, \mathbf{C}) = \mathbf{C} - \{0\}$ there exist no universal functions of $\mathcal{O}(\mathbf{C}^*)$ on $A(\mathbf{C}^*)$ by Remark 2 of Zappa [8, p. 350]. For the complex general linear group $GL(n, \mathbf{C})$, $n \ge 2$, it is not known whether there does exist a universal function of $\mathcal{O}(GL(n, \mathbf{C}))$ on $A(GL(n, \mathbf{C}))$ or not (see Abe-Zappa [3, p. 231]).

According to Zappa [8] let $B(\mathbf{C}^*)$ be the family of all compact sets K of $\mathbf{C}^*$ such that $\mathbf{C}^* - K$ is connected. Here we remark that $B(\mathbf{C}^*)$ is a proper

2000 *Mathematics Subject Classification*. 30E10, 32E30, 32M05.
Key words and phrases. universal function, holomorphically convex hull, multiplicative group.

subfamily of $A(\mathbf{C}^*)$. By the theorem of Zappa [8] there exists a universal function of $\mathcal{O}(\mathbf{C}^*)$ on $B(\mathbf{C}^*)$. Generalizations to the complex general linear group $GL(n, \mathbf{C})$ and to the complex special linear group $SL(n, \mathbf{C})$ are also known by Abe-Zappa [3] and Y. Abe [2].

It is of interest to determine the maximal subfamily of $A(G)$ on which there exists a universal function of $\mathcal{O}(G)$ when a Stein group G is specified. In this paper we prove that for the multiplicative group $\mathbf{C}^*$ the family $B(\mathbf{C}^*)$ is the maximal subfamily of $A(\mathbf{C}^*)$ on which there exists a universal function of $\mathcal{O}(\mathbf{C}^*)$, which is more precise than the theorem of Zappa [8]. We also prove the transcendence of the universal functions of $\mathcal{O}(\mathbf{C}^*)$ on $B(\mathbf{C}^*)$.

2. Maximal subfamily of $A(\mathbf{C}^*)$

Theorem 1. *The family $B(\mathbf{C}^*)$ is the maximal subfamily of $A(\mathbf{C}^*)$ on which there exists a universal function of $\mathcal{O}(\mathbf{C}^*)$.*

Proof. Since there exists a universal function of $\mathcal{O}(\mathbf{C}^*)$ on $B(\mathbf{C}^*)$ by the theorem of Zappa [8], we have only to prove that there exist no universal functions of $\mathcal{O}(\mathbf{C}^*)$ on any subfamily $\mathcal{S}$ of $A(\mathbf{C}^*)$ such that $\mathcal{S} \not\subset B(\mathbf{C}^*)$. We take an arbitrary $K \in \mathcal{S} - B(\mathbf{C}^*)$. Let L_0 and L_∞ be the connected components containing 0 and ∞ respectively of $\mathbf{P}^1 - K$, where $\mathbf{P}^1$ denotes the Riemann sphere. Since $\mathbf{C}^* - K$ has no relatively compact connected component (see Remmert [7, p. 301]), the set $\mathbf{P}^1 - K$ has no connected component other than L_0 and L_∞. Since $\mathbf{C}^* - K$ is not connected, we have that $L_0 \neq L_\infty$. It follows that L_0 is relatively compact in $\mathbf{C}$ and that $\hat{K}_{\mathbf{C}} = K \cup L_0$ (see Remmert [7, p. 301]). Assume that there exists a universal function F of $\mathcal{O}(\mathbf{C}^*)$ on $\mathcal{S}$. Take an arbitrary $k \in \mathbf{C}$. Since the constant function k on $\mathbf{C}^*$ is approximated on K by the functions of the form $F(cz)$, $c \in \mathbf{C}^*$, there exists a sequence $\{c_n\}_{n=1}^\infty \subset \mathbf{C}^*$ such that $\max_{z \in K}|F(c_n z) - k| < 1/n$ for every $n \in \mathbf{N}$. The sequence $\{c_n\}_{n=1}^\infty$ has an accumulation point $c \in \mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$. Replacing by a subsequence we may assume that $\lim_{n\to\infty} c_n = c$ in $\mathbf{P}^1$. First we consider the case where $c \in \mathbf{C}^*$. Since $\max_{z \in K}|F(c_n z) - k| < 1/n$, by letting $n \to \infty$ we have that $F(w) = k$ for every $w \in cK$. Since $\mathbf{C}^* - K$ is not connected, cK is an infinite compact set. By the theorem of identity we have that $F \equiv k$ on $\mathbf{C}^*$. Next we consider the case where $c = 0$. Replacing by a subsequence we may assume that $c_{n+1}\hat{K}_{\mathbf{C}} \subset c_n L_0$ for every $n \in \mathbf{N}$. Then we have that $c_{n+1}\hat{K}_{\mathbf{C}} \subset c_n \hat{K}_{\mathbf{C}}$ and $c_n\hat{K}_{\mathbf{C}} - c_{n+1}\hat{K}_{\mathbf{C}} = (c_n K \cup c_n L_0) - c_{n+1}\hat{K}_{\mathbf{C}} = c_n K \cup Q_n$, where $Q_n := c_n L_0 - c_{n+1}\hat{K}_{\mathbf{C}} = c_n L_0 \cap c_{n+1} L_\infty$. We also have that $\partial Q_n \subset c_n \partial L_0 \cup c_{n+1} \partial L_\infty \subset c_n K \cup c_{n+1} K$. If $x \in c_n K$, then $|F(x) - k| < 1/n$. If $x \in Q_n$, then by the maximum modulus principle we have that

