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On multidimensional inverse scattering under the Stark effect

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ABSTRACT. We study one of multidimensional inverse scattering problems for quantum systems in a constant electric field, by utilization of the Enss-Weder time-dependent method. The main purpose of this paper is to propose some methods of sharpening key estimates in the analysis, which are much simpler than those in the previous works. Our methods give an appropriate class of short-range potentials which can be determined uniquely by scattering operators, that seems natural in terms of direct scattering problems.

1. Introduction

In this paper, we consider one of inverse scattering problems for quantum systems in a constant electric field $E \in \boldsymbol{R}^n$, by applying the Enss-Weder time-dependent method. Throughout this paper, we assume that $n \geq 2$, and suppose $E = e_1 = (1, 0, \ldots, 0)$. The Hamiltonian H under consideration is given by

$$H = H_0 + V; \qquad H_0 = p^2/2 - E \cdot x = p^2/2 - x_1, \tag{1.1}$$

acting on $L^2(\boldsymbol{R}^n)$, where $x = (x_1, x_2, \ldots, x_n) = (x_1, x_\perp) \in \boldsymbol{R}^n$ and $p = -i\nabla = (p_1, p_2, \ldots, p_n) = (p_1, p_\perp)$. We suppose that the potential V is the multiplication operator by the real-valued time-independent function $V(x)$. H_0 is called the free Stark Hamiltonian, and H is called a Stark Hamiltonian. It is well-known that H_0 is essentially self-adjoint on $\mathscr{S}(\boldsymbol{R}^n)$ (see e.g. Avron-Herbst [5]). The self-adjoint realization of H_0 is also denoted by H_0. Under a certain appropriate condition on V, the self-adjointness of H can be guaranteed. As is well-known, if V satisfies a short-range condition under the Stark effect that $|V(x)| \leq C\langle x \rangle^{-\gamma_0}$ holds for some $\gamma_0 > 1/2$, then the wave operators

$$W^\pm = \operatorname*{s-lim}_{t \to \pm\infty} e^{itH} e^{-itH_0} \tag{1.2}$$

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exist (see e.g. [5]). Here $\langle x\rangle = \sqrt{1+x^2}$. Then the scattering operator $S = S(V)$ is defined by

$$S = (W^+)^* W^-. \tag{1.3}$$

Roughly speaking, we are interested in the *widest* class of short-range potentials which can be determined uniquely by scattering operators. In this paper, via the Enss-Weder time-dependent method (see Enss-Weder [8]), we will give a certain appropriate class of short-range potentials which may not necessarily be the widest one but be very close to it, and is natural also in terms of direct scattering problems.

In order to state our results precisely, we make some preparations: We assume that $V(x)$ is represented as a sum of a very short-range part $V^{\mathrm{vs}}(x)$, a short-range part $V^{\mathrm{s}}(x)$ and a long-range part $V^{\mathrm{l}}(x)$ under the Stark effect:

$$V(x) = V^{\mathrm{vs}}(x) + V^{\mathrm{s}}(x) + V^{\mathrm{l}}(x). \tag{1.4}$$

We say that $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$ if $V^{\mathrm{vs}}(x)$ is a real-valued time-independent function and is decomposed into a sum of a singular part $V_1^{\mathrm{vs}}(x)$ and a regular part $V_2^{\mathrm{vs}}(x)$, and V_1^{vs} is compactly supported and belongs to $L^{q_0}(\boldsymbol{R}^n)$, where q_0 satisfies that $q_0 > n/2$ and $q_0 \geq 2$, and $V_2^{\mathrm{vs}} \in C^0(\boldsymbol{R}^n)$ is bounded in $\boldsymbol{R}^n$ and satisfies

$$\int_0^\infty \|V_2^{\mathrm{vs}}(x)F(|x|\geq R)\|_{\mathscr{B}(L^2)}\,dR < \infty. \tag{1.5}$$

Here $F(|x|\geq R)$ is the characteristic function of $\{x\in\boldsymbol{R}^n \mid |x|\geq R\}$. This condition yields

$$\int_0^\infty \|V^{\mathrm{vs}}(x)F(|x|\geq R)(1+K_0)^{-1}\|_{\mathscr{B}(L^2)}\,dR < \infty, \tag{1.6}$$

where

$$K_0 = p^2/2 = -\Delta/2 \tag{1.7}$$

is the free Schrödinger operator. As is well-known, this is equivalent to

$$\int_0^\infty \|V^{\mathrm{vs}}(x)(1+K_0)^{-1}F(|x|\geq R)\|_{\mathscr{B}(L^2)}\,dR < \infty \tag{1.8}$$

(see e.g. Reed-Simon [15]). Under the condition (1.8), the wave operators

$$\Omega^\pm = \operatorname*{s-lim}_{t\to\pm\infty} e^{itK}e^{-itK_0} \tag{1.9}$$

exist, and are asymptotically complete (see e.g. Enss [6]). Here $K = K_0 + V^{\mathrm{vs}}$ is a Schrödinger operator with a short-range potential V^{vs}. Then the scattering operator $\Sigma = \Sigma(V^{\mathrm{vs}})$ is defined by

$$\Sigma = (\Omega^+)^* \Omega^-. \tag{1.10}$$

By virtue of the results of [8], we know that the following holds: Let $V_1, V_2 \in \mathscr{V}^{\mathrm{vs}}$. If $\Sigma(V_1) = \Sigma(V_2)$, then $V_1 = V_2$.

