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Artin L-functions of diagonal hypersurfaces and generalized hypergeometric functions over finite fields

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Abstract. We compute the Artin L-function of a diagonal hypersurface D_λ over a finite field associated to a character of a finite group acting on D_λ, and under some condition, express it in terms of hypergeometric functions and Jacobi sums over the finite field. As an application, we derive certain relations among hypergeometric functions over different finite fields from the Grothendieck-Lefschetz trace formula.

1. Introduction

Let D_λ be a diagonal hypersurface over a finite field $\mathbb{F}_q$ defined by the homogeneous equation

$$X_1^d + \cdots + X_n^d = d\lambda X_1^{h_1} \cdots X_n^{h_n},$$

where $d, n \geq 2$, $h_i \geq 1$, $\sum_i h_i = d$, $\gcd(d, h_1, \ldots, h_n) = 1$ and $\lambda \in \mathbb{F}_q$. Assume that $d \mid q-1$ and let $\mu_d \subset \mathbb{F}_q^\times$ be the group of dth roots of unity. Then a subquotient G of $(\mu_d)^n$ acts on D_λ. The l-adic étale cohomology decomposes into χ-eigenspaces

$$H^*(\overline{D_\lambda}; \overline{\mathbb{Q}_l}) = \bigoplus_{\chi \in \hat{G}} H^*(\overline{D_\lambda}; \overline{\mathbb{Q}_l})(\chi),$$

where $\hat{G} = \mathrm{Hom}(G, \overline{\mathbb{Q}}^\times)$ is the character group. Accordingly, the zeta function decomposes into the Artin L-functions as

$$Z(D_\lambda, t) = \prod_{\chi \in \hat{G}} L(D_\lambda, \chi; t).$$

The first aim of this paper is to describe $L(D_\lambda, \chi; t)$ in terms of hypergeometric functions over finite fields.

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Recall that a classical generalized hypergeometric function ${}_mF_n$ over the complex numbers is defined by a power series

$$ {}_mF_n\left(\begin{matrix} a_1, \dots, a_m \\ b_1, \dots, b_n \end{matrix}; z\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_m)_k}{(1)_k (b_1)_k \cdots (b_n)_k} z^k. $$

Here, $(a)_k = \Gamma(a+k)/\Gamma(a)$ where $\Gamma(s)$ is the gamma function and the parameters a_i, b_j are complex numbers with $b_j \notin \mathbb{Z}_{\leq 0}$. Hypergeometric functions over finite fields are defined independently by Koblitz [12], Greene [9] and McCarthy [15] for $m = n+1$, and by Katz [10] and Otsubo [18] in general. We use Otsubo's definition which coincides with McCarthy's one when $m = n+1$ (see Definition 2.4). Such a function is a map from $\mathbb{F}_q$ to $\overline{\mathbb{Q}}$ whose parameters are characters of $\mathbb{F}_q^\times$, and the definition relies on the analogy between the gamma function and the Gauss sum.

The Artin L-function $L(D_\lambda, \chi; t)$ is a generating function of $N_r(D_\lambda; \chi)$ $(r \geq 1)$, the number of $\mathbb{F}_{q^r}$-rational points associated to χ. For the subvariety D_λ^* of D_λ defined by $X_1 \cdots X_n \neq 0$, Koblitz [12, (3.2)] expressed $N_r(D_\lambda^*; \chi)$ in terms of Jacobi sums (see Proposition 3.1). We extend this result to $N_r(D_\lambda; \chi)$ and express them in terms of hypergeometric functions (Theorem 3.4 and Corollary 3.5). Our result is a refinement of Salerno's result [19, Theorem 4.1] for $\# D_\lambda(\mathbb{F}_{q^r}) = \sum_\chi N_r(D_\lambda; \chi)$. There is also a result of Miyatani [17] for more general hypersurfaces.

When $d = n$ (then $h_i = 1$ for all i), D_λ is called the Dwork hypersuface. In this case, we have a more precise formula. The following is a special case of Theorem 3.7. A character $\chi \in \hat{G}$ is indexed by an element $w = (w_1, \dots, w_n) \in (\mathbb{Z}/d\mathbb{Z})^n$. For $\alpha \in \widehat{\mathbb{F}_q^\times}$ and $r \geq 1$, we write $\tilde{\alpha} = \alpha \circ \mathrm{N}_{\mathbb{F}_{q^r}/\mathbb{F}_q} \in \widehat{\mathbb{F}_{q^r}^\times}$. Fix a character $\varphi_d \in \widehat{\mathbb{F}_q^\times}$ of exact order d. Let $\varepsilon \in \widehat{\mathbb{F}_q^\times}$ and $\mathbb{1} \in \hat{G}$ be the trivial characters.

Theorem 1.1. *Suppose that $\lambda \in \mathbb{F}_q^\times$ and none of w_i is 0. Then*

$$ N_r(D_\lambda; \chi) = (-1)^d j(\varphi_d^w)^r F_{\mathrm{red}}\left(\begin{matrix} \tilde{\varphi}_d^{w_1}, \tilde{\varphi}_d^{w_2}, \dots, \tilde{\varphi}_d^{w_d} \\ \tilde{\varepsilon}, \tilde{\varphi}_d, \dots, \tilde{\varphi}_d^{d-1} \end{matrix}; \lambda^d\right)_{q^r} $$

if $\chi \neq \mathbb{1}$, and if $\chi = \mathbb{1}$ the right-hand side is added by $\dfrac{1-q^{r(d-1)}}{1-q^r}$. Here, F_{red} is the reduced hypergeometric function and $j(\varphi_d^w)$ is a Jacobi sum (see Definition 2.5 and subsection 3.3).

By the Grothendieck-Lefschetz trace formula, when D_λ is non-singular, $L(D_\lambda, \chi; t)$ is essentially the characteristic polynomial of the q-Frobenius F acting on the primitive middle cohomology group $H_{\mathrm{prim}}^{n-2}(\overline{D_\lambda}; \overline{\mathbb{Q}_l})(\chi)$, and $N_r(D_\lambda; \chi)$ is essentially the trace of F^r. The dimension, say k, of the cohomology is determined by Katz (see Lemma 4.1). Therefore, $N_r(D_\lambda; \chi)$ are expressed as polynomials in those for $r = 1, \dots, k$. As a result, we obtain relations among hypergeometric functions over different finite fields. We may regard such relations as analogues of the Davenport-Hasse relation for Gauss sums over different finite fields (see Proposition 2.3).

In the case of Dwork hypersurfaces, such formulas become extremely simple (Theorem 4.2). As a special case when $k = 2$, we have the following relation among finite field analogues of Gauss hypergeometric functions.

Theorem 1.2. *Let $a, b, c \in \mathbb{Z}/d\mathbb{Z}$ satisfy $a, b \notin \{0, c\}$, $c \neq 0$ and*

$$c - a - b = \frac{d(d-1)}{2} \quad \bmod d.$$

For $\lambda \in \mathbb{F}_q^\times - \mu_d$ and $r \geq 1$, we have

$$F\left(\begin{matrix}\tilde{\varphi}_d^a, \tilde{\varphi}_d^b \\ \tilde{\varepsilon}, \tilde{\varphi}_d^c\end{matrix}; \lambda^d\right)_{q^r} = P_r\left(F\left(\begin{matrix}\varphi_d^a, \varphi_d^b \\ \varepsilon, \varphi_d^c\end{matrix}; \lambda^d\right)_q, \frac{1}{2}\left(F\left(\begin{matrix}\varphi_d^a, \varphi_d^b \\ \varepsilon, \varphi_d^c\end{matrix}; \lambda^d\right)_q^2 - F\left(\begin{matrix}\tilde{\varphi}_d^a, \tilde{\varphi}_d^b \\ \tilde{\varepsilon}, \tilde{\varphi}_d^c\end{matrix}; \lambda^d\right)_{q^2}\right)\right).$$

Here, $P_r \in \mathbb{Z}[x, y]$ is the unique polynomial satisfying $P_r(\alpha + \beta, \alpha\beta) = \alpha^r + \beta^r$.

Moreover, we will closely look at the cases $d = 3$ (elliptic curves) and $d = 4$ (K3 surfaces). Then $k \leq 2$ and $k \leq 3$ respectively for any χ. We show, however, that only one hypergeometric function over $\mathbb{F}_q$ is sufficient to express those over $\mathbb{F}_{q^r}$, hence to express $N_r(D_\lambda; \chi)$ and $L(D_\lambda, \chi; t)$ (Example 4.4, Corollary 4.7 and Theorem 4.8).

2. Preliminaries

2.1. Zeta functions and Artin L-functions. In this subsection, we recall the definitions of zeta functions and Artin L-functions, and their properties. For more details, see [20] and [22].

Let $\mathbb{F}_q$ be a finite field with q elements of characteristic p. Let $\mathbb{F}_{q^r}$ be the degree r extension of $\mathbb{F}_q$ in a fixed algebraic closure $\overline{\mathbb{F}_q}$ of $\mathbb{F}_q$. Let V be a variety over $\mathbb{F}_q$ and put

$$N_r(V) = \# V(\mathbb{F}_{q^r}) \qquad (r \in \mathbb{Z}_{>0}).$$

Then, *the zeta function of V* is defined by

$$Z(V, t) = \exp\left(\sum_{r=1}^{\infty} \frac{N_r(V)}{r} t^r\right) \in \mathbb{Q}[[t]].$$

Let G be a finite abelian group, and suppose that G acts on V over $\mathbb{F}_q$. Let F be the q-Frobenius acting on $V(\overline{\mathbb{F}_q})$. We write gF for the composition.

