Hiroshima Math. J.
35 (2005), 47–92

Classification of the incompressible spanning surfaces for prime knots of 10 or less crossings

Osamu Kakimizu

(Received December 25, 2000)
(Revised October 7, 2004)

Abstract. It is known that the incompressible spanning surfaces for a fibred knot are unique. Also for a 2-bridge knot its incompressible spanning surfaces were classified by Hatcher and Thurston. In this paper we shall give the classification of the incompressible spanning surfaces for prime knots of 10 or less crossings, which include many non-fibred and non-2-bridge knots. Furthermore, we determine the associated simplicial complex $IS(K)$ for each prime knot K of 10 or less crossings, which describes the relations between equivalence classes of incompressible spanning surfaces for K.

Introduction

It is known that the incompressible spanning surfaces for a fibred knot are unique in the sense stated below (cf. [17]). Also for a 2-bridge knot its incompressible spanning surfaces were classified by Hatcher and Thurston [8]. In this paper we shall give the classification of the incompressible spanning surfaces for prime knots of 10 or less crossings which include many non-fibred and non-2-bridge knots. Furthermore, we determine the *associated simplicial complex* $IS(K)$ for each prime knot K of 10 or less crossings, which was introduced in [11] to describe the relations between equivalence classes of incompressible spanning surfaces for K.

Let L be an oriented link in the 3-sphere S^3, and let $E(L) = S^3 - \mathrm{Int}\, N(L)$ be its exterior where $N(L)$ is a fixed tubular neighborhood of L. We shall use the term "spanning surface" for L to denote a surface $S = \Sigma \cap E(L)$ where Σ is an oriented surface in S^3 such that $\partial\Sigma = L$, Σ has no closed component and is possibly disconnected and that $\Sigma \cap N(L)$ is a collar of $\partial\Sigma$ in Σ. Two spanning surfaces for L are said to be *equivalent* if they are ambient isotopic in $E(L)$ to each other. A spanning surface S is *incompressible* (resp. *of minimal genus*) if each component of S is π_1-injective in $E(L)$ (resp. the Euler number $\chi(S)$ is maximum among all the spanning surfaces for L). In this paper "link" always means *oriented link*. If L is a knot, then the classification of the incompres-

2000 *Mathematics Subject Classification numbers.* 57M25.
Key words and Phrases. Knots and links in S^3.

sible spanning surfaces is independent of the choice of an orientation of L. We prove the following

Theorem A. (I) *The incompressible spanning surfaces for every prime knot of* 10 *or less crossings are unique except for the following knots* (*see* [15, Appendix C] *for the notation*):

(B)

7_4	8_3	9_5	9_{10}	9_{13}	9_{18}	9_{23}	10_3	10_{11}	10_{16}	10_{18}
2	2	2	4	2	3	2	2	2	4	3
10_{24}	10_{28}	10_{30}	10_{31}	10_{33}	10_{37}	10_{38}	10_{53}	10_{67}	10_{68}	10_{74}
3	2	2	3	4	2	2	2	2	2	3

(II) *Each knot in the table* (B) *has exactly two, three or four equivalence classes of incompressible spanning surfaces according to the number written under the knot, moreover any of them is of minimal genus.*

We note that a composite knot $5_2\#5_2$ has infinitely many non-equivalent minimal genus spanning surfaces by Eisner [2]. Also in the case of 11 or more crossings there are many prime knots which have infinitely many non-equivalent incompressible spanning surfaces ([14], [7], [10] and others). For each prime knot K of 10 or less crossings, it is easy to find a minimal genus spanning surface S whose genus equal to one half of the degree of the Alexander polynomial of K; hence if K is out of the list in (B), then S is a unique incompressible spanning surface for K. Each knot in (B) is a 2-bridge knot except for the last four, 10_{53}, 10_{67}, 10_{68}, 10_{74}, and the assertion (II) for these 2-bridge knots follows from Hatcher and Thurston [8]. The classification of the incompressible spanning surfaces for the remaining four knots forms one of the main parts of the paper: The concrete classification will be given in §6.

Let L be a non-split link, and let $\mathscr{IS}(L)$ denote the set of equivalence classes of incompressible spanning surfaces for L. For an incompressible spanning surface S, $[S]$ denotes its equivalence class. In [11] we associated a simplicial complex $IS(L)$ with L as follows: The set of vertices is $\mathscr{IS}(L)$. Vertices $\sigma_0, \sigma_1, \dots, \sigma_k \in \mathscr{IS}(L)$ span a k-simplex if there are representatives $S_i \in \sigma_i$ $(0 \le i \le k)$ so that $S_i \cap S_j = \emptyset$ for all $i < j$. Note that if $\mathscr{IS}(L)$ consists only one equivalence class, then $IS(L)$ is a point. We now quote the following result.

Theorem 0.1 ([11, Th. A]). *The associated complex $IS(L)$ is connected.*

It follows that if $\mathscr{IS}(L)$ consists of two equivalence classes σ and σ', then $IS(L) = \overset{\sigma}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\sigma'}{\bullet}$. The complex $IS(L)$ describes the relations between equivalence classes in $\mathscr{IS}(L)$. We will determine the complexes for the prime knots of 10 or less crossings. It seems that even for the 2-bridge knots in (B), their complexes can not be determined by the results in [8]. We prove the following

Theorem C. (1) *For the knots* $K = 9_{18}, 10_{18}, 10_{24}$ *and* 10_{31},

$$IS(K) = \overset{\sigma_{10}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\sigma_{11}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\sigma_{01}}{\bullet}$$

(the definition of $\sigma_{ij} = [S(i, j)]$ *is given in* § 6*).*

(2)

$$IS(10_{74}) = \overset{[S']}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{[S]}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{[T']}{\bullet}$$

where S', S *and* T *are given by Figure* 6.15.

(3) *For the knots* $K = 9_{10}, 10_{16}$ *and* 10_{33},

$$IS(K) = \overset{\sigma_{101}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\sigma_{111}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\sigma_{000}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\sigma_{010}}{\bullet}$$

(the definition of $\sigma_{ijk} = [S(i, j, k)]$ *is given in* § 6*).*

Our approach is based on the works of Gabai [3], [6] and Kobayashi [13]. Gabai introduced and developed the theory of sutured manifolds which is powerful in the study of knots and links. Using this theory Kobayashi gave a sufficient condition of the minimal genus spanning surfaces for the given link being unique, and then showed that the knots in the table (B) is the prime knots of $\leq$10 crossings whose minimal spanning surfaces are not unique. We extend Kobayashi's method in some directions.

We explain our method in brief. For a given S with $[S] \in \mathscr{IS}(L)$, let $\mathscr{IS}(L, S)$ denote the set of $\eta \in \mathscr{IS}(L)$ such that $\eta \neq [S]$ and there is a representative $F \in \eta$ with $F \cap S = \varnothing$, i.e. there is an edge in $IS(L)$ which connects η and $[S]$. Then, as corollaries of Theorem 0.1 we have

Corollary 0.2. $\mathscr{IS}(L) = \{[S]\}$ *if and only if* $\mathscr{IS}(L, S) = \varnothing$.

Corollary 0.3. *Suppose that there exists an* S' *with* $[S'] \in \mathscr{IS}(L, S)$. *Then* $IS(L) = \overset{[S]}{\bullet}\!\!-\!\!\!-\!\!\overset{[S']}{\bullet}$ *if and only if*

$$(*) \qquad \mathscr{IS}(L, S) = \{[S']\} \qquad \text{and} \qquad \mathscr{IS}(L, S') = \{[S]\}.$$

Thus, roughly speaking, Theorem 0.1 means that the problem of deciding $IS(L)$ can be reduced to the study of "essential γ-surfaces" in the "complementary sutured manifolds" for some incompressible spanning surfaces. In § 1 and § 2 we give the definitions of these notions and investigate their basic properties. The assertion (I) in Theorem A follows from Corollary 0.2 by checking that each complementary sutured manifold for the given minimal genus spanning surface has no essential γ-surface, and this work of checking essentially due to Kobayashi [12] (see §6). To prove the assertion (II) for non-2-bridge knots 10_{53}, 10_{67}, 10_{68}, we must apply Corollary 0.3 in the case that S is a plumbing of two surfaces S_1 and S_2. In §3, under the assumptions that both S_i are unique incompressible spanning surfaces for $L_i = \partial S_i$ and that neither L_1 nor L_2 are fibred (in this case there exists an S', a "dual" of S, with $[S'] \in \mathscr{IS}(L, S)$), we give a criterion to satisfy the condition $(*)$ in Corollary 0.3 (Theorem 3.12). The assertion (II) for 10_{74} is shown as a corollary to Theorem C (2). To prove the assertions in Theorem C we give, under the same assumptions as above, a sufficient condition that $\mathscr{IS}(L)$ consists of three equivalence classes (Theorem 3.15). Moreover, in §5 we treat some kind of iterated plumbings and determine the associated complexes (Theorems 5.4 and 5.10). These theorems form the main parts of this paper in a technical sense, and they are formulated in terms of "marked sutured manifolds". In §4 we consider decompositions of a marked sutured manifold. In §6 by applying the method developed in §§3, 4 and 5 we give a proof of Theorems A and C.

The author thanks Tsuyoshi Kobayashi, Hiroshi Goda and Makoto Sakuma for valuable conversations and helpful comments.

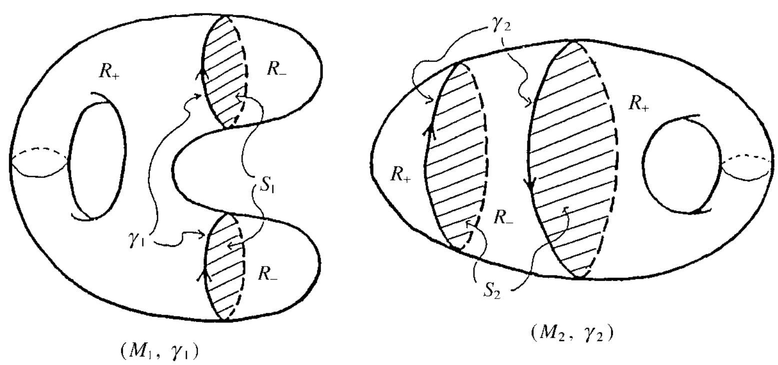
1. Essential γ-surfaces

A *sutured manifold* (M, γ) is a compact oriented 3-manifold M together with a subset $\gamma \subset \partial M$ which is a union of finitely many pairwise disjoint annuli. For each component of γ a *suture*, i.e. an oriented core circle, is fixed, and $s(\gamma)$ denotes the set of sutures. Moreover every component of $R(\gamma) = \partial M - \operatorname{Int} \gamma$ is oriented so that the orientations on $R(\gamma)$ are coherent with respect to $s(\gamma)$. Let $R_+(\gamma)$ (resp. $R_-(\gamma)$) denotes the union of those components of $R(\gamma)$ whose normal vectors point out of (resp. into) M. In the case that (M, γ) is homeomorphic to $(F \times [0,1],\ \partial F \times [0,1])$ where F is a compact oriented 2-manifold, (M, γ) is called a *product sutured manifold*.

A properly embedded compact oriented 2-manifold (possibly disconnected) $S \subset M$ is said to be a *γ-surface* if S has no closed component, the oriented boundary ∂S is contained in $\operatorname{Int} \gamma$ and isotopic to $s(\gamma)$ in γ. A γ-surface S is *parallel* to a surface in $R(\gamma)$ if there is an embedding $e : (S, \partial S) \times [0,1] \to (M, \gamma)$ so that $e_0 = \mathrm{id} : S \to S$ and $e_1(S) \subset R(\gamma)$: Note that $e_1(S)$ is a union of some components of $R(\gamma)$. A γ-surface S is *essential* if S is incompressible in M and not parallel to a surface in $R(\gamma)$. A *γ-isotopy* of M is an isotopy $\{h_t\}$ of M such that $h_0 = \mathrm{id}$, $h_t|R(\gamma) = \mathrm{id}$ and $h_t(\gamma) = \gamma$ for all $0 \le t \le 1$. Two γ-surfaces in M are *equivalent* if they are ambient isotopic to each other by a γ-isotopy. Let $\mathscr{E}(M, \gamma)$ denote the set of equivalence classes of essential γ-surfaces in M. For an essential γ-surface S, $[S]_\gamma$ denotes its equivalence class. It is easy to see the following

Lemma 1.1. *Let (M, γ) be a sutured manifold, and let S be a γ-surface. Suppose that ∂M is connected and that S is parallel to a surface in $R(\gamma)$ by an embedding $e : (S, \partial S) \times [0,1] \to (M, \gamma)$ with $e_0 = \mathrm{id}_S$ and $e_1(S) \subset R(\gamma)$. Then $e_1(S) = R_+(\gamma)$ or $R_-(\gamma)$.*

Example. (Figure 1.1 (1), (2))



(1) γ_1-surface S_1 is parallel to $R_-(\gamma_1)$

(2) S_2 is an essential γ_2-surface

Fig. 1.1

We will state the definitions of two kind of operations on sutured manifolds (cf. [3], [6]). Let (M, γ) be a sutured manifold. A *product disk* $\Delta \subset M$ is a properly embedded disk such that $\partial \Delta$ intersects $s(\gamma)$ transversely in two points. For a product disk $\Delta \subset M$, we get a new sutured manifold (M', γ') in the way shown in Figure 1.2. This decomposition

$$(M, \gamma) \overset{\Delta}{\to} (M', \gamma')$$

is called a *product decomposition*. We note that

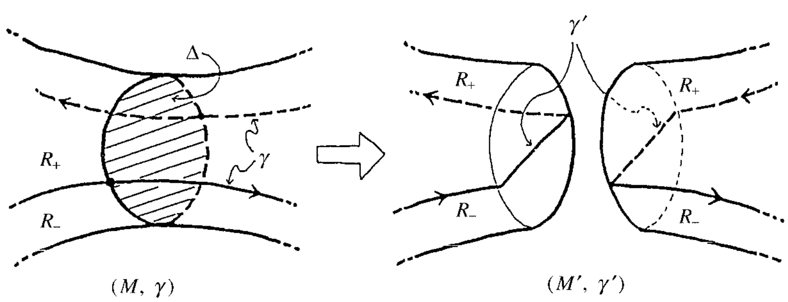


Fig. 1.2

(1.2) each component of $R(\gamma)$ is incompressible if and only if so is each component of $R(\gamma')$.

We next introduce another kind of operation. An *octagon* $\Omega \subset M$ is a properly embedded disk such that $\partial\Omega$ intersects $s(\gamma)$ transversely in four points. Then we have two decompositions of (M, γ):

$$(M_\alpha, \gamma_\alpha) \stackrel{\alpha(\Omega)}{\longleftarrow} (M, \gamma) \stackrel{\beta(\Omega)}{\longrightarrow} (M_\beta, \gamma_\beta)$$

in the way shown in Figure 1.3, and we call them *octagonal decompositions*. Note that both M_α and M_β are homeomorphic to the manifold obtained by cutting M along Ω.

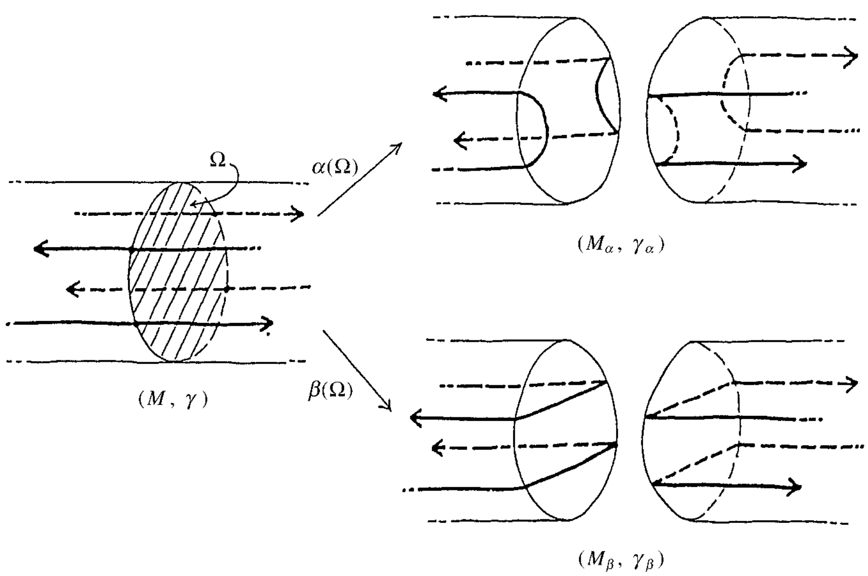


Fig. 1.3

Now let $(M, \gamma) \stackrel{\Delta}{\rightarrow} (M', \gamma')$ be a product decomposition of a sutured manifold where Δ is a product disk, and let $S \subset M$ be an essential γ-surface. Suppose that M is irreducible. Then we can move S by a γ-isotopy so that $\partial S = s(\gamma)$ and that $S \cap \Delta$ is a single arc A connecting the two points of $\partial\Delta \cap s(\gamma)$. By cutting S along A, we obtain a γ'-surface $S_\Delta \subset M'$.

Lemma 1.3. *Let* $(M, \gamma) \stackrel{\Delta}{\rightarrow} (M', \gamma')$ *be a product decomposition. Suppose that* M *is irreducible and* $\partial M'$ *is connected. Then for each essential* γ*-surface* $S \subset M$*, the* γ'*-surface* $S_\Delta \subset M'$ *is also essential. Moreover if two essential* γ*-surfaces* S *and* $S' \subset M$ *are equivalent, then so are* S_Δ *and* S'_Δ.

Proof. First note that M' is irreducible and ∂M is connected since M is irreducible and $\partial M'$ is connected. Let $S \subset M$ be an essential γ-surface.