We say that $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}(\tilde{\gamma}_0, \tilde{\gamma}_1)$ with $\tilde{\gamma}_0 \geq 1/2$ and $\tilde{\gamma}_1 \geq 1$ if $V^{\mathrm{s}}(x)$ is a real-valued time-independent function, belongs to $C^1(\boldsymbol{R}^n)$ and satisfies

$$|(\partial^\beta V^{\mathrm{s}})(x)| \leq C_\beta \langle x \rangle^{-\gamma_{|\beta|}}, \qquad |\beta| \leq 1, \tag{1.11}$$

with some γ_0, γ_1 such that $\tilde{\gamma}_0 < \gamma_0 \leq 1$ and $\tilde{\gamma}_1 < \gamma_1 \leq 1 + \gamma_0$. In Weder [17] and Adachi-Maehara [3], γ_0 and γ_1 were represented as γ and $1 + \alpha$, respectively. However, in this paper, we will use the above notation for the sake of clarification of our theory. If K is equal to $K_0 + V^{\mathrm{vs}} + V^{\mathrm{s}}$, then $\Omega^\pm$ in (1.9) do not exist generally, because of the long-range condition $\gamma_0 \leq 1$ for K_0. On the other hand, when H is equal to $H_0 + V^{\mathrm{vs}} + V^{\mathrm{s}}$, by virtue of the short-range condition $\gamma_0 > 1/2$ for H_0, it can be shown that $W^\pm$ in (1.2) exist, and are asymptotically complete (see e.g. Herbst [11] and Yajima [19]). Hence, we can introduce S in (1.3) instead of Σ in (1.10), as the scattering operator. Thus we mainly consider the case where $\tilde{\gamma}_0 = 1/2$. We note that if $a < b$, then $\mathscr{V}^{\mathrm{s}}(\tilde{\gamma}_0, b) \subsetneq \mathscr{V}^{\mathrm{s}}(\tilde{\gamma}_0, a)$. The condition $\gamma_1 > 1$ is necessary for introducing the v-dependent Graf-type modifier $M^{\mathrm{s}}_{G,v}(t) = e^{-i\int_0^t V^{\mathrm{s}}(vs + e_1 s^2/2)\,ds}$, which was first introduced in [3]. Hence we assume $\tilde{\gamma}_1 \geq 1$ beforehand.

Finally we say that $V^1 \in \mathscr{V}^1_D(\tilde{\gamma}_{D,0})$ with $\tilde{\gamma}_{D,0} \geq 1/4$ if $V^1(x)$ is a real-valued time-independent function, belongs to $C^2(\boldsymbol{R}^n)$ and satisfies

$$|(\partial^\beta V^1)(x)| \leq C_\beta \langle x \rangle^{-\gamma_D - |\beta|/2}, \qquad |\beta| \leq 2, \tag{1.12}$$

with some γ_D such that $\tilde{\gamma}_{D,0} < \gamma_D \leq 1/2$. If H is equal to $H_0 + V^{\mathrm{vs}} + V^{\mathrm{s}} + V^1$, then $W^\pm$ in (1.2) do not exist generally, because of the long-range condition $\gamma_D \leq 1/2$ for H_0. However, by virtue of the condition $\gamma_D > 1/4$, the Dollard-type modified wave operators

$$W_D^\pm = \operatorname*{s-lim}_{t \to \pm\infty} e^{itH} e^{-itH_0} e^{-i\int_0^t V^1(ps + e_1 s^2/2)\,ds} \tag{1.13}$$

exist, and are asymptotically complete (see e.g. Jensen-Yajima [13], White [18] and Adachi-Tamura [4]). $\mathscr{V}^1_D(\tilde{\gamma}_{D,0})$'s are classes of long-range potentials which are appropriate for the introduction of $W_D^\pm$.

We first consider the short-range case, that is, the case where $V^1 = 0$. As mentioned above, then $W^\pm$ in (1.2) exist, and $S = S(V)$ in (1.3) can be defined. The following result is one of those which we will report on in this paper:

THEOREM 1.1. *Let* $V_1, V_2 \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}(1/2, 5/4)$. *If* $S(V_1) = S(V_2)$, *then* $V_1 = V_2$.