For $\chi \in \hat{G}$ and $r \in \mathbb{Z}_{>0}$, put

$$\Lambda(g^{-1}F^r) := \#\{x \in V(\overline{\mathbb{F}_q}) \mid g^{-1}F^r(x) = x\},$$

$$N_r(V;\chi) := \frac{1}{\#G}\sum_{g \in G}\chi(g)\Lambda(g^{-1}F^r) \in \overline{\mathbb{Q}}.$$

The Artin L-function of V associated to χ is defined by

$$L(V,\chi;t) = \exp\left(\sum_{r=1}^{\infty}\frac{N_r(V;\chi)}{r}t^r\right) \in \overline{\mathbb{Q}}[[t]].$$

Since $N_r(V) = \sum_{\chi \in \hat{G}} N_r(V;\chi)$, we have $Z(V,t) = \prod_{\chi \in \hat{G}} L(V,\chi;t)$. Let G_0 be a finite abelian group which acts on V and suppose that $G \subset G_0$. Then, the following holds (cf. [20, (11)]):

$$N_r(V;\chi) = \sum_{\substack{\chi_0 \in \widehat{G_0} \\ \chi_0|_G = \chi}} N_r(V;\chi_0). \tag{2.1}$$

2.2. Gauss and Jacobi sums. For any $\eta \in \widehat{\mathbb{F}_q^\times} = \mathrm{Hom}(\mathbb{F}_q^\times, \overline{\mathbb{Q}}^\times)$, we set $\eta(0) = 0$ and put

$$\delta(\eta) = \begin{cases} 1 & (\text{if } \eta = \varepsilon), \\ 0 & (\text{if } \eta \neq \varepsilon). \end{cases}$$

Fix a non-trivial additive character $\psi \in \mathrm{Hom}(\mathbb{F}_q, \overline{\mathbb{Q}}^\times)$. For $\eta, \eta_1, \ldots, \eta_n \in \widehat{\mathbb{F}_q^\times}$ with $n \geq 1$, define the Gauss sum $g(\eta)$ and the Jacobi sum $j(\eta_1, \ldots, \eta_n)$, which are finite field analogues of the gamma and beta functions respectively, by

$$g(\eta) = -\sum_{x \in \mathbb{F}_q^\times}\eta(x)\psi(x) \in \mathbb{Q}(\mu_{p(q-1)}),$$

$$j(\eta_1, \ldots, \eta_n) = (-1)^{n-1}\sum_{\substack{x_i \in \mathbb{F}_q^\times \\ x_1 + \cdots + x_n = 1}}\eta_1(x_1)\cdots\eta_n(x_n) \in \mathbb{Q}(\mu_{q-1}).$$

Since $\sum_{x \in \mathbb{F}_q}\psi(x) = 0$ and $\psi(0) = 1$, we have

$$g(\varepsilon) = 1.$$

Define

$$g^\circ(\eta) := q^{\delta(\eta)}g(\eta).$$

The following identities are well-known (cf. [18, Proposition 2.2]). For each $\eta \in \widehat{\mathbb{F}_q^\times}$,

$$g(\eta)g^\circ(\eta^{-1}) = \eta(-1)q, \tag{2.2}$$

and if $\eta \neq \varepsilon$ then

$$|g(\eta)| = \sqrt{q}. \tag{2.3}$$

For $n \geq 1$,

$$j(\underbrace{\varepsilon, \dots, \varepsilon}_{n}) = \frac{1 - (1-q)^n}{q}. \tag{2.4}$$

For $\eta_1, \dots, \eta_n \in \widehat{\mathbb{F}_q^\times}$ with not all $\eta_i = \varepsilon$,

$$j(\eta_1, \dots, \eta_n) = \frac{g(\eta_1)\cdots g(\eta_n)}{g^\circ(\eta_1 \cdots \eta_n)}. \tag{2.5}$$

In particular, if $\eta_1 \cdots \eta_n = \varepsilon$, then by (2.2),

$$j(\eta_1, \dots, \eta_n) = \eta_n(-1) j(\eta_1, \dots, \eta_{n-1}) \qquad (n \geq 2). \tag{2.6}$$

We prepare the following lemma obtained by an easy change of variables.

Lemma 2.1. *For $\eta_1, \dots, \eta_{n+1} \in \widehat{\mathbb{F}_q^\times}$, we have*

$$\sum_{\substack{x_i, y \in \mathbb{F}_q^\times \\ x_1 + \cdots + x_n = y}} \eta_1(x_1) \cdots \eta_n(x_n) \eta_{n+1}(y)$$

$$= \begin{cases} (-1)^n (1-q) j(\eta_1, \dots, \eta_n) & (\eta_1 \cdots \eta_{n+1} = \varepsilon), \\ 0 & (\eta_1 \cdots \eta_{n+1} \neq \varepsilon). \end{cases}$$

We will use the Davenport-Hasse multiplication formula.

Proposition 2.2 (cf. [2, 11.3]). *Let $m \in \mathbb{Z}_{>0}$ and suppose that $m \mid q-1$. For any $\eta \in \widehat{\mathbb{F}_q^\times}$, we have*

$$\frac{\prod_{i=0}^{m-1} g(\varphi_m^i \eta)}{\prod_{i=0}^{m-1} g(\varphi_m^i)} \eta(m^m) = g(\eta^m),$$

where φ_m is a character of exact order m.

Let $r \geq 1$ be an integer and let $\mathrm{N}_{\mathbb{F}_{q^r}/\mathbb{F}_q}$ be the norm map. For $\eta \in \widehat{\mathbb{F}_q^\times}$, put $\tilde{\eta} = \eta \circ \mathrm{N}_{\mathbb{F}_{q^r}/\mathbb{F}_q} \in \widehat{\mathbb{F}_{q^r}^\times}$. Then, the following is well known as the Davenport-Hasse theorem.

Proposition 2.3 (cf. [22]). *For each $r \geq 1$, we have*

$$g(\tilde{\eta}) = g(\eta)^r.$$

2.3. Hypergeometric functions over finite fields. In this subsection, we recall hypergeometric functions over finite fields. We follow the definitions of Otsubo [18].

For $\alpha, \nu \in \widehat{\mathbb{F}_q^\times}$, we put

$$(\alpha)_\nu := \frac{g(\alpha\nu)}{g(\alpha)},$$

$$(\alpha)_\nu^\circ := \frac{g^\circ(\alpha\nu)}{g^\circ(\alpha)} = q^{\delta(\alpha\nu)-\delta(\alpha)}(\alpha)_\nu.$$

In particular, $(\varepsilon)_\nu = g(\nu)$ and $(\alpha)_\varepsilon = (\alpha)_\varepsilon^\circ = 1$. By Proposition 2.2, we have, for any $m \mid q-1$,

$$\begin{aligned} (\alpha^m)_{\nu^m} &= \prod_{i=0}^{m-1} (\alpha\varphi_m^i)_\nu \cdot \nu(m^m), \\ (\alpha^m)_{\nu^m}^\circ &= \prod_{i=0}^{m-1} (\alpha\varphi_m^i)_\nu^\circ \cdot \nu(m^m). \end{aligned} \tag{2.7}$$

Definition 2.4. For $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \in \widehat{\mathbb{F}_q^\times}$ and $\lambda \in \mathbb{F}_q$, define

$$F\left(\begin{matrix}\alpha_1, \dots, \alpha_m \\ \beta_1, \dots, \beta_n\end{matrix}; \lambda\right)_q := \frac{1}{1-q} \sum_{\nu \in \widehat{\mathbb{F}_q^\times}} \frac{(\alpha_1)_\nu \cdots (\alpha_m)_\nu}{(\beta_1)_\nu^\circ \cdots (\beta_n)_\nu^\circ} \nu(\lambda).$$

(We often omit writing q of $F(\cdots)_q$).

We only consider the case when $m = n$ in this paper, and then, the values of F are in $\mathbb{Q}(\mu_{q-1})$ (see [18, Lemma 2.5 (iii)]). For comparisons with definitions of Koblitz [12], Greene [9], Katz [10] and McCarthy [15], see [18, Remark 2.13]. As a special case, it is known that

$$F\left(\begin{matrix}\alpha \\ \varepsilon\end{matrix}; \lambda\right) = \alpha^{-1}(1-\lambda) \tag{2.8}$$

for $\alpha \neq \varepsilon$ and $\lambda \neq 0$ (cf. [18, Corollary 3.4]).

We define reduced hypergeometric functions.

Definition 2.5. Let $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_l \in \widehat{\mathbb{F}_q^\times}$ and assume that $\{\alpha_1, \dots, \alpha_m\}$ and $\{\beta_1, \dots, \beta_n\}$ have an empty intersection. Then, we put

$$F_{\mathrm{red}}\left(\begin{matrix}\alpha_1, \dots, \alpha_m, \gamma_1, \dots, \gamma_l \\ \beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_l\end{matrix}; \lambda\right)_q = F\left(\begin{matrix}\alpha_1, \dots, \alpha_m \\ \beta_1, \dots, \beta_n\end{matrix}; \lambda\right)_q.$$

When we reduce a hypergeometric function over finite fields, remainder terms appear as the following.

Lemma 2.6 ([18, Theorem 3.2]). *In the situation of Definition* 2.5, *suppose that* $\gamma_i \neq \gamma_j$ *for all* $1 \leq i < j \leq l$. *Then,*

$$F\left(\begin{matrix}\alpha_1, \dots, \alpha_m, \gamma_1, \dots, \gamma_l \\ \beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_l\end{matrix}; \lambda\right)_q$$

$$= q^\delta F_{\mathrm{red}}\left(\begin{matrix}\alpha_1, \dots, \alpha_m, \gamma_1, \dots, \gamma_l \\ \beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_l\end{matrix}; \lambda\right)_q + \frac{q^\delta}{q} \sum_{j=1}^{l} \frac{\prod_{i=1}^{m} (\alpha_i)_{\gamma_j^{-1}}}{\prod_{i=1}^{n} (\beta_i)^{\circ}_{\gamma_j^{-1}}} \gamma_j^{-1}(\lambda).$$

Here, $\delta = 1$ *when* $\gamma_j = \varepsilon$ *for some* j *and* $\delta = 0$ *otherwise.*

3. Artin L-functions and hypergeometric functions

3.1. Diagonal hypersurfaces. Let $h_i \geq 1$ $(i = 1, \dots, n)$ be integers with $h_1 + \cdots + h_n = d$ and $\gcd(d, h_1, \dots, h_n) = 1$, and let $\lambda \in \mathbb{F}_q$. We consider the diagonal hypersurface D_λ in $\mathbb{P}^{n-1}$ over $\mathbb{F}_q$ defined by the homogeneous equation

$$D_\lambda : X_1^d + \cdots + X_n^d = d\lambda X_1^{h_1} \cdots X_n^{h_n}. \tag{3.1}$$

Note that D_λ is non-singular if and only if $(\prod_{i=1}^{n} h_i^{h_i})\lambda^d \neq 1$. Let D_λ^* denote the subvariety of D_λ defined by $X_1 \cdots X_n \neq 0$.