Clearly S_Δ is incompressible in M'. Suppose that S_Δ is parallel to a surface in $R(\gamma')$. Then there is an embedding $e : (S_\Delta, \partial S_\Delta) \times [0,1] \to (M', \gamma')$ so that $e_0 = \mathrm{id} : S_\Delta \to S_\Delta$ and $e_1(S_\Delta) \subset R(\gamma')$. Since $\partial M'$ is connected, $e_1(S_\Delta) = R_+(\gamma')$ or $R_-(\gamma')$ by Lemma 1.1. We assume that $e_1(S_\Delta) = R_+(\gamma')$. Since $R_+(\gamma')$ is obtained by cutting $R_+(\gamma)$ along the arc $\partial\Delta \cap R_+(\gamma)$, the embedding $e : (S_\Delta, \partial S_\Delta) \times [0,1] \to (M', \gamma')$ can be extend to an embedding $\tilde{e} : (S, \partial S) \times [0,1] \to (M, \gamma)$ so that $\tilde{e}_0 = \mathrm{id} : S \to S$ and $\tilde{e}_1(S) = R_+(\gamma)$. Hence S is parallel to a surface in $R(\gamma)$, and this is a contradiction. Thus S_Δ is essential.

Next we suppose that S and $S' \subset M$ are two essential γ-surfaces and they are equivalent. We may assume that $\partial S = \partial S' = s(\gamma)$, $S \cap \Delta = S' \cap \Delta$ and $A = S \cap \Delta$ is an arc connecting the two points of $\partial\Delta \cap s(\gamma)$. Let $h : M \times [0,1] \to M$ be a γ-isotopy such that $h_0 = \mathrm{id}$ and $h_1(S) = S'$. By the definition of γ-isotopy and the above assumption, we may assume that $h_t|\partial M = \mathrm{id}$ $(0 \le t \le 1)$. Consider the restriction $h : S \times [0,1] \to M$. By the standard method as in the proof of [9, Lemma 6.5], we can move $h|S \times [0,1]$ to a homotopy $g : S \times [0,1] \to M$ so that $g_0 = \mathrm{id}$, $g_1 = h_1|S$, $g_t|\partial M = \mathrm{id}$ and $g_t|A = \mathrm{id}$ $(0 \le t \le 1)$. From this we have a homotopy $g' : S_\Delta \times [0,1] \to M'$ so that $g'_0 = \mathrm{id}$, $g'_1(S_\Delta) = S'_\Delta$ and $g'_t|\partial S_\Delta = \mathrm{id}$ $(0 \le t \le 1)$. Hence by Waldhausen [16, Cor 5.5], we get a γ'-isotopy $\{h'_t\}$ of M' which carries S_Δ to S'_Δ. Lemma 1.3 is proved. □

Thus under the same assumption as in Lemma 1.3, we can define a map

$$\mathscr{E}_\Delta : \mathscr{E}(M,\gamma) \to \mathscr{E}(M',\gamma'), \qquad [S]_\gamma \mapsto [S_\Delta]_{\gamma'}.$$

Moreover we can easily verify the following

Proposition 1.4. *Let $(M,\gamma) \xrightarrow{\Delta} (M',\gamma')$ be a product decomposition. Suppose that M is irreducible and $\partial M'$ is connected. Then the map $\mathscr{E}_\Delta : \mathscr{E}(M,\gamma) \to \mathscr{E}(M',\gamma')$ is bijective.*

Now we consider another situation. Let $(M,\gamma) \xrightarrow{\Delta} (M',\gamma')$ be a product decomposition where M is irreducible. Suppose that (M',γ') is a disjoint union of two connected sutured manifolds (M_1,γ_1) and (M_2,γ_2). Suppose further that (M_2,γ_2) is a product sutured manifold. Let $S \subset M$ be an essential γ-surface, and assume that $S \cap \Delta$ is a single arc. Then we obtain a γ_1-surface $S_{\Delta,1} = S \cap M_1 \subset M_1$. It is easy to see that $S_{\Delta,1}$ is also essential. Moreover if two essential γ-surfaces S and $S' \subset M$ are equivalent, then so are $S_{\Delta,1}$ and $S'_{\Delta,1}$. Hence a map

$$\mathscr{E}_{\Delta,1} : \mathscr{E}(M,\gamma) \to \mathscr{E}(M_1,\gamma_1), \qquad [S]_\gamma \mapsto [S_{\Delta,1}]_{\gamma_1}$$

is well defined. We can easily verify the following

Proposition 1.5. *Let $(M,\gamma) \xrightarrow{\Delta} (M',\gamma')$ be a product decomposition. Suppose that M is irreducible and that (M',γ') has two components (M_1,γ_1) and*

(M_2, γ_2). *Suppose further that* (M_2, γ_2) *is a product sutured manifold and* ∂M_1 *is connected. Then the map* $\mathscr{E}_{\Delta,1} : \mathscr{E}(M, \gamma) \to \mathscr{E}(M_1, \gamma_1)$ *is bijective.*

Now we consider octagonal decompositions.

Proposition 1.6. *Let*

$$(M_\alpha, \gamma_\alpha) \overset{\alpha(\Omega)}{\longleftarrow} (M, \gamma) \overset{\beta(\Omega)}{\longrightarrow} (M_\beta, \gamma_\beta)$$

be octagonal decompositions, where $\Omega \subset M$ *is an octagon. Suppose that* M *is irreducible and* $\partial M_\alpha = \partial M_\beta$ *is connected. If* (M, γ) *has an essential* γ*-surface, then* $(M_\alpha, \gamma_\alpha)$ *or* (M_β, γ_β) *has an essential* γ_α*- or* γ_β*-surface.*

Proof. We note that ∂M is connected and M_α and M_β are irreducible by the assumption. Let $S \subset M$ be an essential γ-surface. By using the cut and paste argument, we can move S by a γ-isotopy so that $\partial S = s(\gamma)$ and $S \cap \Omega$ is a union of two arcs. There are two cases as shown in Figure 1.4, where $\{a, b, c, d\} = \partial\Omega \cap s(\gamma)$. We assume that the case (α) holds

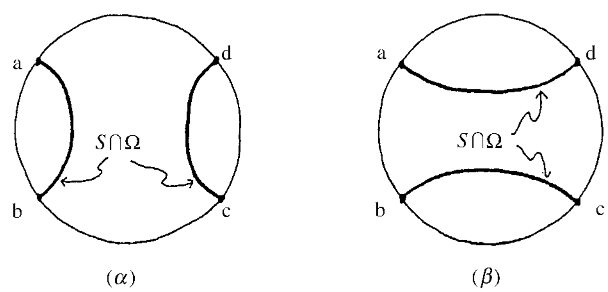


Fig. 1.4

Then we get a γ_α-surface $S_\alpha \subset M_\alpha$ by cutting S along $S \cap \Omega$. Clearly S_α is incompressible in M_α. We will show that S_α is not parallel to a surface in $R(\gamma_\alpha)$. If not, then there is an embedding $e : (S_\alpha, \partial S_\alpha) \times [0, 1] \to M_\alpha$ so that $e_0 = \mathrm{id} : S_\alpha \to S_\alpha$ and $e_1(S_\alpha) \subset R(\gamma_\alpha)$. By Lemma 1.1, $e_1(S_\alpha) = R_+(\gamma_\alpha)$ or $e_1(S_\alpha) = R_-(\gamma_\alpha)$. We assume that $e_1(S_\alpha) = R_+(\gamma_\alpha)$. In this case there are two possibility (i) and (ii) near the cutting disk Ω as shown in Figure 1.5. However the case (ii) is impossible. In fact, for the γ-surface S there is a compressing disk B as shown in Figure 1.6, and this contradicts to the assumption that S is incompressible. On the other hand, if the case (i) holds, then $e : (S_\alpha, \partial S_\alpha) \times [0, 1] \to (M_\alpha, \gamma_\alpha)$ can be extended to an embedding $\tilde{e} : (S, \partial S) \times [0, 1] \to (M, \gamma)$ so that $\tilde{e}_0 = \mathrm{id}_S$ and $\tilde{e}_1(S) = R_+(\gamma)$, and this is also a contradiction. Thus Proposition 1.6 is proved. $\square$

We close this section by showing the following lemma which is used in the latter sections.

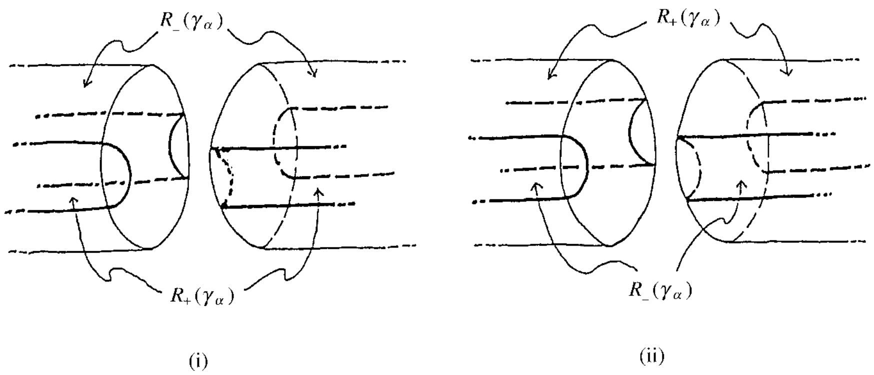


Fig. 1.5

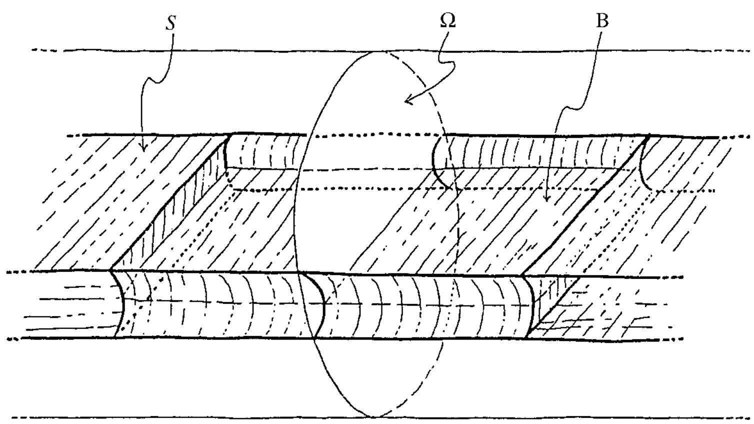


Fig. 1.6

Lemma 1.7. *Let X be connected Haken 3-manifold such that ∂X is a union of incompressible tori. Let Y be a compact irreducible 3-submanifold of X (possibly disconnected) such that each component of $\mathrm{Fr}(Y)$ is a properly embedded incompressible surface in X. Let F and F' be two properly embedded orientable incompressible surfaces in X (possibly disconnected) which satisfy the following properties (1)–(4). Then there is an isotopy $\{h_t\}$ of X keeping Y fixed so that $h_0 = \mathrm{id}$ and $h_1(F) = F'$:*

(1) $F \cup F' \subset X - Y$.

(2) *Each component of ∂X contains at most one component of ∂F, and F has no closed component.*

(3) *There is a homotopy $f : F \times [0,1] \to X$ such that $f_0 = \mathrm{id} : F \to F$, $f_1 : F \to F'$ is a homeomorphism and $f(\partial F \times [0,1]) \subset \partial X$.*

(4) *There is no component of F which is parallel to a component of $\mathrm{Fr}(Y)$.*

Proof. In the case when F is connected, the lemma is (essentially) proved in [12, Lemma 2.4]. We prove the lemma in general cases by induction on the number of the components of F. Let $F_1, \dots, F_k$ denote the components of F. Then F' has the same number of components $F_1', \dots, F_k'$ by (3). We may assume that each restriction $f : F_i \times [0,1] \to X$ of the homotopy in (3) gives a homotopy between F_i and F_i' with $f(\partial F_i \times [0,1]) \subset \partial X$ $(1 \le i \le k)$.

We assume that the lemma holds for $k \le n$, and we will prove it for $k = n$ $(n \ge 2)$. By the assumption, there is an isotopy $\{e_t\}$ of X keeping Y fixed so that $e_0 = \mathrm{id}$ and $e_1(\bigcup_{1 \le i \le n-1} F_i) = \bigcup_{1 \le i \le n-1} F_i'$. Put $F^* = e_1(F_n)$. Then $F^* \cup F_n' \subset X - (Y \cup \bigcup_{1 \le i \le n-1} F_i')$ and $f^* = f \circ ((e_1|F_n)^{-1} \times \mathrm{id}_{[0,1]})$: $F^* \times [0,1] \to X$ is a homotopy between F^* and F_n' with $f^*(\partial F^* \times [0,1]) \subset \partial X$. Take a regular neighborhood Y' of $\bigcup_{1 \le i \le n-1} F_n'$ in X with $Y' \cap (Y \cup F^* \cup F_n') = \emptyset$. We see that F^* is not parallel to any component of $\mathrm{Fr}(Y) \cup \mathrm{Fr}(Y')$. In fact $F^* = e_1(F_n)$ is not parallel to any component of $\mathrm{Fr}(Y)$ by (4). Also if F^* is parallel to a component of $\mathrm{Fr}(Y')$, then F_n is parallel to F_n' for some $1 \le i \le n-1$, and hence F_n is ambient isotopic to F_i. Since $\partial F_n \ne \emptyset$ by (2), some component of ∂X contains at least two components of ∂F, ∂F_n and ∂F_i. This contradicts (2). Now we can apply the lemma in the case that F is connected to $(X, Y, F, F') = (X, Y \cup Y', F^*, F_n')$. Hence we get an isotopy $\{e_t'\}$ of X keeping $Y \cup Y'$ fixed so that $e_0' = \mathrm{id}$ and $e'(F_1^*) = F_n'$. By connecting two isotopies $\{e_t\}$ and $\{e_t'\}$, we have the desired isotopy $\{h_t\}$, and Lemma 1.7 is proved. □

2. Complementary sutured manifolds and Murasugi sums

We assume that the 3-sphere S^3 is oriented. Let $L \subset S^3$ be an (oriented) link and $S \subset E(L)$ a spanning surface for L. Let $(N(S), \delta) = (S \times [-1,1], \partial S \times [-1,1])$ be the product sutured manifold *associated* to S where $S \times [-1,1] \subset E(L)$ is a regular neighborhood of S. We assume that the orientation of ∂S induces that of $s(\delta)$ so that $R_-(\delta) = S \times \{-1\}$ and $R_+(\delta) = S \times \{1\}$. The *complementary sutured manifold* for S is the sutured manifold $(M, \gamma) = (\mathrm{Cl}(E(L) - N(S)), \mathrm{Cl}(\partial E(L) - \delta))$ with $R_-(\gamma) = R_+(\delta)$ and $R_+(\gamma) = R_-(\delta)$. If L is non-split, then $E(L)$ and M are irreducible. We also note that ∂M is connected if and only if so is S. Let $\mathscr{IS}(L)$ denote the set of equivalence classes of incompressible spanning surfaces for L. For a given incompressible spanning surface S, let $\mathscr{IS}(L, S)$ denote the set of $\eta \in \mathscr{IS}(L)$ such that $\eta \ne [S]$ and there is a representative $F \in \eta$ with $F \cap S = \emptyset$.

Proposition 2.1. *Let L be a non-split link, S a connected incompressible spanning surface for L and (M, γ) the complementary sutured manifold for S. Then the inclusion $M \subset E(L)$ induces a well defined map $\iota_S : \mathscr{E}(M, \gamma) \to \mathscr{IS}(L, S)$, $[F]_\gamma \mapsto [F]$. Moreover ι_S is bijective.*

Proof. Every essential γ-surface F is regarded as an incompressible spanning surface for L which is not equivalent to S by the definitions. If two γ-surfaces F_0 and F_1 are isotopic by a γ-isotopy $\{h_t\}$ of M, then the isotopy can be extended to an isotopy of $E(L)$ since $h_t \mid R(\gamma) = \mathrm{id}$ $(0 \le t \le 1)$ and $N(S) = S \times [-1, 1]$. Thus $M \subset E(L)$ induces a well defined map $\iota_S : \mathscr{E}(M, \gamma) \to \mathscr{IS}(L, S)$, $[F]_\gamma \mapsto [F]$.

For each $\eta \in \mathscr{IS}(L, S)$ there is a representative $F \in \eta$ so that $F \subset M$. Clearly F is an essential γ-surface, and hence ι_S is surjective. Next suppose that two essential γ-surfaces F and F' are ambient isotopic in $E(L)$ by an isotopy $\{e_t\}$. Then we can apply Lemma 1.7 to $(X, Y, F, F') = (E(L), N(S), F, F')$. In fact (1) holds clearly, (2) holds since F is a spanning surface for L, and if we set $f_t = e_t|F$ $(0 \le t \le 1)$, then $f : F \times [0, 1] \to E(L)$ satisfies the desired condition in (3). Moreover if there is a component A of F which is parallel to some component of $\mathrm{Fr}(N(S))$, then $A = F$ and F is parallel to $R_+(\gamma)$ or $R_-(\gamma)$ in M. This contradicts the assumption that F is an essential γ-surface; hence (4) holds. Thus by Lemma 1.7, F and F' are ambient isotopic in $E(L)$ by an isotopy keeping $N(S)$ fixed. This implies that ι_S is injective. Proposition 2.1 is proved. □

An oriented surface $\Sigma \subset S^3$ is a *Murasugi sum* of compact oriented surfaces Σ_1 and $\Sigma_2 \subset S^3$ if there are 3-balls V_1 and $V_2 \subset S^3$ satisfying the following property (see [4, §0], [13, §5]):

$$(2.2) \quad V_1 \cup V_2 = S^3, \qquad V_1 \cap V_2 = \partial V_1 = \partial V_2, \qquad \Sigma_i \subset V_i \quad (i = 1, 2),$$
$$\Sigma = \Sigma_1 \cup \Sigma_2 \qquad \text{and} \qquad D = \Sigma_1 \cap \Sigma_2 \text{ is a } 2n\text{-gon.}$$

When D is a 4-gon the Murasugi sum is also called a *plumbing* of Σ_1 and Σ_2. Put $L = \partial\Sigma$, $L_i = \partial\Sigma_i$, $S = \Sigma \cap E(L)$ and $S_i = \Sigma_i \cap E(L_i)$. Then we will also say that S is a *Murasugi sum* of S_1 and S_2. Note that $\Sigma' = (\Sigma - D) \cup D'$ is an oriented surface with $\partial\Sigma' = L$ where $D' = \partial V_1 - \mathrm{Int}\, D$. By a tiny isotopy of S^3 keeping L fixed we can move Σ' so that $\Sigma' \cap \Sigma \cap E(L) = \varnothing$ (see Figure 3.4). We will say that Σ' (resp. $S' = \Sigma' \cap E(L)$) is a *dual* of Σ (resp. S). Note that Σ' (resp. S') is also a Murasugi sum of Σ_1' and Σ_2' (resp. S_1' and S_2') where $\Sigma_i' = (\Sigma_i - D) \cup D'$ and $S_i' = \Sigma_i' \cap E(L_i)$ $(i = 1, 2)$. D' is also called a *dual* of D. Gabai showed that the Murasugi sum operations hold the following natural properties:

Proposition 2.3 ([4], [5]). (i) *S is of minimal genus if and only if so are both S_1 and S_2.*

(ii) *S is incompressible if so are both S_1 and S_2.*

(iii) *L is a fibred link with fibre S if and only if both L_1 and L_2 are fibred links with fibres S_1 and S_2 respectively.*

Now we show the following

Proposition 2.4. *Let L be a non-split oriented link and S a connected incompressible spanning surface for L. Suppose that S is a Murasugi sum of S_1 and S_2, where S_i is a spanning surface for an oriented link L_i $(i = 1, 2)$. Suppose further that L_2 is a fibred link with fibre S_2. Then L_1 is non-split, and S_1 is connected and incompressible. Moreover there is a bijection*

$$\varphi : \mathscr{IS}(L, S) \to \mathscr{IS}(L_1, S_1).$$

Proof. Clearly the connectedness of S implies that of S_1. Let (M, γ) be the complementary sutured manifold for S. Since S is connected so is ∂M. By the same argument as in the proofs of [6, Th. 3.1] and [13, Th. 5.1], there is a finite sequence of product decompositions

$$(M, \gamma) \overset{\Delta_1}{\to} (N_1, \delta_1) \overset{\Delta_2}{\to} \cdots \overset{\Delta_n}{\to} (N_n, \delta_n)$$

so that (N_n, δ_n) is homeomorphic to the complementary sutured manifold for S_1 and that ∂N_i are all connected. Since S is incompressible in $E(L)$, each component of $R(\gamma)$ is incompressible in M. Hence each component of $R(\delta_n)$ is incompressible in N_n by (1.2), and then S_1 is incompressible in $E(L_1)$.