Theorem 1.1 was first proved by Weder [17] for $\mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}(3/4, 1)$. However, the short-range parts V^{s}'s with $1/2 < \gamma_0 \le 3/4$ cannot be treated by the argument of [17] unfortunately. Later Nicoleau [14] proved this theorem under the condition that the short-range parts V^{s}'s belong to $C^{\infty}(\boldsymbol{R}^n)$ and satisfy

$$|(\partial^{\beta} V^{\mathrm{s}})(x)| \le C_{\beta} \langle x \rangle^{-\gamma_0 - |\beta|} \tag{1.14}$$

with some $\gamma_0 > 1/2$, and the additional condition $n \ge 3$. [14] is the first work which treated the case where $1/2 < \gamma_0 \le 3/4$ and suggested that the possible threshold with respect to $\tilde{\gamma}_0$ is equal to $1/2$. Here we note that $\gamma_1 = 1 + \gamma_0$ is supposed in [14]. After that, Adachi-Maehara [3] proved this theorem for $\mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}(1/2, 3/2)$ under the condition not $n \ge 3$ but $n \ge 2$ (see also Adachi-Kamada-Kazuno-Toratani [2] as for the case where time-dependent electric fields are decaying in $|t|$, and Valencia-Weder [16] as for the many body case in a constant electric field). In [3], for the sake of relaxing the smoothness condition on V^{s}'s, the v-dependent Graf-type modifier $M^{\mathrm{s}}_{G,v}(t) = e^{-i\int_0^t V^{\mathrm{s}}(vs + e_1 s^2/2)\,ds}$ was introduced instead of the Dollard-type modifier $e^{-i\int_0^t V^{\mathrm{s}}(p_{\perp} s + e_1 s^2/2)\,ds}$ which was utilized in [14]. Hence, how small $\tilde{\gamma}_1$ of $\mathscr{V}^{\mathrm{s}}(1/2, \tilde{\gamma}_1)$ in Theorem 1.1 can be taken has become a problem to be studied. In Adachi-Fujiwara-Ishida [1], which treated also the case where the electric fields are time-dependent, $\tilde{\gamma}_1$ was taken as $(15 - \sqrt{17})/8$, which is greater than $5/4$, by improving the estimates in a series of lemmas obtained in [3]. And, recently Ishida [12] stated that $\tilde{\gamma}_1$ could be taken as 1. In the direct scattering theory under the Stark effect, we often suppose that smooth short-range potentials V^{s}'s satisfy

$$|(\partial^{\beta} V^{\mathrm{s}})(x)| \le C_{\beta} \langle x \rangle^{-\gamma_0 - |\beta|/2} \tag{1.15}$$

with some $\gamma_0 > 1/2$ (see e.g. [13], [18] and [4]). From this viewpoint, one can expect that the possible threshold with respect to $\tilde{\gamma}_1$ is equal to $1/2 + 1/2 = 1$. But, since in [12] some misapplications of the Hölder inequality were made on key points in the argument unfortunately (for the detail, see Remark 2.1 in §2), it seems difficult to say that the results of [12] were obtained rigorously. Nevertheless, [12] does include a nice device for improving the results obtained in the previous works. In this paper, we utilize the device due to [12] as well as our methods which sharpen the estimates in a series of useful lemmas in obtaining the main results. By virtue of these, $\tilde{\gamma}_1$ in Theorem 1.1 can be taken as $5/4$ at most.

We next consider the long-range case, that is, the case where $V^{\mathrm{l}} \ne 0$. As mentioned above, since $\tilde{\gamma}_{D,0} \ge 1/4$, if $V^{\mathrm{l}} \in \mathscr{V}^{\mathrm{l}}_D(\tilde{\gamma}_{D,0})$, then the Dollard-type modified wave operators $W_D^{\pm}$ exist, and the Dollard-type modified scattering operator $S_D = S_D(V^1; V^{\mathrm{vs}} + V^{\mathrm{s}}) = S_D(V^1; V - V^1)$ is defined by

$$S_D = (W_D^+)^* W_D^-. \tag{1.16}$$

Then we also obtain the following result:

Theorem 1.2. *Suppose that $V^1 \in \mathscr{V}_D^1(3/8)$ is given. Let $V_1, V_2 \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}(1/2, 5/4)$. If $S_D(V^1; V_1) = S_D(V^1; V_2)$, then $V_1 = V_2$. Moreover, any one of the Dollard-type modified scattering operators S_D determines uniquely the total potential V.*

Theorem 1.2 was first proved by [3] for $\mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}(1/2, 3/2)$ under the condition that V^1 belongs to $C^2(\boldsymbol{R}^n)$ and satisfies

$$|(\partial^\beta V^1)(x)| \le C_\beta \langle x \rangle^{-\gamma_D - \mu|\beta|}, \qquad |\beta| \le 2 \tag{1.17}$$

with $0 < \gamma_D \le 1/2$ and $1 - \gamma_D < \mu \le 1$ (see also [2] as for the case where time-dependent electric fields are decaying in $|t|$, and [16] as for the many body case in a constant electric field). The condition $1 - \gamma_D < \mu$, that is, $\gamma_D + \mu > 1$ yields the existence of the Graf-type modified wave operators

$$W_G^{\pm} = \operatorname*{s-lim}_{t \to \pm\infty} e^{itH} e^{-itH_0} e^{-i\int_0^t V^1(e_1 s^2/2)ds} \tag{1.18}$$