Define a variety D_λ^{aff} in $\mathbb{A}^n$ over $\mathbb{F}_q$ by

$$D_\lambda^{\mathrm{aff}} : x_1^d + \cdots + x_n^d = d\lambda x_1^{h_1} \cdots x_n^{h_n},$$

and let $D_\lambda^{*\mathrm{aff}}$ denote the subvariety of D_λ^{aff} defined by $x_1 \cdots x_n \neq 0$.

Suppose that $d \mid q - 1$, so that $\mu_d \subset \mathbb{F}_q^\times$. Define groups by

$$\tilde{G}_0 := \mu_d^n \supset \tilde{G} := \{\xi \in \tilde{G}_0 \mid \xi^h = 1\}.$$

Here, we write $h = (h_1, \dots, h_n)$ and $\xi^h = \xi_1^{h_1} \cdots \xi_n^{h_n}$ for $\xi = (\xi_1, \dots, \xi_n)$. Let $\Delta \subset \tilde{G}$ be the diagonal subgroup and define

$$G_0 := \tilde{G}_0/\Delta \supset G := \tilde{G}/\Delta.$$

Let $\tilde{G}$ act on D_λ^{aff} and $D_\lambda^{*\mathrm{aff}}$ over $\mathbb{F}_q$ by

$$\xi \cdot (x_1, \dots, x_n) = (\xi_1 x_1, \dots, \xi_n x_n).$$

This induces an action of G on D_λ and D_λ^* through the natural map $\mathbb{A}^n \backslash 0 \to \mathbb{P}^{n-1}$. Similarly, $\tilde{G}_0$ acts on D_0^{aff} and this action induces an action of G_0 on the Fermat hypersurface D_0.

Fix a generator φ of $\widehat{\mathbb{F}_q^{\times}}$. We have the following commutative diagrams:

$$\begin{array}{ccc} \tilde{G} & \hookrightarrow & \tilde{G}_0 \\ \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\ G & \hookrightarrow & G_0, \end{array} \qquad \begin{array}{ccc} \hat{\tilde{G}} & \twoheadleftarrow & \widehat{\tilde{G}_0} \\ \uparrow & & \uparrow \\ \hat{G} & \twoheadleftarrow & \widehat{G_0}. \end{array}$$

From now on, we regard $\hat{G}$ (resp. $\widehat{G_0}$) as a subgroup of $\hat{\tilde{G}}$ (resp. $\widehat{\tilde{G}_0}$). We have an isomorphism

$$(\mathbb{Z}/d\mathbb{Z})^n \xrightarrow{\cong} \widehat{\tilde{G}_0}; \qquad w = (w_1, \dots, w_n) \mapsto \chi_0^w,$$

where $\chi_0^w(\xi) := \varphi(\xi^w)$. Put

$$\chi^w = \chi_0^w|_{\tilde{G}}.$$

If we put

$$W := \{w \in (\mathbb{Z}/d\mathbb{Z})^n \mid w_1 + \cdots + w_n = 0\},$$

then

$$w \in W \Leftrightarrow \chi_0^w \in \widehat{G_0} \Leftrightarrow \chi^w \in \hat{G}.$$

Note that, for $w, w' \in (\mathbb{Z}/d\mathbb{Z})^n$,

$$\chi^w = \chi^{w'} \Leftrightarrow w - w' = mh \text{ for some } m \in \{0, 1, \dots, d-1\}.$$

Here, note that we regard mh as an element of $(\mathbb{Z}/d\mathbb{Z})^n$. We write $w \sim w'$ when $\chi^w = \chi^{w'}$. Let $\mathbb{1} \in \hat{G}$ be the unit character (i.e. $\mathbb{1} = \chi^w$ with $w \sim 0$).

3.2. Number of rational points with characters. Recall that for $w \in W$ and $r \geq 1$,

$$N_r(D_\lambda; \chi^w) = \frac{1}{\#G} \sum_{\xi \in G} \chi^w(\xi) \#\{X \in D_\lambda(\overline{\mathbb{F}_q}) \mid \xi^{-1} F^r(X) = X\},$$

where F is the q-Frobenius. Note that $\#G = d^{n-2}$. Without loss of generality, we only consider the case $r = 1$. For any $m \mid q-1$, put

$$\varphi_m = \varphi^{(q-1)/m},$$

a character of exact order m.

For the convenience of the reader, let us give a proof of the following result of Koblitz [12, (3.2)], which was stated without proof.

Proposition 3.1. *For $\lambda \neq 0$ and $w \in W$, we have*

$$N_1(D_\lambda^*;\chi^w) = \frac{(-1)^n}{1-q}\sum_{v\in\widehat{\mathbf{F}_q^\times}} j(\varphi_d^{w_1}v^{h_1},\dots,\varphi_d^{w_n}v^{h_n})v^d(d\lambda).$$

Proof. Note that if $x \in \{x \in D_\lambda^{*\mathrm{aff}}(\overline{\mathbf{F}_q}) \mid \xi^{-1}F(x) = ax\}$ for some $a \in \overline{\mathbf{F}_q}^\times$ and $\xi \in \tilde{G}$, then $a^{-1/(q-1)}x \in \{x \in D_\lambda^{*\mathrm{aff}}(\overline{\mathbf{F}_q}) \mid \xi^{-1}F(x) = x\}$, and that if $x \in D_\lambda^{*\mathrm{aff}}(\overline{\mathbf{F}_q})$ is fixed by $\xi^{-1}F$ for $\xi \in \tilde{G}$, then for $c \in \overline{\mathbf{F}_q}^\times$, $cx \in D_\lambda^{*\mathrm{aff}}(\overline{\mathbf{F}_q})$ is fixed by $\xi^{-1}F$ if and only if $c \in \mathbf{F}_q^\times$. By these, we have

$$N_1(D_\lambda^*;\chi^w) = \frac{1}{q-1}N_1(D_\lambda^{*\mathrm{aff}};\chi^w). \tag{3.2}$$

We have

$$\begin{aligned}
&\#\{x \in D_\lambda^{*\mathrm{aff}}(\overline{\mathbf{F}_q}) \mid \xi^{-1}F(x) = x\}\\
&\quad = \#\{x \in (\overline{\mathbf{F}_q}^\times)^n \mid x_1^d + \cdots + x_n^d = d\lambda x^h,\ x^{q-1} = \xi\}\\
&\quad = \frac{1}{d}\#\{x \in (\overline{\mathbf{F}_q}^\times)^n \mid (x_1^d + \cdots + x_n^d)^d = (d\lambda)^d x^{dh},\ x^{q-1} = \xi\}. \qquad (3.3)
\end{aligned}$$

Here we used the fact that $\#\{x \in (\overline{\mathbf{F}_q}^\times)^n \mid x_1^d + \cdots + x_n^d = cd\lambda x^h,\ x^{q-1} = \xi\}$ does not depend on $c \in \mu_d$ by the assumption $\gcd(d,h_1,\dots,h_n) = 1$.

If we put $u_i = x_i^d$, then $u_i \in \mathbf{F}_q^\times \Leftrightarrow x_i^{q-1} \in \mu_d$. Thus, the last member of (3.3) is equal to

$$d^{n-1}\#\{u \in (\mathbf{F}_q^\times)^n \mid (u_1 + \cdots + u_n)^d = (d\lambda)^d u^h,\ u^l = \xi\},$$

where $l := (q-1)/d$.

Fix $\xi \in \tilde{G}$, and define the function $f_\xi : \mathbf{F}_q^\times \to \mathbf{Z}_{\geq 0}$ by

$$f_\xi(t) := \#\{u \in (\mathbf{F}_q^\times)^n \mid (u_1 + \cdots + u_n)^d = tu^h,\ u^l = \xi\}.$$

By the discrete Fourier transform on $\mathbf{F}_q^\times$, we have

$$f_\xi(t) = \frac{1}{q-1}\sum_{v\in\widehat{\mathbf{F}_q^\times}} \hat{f}(v)v(t),$$

where

$$\hat{f}(v) := \sum_{t\in\mathbf{F}_q^\times} f_\xi(t)v^{-1}(t) = \sum_{\substack{u\in(\mathbf{F}_q^\times)^n\\ u^l=\xi}} v^{-1}\left(\frac{(u_1+\cdots+u_n)^d}{u^h}\right).$$

Therefore, letting $t = (d\lambda)^d$, we have

$$N_1(D_\lambda^{*\mathrm{aff}};\chi^w) = \sum_{\xi \in \tilde{G}} \chi^w(\xi) f_\xi(t)$$

$$= \frac{1}{q-1} \sum_{\xi \in \tilde{G}} \sum_{v \in \widehat{\mathbb{F}_q^\times}} \sum_{\substack{u \in (\mathbb{F}_q^\times)^n \\ u^l = \xi}} \chi^w(\xi) v^{-1}\left(\frac{(u_1 + \cdots + u_n)^d}{u^h}\right) v(t)$$

$$= \frac{1}{q-1} \sum_{\xi} \sum_{v} \sum_{\substack{u \\ u^l = \xi}} \chi_0^w(u^l) v(u^h) v^{-d}(u_1 + \cdots + u_n) v(t)$$

$$= \frac{1}{q-1} \sum_{v} \sum_{\substack{u \\ u^l \in \tilde{G}}} \varphi_d(u^w) v(u^h) v^{-d}(u_1 + \cdots + u_n) v(t).$$