Since L is non-split, M is irreducible. This implies that N_n is irreducible, and hence L_1 is non-split. By Proposition 2.1 there are bijection $\iota_S : \mathscr{E}(M, \gamma) \to \mathscr{IS}(L, S)$ and $\iota_{S_1} : \mathscr{E}(N_n, \delta_n) \to \mathscr{IS}(L_1, S_1)$. Thus by applying Proposition 1.4 to each step of the above sequence of product decompositions, we get a bijection $\varphi : \mathscr{IS}(L, S) \to \mathscr{IS}(L_1, S_1)$. $\square$

Also Boileau and Gabai showed the following

Proposition 2.5 ([6, Cor. 3.2]). *Let L be a non-split link and S a spanning surface for L. Suppose that S is a Murasugi sum of S_1 and S_2 where S_i is an incompressible spanning surface for a link L_i $(i = 1, 2)$. Suppose further that L_i is not a fibred link $(i = 1, 2)$. Then S and its dual S' are not equivalent.*

Propositions 2.4 and 2.5 together with Corollary 0.2 in the introduction imply the following

Corollary 2.6. *Let L be a non-split link and S a connected spanning surface for L which is a Murasugi sum of S_1 and S_2 where S_i is a incompressible spanning surface for a link L_i $(i = 1, 2)$. Then $\mathscr{IS}(L) = \{[S]\}$ if and only if one of L_1 and L_2, say L_1, is a fibred link and $\mathscr{IS}(L_2) = \{[S_2]\}$.*

3. Plumbings and marked sutured manifolds

Let L be a non-split link and S its spanning surface. We suppose that

(3.1) S is connected, and S is a plumbing of S_1 and S_2 which are unique incompressible spanning surfaces for links L_1 and L_2 respectively.

Hence S_1, S_2 and S are of minimal genus by Proposition 2.3 (i). If L_1 or L_2 is fibred, then we have $\mathscr{IS}(L) = \{[S]\}$ by Corollary 2.6. On the other hand, in the case

(3.2) neither L_1 nor L_2 are fibred,

$\mathscr{IS}(L)$ contains at least two distinct equivalence classes $[S]$ and $[S']$ where S' is a dual of S by Proposition 2.5. If the plumbing is in the form shown in Figure 3.1 in addition, then S can be regarded as a connected sum of S_1 and S_2, and hence $\mathscr{IS}(L)$ contains infinitely many equivalence classes of minimal genus by Eisner [2]. In this section, under the assumptions (3.1) and (3.2), we will give a necessary and sufficient condition that $\mathscr{IS}(L)$ consists of exactly two equivalence classes $[S]$ and $[S']$ (Theorem 3.12), and also give a sufficient condition that $\mathscr{IS}(L)$ consists of three equivalence classes (Theorem 3.15). These conditions will be formulated in terms of marked sutured manifolds.

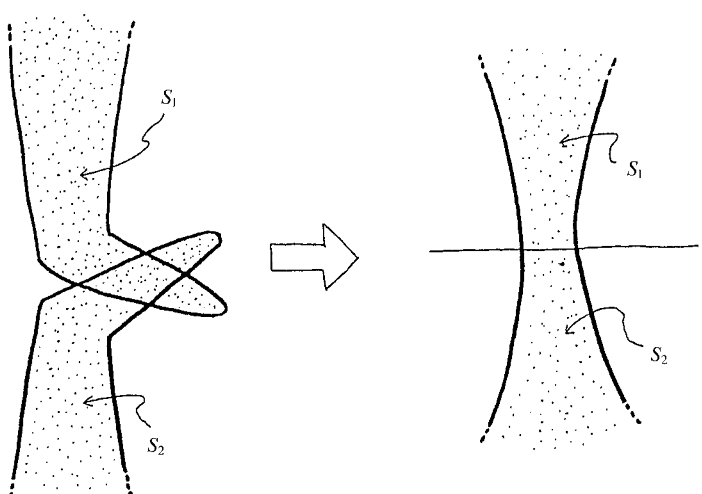


Fig. 3.1

A *marked sutured manifold* (M, γ, A) is a sutured manifold (M, γ) together with a properly embedded arc $A \subset R(\gamma)$, and we call A a *mark* on (M, γ). If there is a product disk $\Delta \subset M$ with A as an edge (see Figure 3.2), then (M, γ, B) is also a marked sutured manifold where B is the opposite edge of A,

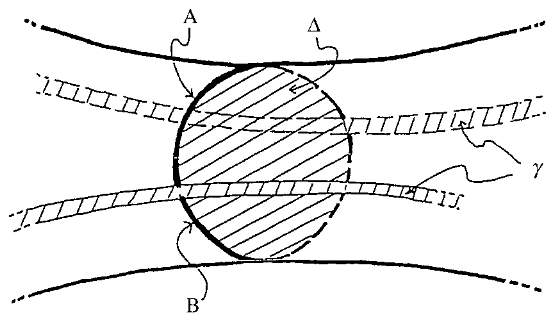


Fig. 3.2

and we call B an *opposite mark* of A relative to Δ. The following lemma is easily verified.

Lemma 3.3. *Let (M,γ,A) be a marked sutured manifold. Suppose that M is irreducible and each component of $R(\gamma)$ is incompressible. If there is a product disk with A as an edge, then the ambient isotopy types of such disks are unique, and hence so are the isotopy types in $R(\gamma)$ of opposite marks of A.*

Let L be a non-split link and S its spanning surface. Suppose that S is a plumbing of S_1 and S_2 where S_i is a spanning surface for a link L_i $(i=1,2)$. We call $D=S_1\cap S_2$ the *plumbing disk*. Let (M_1,γ_1) and (M_2,γ_2) (resp. (N_1,δ_1) and (N_2,δ_2)) be the complementary sutured manifolds (resp. the associated product sutured manifolds) for S, S_1 and S_2 respectively. We will make marked sutured manifolds (M_i,γ_i,A_i) and (N_i,δ_i,A_i) $(i=1,2)$ as follows: We first consider (M_1,γ_1) and (N_1,δ_1). Let I_1 be a core arc of D relative to the embedding $D\subset S_1$, i.e. I_1 is a properly embedded arc in S_1 so that D is a regular neighborhood of I_1 in S_1. Push out I_1 from S_1 to the side on which S_2 is attached, and consider this arc A_1 to be properly embedded in $R(\gamma_1)=R(\delta_1)$. Thus we get marked sutured manifolds (M_1,γ_1,A_1) and (N_1,δ_1,A_1). By the same way we also get (M_2,γ_2,A_2) and (N_2,δ_2,A_2) (see Figure 3.3). These markings correspond to the way of plumbing of S_1 and S_2.

Proposition 3.4. *Let L be a non-split link and S a spanning surface for L. Suppose that* (3.1) *and* (3.2) *hold. Let (M_i,γ_i,A_i) $(i=1,2)$ be the marked complementary sutured manifolds associated with the plumbing $S=S_1\cup S_2$. Suppose that there is no product disk in M_i with A_i as an edge for each $i=1,2$. Then $\mathscr{IS}(L,S)=\{[S']\}$ where S' is a dual of S.*

Proof. By the definition of the plumbing $S=S_1\cup S_2$, there are 3-balls $V_1,\ V_2\subset S^3$ such that $S^3=V_1\cup V_2,\ V_1\cap V_2=\partial V_1=\partial V_2,\ S_i\subset V_i\ (i=1,2)$

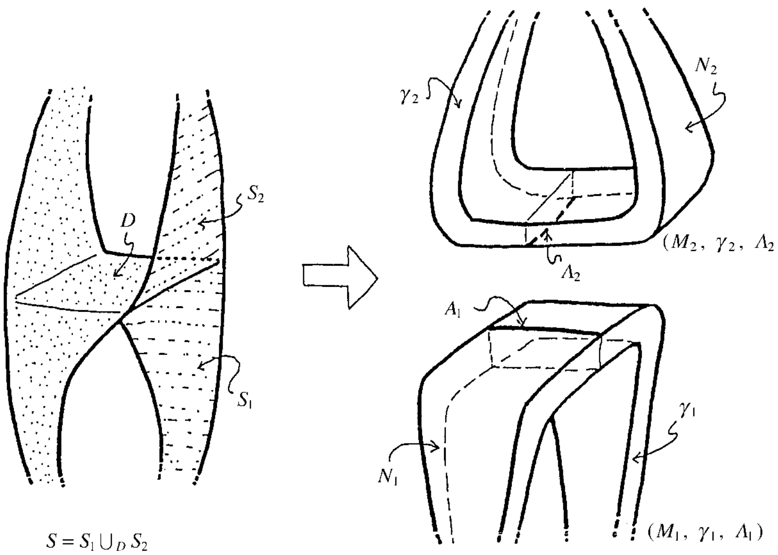


Fig. 3.3

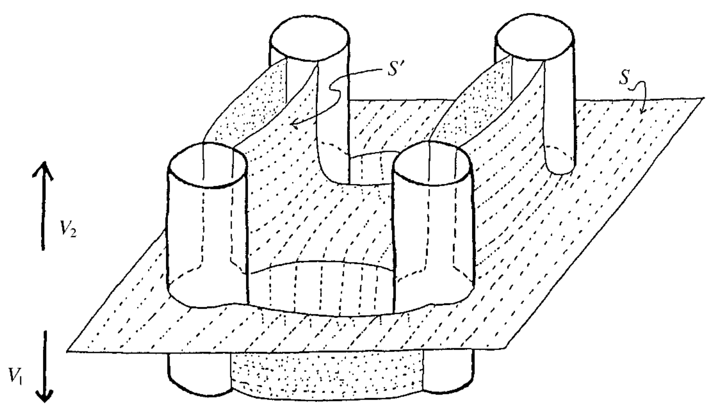


Fig. 3.4

and $D = S_1 \cap S_2 \subset \partial V_1$ is a 4-gon. Also S' is a plumbing of two surfaces S_1' and S_2' such that $S_i' \subset V_i$, $S_1' \cap S_2' = D' \subset \partial V_1$ and $\mathrm{Cl}(S_i' - D')$ is a parallel copy of $\mathrm{Cl}(S_i - D)$ (see Figure 3.4).

Let (M, γ), (M_1, γ_1) and (M_2, γ_2) be the complementary sutured manifolds for S, S_1 and S_2 respectively. We may assume as in the proof of [6, Th. 3.1] that (Figure 3.5)

$$(3.5) \qquad V_1 \cap R_-(\gamma) = R_-(\gamma_1), \qquad V_1 \cap R_+(\gamma) = R_+(\gamma_1) - \mathrm{Int}\, N(D),$$

$$V_2 \cap R_-(\gamma) = R_-(\gamma_2) - \mathrm{Int}\, N(D), \qquad V_2 \cap R_+(\gamma) = R_+(\gamma_2).$$

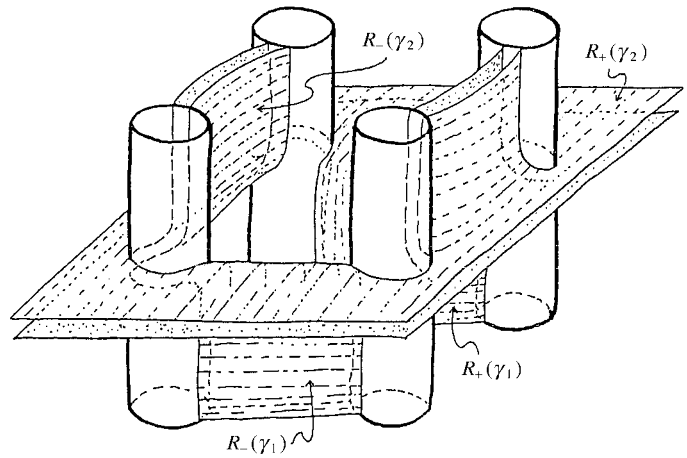


Fig. 3.5

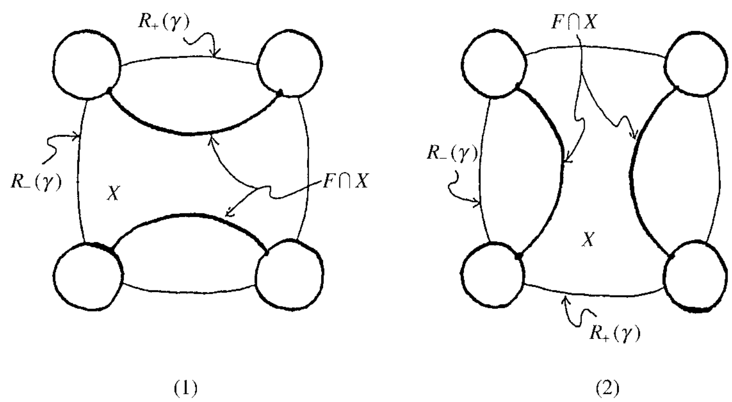


Fig. 3.6

Now let F be an incompressible spanning surface for L such that $[F] \in \mathscr{IS}(L, S)$. We can regard F as a γ-surface in M. Consider the disk $X = M \cap \partial V_1$: X is properly embedded in M and ∂X intersects $s(\gamma)$ transversely in four points. By moving F by a γ-isotopy, we may assume that F intersects X transversely in two arcs, and there are two cases as shown in Figure 3.6. We may assume without loss of generality that the case (1) holds. Put $M_i' = M \cap V_i$ and $F_i = F \cap V_i$ $(i = 1, 2)$. Then we get a sutured manifold (M_2', γ_2') so that F_2 becomes an incompressible γ_2'-surface in M_2'. Note that (M_2', γ_2') is homeomorphic to (M_2, γ_2). By the assumption that S_2 is a unique incompressible spanning surface for L_2, F_2 is parallel to a surface in $R(\gamma_2')$ in M_2'. Let $e : (F_2, \partial F_2) \times [0, 1] \to (M_2', \gamma_2')$ be an embedding so that $e_0 = \mathrm{id} : F_2 \to F_2$ and $e_1(F_2)$ is a union of some components of $R(\gamma_2')$. Since S is connected, so are S_1 and S_2 and also $R_+(\gamma_2')$ and $R_-(\gamma_2')$. Thus $e_1(F_2) = R_+(\gamma_2')$ or $R_-(\gamma_2')$.

Now let (P, λ) denote the sutured manifold

$$(\mathrm{Cl}(E(L) - N(S \cup S')), \mathrm{Cl}(\partial E(L) - N(\partial S \cup \partial S')))$$
$$= (\mathrm{Cl}(M - N(S')), \mathrm{Cl}(\gamma - N(\partial S')))$$

Then (P, λ) has two components (P_1, λ_1) and (P_2, λ_2) which satisfy the following properties: (for $i = 1, 2$) $P_i \cap \partial V_1$ is a union of two disks Δ_i, Δ_i' which are product disks in P_i, and $\Delta_i \cup \Delta_i'$ decomposes (P_i, λ_i) into two sutured manifolds $(P_{i,\alpha}, \lambda_{i,\alpha})$ and $(P_{i,\beta}, \lambda_{i,\beta})$ so that

(3.6.i) $(P_{i,\alpha}, \lambda_{i,\alpha})$ is homeomorphic to the product sutured manifold $(\mathrm{Cl}(S_{3-i} - D) \times [-1, 1], (\partial \mathrm{Cl}(S_{3-i} - D)) \times [-1, 1])$ and $(P_{i,\beta}, \lambda_{i,\beta})$ is homeomorphic to (M_i, γ_i) (Figure 3.7).