(see Zorbas [20] and Graf [9]) as well as the Dollard-type wave operators $W_D^{\pm}$ without the additional condition $\gamma_D > 1/4$. For $V^1 \in \mathscr{V}_D^1(\tilde{\gamma}_{D,0})$, $W_G^{\pm}$ do not exist generally because $\gamma_D + 1/2 \le 1$. Hence, the case where $V^1 \in \mathscr{V}_D^1(\tilde{\gamma}_{D,0})$ was not treated in [3]. Later Theorem 1.2 was proved by [1] for $\mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}(1/2, \tilde{\gamma}_1)$ with $\tilde{\gamma}_1 = (29 - \sqrt{41})/16$, which is greater than $(15 - \sqrt{17})/8$, under the assumption that $V^1 \in \mathscr{V}_D^1(3/8)$, which is the same as the one in our Theorem 1.2. And, recently [12] stated $\tilde{\gamma}_1$ could be taken as 1, but, as mentioned above, it seems difficult to say that the results of [12] were obtained rigorously. We report that also in Theorem 1.2, $\tilde{\gamma}_1$ can be taken as 5/4 at most.

Now we will consider the case where $\tilde{\gamma}_1$ is smaller than 5/4, in particular, $\tilde{\gamma}_1 = 1$, which was considered in [12]. To this end, we need the C^2-regularity of V^{s}'s, which is stronger than the C^1-regularity of V^{s}'s imposed in Theorems 1.1 and 1.2. We will introduce the following subclasses of $\mathscr{V}^{\mathrm{s}}(1/2, \tilde{\gamma}_1)$:

We say that $V^{\mathrm{s}} \in \tilde{\mathscr{V}}^{\mathrm{s}}(1/2, \tilde{\gamma}_1, \tilde{\gamma}_2)$ with $\tilde{\gamma}_1 \ge 1$ and $\tilde{\gamma}_2 \ge 1$ if $V^{\mathrm{s}}(x)$ is a real-valued time-independent function, belongs to $C^2(\boldsymbol{R}^n)$ and satisfies

$$|(\partial^\beta V^{\mathrm{s}})(x)| \le C_\beta \langle x \rangle^{-\gamma_{|\beta|}}, \qquad |\beta| \le 2, \tag{1.19}$$

with some γ_0, γ_1, γ_2 such that $\tilde{\gamma}_0 = 1/2 < \gamma_0 \le 1$, $\tilde{\gamma}_1 < \gamma_1 \le \gamma_0 + 1$ and $\tilde{\gamma}_2 < \gamma_2 \le \gamma_1 + 1$. Since we need the condition $\gamma_2 > 1$ in our analysis (see §4), we assume $\tilde{\gamma}_2 \ge 1$ beforehand.

Now we will state that Theorems 1.1 and 1.2 with replacing $\mathscr{V}^{s}(1/2,5/4)$ by $\tilde{\mathscr{V}}^{s}(1/2,1,5/4)$ also hold:

THEOREM 1.3. *Let* $V_1, V_2 \in \mathscr{V}^{vs} + \tilde{\mathscr{V}}^{s}(1/2,1,5/4)$. *If* $S(V_1) = S(V_2)$, *then* $V_1 = V_2$.

THEOREM 1.4. *Suppose that* $V^1 \in \mathscr{V}_D^1(3/8)$ *is given. Let* $V_1, V_2 \in \mathscr{V}^{vs} + \tilde{\mathscr{V}}^{s}(1/2,1,5/4)$. *If* $S_D(V^1; V_1) = S_D(V^1; V_2)$, *then* $V_1 = V_2$. *Moreover, any one of the Dollard-type modified scattering operators* S_D *determines uniquely the total potential* V.

In their proofs, the Dollard-type modifier $M_D^s(t) = e^{-i\int_0^t V^s(ps+e_1 s^2/2)ds}$, which is slightly different from $e^{-i\int_0^t V^s(p_\perp s+e_1 s^2/2)ds}$ introduced in [14], will be substituted for the v-dependent Graf-type modifier $M_{G,v}^s(t) = e^{-i\int_0^t V^s(vs+e_1 s^2/2)ds}$ used in the proofs of Theorems 1.1 and 1.2. $M_D^s(t)$ has an advantage over $M_{G,v}^s(t)$ for V^s of C^2 by virtue of the Baker-Campbell-Hausdorff formula, although it may not necessarily do so for V^s of only C^1. We will report on it in this paper.

The plan of this paper is as follows: In §2, we consider the case where $V^s \in \mathscr{V}^{s}(1/2,5/4)$ and $V^1 = 0$. In §3, we consider the case where $V^s \in \mathscr{V}^{s}(1/2,5/4)$ and $V^1 \neq 0$. In §4, we consider the case where $V^s \in \tilde{\mathscr{V}}^{s}(1/2,1,5/4)$.