$$\begin{aligned}|F(x)-k| &\le \max_{w\in\partial Q_n}|F(w)-k| \\ &\le \max\left\{\max_{w\in c_nK}|F(w)-k|,\ \max_{w\in c_{n+1}K}|F(w)-k|\right\} \\ &< \max\{1/n, 1/(n+1)\} = 1/n.\end{aligned}$$

Thus we have that $|F(x)-k|<1/n$ for every $x\in c_n\hat{K}_{\mathbf{C}} - c_{n+1}\hat{K}_{\mathbf{C}}$ and $n\in\mathbf{N}$. Since we can verify that $c_n\hat{K}_{\mathbf{C}} - \{0\} = \bigcup_{\nu=n}^{\infty}(c_\nu\hat{K}_{\mathbf{C}} - c_{\nu+1}\hat{K}_{\mathbf{C}})$, it holds that $|F(x)-k|<1/n$ for every $x\in c_n\hat{K}_{\mathbf{C}}-\{0\}$ and $n\in\mathbf{N}$. Since $c_n\hat{K}_{\mathbf{C}}$, $n\in\mathbf{N}$, are compact neighborhoods of 0 in $\mathbf{C}$, we have that $\lim_{z\to 0}F(z)=k$. Finally we consider the case where $c=\infty$. Applying the argument above to the function $\zeta\mapsto F(1/\zeta)$ and the compact set $K^{-1}=\{\zeta\in\mathbf{C}\mid 1/\zeta\in K\}$, we obtain that $\lim_{z\to\infty}F(z)=k$. Thus we proved that one of the conditions $F(z)\equiv k$ on $\mathbf{C}^*$, $\lim_{z\to 0}F(z)=k$ or $\lim_{z\to\infty}F(z)=k$ are satisfied for any $k\in\mathbf{C}$. But these three conditions are satisfied for at most different two constants $k=k_1,k_2$. It is a contradiction. □

3. Transcendence of universal functions

We have the following fact on the transcendence of the universal functions of $\mathcal{O}(\mathbf{C}^*)$ on $B(\mathbf{C}^*)$. We denote by z the coordinate of $\mathbf{C}$.