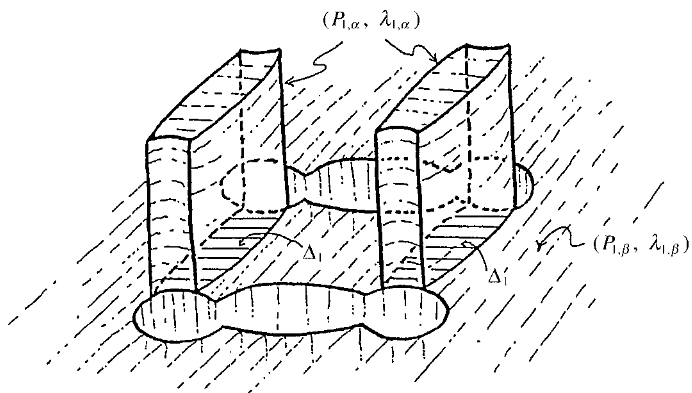


Fig. 3.7. (P_1, λ_1)

Case 1: $e_1(F_2) = R_-(\gamma_2')$. In this case, using the product structures of $e(F_2 \times [0, 1]) \subset M_2'$ and $(P_{2,\alpha}, \lambda_{2,\alpha})$, we can push $(F, \partial F)$ into (P_1, λ_1) by a γ-isotopy of M. Thus F becomes a λ_1-surface in P_1. By (3.6.1) and Proposition 1.5, there is a bijection $\mathscr{E}(P_1, \lambda_1) \to \mathscr{E}(P_{1,\beta}, \lambda_{1,\beta})$. By the assumption that S_1 is a unique incompressible spanning surface for L_1, we have $\mathscr{IS}(L_1, S_1) = \emptyset$, and hence $\mathscr{E}(P_{1,\beta}, \lambda_{1,\beta}) = \emptyset$ by Proposition 2.1. Thus $\mathscr{E}(P_1, \lambda_1) = \emptyset$, and F is parallel to a surface in $R(\lambda_1)$. Moreover, since S and S' are connected, so are $R_+(\lambda_1)$ and $R_-(\lambda_1)$. Hence F is parallel to $R_+(\lambda_1)$ or $R_-(\lambda_1)$. From this together with the assumption that $[F] \neq [S]$, we have $[F] = [S']$.

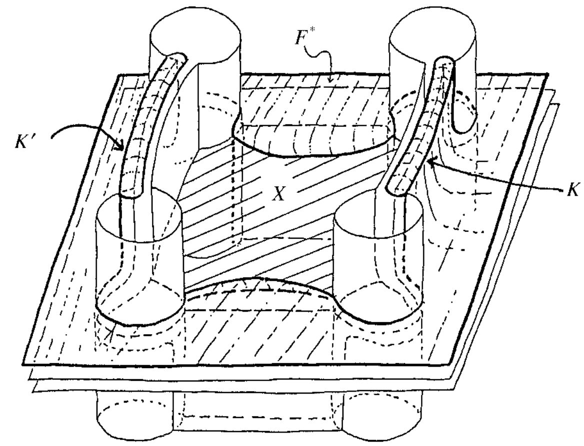


Fig. 3.8. (M^*, γ^*)

Case 2: $e_1(F_2) = R_+(\gamma_2')$. Let $W = \partial V_2 \times [0,1]$ be a thin collar of ∂V_2 in V_2 with $\partial V_2 = \partial V_2 \times \{0\}$, and put $W_1 = V_1 \cup W$ and $W_2 = \mathrm{Cl}(V_2 - W)$. Then we have $S^3 = W_1 \cup W_2$ and $W_1 \cap W_2 = \partial W_1 = \partial W_2 = \partial V_2 \times \{1\}$. In this case, moving F by a γ-isotopy of M keeping M_1 fixed whose restriction to F_2 is $e \,|\, (F_2, \partial F_2) \times [0, 1-r]$ for a small positive number r, we may assume that $D^* = F \cap W$ is a parallel copy of D such that $D^* \cap \partial V_2 = F \cap X$ and $D^* \cap \partial W_1$ $(= F \cap \partial W_1)$ is a parallel copy of $R_+(\gamma) \cap \partial W_1$, and hence $D^* \cap \partial W_1$ is a union of two arcs, say K and K'. Let (M^*, γ^*) denote the sutured manifold $(M \cup W_2, \mathrm{Cl}(\partial W_1 - M) \cup (\gamma \cap W_1))$. Then (M^*, γ^*) is homeomorphic to (M_1, γ_1). Let F^* denote the γ^*-surface in M^* which is obtained by adding two thin rectangles to $F \cap W_1$ along K and K' respectively (see Figure 3.8). Since F is incompressible in M, so is F^* in M^*. Hence by the assumption that S_1 is a unique incompressible spanning surface for L_1, F^* is parallel to a surface in $R(\gamma^*)$. From this together with the connectedness of S_1, there is an embedding $e^* : (F^*, \partial F^*) \times [0,1] \to (M^*, \gamma^*)$ so that $e_0^* = \mathrm{id}$ and $e_1^*(F^*) = R_+(\gamma^*)$ or $R_-(\gamma^*)$.

Subcase 2.1: $e_1^*(F^*) = R_+(\gamma^*)$. In this case the embeddings $e : F_2 \times [0,1] \to M_2'$ and $e^* : F^* \times [0,1] \to M^*$ can be taken so that $e((F \cap W_2) \times [0,1]) \subset M \cap W_2$, $e^*((F \cap W_1) \times [0,1]) \subset M \cap W_1$ and $e \,|\, (F \cap \partial W_1) \times [0,1] = e^* \,|\, (F \cap \partial W_1) \times [0,1]$. By connecting these two embeddings, we get an embedding $\bar{e} : (F, \partial F) \times [0,1] \to (M, \gamma)$ so that $\bar{e}_0 = \mathrm{id}$ and $\bar{e}_1(F) = R_+(\gamma)$. This contradicts the assumption that $[F] \in \mathscr{IS}(L, S)$.

Subcase 2.2: $e_1^*(F^*) = R_-(\gamma^*)$. Recall that $F \cap X$ is a union of two arcs where $X = M \cap \partial V_2$, and let A denote one of them. The arc A cuts off a rectangle Δ_+ from X such that Δ_+ is disjoint from another component of $F \cap X$, $A_+ = \Delta_+ \cap R(\gamma)$ is the opposite edge of A and $A_+ \subset R_+(\gamma)$. We can regard Δ_+ as an disk in M^* and $A_+ \subset R_+(\gamma^*)$ as a mark on (M^*, γ^*). By the definition of (M^*, γ^*), there is a homeomorphism $h : (M^*, \gamma^*) \to (M_1, \gamma_1)$ so that $h(A_+) = A_1$. We will construct a product disk Δ in M^* with A_+ as an edge. Note that A is a properly embedded arc in F^*. For the embedding $e^* : F^* \times [0,1] \to M^*$, $\Delta_- = e^*(A \times [0,1])$ is a rectangle such that $\Delta_- \cap \Delta_+ = A$, $A_- = e^*(A \times \{1\})$ is the opposite edge of A and $\Delta_- \cap R(\gamma^*) = A_- \subset R_-(\gamma^*)$. Thus $\Delta = \Delta_- \cup \Delta_+$ is a product disk in M^* with A_+ as an edge. This contradicts the assumption on the marked sutured manifold (M_1, γ_1, A_1).

The proof of Proposition 3.4 is completed. □

Now we will consider the case that one of the marked sutured manifolds (M_i, γ_i, A_i) $(i = 1, 2)$ has a product disk with the mark as an edge. Suppose that (M_1, γ_1, A_1) has a product disk Δ with A_1 as an edge, and let B_1 denote an opposite mark of A_1 relative to Δ. Let $\iota : S_1 \times [-1, 1] \to S^3$ be a bicollar map with $N_1 = \text{Image}\, \iota \cap E(L_1)$. We may assume that $V_2 = \iota(D \times [0,1])$. We take a regular neighborhood $N(\Delta)$ of Δ in M_1 so that $N(\Delta) \cap R(\gamma_1)$ is a union of two rectangles D_+ and E_- which have A_1 and B_1 as cores respectively. We regard A_1 and B_1 as marks on (N_1, δ_1), and D_+, E_- as disks in $R(\delta_1)$ where (N_1, δ_1) is the product sutured manifold associated with S_1. Then D_+ corresponds to the plumbing disk $D = S_1 \cap S_2$ by projecting D_+ onto S_1. Let E denote the disk on S_1 obtained by projecting E_- onto S_1. Let $D \times [0,1] \subset N_1$ and $E \times [-1, 0] \subset N_1$ be embeddings with $D = D \times \{0\}$, $D_+ = D \times \{1\}$, $E = E \times \{0\}$ and $E_- = E \times \{-1\}$ respectively. Consider the set $N'(\Delta) = N(\Delta) \cup D \times [0,1] \cup E \times [-1, 0]$. Then $\text{Int}\, N'(\Delta)$ is still an open 3-ball. Since the 3-ball V_2 is contained in $N'(\Delta)$, we have $S_2 \cap \partial N'(\Delta) = D$. Hence there is a 3-ball $Q_2 \subset N'(\Delta)$ so that $Q_2 \cap \partial N'(\Delta) = E$ and $T_2 = (S_2 - D) \cup E \subset Q_2$. Thus two 3-balls Q_2 and $Q_1 = S^3 - \text{Int}\, Q_2$ satisfy the following properties:

$$(3.7) \quad Q_1 \cup Q_2 = S^3, \qquad Q_1 \cap Q_2 = \partial Q_1 = \partial Q_2, \qquad S_1 \subset Q_1,\ T_2 \subset Q_2 \qquad \text{and}$$
$$E = S_1 \cap T_2 \qquad \text{is a 4-gon.}$$

This means that S has another decomposition which consists of a plumbing of S_1 and T_2 with the plumbing disk E. We set $T = S_1 \cup_E T_2$ for convenience, and T' denotes its dual relative to E; hence $[T] = [S] \in \mathscr{I}S(L)$. We also note that the link ∂T_2 is equivalent to L_2, and that T_2 is equivalent to S_2 as spanning surfaces for L_2. Thus the conditions (3.1) and (3.2) hold if we replace S and S_2 by T and T_2 respectively.

Proposition 3.8. *Let L be a non-split link and S its spanning surface. Suppose that* (3.1) *and* (3.2) *hold, and let (M_i, γ_i, A_i) $(i = 1, 2)$ be the marked* *sutured manifolds associated with the plumbing* $S = S_1 \cup_D S_2$. *Suppose further that* (M_1, γ_1, A_1) *has a product disk* Δ *with* A_1 *as an edge, and let* $T = S_1 \cup_E T_2$ *denote the plumbing corresponding to an opposite mark* B_1. *Let* S' *and* T' *denote duals of* S *and* T *respectively. Then* S' *can not be disjoint from* T' *by any ambient isotopy of* $E(L)$. *In particular,* S' *is not equivalent to* T'.

Proof. Let (M, γ), (M_1, γ_1) and (M_2, γ_2) be the complementary sutured manifolds for S, S_1 and S_2 respectively. Let V_1 and V_2 be as in the beginning of the proof of Proposition 3.4, and assume (3.5). We further assume the notation $N(\Delta)$, D_+, E_- etc. and the condition (3.7). Note that $D_+ \subset R_+(\gamma)$ and $E_- \subset R_-(\gamma)$ by (3.5). We identify $N(\Delta)$ with $\Delta \times [-1, 1]$ and suppose that $s(\gamma) \cap \partial N(\Delta) = (s(\gamma) \cap \partial\Delta) \times [-1, 1]$. Consider the product decomposition $(M_1, \gamma_1) \xrightarrow{\Delta} (\tilde{M}_1, \tilde{\gamma}_1)$. We may identify $\tilde{M}_1$ with $\mathrm{Cl}(M_1 - N(\Delta))$ and suppose that $s(\tilde{\gamma}_1) \cap (\Delta \times \{\pm 1\}) = a \times \{\pm 1\}$, where $a \subset \Delta$ is a properly embedded arc connecting the two points of $s(\gamma) \cap \partial\Delta$. We also get a sutured manifold $(\tilde{M}_2, \tilde{\gamma}_2)$ such that $\tilde{M}_2 = M \cap N'(\Delta)$ and $s(\tilde{\gamma}_2) = (s(\gamma) \cap N(\Delta)) \cup (a \times \{-1, 1\})$. Note that $(\tilde{M}_2, \tilde{\gamma}_2)$ is ambient isotopic to (M_2, γ_2).

Since S' and T' are disjoint from S, we regard S' and T' as γ-surfaces in M. Now suppose that S' can be disjoint from T' by an isotopy of $E(L)$. We can take this isotopy to be keeping S fixed by Lemma 1.7; hence as a γ-surface S' can be disjoint from T' by a γ-isotopy of M. By the construction of S' and T', we can assume that

$$\partial S' = \partial T' = s(\gamma),$$

$$S' \cap (\Delta \times \{\pm 1\}) = T' \cap (\Delta \times \{\pm 1\}) = a \times \{\pm 1\},$$

$$S' \cap T' = s(\gamma) \cup (a \times \{-1, 1\})$$

(Figure 3.9).

Set $\tilde{S}_i = S' \cap \tilde{M}_i$ and $\tilde{T}_i = T' \cap \tilde{M}_i$ for $i = 1$ and 2. Then $\tilde{S}_i$ and $\tilde{T}_i$ become $\tilde{\gamma}_i$-surfaces in $\tilde{M}_i$. Moreover we claim that

(3.9) (1) $\tilde{S}_1$ and $\tilde{T}_1$ are parallel to $R_+(\tilde{\gamma}_1)$ and $R_-(\tilde{\gamma}_1)$ in $\tilde{M}_1$ respectively, and

(2) $\tilde{S}_2$ and $\tilde{T}_2$ are parallel to $R_-(\tilde{\gamma}_2)$ and $R_+(\tilde{\gamma}_2)$ in $\tilde{M}_2$ respectively.

We now consider two cases.

Case 1. $\tilde{M}_1$ is connected. By the assumption that S' can be disjoint from T' by a γ-isotopy, using Waldhausen [16, Prop. 5.4], we see that $\tilde{S}_i$ is parallel to $\tilde{T}_i$ in $\tilde{M}_i$ for $i = 1$ or 2. On the other hand, since (M_1, γ_1) is not a product sutured manifold by (3.2), neither is $(\tilde{M}_1, \tilde{\gamma}_1)$. It follows from this together with (3.9)(1) that $\tilde{S}_1$ can not be parallel to $\tilde{T}_1$ in $\tilde{M}_1$. Also $(\tilde{M}_2, \tilde{\gamma}_2)$ is not a

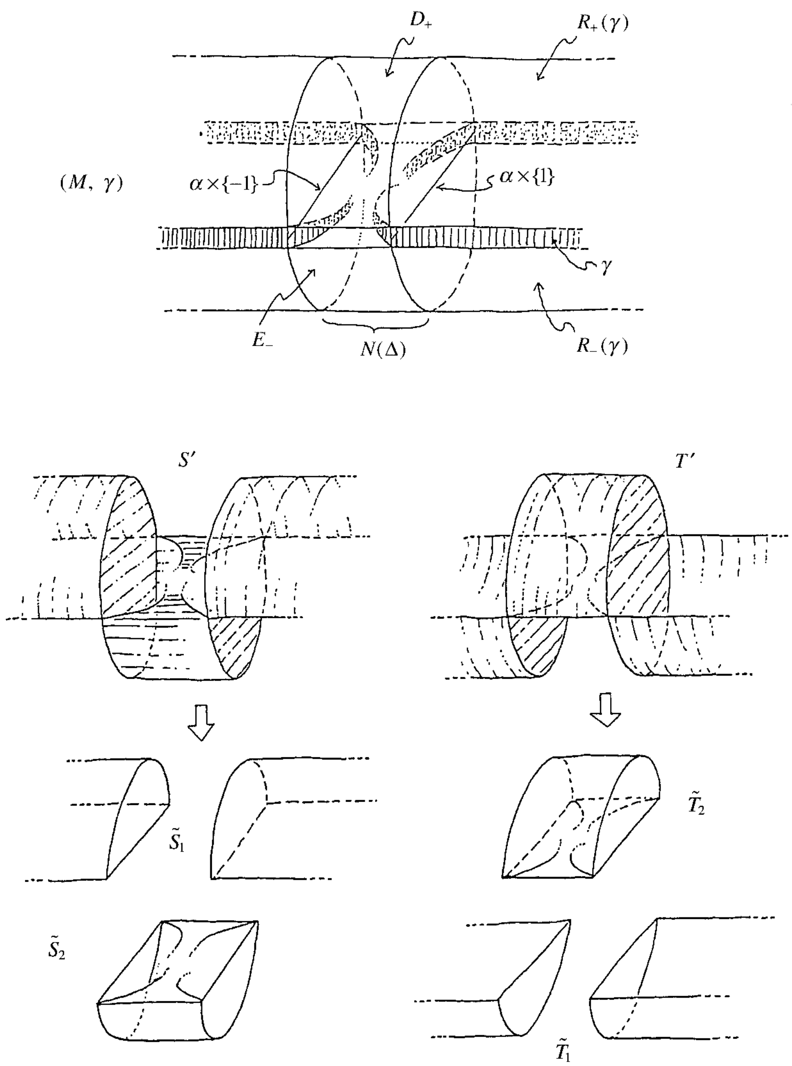


Fig. 3.9

product sutured manifold by (3.2). Hence (3.9)(2) implies that $\tilde{S}_2$ can not be parallel to $\tilde{T}_2$ in $\tilde{M}_2$. This is a contradiction.

Case 2. $\tilde{M}_1$ is disconnected. Let (M_1^-, γ_1^-) and (M_1^+, γ_1^+) be the two components of $(\tilde{M}_1, \tilde{\gamma}_1)$ with $\Delta \times \{\pm 1\} \subset \partial M_1^{\pm}$. Since $(\tilde{M}_1, \tilde{\gamma}_1)$ is not a product sutured manifold, neither is one of $(M_1^{\pm}, \gamma_1^{\pm})$. If both of $(M_1^{\pm}, \gamma_1^{\pm})$ are not product sutured manifolds, then the same argument as in the case 1 implies a contradiction. Hence we suppose that (M_1^-, γ_1^-) is a product sutured manifold. In this case we can move S' by a γ-isotopy to eliminate the intersection arc $a \times \{-1\}$; hence we suppose that $S' \cap T' = s(\gamma) \cup (a \times \{1\})$. Consider the product decomposition $(M, \gamma) \xrightarrow{\Delta \times \{1\}} (M^-, \gamma^-) \cup (M_1^+, \gamma_1^+)$; the resulting manifold has two components, and set $S^- = S' \cap M^-$ and $T^- = T' \cap M^-$. Then S^- and T^- become γ^--surfaces, and they are parallel to $R_-(\gamma^-)$ and $R_+(\gamma^-)$ in M^- respectively. Also by considering the product decomposition $(M^-, \gamma^-) \xrightarrow{\Delta \times \{-1\}} (\tilde{M}_2, \tilde{\gamma}_2) \cup (M_1^-, \gamma_1^-)$ and by the assumption that $(\tilde{M}_2, \tilde{\gamma}_2)$ is not a product sutured manifold, we see that (M^-, γ^-) is not a product sutured manifold. It follows that S^- is not parallel to T^- in M^-. Hence by the same argument as in the case 1, we get a contradiction.