In the following sections, $\|\cdot\|$ and $(\cdot,\cdot)$ stand for the L^2-norm and the L^2-inner product, respectively, for the sake of brevity.

2. The case where $V^s \in \mathscr{V}^{s}(1/2,5/4)$ and $V^1 = 0$

Throughout this section, we suppose $V^1 = 0$. The main purpose of this section is showing the following reconstruction formula, which yields Theorem 1.1.

THEOREM 2.1. *Let* $\hat{v} \in \boldsymbol{R}^n$ *be given such that* $|\hat{v}| = 1$ *and* $|\hat{v}\cdot e_1| < 1$. *Put* $v = |v|\hat{v}$. *Let* $\eta > 0$ *be given, and* $\Phi_0, \Psi_0 \in L^2(\boldsymbol{R}^n)$ *be such that* $\hat{\Phi}_0, \hat{\Psi}_0 \in C_0^\infty(\boldsymbol{R}^n)$ *with* $\operatorname{supp}\hat{\Phi}_0, \operatorname{supp}\hat{\Psi}_0 \subset \{\xi \in \boldsymbol{R}^n \mid |\xi| < \eta\}$. *Put* $\Phi_v = e^{iv\cdot x}\Phi_0$ *and* $\Psi_v = e^{iv\cdot x}\Psi_0$. *Let* $V^{vs} \in \mathscr{V}^{vs}$ *and* $V^s \in \mathscr{V}^{s}(1/2,5/4)$. *Then the following holds:*

$$\lim_{|v|\to\infty} |v|(i[S, p_j]\Phi_v, \Psi_v) = \int_{-\infty}^{\infty} [(V^{vs}(x+\hat{v}\tau)p_j\Phi_0, \Psi_0) - (V^{vs}(x+\hat{v}\tau)\Phi_0, p_j\Psi_0) + i((\partial_j V^s)(x+\hat{v}\tau)\Phi_0, \Psi_0)]d\tau \tag{2.1}$$

for $1 \le j \le n$.

We will make preparations for the proof of Theorem 2.1. We first need the following proposition due to Enss [7] (see Proposition 2.10 of [7]):

PROPOSITION 2.2. *For any* $f \in C_0^\infty(\boldsymbol{R}^n)$ *with* $\operatorname{supp} f \subset \{x \in \boldsymbol{R}^n \mid |x| < \eta\}$ *for some* $\eta > 0$*, and any* $l \in \boldsymbol{N}$*, there exists a constant* C_l *dependent on* f *only such that*

$$\|F(x \in \mathscr{M}')e^{-itK_0}f(p - v)F(x \in \mathscr{M})\|_{\mathscr{B}(L^2)} \le C_l(1 + r + |t|)^{-l} \tag{2.2}$$

for $v \in \boldsymbol{R}^n$*,* $t \in \boldsymbol{R}$ *and measurable sets* $\mathscr{M}$*,* $\mathscr{M}'$ *with the property that* $r = \operatorname{dist}(\mathscr{M}', \mathscr{M} + vt) - \eta|t| \ge 0$*. Here* $F(x \in \mathscr{M})$ *stands for the characteristic function of* $\mathscr{M}$*.*

The following lemma was already obtained in [17] (see also [3]):

LEMMA 2.3. *Let* v *and* Φ_v *be as in Theorem* 2.1. *Then*

$$\int_{-\infty}^{\infty} \|V^{\mathrm{vs}}(x)e^{-itH_0}\Phi_v\| dt = O(|v|^{-1}) \tag{2.3}$$

holds as $|v| \to \infty$ *for* $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$.

In the proof of this lemma, the estimate of $|vt + e_1t^2/2|$ plays an important role. Here we recall the argument about the estimate of $|vt + e_1t^2/2|$ in [3]: Put $\delta = |\hat{v} \cdot e_1| < 1$. $|vt + e_1t^2/2|^2 = |v|^2t^2 + t^4/4 + v \cdot e_1t^3$ can be estimated as

$$\begin{aligned} |vt + e_1t^2/2|^2 &\ge |v|^2|t|^2 + |t|^4/4 - \delta|v|\,|t|^3 \\ &= |t|^2(|t| - 2\delta|v|)^2/4 + (1 - \delta^2)|v|^2|t|^2 \\ &\ge (1 - \delta^2)|v|^2|t|^2. \end{aligned} \tag{2.4}$$

(2.4) is used in the proof of Lemma 2.3. Here we note that

$$\begin{aligned} |vt + e_1t^2/2|^2 &\ge |t|^2(\delta|t| - 2|v|)^2/4 + (1 - \delta^2)|t|^4/4 \\ &\ge (1 - \delta^2)|t|^4/4 \end{aligned} \tag{2.5}$$

can be also obtained. Based on the above estimates, we conclude that

$$|vt + e_1t^2/2| \ge \max\{\sqrt{1 - \delta^2}|v|\,|t|, (\sqrt{1 - \delta^2}/2)|t|^2\} \tag{2.6}$$

holds.