Since

$$\frac{1}{d}\sum_{m=0}^{d-1} \varphi_d(u^{w+mh}) = \begin{cases} \varphi_d(u^w) & (u^{lh} = 1), \\ 0 & (u^{lh} \neq 1), \end{cases}$$

and $u^{lh} = 1 \Leftrightarrow u^l \in \tilde{G}$, $N(D_\lambda^{*\mathrm{aff}};\chi^w)$ is equal to

$$\frac{1}{d(q-1)} \sum_{v \in \widehat{\mathbb{F}_q^\times}} \sum_{m=0}^{d-1} \sum_{u \in (\mathbb{F}_q^\times)^n} \varphi_d(u^{w+mh}) v(u^h) v^{-d}(u_1 + \cdots + u_n) v(t)$$

$$= \frac{1}{d(q-1)} \sum_{m=0}^{d-1} \sum_{v} \sum_{u \in (\mathbb{F}_q^\times)^n} \varphi_d(u^w) \varphi_d^m v(u^h) (\varphi_d^m v)^{-d}(u_1 + \cdots + u_n) \varphi_d^m v(t)$$

$$= \frac{1}{q-1} \sum_{v} \sum_{u \in (\mathbb{F}_q^\times)^n} \varphi_d^{w_1} v^{h_1}(u_1) \cdots \varphi_d^{w_n} v^{h_n}(u_n) v^{-d}(u_1 + \cdots + u_n) v(t)$$

$$= (-1)^{n-1} \sum_{v} j(\varphi_d^{w_1} v^{h_1}, \ldots, \varphi_d^{w_n} v^{h_n}) v(t).$$

The first equality follows by $\varphi_d^m(t) = 1$, the second equality follows by replacing $\varphi_d^m v$ with v, and the last equality follows from Lemma 2.1 and that $w \in W$. Thus, by (3.2), the proposition follows.

3.3. Hypergeometric expressions. For brevity, we write

$$j(\varphi_d^w) = j(\varphi_d^{w_1}, \ldots, \varphi_d^{w_n}) = \frac{g(\varphi_d^{w_1}) \cdots g(\varphi_d^{w_n})}{q}.$$

From now on, we suppose that $dh_i \mid q-1$ for all $i = 1, \dots, n$. For each $w \in W$, put

$$F^w(\lambda) := F\left(\begin{matrix}\varphi_{dh_i}^{w_i}, \varphi_{dh_i}^{w_i+d}, \dots, \varphi_{dh_i}^{w_i+d(h_i-1)} \\ \varepsilon, \varphi_d, \dots, \varphi_d^{d-1}\end{matrix}; \left(\prod_{j=1}^{n} h_j^{h_j}\right)\lambda^d\right)$$

(i runs through $1, \dots, n$). Here, we understand that w_i means its representative in $\{0, \dots, d-1\}$.

Theorem 3.2. *Suppose that $dh_i \mid q-1$ for all $i = 1, \dots, n$. For any $\lambda \neq 0$ and $w \in W$,*

$$N_1(D_\lambda^*; \chi^w) = \begin{cases} \dfrac{(-1)^n}{q^{\delta(m)}} j(\varphi_d^w)^{1-\delta(m)} F^w(\lambda) + (-1)^{n-1} \dfrac{(1-q)^{n-1}}{q} & (\textit{if } \ w = mh), \\ (-1)^n j(\varphi_d^w) F^w(\lambda) & (\textit{if } \ w \not\sim 0), \end{cases}$$

where $\delta(m) = 1$ if $m = 0$ and $\delta(m) = 0$ otherwise.

Proof. Since $\sum w_i = 0$ and $\sum h_i = d$, we have

$$\varphi_d^{w_i} v^{h_i} = \varepsilon \text{ for all } i \Leftrightarrow w = mh \text{ for some } m \in \{0, \dots, d-1\} \text{ and } v = \varphi_d^{-m}.$$

Thus, if $w \not\sim 0$ then, by (2.5),

$$\sum_v j(\varphi_d^{w_1} v^{h_1}, \dots, \varphi_d^{w_n} v^{h_n}) v^d(d\lambda) = \sum_v \frac{g(\varphi_d^{w_1} v^{h_1}) \cdots g(\varphi_d^{w_n} v^{h_n})}{g^\circ(v^d)} v^d(d\lambda),$$

and if $w = mh$ for some m then, by (2.4), (2.5) and $g^\circ(\varepsilon) = q$,

$$\begin{aligned} &\sum_v j(\varphi_d^{w_1} v^{h_1}, \dots, \varphi_d^{w_n} v^{h_n}) v^d(d\lambda) \\ &\quad = \sum_{v \neq \varphi_d^{-m}} \frac{g(\varphi_d^{w_1} v^{h_1}) \cdots g(\varphi_d^{w_n} v^{h_n})}{g^\circ(v^d)} v^d(d\lambda) + \frac{1-(1-q)^n}{q} \varphi_d^{-md}(d\lambda) \\ &\quad = \sum_v \frac{g(\varphi_d^{w_1} v^{h_1}) \cdots g(\varphi_d^{w_n} v^{h_n})}{g^\circ(v^d)} v^d(d\lambda) - \frac{(1-q)^n}{q}. \end{aligned}$$

By (2.7), we have

$$g^\circ(v^d) = g^\circ(\varepsilon)(\varepsilon)_{v^d}^\circ = q(\varepsilon)_v^\circ (\varphi_d)_v^\circ \cdots (\varphi_d^{d-1})_v^\circ v(d^d).$$

For each $i = 1, \dots, n$, we have similarly

$$\begin{aligned} g(\varphi_d^{w_i} v^{h_i}) &= g((\varphi_{dh_i}^{w_i} v)^{h_i}) = g(\varphi_d^{w_i})(\varphi_{dh_i}^{w_i h_i})_{v^{h_i}} \\ &= g(\varphi_d^{w_i})(\varphi_{dh_i}^{w_i})_v (\varphi_{dh_i}^{w_i+d})_v \cdots (\varphi_{dh_i}^{w_i+d(h_i-1)})_v v(h_i^{h_i}). \end{aligned}$$

Thus,

$$\sum_{v} \frac{g(\varphi_d^{w_1} v^{h_1}) \cdots g(\varphi_d^{w_n} v^{h_n})}{g^\circ(v^d)} v^d(d\lambda)$$

$$= q^{-1} \prod_{k=1}^{n} g(\varphi_d^{w_k}) \sum_{v} \frac{\prod_{i=1}^{n} (\varphi_{dh_i}^{w_i})_v (\varphi_{dh_i}^{w_i+d})_v \cdots (\varphi_{dh_i}^{w_i+d(h_i-1)})_v}{(\varepsilon)_v^\circ (\varphi_d)_v^\circ \cdots (\varphi_d^{d-1})_v^\circ} v\left(\left(\prod_{j=1}^{n} h_j^{h_j}\right)\lambda^d\right)$$

$$= \frac{(1-q)}{q} \prod_{k=1}^{n} g(\varphi_d^{w_k}) \cdot F^w(\lambda).$$

Therefore, we obtain the theorem by Proposition 3.1 and (2.5), where note that $mh = 0 \Leftrightarrow m = 0$ by the assumption that $\gcd(d, h_1, \dots, h_n) = 1$.

Next, we consider the case where $\lambda = 0$. Note that $\tilde{G}$ (resp. G) acts on D_0^{aff} (resp. D_0) through the inclusion $\tilde{G} \hookrightarrow \tilde{G}_0$ (resp. $G \hookrightarrow G_0$).
The following is proved by Weil [23].

Lemma 3.3 (cf. [12, (2.12)]). *For each $w \in W$,*

$$N_1(D_0; \varphi^w) = \begin{cases} 0 & (\textit{if some but not all } w_i = 0), \\ \dfrac{1-q^{n-1}}{1-q} & (\textit{if } w = 0), \\ (-1)^n j(\varphi_d^w) & (\textit{if all } w_i \neq 0). \end{cases}$$

Theorem 3.4. *Suppose that $dh_i \mid q-1$ for all $i = 1, \dots, n$. For any $\lambda \neq 0$ and $w \in W$, we have*

$$N_1(D_\lambda; \chi^w)$$
$$= \begin{cases} \dfrac{(-1)^n}{q^{\delta(m)}} j(\varphi_d^w)^{1-\delta(m)} F^w(\lambda) + \dfrac{1-q^{n-1}}{1-q} + \dfrac{(-1)^{n-1}}{q} + C & (\textit{if } w = mh), \\ (-1)^n j(\varphi_d^w) F^w(\lambda) + C & (\textit{if } w \not\sim 0). \end{cases}$$

Here, $\delta(m)$ is as in Theorem 3.2 and

$$C := (-1)^{n-1} \sum_{\substack{w' \sim w \\ w'_i = 0 \text{ for some but not all } i}} j(\varphi_d^{w'}).$$

Proof. It is trivial that

$$N_1(D_\lambda; \chi^w) - N_1(D_\lambda^*; \chi^w) = N_1(D_0; \chi^w) - N_1(D_0^*; \chi^w).$$

By (2.1), we have, for each $w \in W$,

$$N_1(D_0^*;\chi^w) = \sum_{w' \sim w} N_1(D_0^*;\chi_0^{w'}),$$

$$N_1(D_0;\chi^w) = \sum_{w' \sim w} N_1(D_0;\chi_0^{w'}).$$

Hence,

$$N_1(D_\lambda;\chi^w) = N_1(D_\lambda^*;\chi^w) + \sum_{w' \sim w} (N_1(D_0;\chi_0^{w'}) - N_1(D_0^*;\chi_0^{w'})). \tag{3.4}$$