Thus S' can not be disjoint from T' by any ambient isotopy of $E(L)$, and Proposition 3.8 is proved. $\square$

From Propositions 3.4 and 3.8 we have

Corollary 3.10. *Let L be a non-split link and S a spanning surface for L. Suppose that* (3.1) *and* (3.2) *hold. Let (M_i, γ_i, A_i) $(i = 1, 2)$ be the marked complementary sutured manifolds associated with the plumbing $S = S_1 \cup S_2$. Then $\mathscr{IS}(L, S) = \{[S']\}$ (where S' is a dual of S), if and only if there is no product disk in M_i with A_i as an edge for $i = 1$ and* 2.

Now let (M_i, γ_i, A_i) (resp. (N_i, δ_i, A_i)) $(i = 1, 2)$ be the marked complementary sutured manifolds (resp. the marked product sutured manifolds) associated with the plumbing $S = S_1 \cup_D S_2$. Let $S' = S_1' \cup_{D'} S_2'$ be a dual of S. We consider the marked sutured manifolds associated with this plumbing. Let Γ_i be a product disk in N_i with A_i as an edge and A_i' an opposite mark on (N_i, δ_i) relative to Γ_i $(i = 1, 2)$. It is easy to see that

(3.11) (a) the marked product sutured manifolds associated with $S' = S_1' \cup_{D'} S_2'$ are homeomorphic to (N_i, δ_i, A_i') $(i = 1, 2)$, and
(b) the marked complementary sutured manifolds associated with $S' = S_1' \cup_{D'} S_2'$ are homeomorphic to (M_i, γ_i, A_i') $(i = 1, 2)$.

Thus Corollaries 0.3 and 3.10 imply the following

Theorem 3.12. *Let L be a non-split link and S its spanning surface. Suppose that* (3.1) *and* (3.2) *hold. Let (M_i, γ_i, A_i) and (M_i, γ_i, A_i') $(i = 1, 2)$ the marked complementary sutured manifolds associated with S and its dual S' respectively. Then $\mathscr{IS}(L) = \{[S], [S']\}$ if and only if the following conditions* $(*1)$–$(*2)$ *hold:*

$(*1)$ *There is no product disk in M_1 with A_1 or A_1' as an edge.*
$(*2)$ *There is no product disk in M_2 with A_2 or A_2' as an edge.*

Now we return to the case that (M_1, γ_1, A_1) has a product disk with A_1 as an edge.

Proposition 3.13. *Under the same assumptions as in Proposition* 3.8, *we have* $\mathscr{IS}(L, S) = \{[S'], [T']\}$.

Proof. Let F be an incompressible spanning surface for L such that $[F] \in \mathscr{IS}(L, S)$. We use the same notation (M, γ), (M_i, γ_i) etc. in the proof of Proposition 3.4 and assume (3.5). We may assume that F intersects the disk $X = M \cap \partial V_1$ transversely in two arcs and that the case (1) in Figure 3.6 holds. The arguments in the proof of Proposition 3.4 remain valid in the subcase 2.1. Thus we will start with the same situation as in the subcase 2.2: For the embeddings $e : (F_2, \partial F_2) \times [0, 1] \to (M_2', \gamma_2')$ and $e^* : (F^*, \partial F^*) \times [0, 1] \to (M^*, \gamma^*)$, we assume that $e_1(F_2) = R_+(\gamma_2')$ and $e_1^*(F^*) = R_-(\gamma^*)$.

Note that $D^* = F \cap W = F_2 \cap F^*$ is a disk, and consider two 3-balls $Z_+ = e(D^* \times [0, 1])$ and $Z_- = e^*(D^* \times [0, 1])$. By the constructions of e and e^*, we may assume that $Z_+ \cap Z_- = D^*$ and $Z_+ \subset W_1$. Put $Z = Z_+ \cup Z_-$. Then $D_+ = e(D^* \times \{1\}) \subset R_+(\gamma^*)$ and $E_- = e^*(D^* \times \{1\}) \subset R_-(\gamma^*)$ are rectangles such that (Z, E_-, D_+) is homeomorphic to $D^* \times ([-1, 1], -1, 1)$. A core arc A_+ of $D_+ \subset R_+(\gamma^*)$ is a mark on (M^*, γ^*) and (M^*, γ^*, A_+) is homeomorphic to (M_1, γ_1, A_1). Hence a core arc A_- of $E_- \subset R_-(\gamma^*)$ is a dual mark of A_+, and (M^*, γ^*, A_-) is homeomorphic to (M_1, γ_1, B_1) by Lemma 3.3. Thus we identify (M^*, γ^*, A_+) and (M^*, γ^*, A_-) with (M_1, γ_1, A_1) and (M_1, γ_1, B_1) respectively, and (Z, E_-, D_+) with $(N(\Delta), E_-, D_+)$ (which is indicated just before Proposition 3.8). By the assumption that $e_1(F_2) = R_+(\gamma_2')$, F_2 is regarded as a parallel copy of $R_+(\gamma_2')$ in (M_2', γ_2') (Figure 3.10). Take a parallel copy $\tilde{F}_2$ of $R_-(\gamma_2')$ in (M_2', γ_2') which is a γ_2'-surface. Then replacing F_2 with $\tilde{F}_2$, we get a γ-surface $\tilde{F}$ from F; $\tilde{F} = (F - F_2) \cup \tilde{F}_2$. Since $e_1^*(F^*) = R_-(\gamma^*)$, $\tilde{F}$ is a parallel copy of $R_-(\gamma)$, and then we can regard $\tilde{F}$ as $T = S_1 \cup_E T_2$ (see Figure 3.10). Hence F is a dual of T relative to E, and $[F] = [T']$. Proposition 3.13 is proved. □

We proceed with our study under the assumptions in Proposition 3.8. As noted previously the link ∂T_2 is equivalent to L_2, and T_2 is equivalent to S_2 as spanning surfaces for L_2. Hence the complementary sutured manifold for T_2 is homeomorphic to (M_2, γ_2). Let (M_i, γ_i, A_i) $(i = 1, 2)$ be the marked complementary sutured manifolds associated with the plumbing $S = S_1 \cup_D S_2$, and let (M_i, γ_i, B_i) $(i = 1, 2)$ be those ones associated with the plumbing $T = T_1 \cup_E T_2$ $(T_1 = S_1, T = S)$. By the definition of T_2 and the plumbing $T = T_1 \cup_E T_2$ (cf. (3.7)), we see that (M_2, γ_2, B_2) is homeomorphic to (M_2, γ_2, A_2'). Let $S' = S_1' \cup_{D'} S_2'$ and $T' = T_1' \cup_{E'} T_2'$ denote duals of S and T respectively. Then

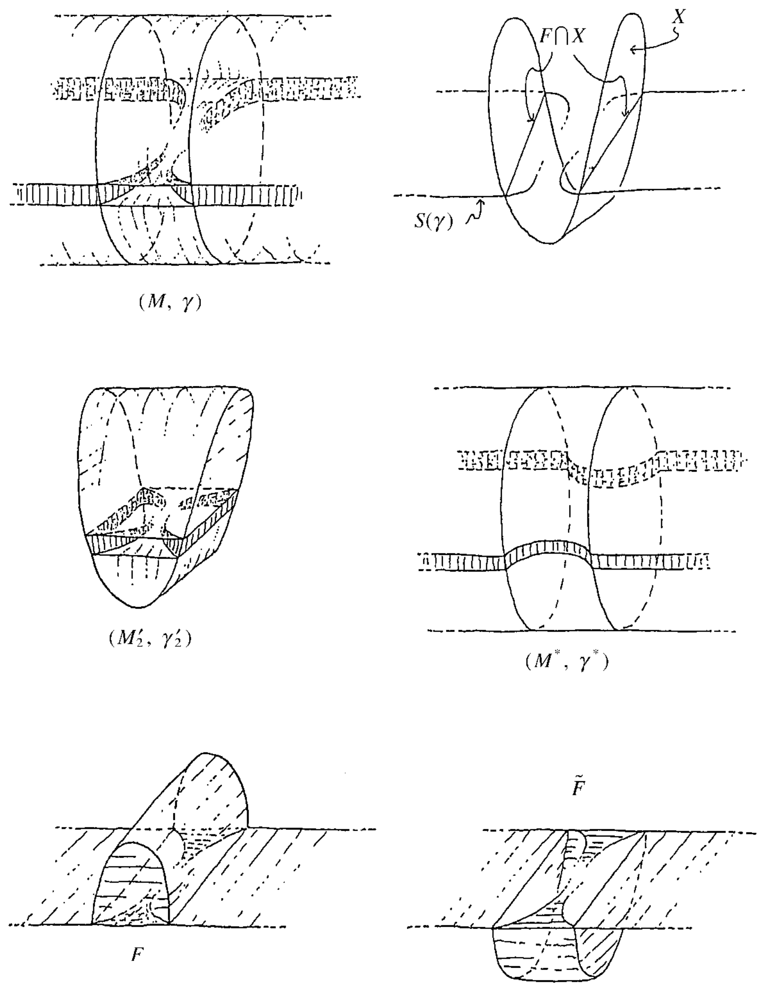


Fig. 3.10

(3.14) (i) the marked complementary sutured manifolds associated with $S' = S_1' \cup_{D'} S_2'$ are homeomorphic to (M_i, γ_i, A_i') $(i = 1, 2)$, and
(ii) the marked complementary sutured manifolds associated with $T' = T_1' \cup_{E'} T_2'$ are homeomorphic to (M_1, γ_1, B_1') and (M_2, γ_2, A_2),

where (M_i, γ_i, A_i') $(i = 1, 2)$ are the marked sutured manifolds given in (3.9)(b), and (M_1, γ_1, B_1') is obtained by the same way from (M_1, γ_1, B_1). Thus Corollary 3.10 and Proposition 3.13 together with Theorem 0.1 imply the following

Theorem 3.15. *Let L be a non-split link and S its spanning surface. Suppose that* (3.1) *and* (3.2) *hold, and let* (M_i, γ_i, A_i) $(i = 1, 2)$ *be the marked* *sutured manifolds associated with the plumbing* $S = S_1 \cup_D S_2$. *Suppose further that* (M_1, γ_1, A_1) *has a product disk* $\varDelta$ *with* A_1 *as an edge, and let* $T = S_1 \cup_E T_2$ *denote the plumbing corresponding to an opposite mark* B_1. *Let* S' *and* T' *denote duals of* S *and* T *respectively. Then*

$$\mathscr{I}S(L) = \{[S], [S'], [T']\}$$

and

$$IS(L) = \overset{[S']}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{[S]}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{[T']}{\bullet},$$

if and only if the following conditions $(**1)$–$(**2)$ *hold:*
$(**1)$ *There is no product disk in* M_1 *with* A_1' *or* B_1' *as an edge.*
$(**2)$ *There is no product disk in* M_2 *with* A_2 *or* A_2' *as an edge.*

4. Decompositions of a marked sutured manifold

In this section we deal with a method deciding whether a given marked sutured manifold has a product disk with the mark as an edge.

Let (M, γ, A) be a marked sutured manifold, i.e. A is a properly embedded arc in $R(\gamma)$. Suppose that there is a product disk $\varDelta \subset M$ with $\varDelta \cap A = \emptyset$, and let $(M, \gamma) \overset{\varDelta}{\to} (M', \gamma')$ be the corresponding product decomposition. Then we can regard A as a mark on (M', γ'), and hence we have a *product decomposition* of a marked sutured manifold:

$$(M, \gamma, A) \overset{\varDelta}{\to} (M', \gamma', A).$$

We now assume that there is a product disk $\Gamma \subset M$ with A as an edge in addition. In the case that $\Gamma \cap \varDelta = \emptyset$, Γ is also a product disk in M' with A as an edge. If $\Gamma \cap \varDelta \neq \emptyset$, then by using the standard cut and paste argument, we can get another product disk $\Gamma' \subset M'$ with A as an edge. Also the converse is easily verified. Thus we have the following

Lemma 4.1. *Let* $(M, \gamma, A) \overset{\varDelta}{\to} (M', \gamma', A)$ *be a product decomposition of a marked sutured manifold. Then* M *has a product disk with* A *as an edge, if and only if so does* M'.

Next we consider an octagon $\Omega \subset M$ (see §1) satisfying $\Omega \cap A = \emptyset$. We assume that $A \subset R_-(\gamma)$. Then we associate the following three kind of decompositions of (M, γ, A) with Ω as shown in Figure 4.1:

$$\begin{array}{ccc} (M_a, \gamma_a, A) & \xleftarrow{a(\Omega)} (M, \gamma, A) \xrightarrow{c(\Omega)} & (M_c, \gamma_c, A) \\ & \Big\downarrow{\scriptstyle b(\Omega)} & \\ & (M_b, \gamma_b, A). & \end{array}$$

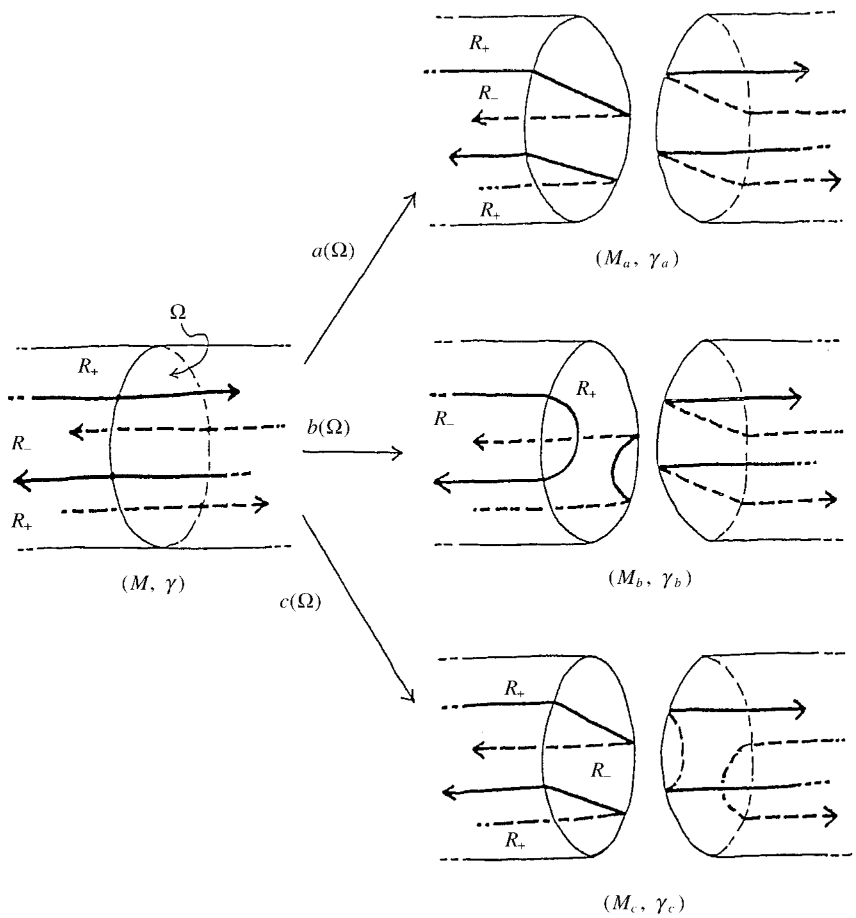


Fig. 4.1

Note that M_a, M_b and M_c are homeomorphic to the manifold obtained by cutting M along Ω. In the case of $A \subset R_+(\gamma)$, by replacing the situations of $R_-(\gamma)$ and $R_+(\gamma)$, we define the three decompositions in the same way.

Proposition 4.2. *Let (M, γ, A) be a marked sutured manifold with $A \subset R_-(\gamma)$ (resp. $A \subset R_+(\gamma)$), and let $\Omega \subset M$ be an octagon with $\Omega \cap A = \emptyset$. Suppose that there is a product disk $\Gamma \subset M$ with A as an edge. Then one of the following assertions* (1)–(3) *holds:*

(1) *There is a product disk $\Gamma_a \subset M_a$ with A as an edge.*
(2) *Some component of $R_+(\gamma_b)$ (resp. $R_-(\gamma_b)$) is compressible in M_b.*
(3) *Some component of $R_+(\gamma_c)$ (resp. $R_-(\gamma_c)$) is compressible in M_c.*

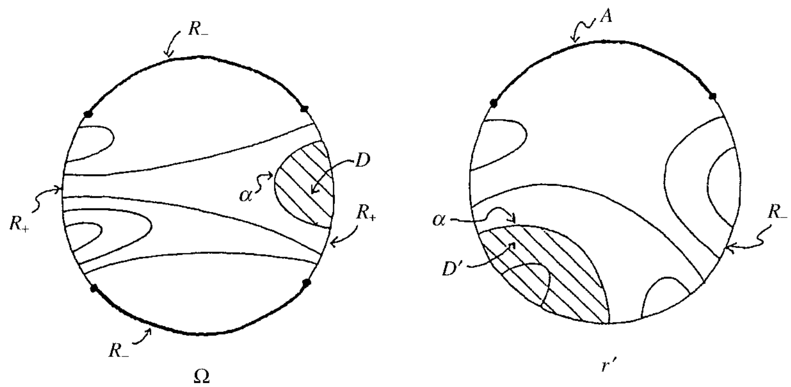


Fig. 4.2

Proof. We prove the proposition in the case of $A \subset R_-(\gamma)$. We take Γ transversely relative to Ω. Then $\Gamma \cap \Omega$ is a union of a finite number of circles and arcs, each arc is properly embedded in Γ and Ω, and $\partial\Gamma \cap \partial\Omega \subset R_+(\gamma)$. By applying the cut and paste operations to Γ in outer most ordering of circles of $\Gamma \cap \Omega$ in Γ, we can replace Γ by a product disk $\Gamma' \subset M$ with A as an edge so that $\Gamma' \cap \Omega$ has no circle component.