The following lemma is an improvement of Lemma 2.2 of [3] and Lemma 3.4 with $\mu = 0$ of [1]. This is one of the keys in this section. Here we note that the class $\mathscr{V}^{\mathrm{s}}(1/2,1)$ of short-range potentials considered in Lemmas 2.4 and 2.5 is wider than the class $\mathscr{V}^{\mathrm{s}}(1/2,5/4)$ supposed in Theorem 2.1:

LEMMA 2.4. *Let v and Φ_v be as in Theorem 2.1, and $\varepsilon > 0$. Then*

$$\int_{-\infty}^{\infty} \|\{V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt + e_1 t^2/2)\} e^{-itH_0} \Phi_v\| dt = O(|v|^{\max\{-1, -2(\gamma_1 - 1) + \varepsilon\}}) \quad (2.7)$$

holds as $|v| \to \infty$ for $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}(1/2,1)$.

PROOF. For the sake of brevity, we put

$$I = \|\{V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt + e_1 t^2/2)\} e^{-itH_0} \Phi_v\|.$$

By virtue of the Avron-Herbst formula

$$e^{-itH_0} = e^{-it^3/6} e^{itx_1} e^{-ip_1 t^2/2} e^{-itK_0} \quad (2.8)$$

(see e.g. [5]) and

$$e^{-iv\cdot x} e^{-itK_0} e^{iv\cdot x} = e^{-iv^2 t/2} e^{-ip\cdot vt} e^{-itK_0}, \quad (2.9)$$

we have

$$\begin{aligned} &V^{\mathrm{s}}(x) e^{-itH_0} \Phi_v \\ &\quad = (e^{-it^3/6} e^{itx_1} e^{-ip_1 t^2/2}) V^{\mathrm{s}}(x + e_1 t^2/2) e^{-itK_0} \Phi_v \\ &\quad = (e^{-it^3/6} e^{itx_1} e^{-ip_1 t^2/2}) e^{iv\cdot x} (e^{-iv^2 t/2} e^{-ip\cdot vt}) V^{\mathrm{s}}(x + vt + e_1 t^2/2) e^{-itK_0} \Phi_0. \end{aligned}$$

Such a relation has been used also in the proof of Lemma 2.3, which is omitted in this paper. Then I can be written as

$$I = \|\{V^{\mathrm{s}}(x + vt + e_1 t^2/2) - V^{\mathrm{s}}(vt + e_1 t^2/2)\} e^{-itK_0} \Phi_0\|.$$

Taking $f \in C_0^\infty(\boldsymbol{R}^n)$ such that $0 \le f \le 1$, $f\hat{\Phi}_0 = \hat{\Phi}_0$ and $\operatorname{supp} f \subset \{\xi \in \boldsymbol{R}^n \mid |\xi| < \eta\}$, we see that $\Phi_0 = f(p)\Phi_0$. Now let us take $g \in C_0^\infty(\boldsymbol{R}^n)$ such that $0 \le g \le 1$, $g(y) = 1$ $(|y| \le 3)$ and $g(y) = 0$ $(|y| \ge 4)$, and introduce

$$\tilde{V}^{\mathrm{s}}_{|v|,t}(x) = V^{\mathrm{s}}(x + vt + e_1 t^2/2) g((\lambda_1 |v| \langle t \rangle)^{-1} x),$$

where $\lambda_1 > 0$ is a small constant which will be determined below. In the definition of $\tilde{V}^{\mathrm{s}}_{|v|,t}(x)$, taking $(\lambda_1 |v| \langle t \rangle)^{-1} x$ as the argument of g, which was taken as $(\lambda_1 |v|\,|t|)^{-1} x$ in [3], is one of our devices, and by virtue of this, we can eliminate the singularity of $\tilde{V}^{\mathrm{s}}_{|v|,t}(x)$ and its derivatives at $t = 0$. Since $\tilde{V}^{\mathrm{s}}_{|v|,t}(0) = V^{\mathrm{s}}(vt + e_1 t^2/2)$ and $\Phi_0 = f(p)\Phi_0$, we have

$$I \le \|\overline{V}^{\mathrm{s}}_{|v|,t}(x) e^{-itK_0} f(p) \Phi_0\| + \|\{\tilde{V}^{\mathrm{s}}_{|v|,t}(x) - \tilde{V}^{\mathrm{s}}_{|v|,t}(0)\} e^{-itK_0} \Phi_0\|, \quad (2.10)$$

where $\overline{V}^{s}_{|v|,t}(x) = V^{s}(x + vt + e_1 t^2/2) - \tilde{V}^{s}_{|v|,t}(x)$. As for the first term of the inequality (2.10), we estimate it in the same way as in [17]:

$$\|\overline{V}^{s}_{|v|,t}(x)e^{-itK_0}f(p)\Phi_0\| \le I_1 + I_2 + I_3,$$

where

$$I_1 = \|\overline{V}^{s}_{|v|,t}(x)F(|x| \ge 3\lambda_1|v|\,|t|)e^{-itK_0}f(p)F(|x| \le \lambda_1|v|\,|t|)\Phi_0\|,$$

$$I_2 = \|\overline{V}^{s}_{|v|,t}(x)F(|x| \ge 3\lambda_1|v|\,|t|)e^{-itK_0}f(p)F(|x| > \lambda_1|v|\,|t|)\Phi_0\|,$$

$$I_3 = \|\overline{V}^{s}_{|v|,t}(x)F(|x| < 3\lambda_1|v|\,|t|)e^{-itK_0}f(p)\Phi_0\|.$$