Letting $u_i = x_i^d$, $v_i = -u_i/u_n$ and $l = (q-1)/d$, we have, by a similar argument as in the proof of Proposition 3.1,

$$\begin{aligned}
&N_1(D_0^{*\mathrm{aff}};\chi_0^w)\\
&\quad = \frac{1}{d^n} \sum_{\xi \in \tilde{G}_0} \chi_0^w(\xi) \#\{x \in (\overline{\mathbb{F}_q}^\times)^n \mid x_1^d + \cdots + x_n^d = 0,\ x^{q-1} = \xi\}\\
&\quad = \sum_{\xi \in \tilde{G}_0} \varphi(\xi^w) \#\{u \in (\mathbb{F}_q^\times)^n \mid u_1 + \cdots + u_n = 0,\ u^l = \xi\}\\
&\quad = \sum_{\substack{u \in (\mathbb{F}_q^\times)^n \\ u_1 + \cdots + u_n = 0}} \varphi_d(u^w)\\
&\quad = \varphi_d^{w_n}(-1) \cdot (q-1)(-1)^{n-2} j(\varphi_d^{w_1}, \ldots, \varphi_d^{w_{n-1}}),
\end{aligned}$$

where note that $\varphi_d^{w_1 + \cdots + w_n} = \varepsilon$ at the last equality. Therefore, by (2.4), (2.6) and (3.2),

$$N_1(D_0^*;\chi_0^w) = \begin{cases} (-1)^n j(\varphi_d^w) & (\text{if } w \neq 0), \\ (-1)^n \dfrac{1 - (1-q)^{n-1}}{q} & (\text{if } w = 0). \end{cases}$$

By Lemma 3.3, we have

$$\begin{aligned}
&N_1(D_0;\chi_0^w) - N_1(D_0^*;\chi_0^w)\\
&\quad = \begin{cases} 0 & (\text{if } w_i \neq 0 \text{ for all } i), \\ \dfrac{1 - q^{n-1}}{1 - q} + (-1)^{n-1} \dfrac{1 - (1-q)^{n-1}}{q} & (\text{if } w = 0), \\ (-1)^{n-1} j(\varphi_d^w) & (\text{otherwise}). \end{cases}
\end{aligned}$$

Thus, by Theorem 3.2 and (3.4), we obtain the result.

Corollary 3.5. *Suppose that $dh_i \mid q-1$ for all $i=1,\dots,n$ and that $w \sim 0$ (i.e. $w = mh$ for some $m \in \{0,\dots,d-1\}$). Then, for any $\lambda \neq 0$,*

$$N_1(D_\lambda;\chi^w) = (-1)^n j(\varphi_d^w)^{1-\delta(m)} F^w(\text{red. } \varphi_d^m;\lambda) + \frac{1-q^{n-1}}{1-q} + C,$$

where $F^w(\text{red. } \varphi_d^m;\lambda)$ is the function obtained by removing one φ_d^m from both the numerator and the denominator parameters in $F^w(\lambda)$. Furthermore, if $gcd(d,h_i)=1$ for all $i=1,\dots,n$, then

$$N_1(D_\lambda;\chi^w) = (-1)^n j(\varphi_d^w)^{1-\delta(m)} F^w_{\text{red}}(\lambda) + \frac{1-q^{n-1}}{1-q}.$$

Proof. Note that $\varphi_{dh_i}^{w_i} = \varphi_d^m$ for all i. Using Lemma 2.6 partially, we have

$$F^w(\lambda) = q^{\delta(m)} F^w(\text{red. } \varphi_d^m;\lambda) + R,$$

where

$$R := \frac{q^{\delta(m)}}{q} \cdot \frac{(\varphi_d^m)_{\varphi_d^{-m}}^{n-1} \prod_{i=1}^n \varphi_d^{-m}(h_i^{h_i}) \prod_{j=1}^{h_i-1} (\varphi_d^m \varphi_{h_i}^j)_{\varphi_d^{-m}}}{\prod_{1\le i\le d, i\neq m} (\varphi_d^i)^\circ_{\varphi_d^{-m}}}.$$

Using (2.7) and (2.5), we have

$$R = \frac{q^{\delta(m)}}{q} \cdot \frac{(\varphi_d^m)^\circ_{\varphi_d^{-m}}}{(\varphi_d^m)_{\varphi_d^{-m}}} \cdot \frac{\prod_{i=1}^n (\varphi_d^{mh_i})_{\varphi_d^{-mh_i}}}{(\varepsilon)^\circ_\varepsilon} = \frac{1}{\prod_{i=1}^n g(\varphi_d^{mh_i})} = \frac{q^{\delta(m)}}{q} j(\varphi_d^w)^{-1+\delta(m)}.$$

Thus, by Theorem 3.4, we obtain that

$$N_1(D_\lambda;\chi^w) = F^w(\text{red. } \varphi_d^m;\lambda) + \frac{1-q^{n-1}}{1-q} + C.$$

If $\gcd(d,h_i)=1$ for all i, then $F^w(\text{red. } \varphi_d^m;\lambda) = F^w_{\text{red}}(\lambda)$ and $C=0$. Here, note that $\varphi_{h_i}^j$ is not of order d for all $i=1,\dots,n$ and $j=1,\dots h_i-1$, and that $w\sim 0$ implies that $w=0$ or $w_i \neq 0$ for all i. Therefore, we have the second equation of the corollary.

3.4. Artin L-functions. Fix an integer $r \ge 1$. For $\eta \in \widehat{\mathbb{F}_q^\times}$, define a character $\tilde{\eta} \in \widehat{\mathbb{F}_{q^r}^\times}$ by

$$\tilde{\eta} := \eta \circ \mathrm{N}_{\mathbb{F}_{q^r}/\mathbb{F}_q}.$$

To define hypergeometric functions over $\mathbb{F}_{q^r}$, fix a generator φ' of $\widehat{\mathbb{F}_{q^r}^\times}$ such that $\varphi'|_{\mathbb{F}_q^\times} = \varphi$. Then, we have $\varphi'^{(q^r-1)/(q-1)} = \tilde{\varphi}$, and

$$\varphi'_d := \varphi'^{(q^r-1)/d} = \widetilde{\varphi_d}$$

for any $d \mid q-1$. For convenience, we also write $\tilde{\varphi}_d = \widetilde{\varphi_d}$. Put as before,

$$F_r^w(\lambda) = F\left(\begin{matrix} \tilde{\varphi}_{dh_i}^{w_i}, \tilde{\varphi}_{dh_i}^{w_i+d}, \dots, \tilde{\varphi}_{dh_i}^{w_i+d(h_i-1)} \\ \tilde{\varepsilon}, \tilde{\varphi}_d, \dots, \tilde{\varphi}_d^{d-1} \end{matrix}; \left(\prod_{j=1}^n h_j^{h_j}\right)\lambda^d\right)_{q^r}.$$

When $w = mh$ for some $m \in \{0, \dots, d-1\}$, define $F_r^w(\text{red. } \tilde{\varphi}_d^m; \lambda)$ similarly as in Corollary 3.5.

Corollary 3.6. *Suppose that $dh_i \mid q-1$ for all $i = 1, \dots, n$ and that $\lambda \neq 0$.*
(i) *If $w = mh$ for some $m \in \{0, \dots, d-1\}$, then*

$$L(D_\lambda, \chi^w; t) = \exp\left(\sum_{r=1}^\infty \frac{(-1)^n}{q^{r\delta(m)}} j(\varphi_d^w)^{r(1-\delta(m))} F_r^w(\text{red. } \tilde{\varphi}_d^m; \lambda) \frac{t^r}{r}\right) \prod_{k=0}^{n-2} \frac{1}{1-q^k t} \cdot C(t),$$

where

$$C(t) := \prod_{\substack{w' \sim w \\ w'_i = 0 \text{ for some but not all } i}} (1 - j(\varphi_d^{w'})t)^{(-1)^n}.$$

(ii) *If $w \not\sim 0$, then*

$$L(D_\lambda, \chi^w; t) = \exp\left(\sum_{r=1}^\infty (-1)^n j(\varphi_d^w)^r F_r^w(\lambda) \frac{t^r}{r}\right) C(t).$$

Proof. By applying Theorem 3.4 and Corollary 3.5 for $\mathbb{F}_{q^r}$ and φ', we obtain the formula for $N_r(D_\lambda; \chi^w)$. For the Jacobi sum, by Proposition 2.3, we have

$$j(\tilde{\varphi}_d^{w_1}, \dots, \tilde{\varphi}_d^{w_n}) = j(\varphi_d^{w_1}, \dots, \varphi_d^{w_n})^r.$$

Thus, the theorem follows formally.

3.5. Dwork hypersurfaces. In this subsection, let D_λ denote the Dwork hypersurfaces of degree d, which is the case when $n = d$ and $h_i = 1$ for all i in (3.1). In this case, for $w, w' \in W$, $\chi^w = \chi^{w'}$ if and only if $w - w' = (m, \dots, m)$ for some $m \in \mathbb{Z}/d\mathbb{Z}$. For $w \in W$, put $\delta(w) = 1$ if $0 \in \{w_1, \dots, w_d\}$ and $\delta(w) = 0$ otherwise.