Now note that there are the following two types I and II of arc $\alpha \subset \Gamma' \cap \Omega$ (see Figure 4.2):

Type I. $\partial\alpha$ is contained in one component of $\partial\Omega \cap R_+(\gamma)$.

Type II. α connects two components of $\partial\Omega \cap R_+(\gamma)$ in Ω.

We show that type I arcs can be eliminate by replacing Γ'. Let $\alpha \subset \Gamma' \cap \Omega$ be a type I arc. Then α cut off a unique disk D from Ω so that $D \cap R_-(\gamma) = \emptyset$. We assume that $D \cap \Gamma' = \alpha$. Also α cut off a unique disk D' from Γ' so that $D' \cap A = \emptyset$ (Figure 4.2). Consider the disk $(\Gamma' - D') \cup D$ and push out its D part from Ω. Then the resulting disk Γ'' is also a product disk with A as an edge, and the arc α (and some other arcs of $\Gamma' \cap \Omega$) is eliminated from the intersection $\Gamma'' \cap \Omega$. By repeating this process we get a product disk $\Gamma^* \subset M$ with A as an edge so that each component of $\Gamma^* \cap \Omega$ is a type II arc (if $\Gamma^* \cap \Omega \neq \emptyset$).

In the case that $\Gamma^* \cap \Omega = \emptyset$, the assertion (1) holds by setting $\Gamma_a = \Gamma^*$. Now we assume that $\Gamma^* \cap \Omega \neq \emptyset$. Then there is an arc component $\beta \subset \Gamma^* \cap \Omega$ which cut off a unique disk D^* from Γ^* so that $D^* \cap \Omega = \beta$. Since $D^* \cap \Omega = \beta$, we can consider that D^* is contained in the manifold obtained by cutting M along Ω. Hence $\partial D^* \subset R_+(\gamma_b)$ or $\partial D^* \subset R_+(\gamma_c)$. Suppose that $\partial D^* \subset R_+(\gamma_b)$. Then by the assumption that $\beta \subset \partial D^*$ is of type II, ∂D^* cannot bound a disk in $R_+(\gamma_b)$. Hence (2) holds. Similarly if $\partial D^* \subset R_+(\gamma_c)$, then (3) holds. Thus Proposition 4.2 is proved. □

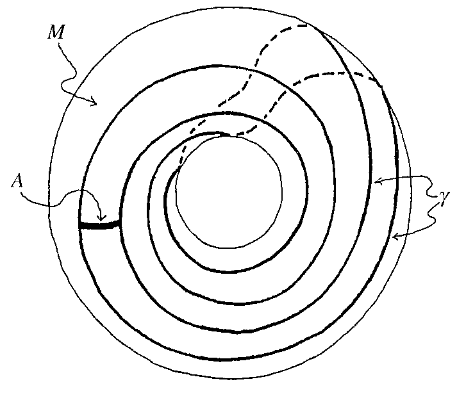


Fig. 4.3

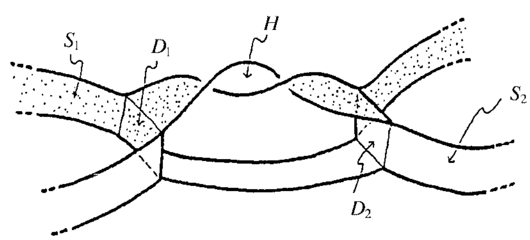


Fig. 5.1

Example 4.3. Let (M, γ, A) be a marked sutured manifold shown in Figure 4.3. Then there is no product disk with A as an edge.

5. Some iterated plumbings

In this section we treat two typical cases of iterated plumbings. As applications of the results in §§3 and 4 we determine the simplicial complexes $IS(L)$ in these cases (Theorems 5.4 and 5.10). Let L be a non-split link and S its spanning surfaces. We first consider the case that

(5.1) S is connected and is an iterated plumbing $S_1 \cup_{D_1} H \cup_{D_2} S_2$ of three surfaces S_1, H and S_2 as shown in Figure 5.1, where H is a Hopf band of $+$ or $-$ type.

We further suppose that

(5.2) S_i is a unique incompressible spanning surface for L_i, and L_i is not fibred for $i = 1$ and 2.

In this case L bounds obvious four surfaces given by performing each plumbing in different ways. Each surface is obtained by replacing one of the plumbing disks D_i with its dual disk D_i' $(i = 1, 2)$, or replacing both D_1 and D_2 with D_1' and D_2' respectively. We denote these surfaces by $S(k_1, k_2)$, where $k_i - 1$ or 0 according to the choice of D_i or D_i' as the plumbing disk $(i = 1, 2)$. For example, $S(1, 1) = S$ and $S(0, 1)$ is a dual of S relative to D_1. Also $S(0, 0) = S_1' \cup_{D_1'} H' \cup_{D_2'} S_2'$ where $S_i' = (S_i - D_i) \cup D_i'$ $(i = 1, 2)$ and $H' = (H - (D_1 \cup D_2)) \cup (D_1' \cup D_2')$. Since H is a Hopf band, as in the case of 2-bridge knots [8], we have

Lemma 5.3. *$S(0, 0)$ is equivalent to $S(1, 1)$.*

Moreover we prove the following

Theorem 5.4. *Let L be a non-split link and S its spanning surface. Suppose that (5.1) and (5.2) hold. Let (M_i, γ_i) be the complementary sutured manifold for S_i $(i = 1, 2)$, and let A_i and A_i' be the markes on (M_i, γ_i) corresponding to the plumbing $S_i \cup_{D_i} H$ and its dual $S_i' \cup_{D_i'} H'$ respectively. Put $\sigma_{jk} = [S(j, k)]$. Then*

$$IS(L) = \overset{\sigma_{01}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\sigma_{11}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\sigma_{10}}{\bullet}$$

if and only if the following conditions $(*1')$–$(*2')$ hold:
$(*1')$ There is no product disk in M_1 with A_1 or A_1' as an edge.
$(*2')$ There is no product disk in M_2 with A_2 or A_2' as an edge.

Proof. We give the proof in the case that H is the Hopf band shown in Figure 5.1. Putting $R = S_1 \cup_{D_1} H$, we have $S = R \cup_{D_2} S_2$. We will prove the theorem by applying Theorem 3.15 to this plumbing. First we check the conditions (3.1) and (3.2). By the assumption (5.2) and Corollary 2.6, R is unique spanning surface for $J = \partial R$, and J is not fibred. Hence (3.1) and (3.2) hold taking R for S_1 and J for L_1.

Let (P, λ, B) be the marked complementary sutured manifold for R corresponding to the plumbing $S = R \cup_{D_2} S_2$. Then there is a product disk Γ in P with B as an edge; C denotes its opposite edge (see Figure 5.2). Let $S = R \cup_E R_2$ denote the plumbing corresponding to the opposite mark C.

We note that $S(1, 0)$ is a dual of $S = R \cup_{D_2} S_2$; write $S(1, 0) = R' \cup_{D_2'} S_2'$ and B' denote the corresponding mark on (P, λ). Let $S'' = R'' \cup_{E'} R_2'$ denote a dual of $S = R \cup_E R_2$ and C' the corresponding mark on (P, λ). If we verify the following conditions (5.5)(1)–(2), then we have

$$IS(L) = \overset{[S(1,0)]}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{[S]}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{[S'']}{\bullet}$$

by Theorem 3.15:

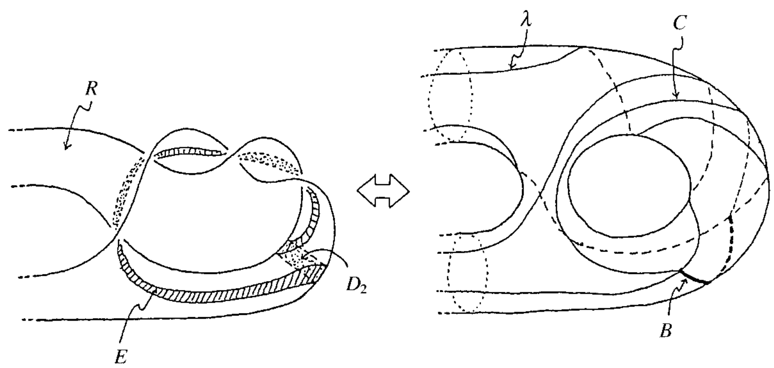


Fig. 5.2

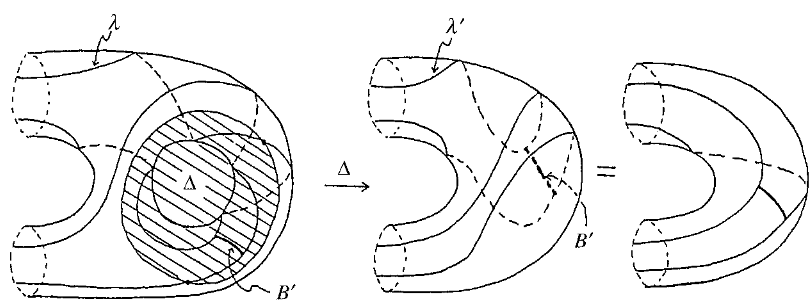


Fig. 5.3

(5.5) (1) There is no product disk in P with B' or C' as an edge.
(2) There is no product disk in M_2 with A_2 or A_2' as an edge.

Note that the condition (5.5)(2) is just the condition $(*2')$. To see (5.5)(1) we consider the product decomposition $(P, \lambda, B') \xrightarrow{\Delta} (P', \lambda', B')$ as shown in Figure 5.3. Then we have $(P', \lambda', B') = (M_1, \gamma_1, A_1)$ as shown. Hence, by the assumption $(*1')$ together with Lemma 4.1, there is no product disk in P with B' as an edge. Similarly, by considering the product decomposition $(P, \lambda, C') \xrightarrow{\nabla} (P'', \lambda'', C')$ as shown in Figure 5.4, we see that there is no product disk in P with C' as an edge. Thus (5.5)(1) follows.

Now to complete the proof of the theorem we must show the following assertion

(5.6) S'' is equivalent to $S(0,1)$.

The plumbing $S = S_1 \cup_{D_1} H \cup_E R_2$ is shown as in Figure 5.5. By moving a tiny isotopy we may assume that $S'' \cap S(0,1) = \varnothing$ as shown in Figure 5.6 (a). We can further assume that

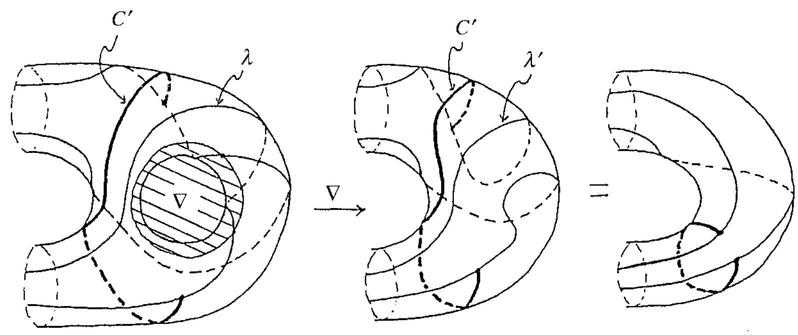


Fig. 5.4

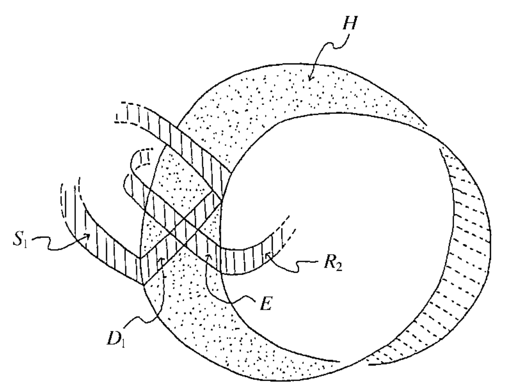


Fig. 5.5

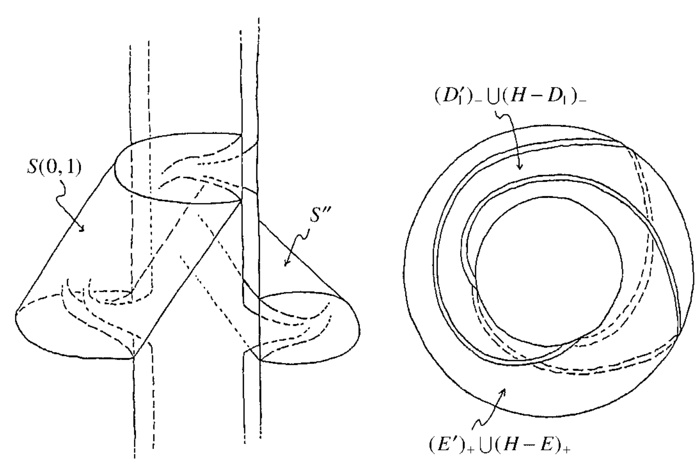


Fig. 5.6

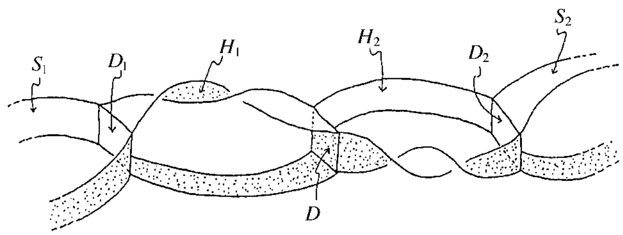


Fig. 5.7

$$S(0,1) = (S_1 - D_1)_- \cup (D_1')_- \cup (H - D_1)_- \cup (R_2 - E)_-,$$

$$S'' = (S_1 - D_1)_+ \cup (E')_+ \cup (H - E)_+ \cup (R_2 - E)_+$$

where $S_1 \times [-1,1]$ is a thin product neighborhood of S_1 and $(S_1 - D_1)_\pm = (S_1 - D_1) \times \{\pm 1\}$, and so on. Hence if we see that there is a product region $A \times [-1,1]$ in $E(L) - (S_1 \cup R_2) \times (-1,1)$ with $A \times \{-1\} = (D_1')_- \cup (H - D_1)_-$ and $A \times \{1\} = (E')_+ \cup (H - E)_+$ where A is an annulus, then S'' and $S(0,1)$ bound a product region in $E(L)$. In fact $(D_1')_- \cup (H - D_1)_-$ and $(E')_+ \cup (H - E)_+$ bound a product region which is shown as the exterior of the solid torus in Figure 5.6 (b). Hence (5.6) follows. Thus Theorem 5.4 is proved. □

Now we consider another type of iterated plumbings. Let L be a non-split link and S its spanning surface which satisfies the condition that

(5.7) S is connected, and is an iterated plumbing $S = S_1 \cup_{D_1} H_1 \cup_D H_2 \cup_{D_2} S_2$ of four surfaces S_1, H_1, H_2 and S_2 as shown in Figure 5.7, where H_i is a Hopf band of $+$ or $-$ type $(i = 1,2)$.

We further assume that

(5.8) S_i is a unique incompressible spanning surface for L_i, and L_i is not fibred for $i = 1,2$.

In this case L bounds eight obvious surfaces which are obtained by performing each plumbing in different ways. Let D_1', D' and D_2' denote dual plumbing disks of D_1, D and D_2 respectively. Each of eight surfaces is denoted by $S(k_1,k,k_2)$ where k_i (resp. k) $= 1$ or 0 according to the choice of D_i or D_i' (resp. D or D') as the plumbing disk. For example, $S(1,1,1) = S$ and $S(1,0,1)$ is a dual of S relative to D. By the assumption that H_1 and H_2 are Hopf bands, we have

LEMMA 5.9. (1) *$S(1,1,1)$, $S(1,0,0)$ and $S(0,0,1)$ are equivalent.*
(2) *$S(0,0,0)$, $S(1,1,0)$ and $S(0,1,1)$ are equivalent.*

Moreover we prove the following

Theorem 5.10. *Let L be a non-split link and S its spanning surface. Suppose that* (5.7) *and* (5.8) *hold. Let (M_i, γ_i) be the complementary sutured manifold for S_i $(i = 1, 2)$, and let A_i and A_i' be the marks on M_i corresponding to the plumbing $S_i \cup_{D_i} H_i$ and its dual. Put $\sigma_{ijk} = [S(i, j, k)]$. Then*

$$IS(L) = \overset{\sigma_{101}}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{\sigma_{111}}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{\sigma_{000}}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{\sigma_{010}}{\bullet}$$

if and only if the following conditions $(*1'')$–$(*2'')$ *hold:*
$(*1'')$ *There is no product disk in M_1 with A_1 or A_1' as an edge.*
$(*2'')$ *There is no product disk in M_2 with A_2 or A_2' as an edge.*

Proof. We will show the theorem only in the case that H_1 and H_2 are the Hopf bands shown in Figure 5.7; the same argument holds in other cases. Put $R_i = S_i \cup_{D_i} H_i$ $(i = 1, 2)$. Then $S = R_1 \cup_D R_2$ and by (5.8) and Corollary 2.6, we see that

(5.11) R_i is a unique incompressible spanning surface for $J_i = \partial R_i$, and J_i is not fibred $(i = 1, 2)$.

Let (P_i, λ_i) be the complementary sutured manifold for R_i $(i = 1, 2)$ and B_i the mark on (P_i, λ_i) corresponding to the plumbing $S = S(1, 1, 1) = R_1 \cup_D R_2$. Also let B_i' be the mark on (P_i, λ_i) corresponding to $S(1, 0, 1) = R_1' \cup_{D'} R_2'$ which is a dual of S relative to D. Consider the marked sutured manifold (P_1, λ_1, B_1). As in the case of (P, λ, B) in the proof of Theorem 5.4, (P_1, λ_1) has a product disk with B_1 as an edge (cf. Figure 5.2); C denotes its opposite mark. Let $S = R_1 \cup_E T_2$ denote the plumbing corresponding to the mark C, and let $\tilde{S}$ denote a dual of S relative to the plumbing disk E. By the same argument as in the proof of (5.5), we claim that, for $i = 1$ and 2, there is no product disk in P_i with B_i' as an edge if and only if there is no product disk in M_i with A_i as an edge. By using this together with Corollary 3.10 and Propositions 3.8 and 3.13, we can show the following assertions:

(5.12) (a) $S(1, 0, 1)$, $S = S(1, 1, 1)$ and $\tilde{S}$ are mutually non-equivalent.
(b) $\mathscr{IS}(L, S) = \{[S(1, 0, 1)], [\tilde{S}]\}$.
(c) $\mathscr{IS}(L, S(1, 0, 1)) = \{[S]\}$ if and only if there is no product disk in M_i with A_i as an edge for $i = 1$ and 2.
(d) $[\tilde{S}] = [S(0, 0, 0)]$.