Theorem 2. *Let $\mathcal{S}$ be a subfamily of $A(\mathbf{C}^*)$. Assume that there exists a compact set $K\in\mathcal{S}$ such that $\#K\ge 2$. Then every function F of the form $F(z)=\sum_{\nu=-k}^{l}a_\nu z^\nu\in\mathbf{C}[z,1/z]$ cannot be a universal function of $\mathcal{O}(\mathbf{C}^*)$ on $\mathcal{S}$.*

Proof. Assume that $F(z)=\sum_{\nu=-k}^{l}a_\nu z^\nu$ is a universal function of $\mathcal{O}(\mathbf{C}^*)$. We take two points $p,q\in K$, $p\ne q$. Let $M_n:=\max_{1/n\le|c|\le n}|F(cp)|$ for every $n\in\mathbf{N}$. Since the function $z\mapsto(M_n+n)(z-q)/(p-q)$ is approximated on K by the functions of the form $F(cz)$, $c\in\mathbf{C}^*$, there exists a sequence $\{c_n\}_{n=1}^{\infty}\subset\mathbf{C}^*$ such that $|F(c_np)-(M_n+n)|<1/n$ and $|F(c_nq)|<1/n$ for every $n\in\mathbf{N}$. If $1/n\le|c|\le n$, then we have that $|F(cp)-(M_n+n)|\ge(M_n+n)-M_n=n\ge 1/n$. Therefore we have that $|c_n|<1/n$ or $|c_n|>n$ for every $n\in\mathbf{N}$. It follows that there exists a subsequence $\{c_{\alpha(n)}\}_{n=1}^{\infty}$ of $\{c_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty}c_{\alpha(n)}=0$ or $\lim_{n\to\infty}c_{\alpha(n)}=\infty$ in $\mathbf{P}^1$. Since $|F(c_{\alpha(n)}p)|>(M_{\alpha(n)}+\alpha(n))-1/\alpha(n)\ge\alpha(n)-1$ for every $n\in\mathbf{N}$, we have that $\lim_{n\to\infty}F(c_{\alpha(n)}p)=\infty$. Since $|F(c_{\alpha(n)}q)|<1/\alpha(n)$ for every $n\in\mathbf{N}$, we have that $\lim_{n\to\infty}F(c_{\alpha(n)}q)=0$. It follows that either $F(0)=\lim_{z\to 0}F(z)$ or $F(\infty)=\lim_{z\to\infty}F(z)$ is indeterminate. It is a contradiction. □

Corollary 3. *Every universal function of $\mathcal{O}(\mathbf{C}^*)$ on $B(\mathbf{C}^*)$ has at least one essential singularity at 0 or ∞.*

References

[1] Y. Abe, Universal holomorphic functions in several variables, Analysis **17** (1997), 71–77.

[2] Y. Abe, Universal functions on complex special linear groups, Communications in difference equations. Proceedings of the Fourth International Conference on Difference Equations, Poznan, Poland, 1998 (S. Elaydi, G. Ladas, J. Popenda and J. Rakowski, eds.), Gordon & Breach, London, 2000, pp. 1–8.

[3] Y. Abe and P. Zappa, Universal functions on complex general linear groups, J. Approximation Theory **100** (1999), 224–232.

[4] G. D. Birkhoff, Démonstration d'un théorème élémentaire sur les fonctions entières, C. R. Acad. Sci. Paris **189** (1929), 473–475.

[5] H. Grauert and R. Remmert, Theory of Stein spaces, Grundl. Math. Wiss. 236, Springer, Berlin-Heidelberg-New York, 1979.

[6] W. Luh, On universal functions, Fourier analysis and approximation theory. Colloquia Mathematica Societatis János Bolyai 19, Budapest, Hungary, 1976 (G. Alexits and P. Turán, eds.), Vol. 2, North-Holland, Amsterdam-New York, 1978, pp. 503–511.

[7] R. Remmert, Classical topics in complex function theory, Springer, New York-Berlin-Heidelberg, 1997.

[8] P. Zappa, On universal holomorphic functions, Boll. Un. Mat. Ital. (7) **2-A** (1988), 345–352.

Makoto Abe
Kumamoto University College of Medical Science
24-1 *Kuhonji* 4-*chome, Kumamoto* 862-0976
Japan
e-mail: mabe@cms.kumamoto-u.ac.jp

Teisuke Jin
Graduate School of Mathematical Sciences
University of Tokyo
8-1 *Komaba* 3-*chome, Meguro, Tokyo* 153-8914
Japan
*e-mail: MAF*01011*@nifty.ne.jp*

Tadashi Shima
Integrated Arts and Sciences
Hiroshima University
Higashi-Hiroshima 739-8521
Japan
e-mail: tadashi@mis.hiroshima-u.ac.jp