Theorem 3.7. *Suppose that $d \mid q-1$. Let $r \geq 1$ and $\lambda \in \mathbb{F}_q^\times$.*
(i) *If $w = (m, \dots, m) \in W$ with $m \in \mathbb{Z}/d\mathbb{Z}$ (i.e. $\chi^w = \mathbb{1}$), then*

$$N_r(D_\lambda; \chi^w) = (-1)^d j(\varphi_d^w)^{(1-\delta(w))r} F_{\text{red}}\left(\begin{matrix} \overbrace{\tilde{\varphi}_d^m, \tilde{\varphi}_d^m, \dots, \tilde{\varphi}_d^m}^{d} \\ \tilde{\varepsilon}, \tilde{\varphi}_d, \dots, \tilde{\varphi}_d^{d-1} \end{matrix}; \lambda^d\right)_{q^r} + \frac{1-q^{r(d-1)}}{1-q^r}.$$

(ii) *If* $\chi^w \neq \mathbb{1}$, *then*

$$N_r(D_\lambda;\chi^w) = (-1)^d (q^{\delta(w)} j(\varphi_d^w))^r F_{\mathrm{red}}\begin{pmatrix} \tilde{\varphi}_d^{w_1}, \tilde{\varphi}_d^{w_2}, \dots, \tilde{\varphi}_d^{w_d} \\ \tilde{\varepsilon}, \tilde{\varphi}_d, \dots, \tilde{\varphi}_d^{d-1} \end{pmatrix}_{q^r}.$$

In particular, if w is a permutation of $(0,1,\dots,d-1)$ *(this occurs only for odd d),*

$$N_r(D_\lambda;\chi^w) = \begin{cases} 0 & (\lambda^d \neq 1), \\ (-1)^{r(d^2-1)(q-1)/(8d)} q^{r(d-1)/2} & (\lambda^d = 1). \end{cases}$$

Proof. By Proposition 2.3, we only have to prove it for $r=1$.
(i) It follows by Corollary 3.5.
(ii) For $w=(w_1,\dots,w_d) \in W$ and $k \in \mathbb{Z}/d\mathbb{Z}$, put

$$n_w(k) = \#\{w_i \mid w_i = k\}.$$

By Theorem 3.4,

$$N_1(D_\lambda;\chi^w) = (-1)^d j(\varphi_d^w) F\begin{pmatrix} \varphi_d^{w_1}, \varphi_d^{w_2}, \dots, \varphi_d^{w_d} \\ \varepsilon, \varphi_d, \dots, \varphi_d^{d-1} \end{pmatrix} + (-1)^{d-1} \sum_{\substack{w' \sim w \\ w'_i = 0 \text{ for some } i}} j(\varphi_d^{w'}). \tag{3.5}$$

Put $S = \{w_1,\dots,w_n\} \subset \mathbb{Z}/d\mathbb{Z}$, then by Lemma 2.6, we have

$$F\begin{pmatrix} \varphi_d^{w_1}, \varphi_d^{w_2}, \dots, \varphi_d^{w_d} \\ \varepsilon, \varphi_d, \dots, \varphi_d^{d-1} \end{pmatrix} = q^{\delta(w)} F_{\mathrm{red}}\begin{pmatrix} \varphi_d^{w_1}, \varphi_d^{w_2}, \dots, \varphi_d^{w_d} \\ \varepsilon, \varphi_d, \dots, \varphi_d^{d-1} \end{pmatrix} + \sum_{c \in S} \frac{\prod_{a \in S} (\varphi_d^a)_{\varphi_d^{-c}}^{n_w(a)-1}}{\prod_{b \notin S} (\varphi_d^b)_{\varphi_d^{-c}}}.$$

Noting that $\prod_{i=1}^d g(\varphi_d^i) = \prod_{i=1}^d g(\varphi_d^{i-c})$ for $c \in \mathbb{Z}/d\mathbb{Z}$ and using (2.5),

$$\begin{aligned} \sum_{c \in S} \frac{\prod_{a \in S} (\varphi_d^a)_{\varphi_d^{-c}}^{n_w(a)-1}}{\prod_{b \notin S} (\varphi_d^b)_{\varphi_d^{-c}}} &= \sum_{c \in S} \prod_{i=1}^d \frac{(\varphi_d^{w_i})_{\varphi_d^{-c}}}{(\varphi_d^i)_{\varphi_d^{-c}}} \\ &= \sum_{c \in S} \prod_{i=1}^d \frac{g(\varphi_d^{w_i - c}) g(\varphi_d^i)}{g(\varphi_d^{w_i}) g(\varphi_d^{i-c})} \\ &= \sum_{c \in S} \prod_{i=1}^d \frac{g(\varphi_d^{w_i - c})}{g(\varphi_d^{w_i})} \end{aligned}$$

$$= j(\varphi_d^w)^{-1} \sum_{c \in S} j(\varphi_d^{w-c})$$

$$= j(\varphi_d^w)^{-1} \sum_{\substack{w' \sim w \\ w'_i = 0 \text{ for some } i}} j(\varphi_d^{w'}).$$

Thus, we obtain the anterior half of (ii) by (3.5).

If $w = (w_1, \dots, w_d)$ is a permutation of $(0, 1, \dots, d-1)$, then

$$F_{\mathrm{red}}\left(\begin{matrix} \varphi_d^{w_1}, \varphi_d^{w_2}, \dots, \varphi_d^{w_d} \\ \varepsilon, \varphi_d, \dots, \varphi_d^{d-1} \end{matrix} ; \lambda^d \right) = \frac{1}{1-q} \sum_{v} v(\lambda^d) = \begin{cases} 0 & (\lambda^d \neq 1), \\ -1 & (\lambda^d = 1). \end{cases}$$

Thus, we have the latter half of (ii) by the anterior half of (ii) and

$$j(\varepsilon, \varphi_d, \dots, \varphi_d^{d-1}) = (-1)^{(d^2-1)(q-1)/(8d)} q^{(d-3)/2},$$

which follows by (2.2) and that d is odd.

Remark 3.8. *For the Dwork hypersurface, McCarthy* [16, Corollary 2.6] *expresses the number* $\#D_\lambda(\mathbb{F}_q)$ *in terms of his hypergeometric functions. Theorem* 3.7 *is a refinement of this result. Goodson (*[6] *and* [7, Theorem 1.2] *for* $d = 4$ *and odd* d*) and Kumabe (*[13] *for* $d = 6$*) show similar results in terms of Greene's hypergeometric functions.*

For each $r \geq 1$ and $w \in W$, put

$$F^w_{r,\mathrm{red}}(\lambda) = F_{\mathrm{red}}\left(\begin{matrix} \tilde{\varphi}_d^{w_1}, \tilde{\varphi}_d^{w_2}, \dots, \tilde{\varphi}_d^{w_d} \\ \tilde{\varepsilon}, \tilde{\varphi}_d, \dots, \tilde{\varphi}_d^{d-1} \end{matrix} ; \lambda^d \right)_{q^r}.$$

Similarly as Corollary 3.6, we have the following corollary of Theorem 3.7.

Corollary 3.9. *Suppose that* $d \mid q-1$. *For any* $\lambda \in \mathbb{F}_q^\times$ *and* $w \in W$, *we have the following:*

(i) *For* $w = (m, \dots, m) \in W$ *with* $m \in \mathbb{Z}/d\mathbb{Z}$,

$$L(D_\lambda, \chi^w; t) = \frac{\exp\left(\sum_{r=1}^\infty j(\varphi_d^w)^{r(1-\delta(w))} F^w_{r,\mathrm{red}}(\lambda) \dfrac{t^r}{r} \right)^{(-1)^d}}{(1-t)(1-qt)\cdots(1-q^{d-2}t)}.$$

(ii) *For* $w \in W$ *with* $\chi^w \neq \mathbb{1}$,

$$L(D_\lambda, \chi^w; t) = \exp\left(\sum_{r=1}^\infty q^{r\delta(w)} j(\varphi_d^w)^r F^w_{r,\mathrm{red}}(\lambda) \frac{t^r}{r} \right)^{(-1)^d}.$$

In particular, for $w \in W$ *which is a permutation of* $(0, 1, \dots, d-1)$,

$$L(D_\lambda, \chi^w; t) = \begin{cases} 1 & (\lambda^d \neq 1), \\ (1 - (-1)^{(q-1)/d} q^{(d-1)/2} t)^{-1} & (\lambda^d = 1). \end{cases}$$

REMARK 3.10. *On the factorization of the zeta functions of Dwork hypersurfaces, Goutet* [8] *gave a factorization of* $Z(D_\lambda, t)$ *considering the actions of the groups* G *and* $\mathfrak{S}_d$.

4. Hypergeometric functions over different finite fields

4.1. l-adic étale cohomology. Let X be a smooth hypersurface in $\mathbb{P}^{n-1}$ over $\mathbb{F}_q$ and write $\overline{X} = X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. Let l be a prime number with $l \neq p$. Write $H^m(\overline{X}, \overline{\mathbb{Q}_l})$ for the l-adic étale cohomology of $\overline{X}$. By the weak Lefschetz theorem (cf. [4, p. 106]), the map

$$H^m(\overline{\mathbb{P}^{n-1}}, \overline{\mathbb{Q}_l}) \to H^m(\overline{X}, \overline{\mathbb{Q}_l})$$

induced by the embedding $X \hookrightarrow \mathbb{P}^{n-1}$ is an isomorphism for $m < n-2$. Thus, by the Poincaré duality and the structure of the l-adic étale cohomology of $\mathbb{P}^{n-1}$, we have

$$H^m(\overline{X}, \overline{\mathbb{Q}_l}) = \begin{cases} 0 & (m \text{ is odd}, \ m \neq n-2), \\ \overline{\mathbb{Q}_l}\left(-\dfrac{m}{2}\right) & (m \text{ is even}, \ m \neq n-2, \ 0 \leq m \leq 2(n-2)). \end{cases}$$

Let $H \subset \mathbb{P}^{n-1}$ be a hyperplane and $h \in H^2(\overline{X}, \overline{\mathbb{Q}_l}(1))$ be the cohomology class associated with $H \cap X$ (independent of H). Then, by the hard Lefschetz theorem (cf. [4, p. 274]), the composition

$$H^{n-4}(\overline{X}, \overline{\mathbb{Q}_l}(-1)) \xrightarrow{\cup h} H^{n-2}(\overline{X}, \overline{\mathbb{Q}_l}) \xrightarrow{\cup h} H^n(\overline{X}, \overline{\mathbb{Q}_l}(1))$$

is an isomorphism, where $\cup$ is the cup product. Thus, we have the primitive decomposition

$$H^{n-2}(\overline{X}, \overline{\mathbb{Q}_l}) \cong \begin{cases} H^{n-2}_{\text{prim}}(\overline{X}, \overline{\mathbb{Q}_l}) & (n \text{ is odd}), \\ H^{n-2}_{\text{prim}}(\overline{X}, \overline{\mathbb{Q}_l}) \oplus \overline{\mathbb{Q}_l}\left(-\dfrac{n-2}{2}\right) & (n \text{ is even}), \end{cases} \tag{4.1}$$

where

$$H^{n-2}_{\text{prim}}(\overline{X}, \overline{\mathbb{Q}_l}) := \operatorname{Ker}(H^{n-2}(\overline{X}, \overline{\mathbb{Q}_l}) \xrightarrow{\cup h} H^n(\overline{X}, \overline{\mathbb{Q}_l}(1))).$$