Now let Q_i denote the dual of $R_i = S_i \cup_{D_i} H_i$ relative to D_i $(i - 1, 2)$. Then $S(0, 1, 0) = Q_1 \cup_D Q_2$, $S(0, 0, 0)$ is its dual relative to D, and we set $S(0, 0, 0) = Q_1' \cup_{D'} Q_2'$. Note that Q_i is equivalent to R_i as a spanning surface for J_i, since H_i is a Hopf band. From this together with (5.11) we have

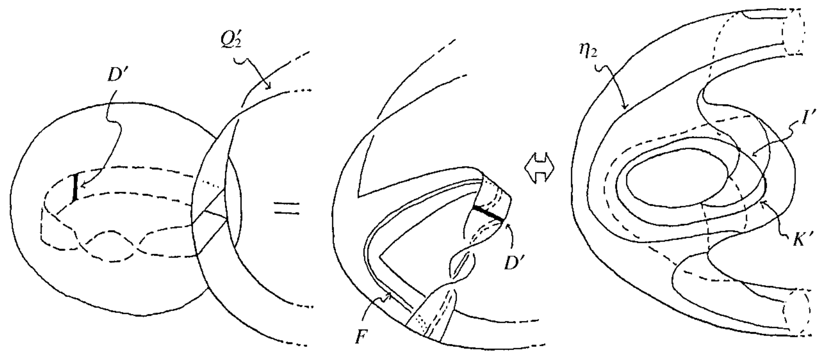


Fig. 5.8

(5.13) Q_i is a unique incompressible spanning surface for $J_i = \partial Q_i$, and J_i is not fibred $(i = 1, 2)$.

Let (U_i, η_i) be the complementary sutured manifold for Q_i $(i = 1, 2)$ and I_i the mark on (U_i, η_i) corresponding to the plumbing $(0, 1, 0) = Q_1 \cup_D Q_2$. Consider the mark I' on (U_2, η_2) corresponding to the plumbing $S(0, 0, 0) = Q_1' \cup_{D'} Q_2'$ (see Figure 5.8). Then (U_2, η_2) has a product disk with I' as an edge, and K' denotes its opposite mark. Let $S(0, 0, 0) = T_1' \cup_F Q_2'$ denote the plumbing corresponding to the mark K', and let $\tilde{S}(0, 0, 0)$ denote a dual of $S(0, 0, 0)$ relative to F. We will show the following assertions:

(5.14) (a) $S(0, 1, 0)$, $S(0, 0, 0)$ and $\tilde{S}(0, 0, 0)$ are mutually non-equivalent.
(b) $\mathscr{I}S(L, S(0, 0, 0)) = \{[S(0, 1, 0)], [\tilde{S}(0, 0, 0)]\}$.
(c) $\mathscr{I}S(L, S(0, 1, 0)) = \{[S(0, 0, 0)]\}$ if and only if there is no product disk in M_i with A_i' as an edge for $i = 1$ and 2.
(d) $[\tilde{S}(0, 0, 0)] = [S]$.

The assertions (a) and (b) follow from (5.13) and Proposition 3.8 directly. To show (c) we first note that $\mathscr{I}S(L, S(0, 1, 0)) = \{[S(0, 0, 0)]\}$ is equivalent to the following condition (c.i) for $i = 1$ and 2 by Corollary 3.10:

(c.i) There is no product disk in (U_i, η_i) with I_i as an edge.

We consider the product decomposition $(U_1, \eta_1, I_1) \xrightarrow{\Delta} (U_1', \eta_1', I_1)$ as shown in Figure 5.9. Then we see that $(U_1', \eta_1', I_1) = (M_1, \gamma_1, A_1')$. Hence (c.1) is equivalent to the condition that there is no product disk in M_1 with A_1' as an edge by Lemma 4.1. Similar argument holds for $i = 2$. Thus (5.14)(c) is proved. To show (d) we note that $Q_2' = H_2' \cup_{D_2'} S_2'$ where $H_2' = (H_2 - (D \cup D_2)) \cup (D' \cup D_2')$ and $S_2' = (S_2 - D_2) \cup D_2'$. Then $S(0, 0, 0) = T_1' \cup_F Q_2' = T_1' \cup_F H_2' \cup_{D_2'} S_2'$. This plumbing is shown as in Figure 5.10.

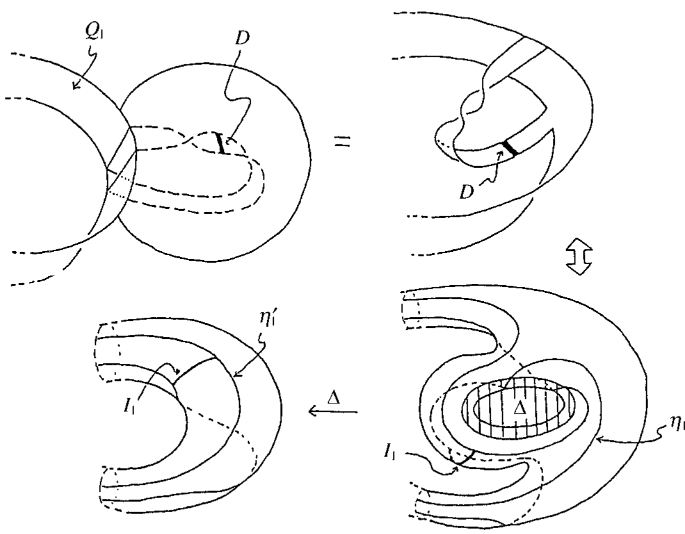


Fig. 5.9

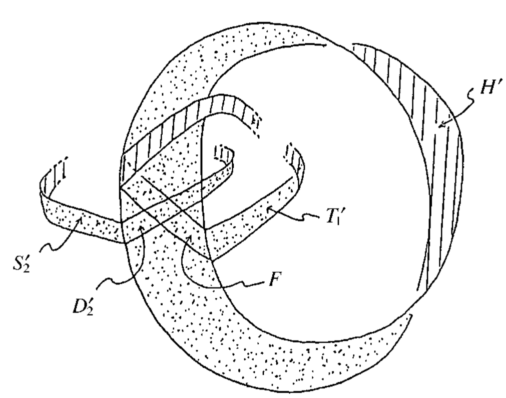


Fig. 5.10

Hence by the same method as in the proof of (5.6), we can verify that $\tilde{S}(0,0,0)$ is equivalent to S.

Now if the complex $IS(L)$ is in the form indicated in the statement of the theorem, then (5.12)(c) and (5.14)(c), the conditions $(*1'')$–$(*2'')$ hold. Conversely we assume the conditions $(*1'')$–$(*2'')$. Then, we note that the four vertices σ_{101}, σ_{111}, σ_{000} and σ_{010} are mutually distinct. In fact it suffices to verify that $\sigma_{101} \neq \sigma_{010}$. This follows from the fact that there is an edge connecting σ_{101} to σ_{111}, however there is no edge connecting σ_{010} to σ_{111}.

Furthermore (5.12) and (5.14) imply that $IS(L)$ is in the desired form. The proof of Theorem 5.10 is now completed. □

6. Proof of Theorems A and C

We are now in a position of proving Theorems A and C stated in the introduction.

Proof of Theorem A (I). Let K be a prime knot of 10 or less crossings. Then it is easy to find a spanning surface S for K whose genus is equal to one half of the degree of the Alexander polynomial of K; hence S is of minimal genus. Let (M, γ) denote the complementary sutured manifold for S. By Corollary 0.2 and Proposition 2.1, if we show that (M, γ) has no essential γ-surface, then S is a unique incompressible spanning surface for K. In [13] Kobayashi proved that the knots in (B) is the list of prime knots of 10 or less crossings whose minimal genus spanning surfaces are not unique. In the process of proving this assertion he essentially showed the following

Proposition 6.1 (Kobayashi). *Under the above assumption, if K is out of the list* (B)*, then (M, γ) has no essential γ-surface.*

Thus the assertion (I) follows from this. We note that Kobayashi [13] used the notion "(M, γ) is an *almost product* sutured manifold" instead of "(M, γ) has no essential γ-surface". These two notions are equivalent for the complementary sutured manifold for a *connected* surface in S^3, and this is the case in our situation.

We will give two typical examoles which explain a method of proving Proposition 6.1.

Example 6.2: 8_{15}. This knot is the first non-fibred and non-2-bridge prime knot in the table of Rolfsen [15, Appendix C]. The knot spans a minimal genus spanning surface S shown in Figure 6.1, and the complementary sutured manifold (M, γ) for S is shown in the figure. Note that M is the exterior of the handlebody of genus four. We apply product decompositions to (M, γ) associated with two disks Δ_1 and Δ_2, and then apply octagonal decompositions associated with Ω as shown. Clearly the resulting two sutured manifolds $(M_\alpha, \gamma_\alpha)$ and (M_β, γ_β) have no essential γ_α- or γ_β-surface respectively. Hence (M, γ) has also no essential γ-surface by Propositions 1.4 and 1.6. □

Example 6.3: 9_{49}. This knot is also non-fibred and non-2-bridge. It spans a minimal genus spanning surface S shown in Figure 6.2. Note that S is a plumbing of two surfaces S_1 and S_2, and S_2 is a Hopf band. Since $L_2 = \partial S_2$ is a fibred link with fibre S_2 (see Lemma 6.4 below), we can use Corollary 2.6. Since S is of minimal genus, S_1 is also a minimal genus spanning surface for

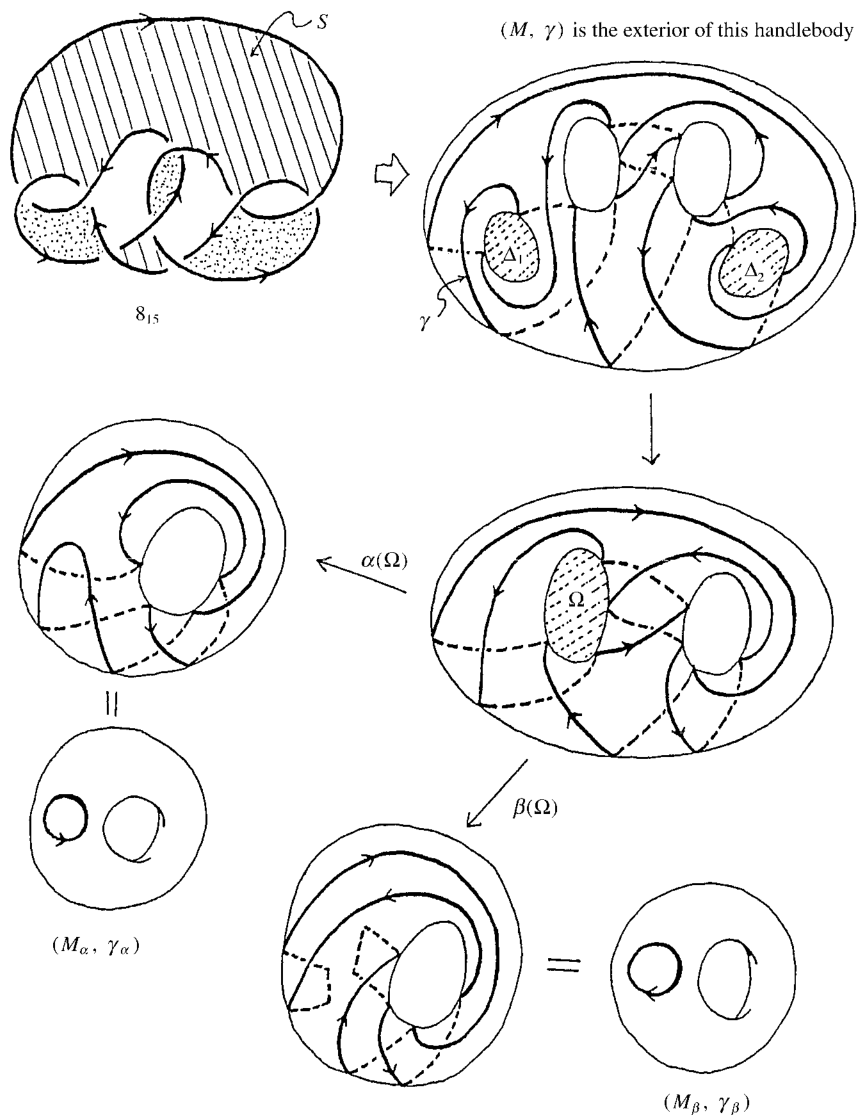


Fig. 6.1. 8_{15}

$L_1 = \partial S_1$ by Proposition 2.3 (i). Hence S is a unique incompressible spanning surface for 9_{49}, if and only if S_1 is a unique incompressible spanning surface for L_1 by Corollary 2.6. Consider the complementary sutured manifold (M_1, γ_1) for S_1. By applying decomposition operations to (M_1, γ_1) as in Example 6.2, we see that (M_1, γ_1) has no essential γ_1-surface. Thus S is a unique incompressible spanning surface for 9_{49}. □

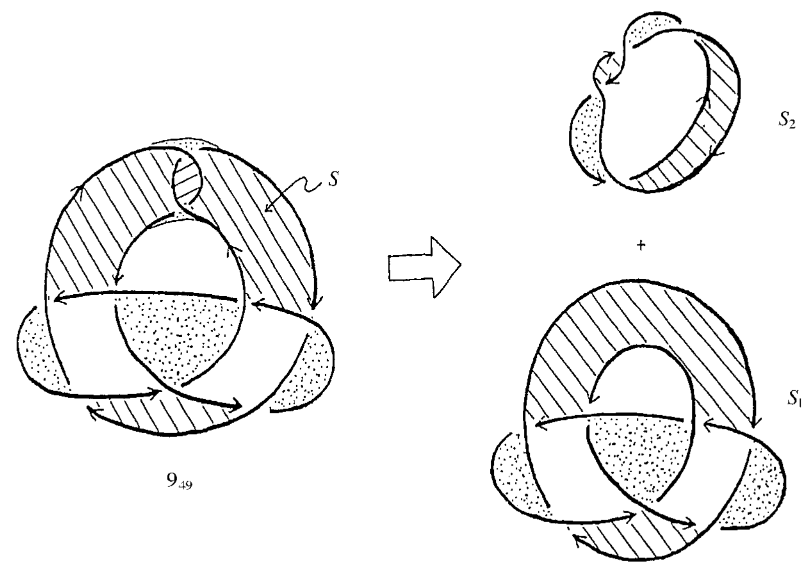


Fig. 6.2. 9_{49}

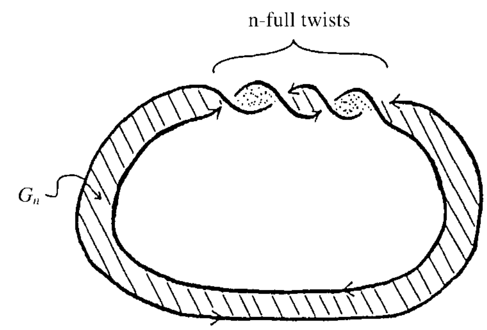


Fig. 6.3

We note the following well known fact which is also useful to the proof of the assertion (I).

Lemma 6.4. *Let G_n be an (n-full twisted) annulus in S^3 shown in Figure* 6.3 $(n \in Z)$. *Then*

(1) *G_n is a unique incompressible spanning surface for $H_n = \partial G_n$ if $n \neq 0$.*

(2) *H_n is a fibred link with fibre G_n if and only if $n = \pm 1$.*

Proof of Theorem C (1), (3). It follows from [1, Appendix C], each knot in the table (B) is a 2-bridge knot except for the last four knots 10_{53}, 10_{67}, 10_{68},

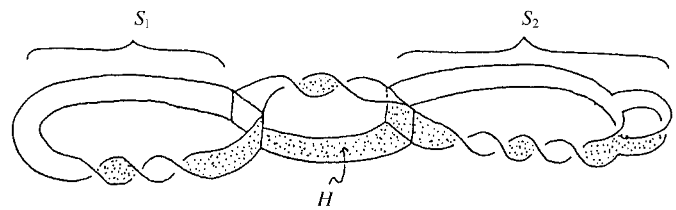


Fig. 6.4

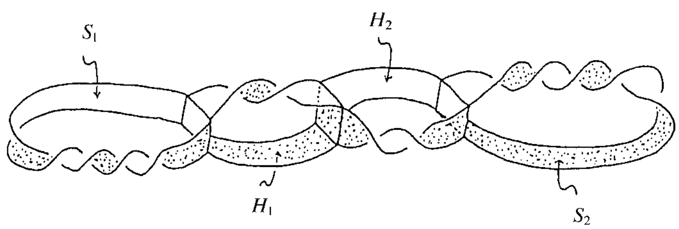


Fig. 6.5

10_{74}. We consider the appropriate continued fraction expansion of the rational number corresponding to each 2-bridge knot. Then the assertion (II) in Theorem A for these 2-bridge knots follows from Hatcher and Thurston [8]. The assertions (1) and (3) in Theorem C are proved by applying Theorems 5.4 and 5.10 to the knots respectively. In fact, for $K = 9_{18}, 10_{18}, 10_{24}, 10_{31}$, we see that K bounds a surface S which is an iterated plumbing of three surfaces S_1, H and S_2 in the form of Figure 5.1. Moreover it satisfies the condition (5.2). In this case K bounds surfaces $S(i, j)$ defined in the beginning of § 5. For example, Figure 6.4 indicates the case of $K = 9_{18}$. Hence the assertion (1) follws from Theorem 5.4. Similarly, for $K = 9_{10}, 10_{16}, 10_{33}$, K bounds a surface S which is an iterated plumbing of four surfaces S_1, H_1, H_2 and S_2 in the form of Figure 5.7. Moreover it satisfies the condition (5.8), and K bounds surfaces $S(i, j, k)$ which are defined just before Lemma 5.9. Figure 6.5 indicates the case of $K = 9_{10}$. Hence the assertion (3) follows from Lemma 5.9 and Theorem 5.10. □

Proof of Theorem A (II) for non-2-bridge knots 10_{53}, 10_{67} and 19_{68}. The proof will be given by using Theorem 3.12.