As for I_1, by virtue of Proposition 2.2 and $\|\overline{V}^{s}_{|v|,t}(x)\|_{\mathscr{B}(L^2)} \le \|V^{s}\|_{L^\infty}$,

$$\begin{aligned} I_1 &\le \|V^{s}\|_{L^\infty}\|F(|x| \ge 3\lambda_1|v|\,|t|)e^{-itK_0}f(p)F(|x| \le \lambda_1|v|\,|t|)\|_{\mathscr{B}(L^2)}\|\Phi_0\| \\ &\le C(1 + \lambda_1|v|\,|t|)^{-2} \end{aligned}$$

holds for $|v| > \eta/\lambda_1$. Here we used $0 \le 1 - g \le 1$. As for I_2, we have

$$\begin{aligned} I_2 &\le \|V^{s}\|_{L^\infty}\|F(|x| > \lambda_1|v|\,|t|)\langle x\rangle^{-2}\|_{\mathscr{B}(L^2)}\|\langle x\rangle^2\Phi_0\| \\ &\le C(1 + \lambda_1|v|\,|t|)^{-2}. \end{aligned}$$

As for I_3, by virtue of the definition of $\tilde{V}^{s}_{|v|,t}(x)$, we have $I_3 = 0$, because $\{1 - g((\lambda_1|v|\langle t\rangle)^{-1}x)\}F(|x| < 3\lambda_1|v|\,|t|) \equiv 0$. Based on the above observations, we obtain

$$\int_{-\infty}^{\infty}\|\overline{V}^{s}_{|v|,t}(x)e^{-itK_0}f(p)\Phi_0\|dt \le C\int_{-\infty}^{\infty}(1 + \lambda_1|v|\,|t|)^{-2}dt = O(|v|^{-1}) \qquad (2.11)$$

by an appropriate change of variables. Now we will estimate the second term of the inequality (2.10) by using the device of [12]. By

$$\tilde{V}^{s}_{|v|,t}(x) - \tilde{V}^{s}_{|v|,t}(0) = \int_0^1 \frac{d}{d\theta}\{\tilde{V}^{s}_{|v|,t}(\theta x)\}d\theta = \left(\int_0^1(\nabla\tilde{V}^{s}_{|v|,t})(\theta x)d\theta\right)\cdot x,$$

we have

$$\begin{aligned} \|\{\tilde{V}^{s}_{|v|,t}(x) - \tilde{V}^{s}_{|v|,t}(0)\}e^{-itK_0}\Phi_0\| &\le \left\|\int_0^1(\nabla\tilde{V}^{s}_{|v|,t})(\theta x)d\theta\right\|_{\mathscr{B}(L^2)}\|xe^{-itK_0}\Phi_0\| \\ &\le \|\nabla\tilde{V}^{s}_{|v|,t}\|_{L^\infty}\|(x + pt)\Phi_0\| \\ &\le \|\nabla\tilde{V}^{s}_{|v|,t}\|_{L^\infty}(\|x\Phi_0\| + |t|\,\|p\Phi_0\|). \end{aligned}$$

Here we used $e^{itK_0}xe^{-itK_0} = x + pt$. Dealing with $\|xe^{-itK_0}\Phi_0\|$ directly without using cut-offs is the device of [12]. Now we will watch $\|\nabla \tilde{V}^{s}_{|v|,t}\|_{L^\infty}$, where

$$(\nabla \tilde{V}^{s}_{|v|,t})(x) = (\nabla V^{s})(x + vt + e_1t^2/2)g((\lambda_1|v|\langle t\rangle)^{-1}x)$$
$$+ V^{s}(x + vt + e_1t^2/2)(\nabla g)((\lambda_1|v|\langle t\rangle)^{-1}x)(\lambda_1|v|\langle t\rangle)^{-1}.$$