Now let $X = D_\lambda$ $(\lambda \in \mathbb{F}_q)$ with $\lambda^d \prod_{i=1}^n h_i^{h_i} \neq 1$. Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_l}$ and identify $\hat{G}$ with $\operatorname{Hom}(G, \overline{\mathbb{Q}_l}^\times)$. Then, we have a decomposition

$$H^m(\overline{D_\lambda}, \overline{\mathbb{Q}_l}) = \bigoplus_{\chi \in \hat{G}} H^m(\overline{D_\lambda}, \overline{\mathbb{Q}_l})(\chi).$$

Here, for a vector space V over $\overline{\mathbb{Q}_l}$ equipped with a G-action, we write

$$V(\chi) = \{v \in V \mid g \cdot v = \chi(g)v \text{ for all } g \in G\}.$$

Since the G-action fixes h, it is compatible with the primitive decomposition (4.1). Noting that $H^{2k}(\overline{D_\lambda}, \overline{\mathbb{Q}_l}(k))$ is generated by h^k if $2k \neq n-2$, we have, for $0 \leq k \leq n-2$ $(2k \neq n-2)$,

$$H^{2k}(\overline{D_\lambda}, \overline{\mathbb{Q}_l})(\chi) = \begin{cases} \overline{\mathbb{Q}_l}(-k) & (\chi = \mathbb{1}), \\ 0 & (\chi \neq \mathbb{1}), \end{cases}$$

and

$$H^{n-2}(\overline{D_\lambda}, \overline{\mathbb{Q}_l})(\chi) = \begin{cases} H^{n-2}_{\mathrm{prim}}(\overline{D_\lambda}, \overline{\mathbb{Q}_l})(\chi) \oplus \overline{\mathbb{Q}_l}\left(-\dfrac{n-2}{2}\right) & (\chi = \mathbb{1}, \ n \text{ is even}), \\ H^{n-2}_{\mathrm{prim}}(\overline{D_\lambda}, \overline{\mathbb{Q}_l})(\chi) & (\text{otherwise}). \end{cases}$$

The dimension of this cohomology group is determined by Katz.

Lemma 4.1 ([11, Lemma 3.1 (i)]). *For any* $w \in W$,

$$\dim H^{n-2}_{\mathrm{prim}}(\overline{D_\lambda}, \overline{\mathbb{Q}_l})(\chi^w) = \#\{m \in \mathbb{Z}/d\mathbb{Z} \mid \delta(w + mh) = 0\},$$

where $\delta(w)$ *is as in Theorem* 3.7.

4.2. Relations among hypergeometric functions over different fields. By the Grothendieck-Lefschetz trace formula (cf. [4, Theorem 2.9]), we have

$$N_r(D_\lambda; \chi) = \sum_{i=0}^{2(n-2)} (-1)^i \operatorname{Tr}((F^r)^* \mid H^i(\overline{D_\lambda}, \overline{\mathbb{Q}}_l)(\chi)). \tag{4.2}$$

Therefore,

$$L(D_\lambda, \chi; t) = \det(1 - F^* t \mid H^{n-2}_{\mathrm{prim}}(\overline{D_\lambda}, \overline{\mathbb{Q}}_l)(\chi))^{(-1)^{n-1}}$$

when $\chi \neq \mathbb{1}$, and the right-hand side is multiplied with $\prod_{i=0}^{n-2}(1 - q^i t)^{-1}$ when $\chi = \mathbb{1}$. We can write

$$\det(1 - F^* t \mid H^{n-2}_{\mathrm{prim}}(\overline{D_\lambda}, \overline{\mathbb{Q}}_l)(\chi)) = \prod_{i=1}^{k} (1 - \alpha_i t), \tag{4.3}$$

where $k = \dim H^{n-2}_{\mathrm{prim}}(\overline{D_\lambda}, \overline{\mathbb{Q}_l})(\chi)$. By the Weil conjecture proved by Deligne [3], we have $|\alpha_i| = q^{(n-2)/2}$ for all i.

Let e_i $(0 \leq i \leq k)$ be the elementary symmetric polynomial of degree i in indeterminates $\alpha_1, \ldots, \alpha_k$ and put

$$p_r = \sum_{i=1}^{k} \alpha_i^r.$$

There exist unique polynomials $P_r(x_1, \dots, x_k) \in \mathbb{Z}[x_1, \dots, x_k]$ $(r \geq 1)$ satisfying

$$p_r = P_r(e_1, \dots, e_k).$$

On the other hand, there exist polynomials $Q_i(x_1, \dots, x_i)$ $(1 \leq i \leq k)$ such that

$$e_i = Q_i(p_1, \dots, p_i) \in \mathbb{Q}[x_1, \dots, x_i].$$

This fact follows for example from Newton's identity

$$ie_i = \sum_{j=1}^{i} (-1)^{j-1} e_{i-j} p_j \qquad (1 \leq i \leq k).$$

If we put

$$R_r(x_1, \dots, x_k) = P_r(Q_1(x_1), Q_2(x_1, x_2), \dots, Q_k(x_1, \dots, x_k)),$$

then

$$p_r = R_r(p_1, \dots, p_k)$$

for any $r \geq 1$.

For α_i as in (4.3), we have

$$N_r(D_\lambda; \chi) = \begin{cases} (-1)^n p_r & (\chi \neq \mathbb{1}), \\ (-1)^n p_r + \dfrac{1 - q^{n-1}}{1 - q} & (\chi = \mathbb{1}). \end{cases} \tag{4.4}$$

Therefore, it follows from Theorem 3.4 that we can write $F_r^w(\lambda)$ in terms of $F_1^w(\lambda), \dots, F_k^w(\lambda)$ and Jacobi sums, where k is determined by Lemma 4.1.

4.3. Dwork hypersurfaces. Let us write down such relations more concretely when D_λ $(\lambda^d \neq 1)$ is the Dwork hypersurface of degree d. For $w \in W$, we have by Lemma 4.1

$$k(w) := \dim H^{d-2}(\overline{D_\lambda}, \overline{\mathbb{Q}_l})(\chi^w) = d - \#\{w_i \mid i = 1, \dots, d\}.$$

Let $F_{r,\mathrm{red}}^w(\lambda)$ be as in subsection 3.5. By Theorem 3.7 and (4.4),

$$j'(\varphi_d^w)^r F_{r,\mathrm{red}}^w(\lambda) = \sum_{i=1}^{k(w)} \alpha_i^r, \qquad j'(\varphi_d^w) := \begin{cases} j(\varphi_d^w)^{1-\delta(w)} & (\chi^w = \mathbb{1}), \\ q^{\delta(w)} j(\varphi_d^w) & (\chi^w \neq \mathbb{1}). \end{cases} \tag{4.5}$$

Hence we obtain the following.

Theorem 4.2. *Suppose that $d \mid q-1$. For any $w \in W$, $\lambda \in \mathbb{F}_q - \mu_d$ and $r \geq 1$, we have*

$$F_{r,\mathrm{red}}^w(\lambda) = R_r(F_{1,\mathrm{red}}^w(\lambda), \dots, F_{k(w),\mathrm{red}}^w(\lambda)).$$

We also have the following consequence of the Weil conjecture.

THEOREM 4.3. *Suppose that* $d \mid q-1$. *For* $w \in W$ *with* $\delta(w) = 0$ *and* $\lambda \in \mathbb{F}_{q^r} - \mu_d$,

$$|F^w_{r,\mathrm{red}}(\lambda)| \le k(w).$$

PROOF. We are reduced to the case when $r = 1$. The statement for $\lambda = 0$ is clear. By assumption, $|j(\varphi^w_d)| = q^{(d-2)/2}$ by (2.3) and (2.5). When $\lambda \neq 0$, the theorem follows from (4.5) and that $|\alpha_i| = q^{(d-2)/2}$ for all i.

Now, let us consider the case when $k(w) = 2$. Then Theorem 4.2 becomes

$$F^w_{r,\mathrm{red}}(\lambda) = P_r\left(F^w_{1,\mathrm{red}}(\lambda), \frac{1}{2}(F^w_{1,\mathrm{red}}(\lambda)^2 - F^w_{2,\mathrm{red}}(\lambda))\right).$$

As an example, we take $w = (a, b, \check{0}, 1, \ldots, \check{c}, \ldots, d-1)$, where $a, b \notin \{0, c\}$, $c \neq 0$ and

$$c - a - b = \frac{d(d-1)}{2} \mod d.$$

Then

$$F^w_{r,\mathrm{red}}(\lambda) = F\begin{pmatrix} \tilde{\varphi}^a_d, \tilde{\varphi}^b_d \\ \tilde{\varepsilon}, \tilde{\varphi}^c_d \end{pmatrix}; \lambda^d \Bigg)_{q^r}.$$

Thus, we obtain Theorem 1.2.