10_{53}. This knot spans a minimal genus spanning surface S shown in Figure 6.6, and S is a plumbing of two surfaces S_1 and S_2 in Figure 6.7. We will show that

$$\mathscr{I}S(10_{53}) = \{[S], [S']\} \qquad \text{(Figure 6.6)}$$

where S' is a dual of S.

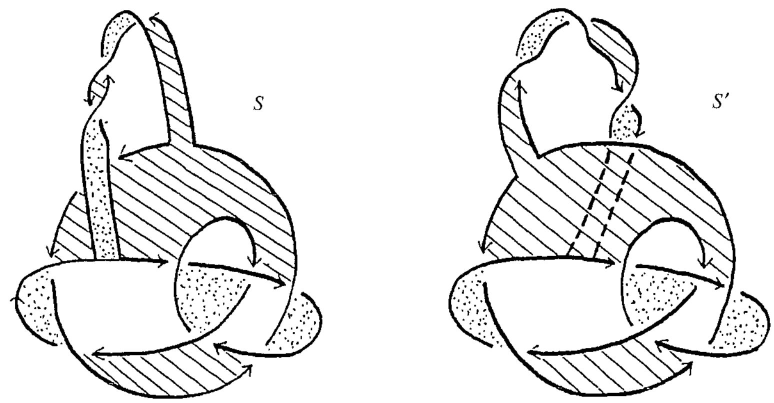


Fig. 6.6. 10_{53}

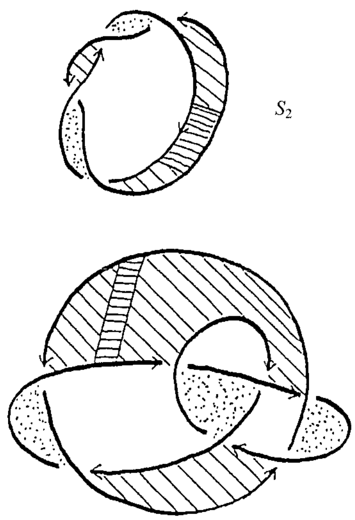
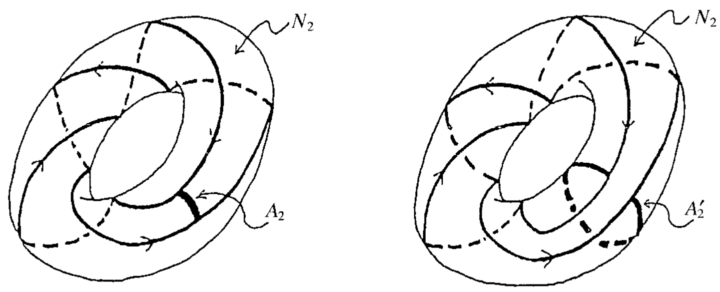


Fig. 6.7

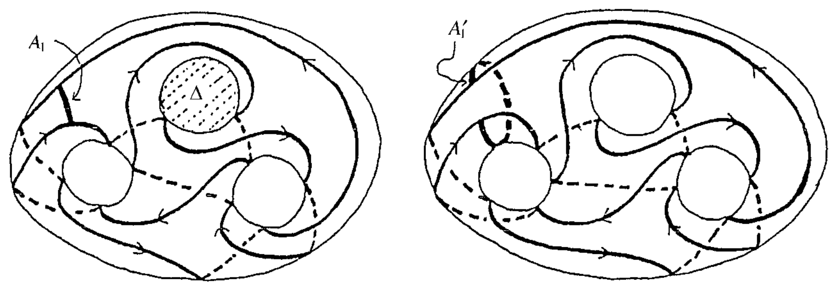
Firstly we show that $S = S_1 \cup S_2$ satisfies the conditions (3.1) and (3.2). Since S is of minimal genus, so are S_1 and S_2 by Proposition 2.3 (i). By Lemma 5.4, S_2 is a unique incompressible spanning surfaces for $L_2 = \partial S_2$ and L_2 is not fibred. Note that S_1 is equal to the mirror image of the surface S_1 in Example 6.3 (Figure 6.2). Hence S_1 is a unique incompressible spanning surface for $L_1 = \partial S_1$ and L_1 is not fibred. Thus S satisfies the conditions (3.1) and (3.2).

We next verify the conditions ($*1$) and ($*2$) in Theorem 3.12. Consider



(M_2, γ_2, A_2) (M_2, γ_2, A_2')

Fig. 6.8



(M_1, γ_1, A_1) (M_1, γ_1, A_1')

Fig. 6.9

the marked complementary sutured manifolds associated with $S = S_1 \cup S_2$ and its dual $S' = S_1' \cup S_2'$. As shown in Figure 6.8, both (M_2, γ_2, A_2) and (M_2, γ_2, A_2') are homeomorphic to the marked sutured manifold (M, γ, A) in Example 4.3 (note that M_2 is the exterior of the solid torus N_2 in Figure 6.8). Hence the condition $(*1)$ holds.

On the other hand (M_1, γ_1, A_1) and (M_1, γ_1, A_1') are shown in Figure 6.9. To see that (M_1, γ_1) has no product disk with A_1 as an edge, we apply decomposition operations defined in §4 to (M_1, γ_1, A_1) as shown in Figure 6.10. Clearly the resulting sutured manifold (M_a, γ_a) has no product disk with A_1 as an edge, and for both (M_b, γ_b) and (M_c, γ_c), each component of $R(\gamma_b)$ and $R(\gamma_c)$ are incompressible in M_b and M_c respectively. Hence, by Propositions 4.1 and 4.2, (M_1, γ_1) has no product disk with A_1 as an edge. In the same way we can verify that (M_1, γ_1) has no product disk with A_1' as an edge. Hence the condition $(*2)$ follows. Thus we have $\mathscr{IS}(10_{53}) = \{[S], [S']\}$ by Theorem 3.12. $\square$

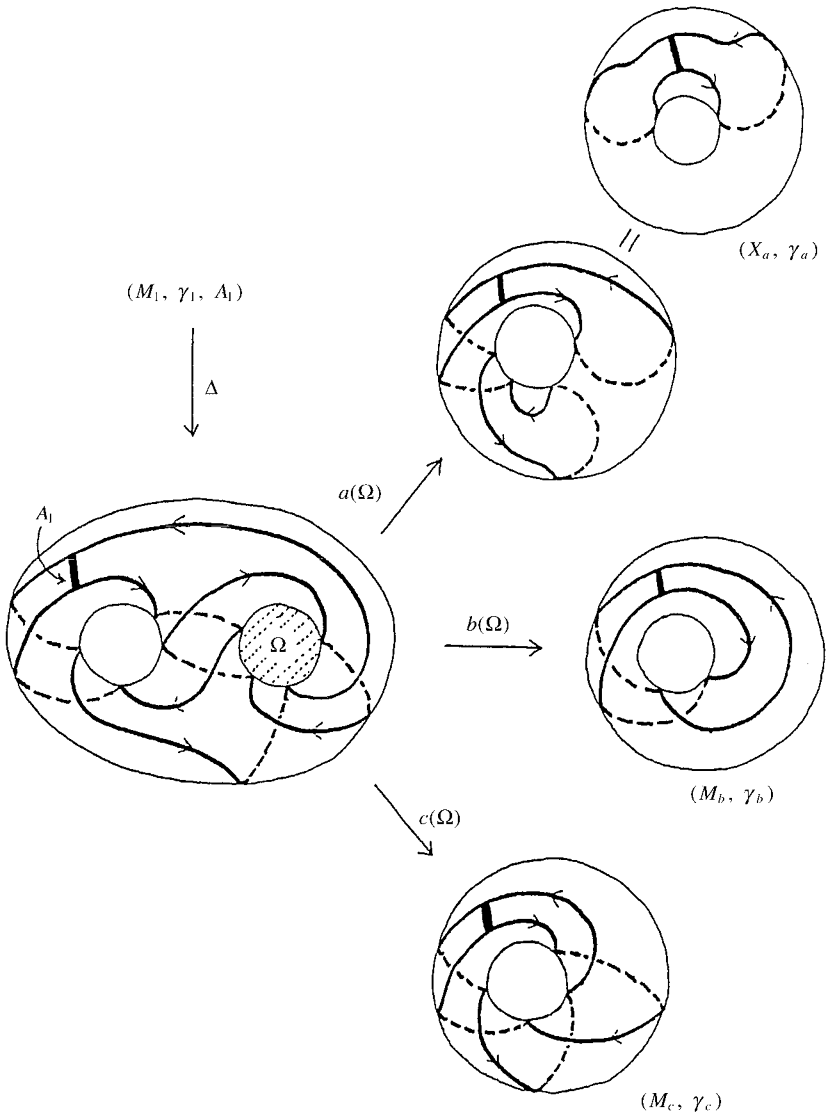


Fig. 6.10

10_{67}. In the same way as the case of 10_{53}, we have

$$\mathscr{I}S(10_{67}) = \{[S], [S']\} \qquad \text{(Figure 6.11)}$$

where S is a minimal genus spanning surface for 10_{67} and S' is its dual relative to the plumbing disk D. $\square$

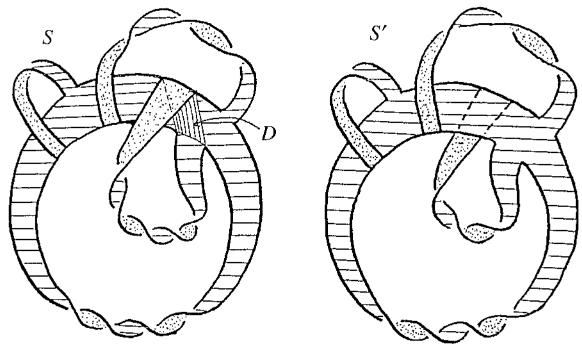


Fig. 6.11. 10_{67}

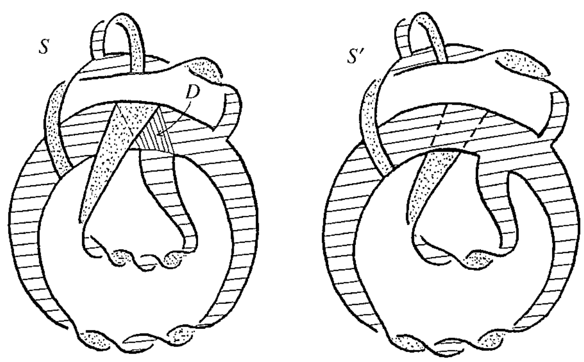


Fig. 6.12. 10_{68}

10_{68}. Similarly we have

$$\mathscr{IS}(10_{68}) = \{[S], [S']\} \qquad \text{(Figure 6.12)}$$

where S is a minimal genus spanning surface for 10_{68} and S' is its dual relative to the plumbing disk D. $\square$

Proof of Theorem C (2). The proof is given by using Theorem 3.15. The knot 10_{74} spans a minimal genus spanning surface S shown in Figure 6.13: S is a plumbing of two surfaces S_1 and S_2 with the plumbing disk D as shown. It is easy to see (as in the case of 10_{53}) that $S = S_1 \cup_D S_2$ satisfies the conditions (3.1) and (3.2). Let (M_1, γ_1, A_1) and (M_2, γ_2, A_2) be the marked complementary sutured manifolds for $S = S_1 \cup_D S_2$. In this case (M_1, γ_1) has a product disk Γ with A_1 as an edge as shown in Figure 6.14. Let B_1 denote the opposite mark on (M_1, γ_1). Then S can be regarded as a plumbing with the plumbing disk E corresponding to the mark B_1; we denote this plumbing by

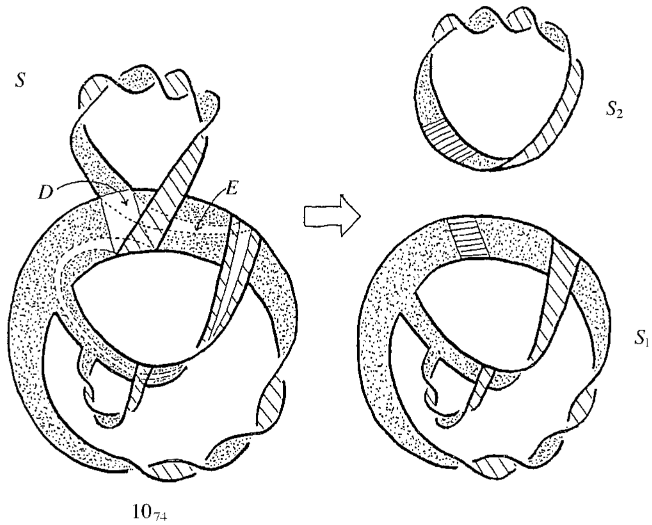


Fig. 6.13. 10_{74}—(1)

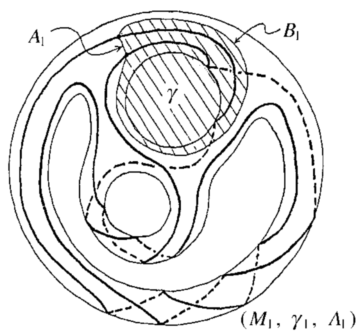


Fig. 6.14

$T = S_1 \cup_E T_2$ (see Figure 6.15). Note that $[S] = [T] \in \mathscr{I}S(10_{74})$. Let S' and T' denote duals of S and T respectively. Then we show the following assertion by checking the conditions $(**1)$ and $(**2)$ in Theorem 3.15.

(6.5) $$\mathscr{I}S(10_{74}) = \{[S], [S'], [T']\} \qquad \text{(Figure 6.15)},$$

and

$$IS(10_{74}) = \overset{[S']}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{[S]}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{[T']}{\bullet}.$$

First we consider the marked sutured manifolds (M_2, γ_2, A_2) and (M_2, γ_2, A_2'). These are the same ones shown in Figure 5.6. Thus there is no

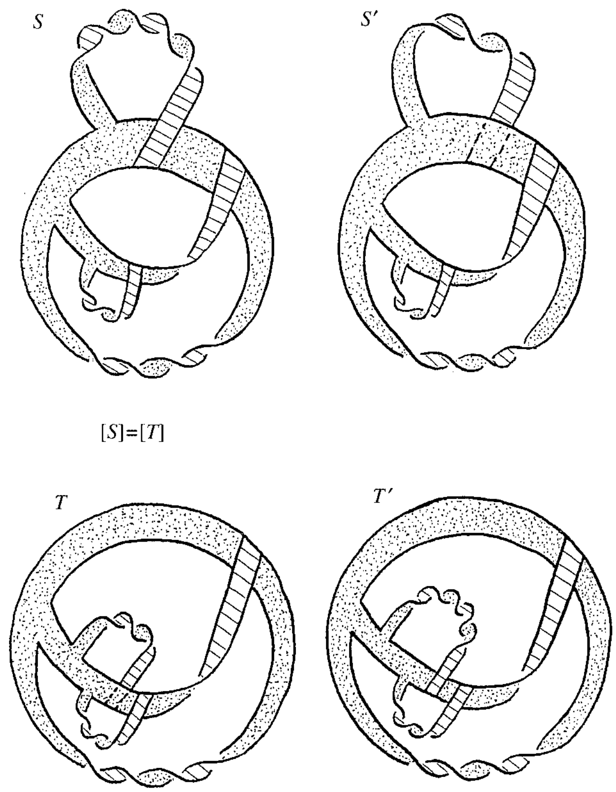


Fig. 6.15. 10_{74}—(2)

product disk in M_2 with A_2 or A_2' as an edge, and the condition $(**2)$ holds. On the other hand, by applying the decomposition operations defined in §4 to the marked sutured manifolds (M_1, γ_1, A_1') and (M_1, γ_1, B_1') (as used in the argument on 10_{53}) and using Propositions 4.1 and 4.2, we can verify the condition $(**1)$. Thus we get the assertion (6.5). □

The proof of Theorems A and C is now completed. □

References

[1] G. Burde and H. Zieschang, Knots, Walter de Gruyter, Berlin, 1985.
[2] J. Eisner, Knots with infinitely many minimal spanning surfaces, Trans. Amer. Math. Soc. **229** (1977), 329–349.
[3] D. Gabai, Foliations and genera of links, Topology **23** (1984), 381–394.
[4] ———, The Murasugi sum is a natural geometric operation, Contemporary Math. AMS **20** (1983), 131–143.

[5] ———, The Murasugi sum is a natural geometric operation II, Contemporary Math. AMS **44** (1985), 93–100.
[6] ———, Detecting fibred links in S^3, Com. Math. Helv. **61** (1986), 519–555.
[7] R. Gustafson, A simple genus one knot with incompressible spanning surfaces of arbitrarily high genus, Pacific J. Math. **96** (1981), 81–98.
[8] A. Hatcher and W. Thurston, Incompressible surfaces in 2-bridge knot complements, Invent. Math. **79** (1985), 225–246.
[9] J. Hempel, 3-Manifolds, Ann. of Math. Studies 86, Princeton Univ. Press, Princeton N.J., 1976.
[10] O. Kakimizu, Doubled knots with infinitely many incompressible spanning surfaces, Bull. London Math. Soc. **23** (1991), 300–302.
[11] ———, Finding disjoint incompressible spanning surfaces for a link, Hiroshima Math. J. **22** (1992), 225–236.
[12] ———, Incompressible spanning surfaces and maximal fibred submanifolds for a knot, Math. Z. **210** (1992), 207–223.
[13] T. Kobayashi, Uniqueness of minimal genus Seifert surfaces for links, Topology and its Appl. **33** (1989), 265–279.
[14] H. Lyon, Incompressible surfaces in knot spaces, Trans. Amer. Math. Soc. **157** (1971), 53–62.
[15] D. Rolfsen, Knots and Links, Mathematical Lecture Series, vol. 7, Publish or Perish Inc., Berkeley, 1976.
[16] F. Waldhausen, On irreducible 3-manifolds which are sufficiently large, Ann. of Math. **87** (1968), 56–88.
[17] W. Whitten, Isotopy types of knot spanning surfaces, Topology **12** (1974), 373–380.

Department of Mathematics
Faculty of Education and Human Sciences
Niigata University
Niigata 950-2181 *Japan*
e-mail: kakimizu@ed.niigata-u.ac.jp