EXAMPLE 4.4. *Let* $d = 3$ *and* $\lambda^3 \neq 1$. *Then* D_λ *is the Hesse elliptic curve and*

$$H^1(\overline{D_\lambda}, \overline{\mathbb{Q}_l}) = H^1_{\mathrm{prim}}(\overline{D_\lambda}, \overline{\mathbb{Q}_l})(\mathbb{1})$$

by Lemma 4.1. *Suppose that* $3 \mid q-1$. *By Theorem* 3.7 *and* (2.6), *we have, for* $m = 1, 2$ *and* $r \ge 1$,

$$\begin{aligned} N_r(D_\lambda) &= N_r(D_\lambda; \mathbb{1}) \\ &= (-1)^{((q-1)/3)mr+1} j(\varphi^m_3, \varphi^m_3)^r F\begin{pmatrix} \tilde{\varphi}^m_3, \tilde{\varphi}^m_3 \\ \tilde{\varepsilon}, \tilde{\varphi}^{2m}_3 \end{pmatrix}; \lambda^3 \Bigg)_{q^r} + 1 + q^r. \end{aligned} \tag{4.6}$$

In this case, $\alpha_1\alpha_2 = q$ *by the Poincaré duality. Therefore, we have*

$$F^w_{r,\mathrm{red}}(\lambda) = P_r\left(F^w_{1,\mathrm{red}}(\lambda), \frac{q}{j(\varphi^m_3, \varphi^m_3)^2}\right)$$

for any $r \ge 1$.

REMARK 4.5. *The complex periods of* D_λ $(\lambda \in \mathbb{C},\ \lambda^3 \neq 1)$ *are computed in* [14, Theorem 1], *one of which is*

$$B\left(\frac{1}{3},\frac{1}{3}\right){}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{1}{3}\\ \frac{2}{3}\end{matrix};\lambda^3\right) - \lambda B\left(\frac{2}{3},\frac{2}{3}\right){}_2F_1\left(\begin{matrix}\frac{2}{3},\frac{2}{3}\\ \frac{4}{3}\end{matrix};\lambda^3\right).$$

Our formula (4.6) *can be regarded as a finite field analogue of this result.*

4.4. Dwork K3 surfaces. We consider the case when D_λ is the Dwork hypersurfaces of degree 4. Suppose that $\lambda^4 \neq 1$, so that D_λ is a K3 surface, and $\lambda \neq 0$. Now, $\hat{G}$ consists of 16 characters, which are up to permutation of indices $w_1, \dots, w_4$:

$\mathbb{1}$,

$\chi^{(1,1,3,3)}$ (3 characters, note that $\chi^{(1,1,3,3)} = \chi^{(3,3,1,1)}$),

$\chi^{(1,2,2,3)}$ (12 characters).

Note that the number $N_r(D_\lambda; \chi^w)$ does not change under permutations of indices. Recall that

$$k(0,0,0,0) = 3, \qquad k(1,1,3,3) = 2, \qquad k(1,2,2,3) = 1.$$

PROPOSITION 4.6. *Suppose that* $4 \mid q-1$. *Let* $\lambda \in \mathbb{F}_q^\times$ *(we do not assume* $\lambda \notin \mu_4$*).*

(i) *For* $m \in \{1,2,3\}$,

$$\begin{aligned} N_r(D_\lambda; \mathbb{1}) &= F\left(\begin{matrix}\tilde{\varepsilon},\tilde{\varepsilon},\tilde{\varepsilon}\\ \tilde{\varphi}_4,\tilde{\varphi}_4^2,\tilde{\varphi}_4^3\end{matrix};\lambda^4\right)_{q^r} + 1 + q^r + q^{2r} \\ &= j(\varphi_4^m,\varphi_4^m,\varphi_4^m,\varphi_4^m)^r F_{\mathrm{red}}\left(\begin{matrix}\tilde{\varphi}_4^m,\tilde{\varphi}_4^m,\tilde{\varphi}_4^m,\tilde{\varphi}_4^m\\ \tilde{\varepsilon},\tilde{\varphi}_4,\tilde{\varphi}_4^2,\tilde{\varphi}_4^3\end{matrix};\lambda^4\right)_{q^r} + 1 + q^r + q^{2r}. \end{aligned}$$

(ii)

$$N_r(D_\lambda; \chi^{(1,1,3,3)}) = (\varphi_2^r(1-\lambda^2) + \varphi_2^r(1+\lambda^2))q^r.$$

(iii)

$$N_r(D_\lambda; \chi^{(1,2,2,3)}) = (-1)^{r(q-1)/4}\varphi_2^r(1-\lambda^4)q^r.$$

PROOF. (i) Immediate consequence of Theorem 3.7.
(ii) By loc. cit., we have

$$N_r(D_{\lambda;4}; \chi^{(1,1,3,3)}) = j(\varphi_4,\varphi_4,\varphi_4^3,\varphi_4^3)^r F\left(\begin{matrix}\tilde{\varphi}_4,\tilde{\varphi}_4^3\\ \tilde{\varepsilon},\tilde{\varphi}_4^2\end{matrix};\lambda^4\right)_{q^r}.$$

First, $j(\varphi_4, \varphi_4, \varphi_4^3, \varphi_4^3) = q$ by (2.5) and (2.2). Secondly, using (2.7), (2.2) and (2.5), we have for $v \in \widehat{\mathbb{F}_{q^r}^\times}$,

$$\frac{(\tilde{\varphi}_4)_v(\tilde{\varphi}_4^3)_v}{(\tilde{\varepsilon})_v^\circ(\tilde{\varphi}_4^2)_v^\circ} = \frac{(\tilde{\varphi}_4^2)_{v^2}}{(\tilde{\varepsilon})_{v^2}^\circ} = j(v^{-2}, \tilde{\varphi}_4^2),$$

and hence,

$$\begin{aligned} F\left(\begin{matrix}\tilde{\varphi}_4, \tilde{\varphi}_4^3\\ \tilde{\varepsilon}, \tilde{\varphi}_4^2\end{matrix}; \lambda^4\right)_{q^r} &= \frac{1}{1-q^r}\sum_v j(v^{-2}, \tilde{\varphi}_4^2)v(\lambda^4) \\ &= \frac{1}{q^r-1}\sum_v \sum_{u \in \mathbb{F}_{q^r}^\times} v^{-2}(u)\tilde{\varphi}_4^2(1-u)v(\lambda^4) \\ &= \frac{1}{q^r-1}\sum_u \tilde{\varphi}_4^2(1-u)\sum_v v\left(\frac{\lambda^4}{u^2}\right) \\ &= \tilde{\varphi}_4^2(1-\lambda^2) + \tilde{\varphi}_4^2(1+\lambda^2). \end{aligned}$$

Noting that $\tilde{\varphi}_4^2(\lambda) = \varphi_2(\lambda)^{(1-q^r)/(1-q)} = \varphi_2^r(\lambda)$, the formula follows.

(iii) By Theorem 3.7, we have

$$N_r(D_\lambda; \chi^{(1,2,2,3)}) = j(\varphi_4, \varphi_4^2, \varphi_4^2, \varphi_4^3)^r F\left(\begin{matrix}\tilde{\varphi}_4^2\\ \tilde{\varepsilon}\end{matrix}; \lambda^4\right)_{q^r},$$

and the formula follows by (2.8), (2.5) and (2.2).

Corollary 4.7. *Suppose that* $4 \mid q-1$. *Let* $m \in \{1,2,3\}$ *and put*

$$Q(t) = 1 - p_1 t + \frac{p_1^2 - p_2}{2}t^2 - \frac{p_1^3 - 3p_1p_2 + 2p_3}{6}t^3,$$

where

$$p_r = j(\varphi_4^m, \varphi_4^m, \varphi_4^m, \varphi_4^m)^r F_{r,\mathrm{red}}^{(m,m,m,m)}(\lambda).$$

Then

$$Z(D_\lambda, t) = \frac{1}{(1-t)(1-qt)(1-uqt)^3(1-u'qt)^3(1-vqt)^{12}Q(t)(1-q^2t)},$$

where $u = \varphi_2(1-\lambda^2)$, $u' = \varphi_2(1+\lambda^2)$ *and* $v = (-1)^{(q-1)/4}uu'$.

In fact, $F_{r,\mathrm{red}}^w(\lambda)$ hence $Q(t)$ and $Z(D_\lambda, t)$ are determined only by $F_{1,\mathrm{red}}^w(\lambda)$.

Theorem 4.8. *Suppose that* $4 \mid q-1$. *Let* $P_r(x_1, x_2)$ *be the polynomial defined in subsection* 4.2 *for* $k = 2$. *Let* $w = (m,m,m,m)$ $(m \in \{1,2,3\})$.

Then, for any $r \geq 1$,

$$F^w_{r,\mathrm{red}}(\lambda) = \left(\frac{\varphi_2(1-\lambda^4)q}{j(\varphi_4^w)}\right)^r + P_r\left(F^w_{1,\mathrm{red}}(\lambda) - \frac{\varphi_2(1-\lambda^4)q}{j(\varphi_4^w)}, \left(\frac{q}{j(\varphi_4^w)}\right)^2\right).$$

Proof. Since $\det(1 - F^*t \mid H^2(\overline{D_\lambda}, \mathbb{Q}_l)) \in \mathbb{Q}[t]$ (independent of l), it follows that $Q(t) \in \mathbb{Q}[t]$. Hence we can write

$$Q(t) = (1-\alpha_1 t)(1-\alpha_2 t)(1-\alpha_3 t)$$

with $\alpha_1 \in \mathbb{R}$, $\overline{\alpha_2} = \alpha_3$ and $|\alpha_i| = q$ for all i. It is known that the highest term of $\det(1 - F^*t \mid H^2(\overline{D_\lambda}, \mathbb{Q}_l))$ is $q^{22}t^{22}$. One can show this fact by reducing to the case of the Fermat quartic surface ($\lambda = 0$) similarly as in the proof of Lemma 4.1. Hence $\alpha_1\alpha_2\alpha_3 = \alpha_1 q^2 = uu'q^3$, i.e. $\alpha_1 = \varphi_2(1-\lambda^4)q$ (this fact is also proved in [1]). Since

$$F^w_{r,\mathrm{red}}(\lambda) = \frac{\alpha_1^r + \alpha_2^r + \alpha_3^r}{j(\varphi_4^w)^r},$$

we obtain the theorem.

Remark 4.9. *When* $m = 2$, *we have* $j(\varphi_4^w) = q$.

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