For a while, suppose $|t| \geq 1$. Then $\langle t\rangle = \sqrt{1+t^2} \leq \sqrt{t^2+t^2} = \sqrt{2}|t|$ holds. Note that $x \in \operatorname{supp}\{g(\cdot/(\lambda_1|v|\langle t\rangle))\}$ satisfies $|x| \leq 4\lambda_1|v|\langle t\rangle$. For such x's,

$$\begin{aligned}|x + vt + e_1t^2/2|^2 &= |x + vt|^2 + t^4/4 + t^2(x + vt)\cdot e_1\\ &\geq (|vt| - |x|)^2 + |t|^4/4 - |t|^2(|x_1| + \delta|v|\,|t|)\\ &\geq (|v|\,|t| - 4\lambda_1|v|\langle t\rangle)^2 + |t|^4/4 - |t|^2(4\lambda_1|v|\langle t\rangle + \delta|v|\,|t|)\\ &\geq \{(1 - 4\sqrt{2}\lambda_1)|v|\,|t|\}^2 + |t|^4/4 - |t|^2\{(4\sqrt{2}\lambda_1 + \delta)|v|\,|t|\}\end{aligned}$$

holds. If λ_1 is so small that $4\sqrt{2}\lambda_1 \leq (1-\delta)/4$, then

$$\begin{aligned}|x + vt + e_1t^2/2|^2 &\geq ((3+\delta)/4)^2|v|^2|t|^2 + |t|^4/4 - ((1+3\delta)/4)|v|\,|t|^3\\ &= |t|^2\{|t| - ((1+3\delta)/2)|v|\}^2/4 + ((1-\delta^2)/2)|v|^2|t|^2\\ &\geq ((1-\delta^2)/2)|v|^2|t|^2\end{aligned}$$

holds (cf. (2.4)). We also have

$$\begin{aligned}|x + vt + e_1t^2/2|^2 &\geq |t|^2(((1+3\delta)/\{2(3+\delta)\})|t| - ((3+\delta)/4)|v|)^2\\ &\quad + (2(1-\delta^2)/(3+\delta)^2)|t|^4\\ &\geq (2(1-\delta^2)/(3+\delta)^2)|t|^4\end{aligned}$$

(cf. (2.5)). Hence, by taking λ_1 as $(1-\delta)/(16\sqrt{2})$,

$$|x + vt + e_1t^2/2| \geq \max\{c_1|v|\,|t|, c_2|t|^2\} \tag{2.12}$$

holds for $|t| \geq 1$, where $c_1 = \sqrt{2(1-\delta^2)}/2$ and $c_2 = \sqrt{2(1-\delta^2)}/(3+\delta)$. The estimate (2.12) yields

$$|x + vt + e_1t^2/2| \geq (c_1|v|\,|t|)^{\nu}(c_2|t|^2)^{1-\nu} = c_1^{\nu}c_2^{1-\nu}|v|^{\nu}|t|^{2-\nu} \tag{2.13}$$

for $0 \leq \nu \leq 1$. Throughout this paper, we will frequently use this interpolation estimate on $|x + vt + e_1t^2/2|$ parameterized by ν's such that $0 \leq \nu \leq 1$ in our optimization arguments. Taking account of the boundedness of $\nabla \tilde{V}^{s}_{|v|,t}$ for

$|t| < 1$, we see that

$$\begin{aligned}\|\nabla\tilde{V}^{s}_{|v|,t}\|_{L^\infty} &\le C_1(1+|v|^{\nu_1}|t|^{2-\nu_1})^{-\gamma_1} + C_2(1+|v|^{\nu_2}|t|^{2-\nu_2})^{-\gamma_0}|v|^{-1}(1+|t|)^{-1}\\ &\le C_1'(1+|v|^{\nu_1/(2-\nu_1)}|t|)^{-\gamma_1(2-\nu_1)} + C_2'(1+|v|^{\nu_2/(2-\nu_2)}|t|)^{-\gamma_0(2-\nu_2)}\\ &\quad\times |v|^{\nu_2/(2-\nu_2)-1}(|v|^{\nu_2/(2-\nu_2)} + |v|^{\nu_2/(2-\nu_2)}|t|)^{-1}\\ &\le C_1'(1+|v|^{\nu_1/(2-\nu_1)}|t|)^{-\gamma_1(2-\nu_1)}\\ &\quad + C_2'|v|^{\nu_2/(2-\nu_2)-1}(1+|v|^{\nu_2/(2-\nu_2)}|t|)^{-\gamma_0(2-\nu_2)-1} \end{aligned} \tag{2.14}$$

holds for $0 \le \nu_1, \nu_2 \le 1$ and $|v| \ge 1$. Then

$$\begin{aligned}&\int_{-\infty}^{\infty}\|\nabla\tilde{V}^{s}_{|v|,t}\|_{L^\infty}\|x\Phi_0\|dt\\ &\quad\le C_1'\int_{-\infty}^{\infty}(1+|v|^{\nu_1/(2-\nu_1)}|t|)^{-\gamma_1(2-\nu_1)}dt\\ &\qquad + C_2'|v|^{\nu_2/(2-\nu_2)-1}\int_{-\infty}^{\infty}(1+|v|^{\nu_2/(2-\nu_2)}|t|)^{-\gamma_0(2-\nu_2)-1}dt\\ &\quad= O(|v|^{-\nu_1/(2-\nu_1)}) + O(|v|^{\nu_2/(2-\nu_2)-1-\nu_2/(2-\nu_2)})\\ &\quad= O(|v|^{-\nu_1/(2-\nu_1)}) + O(|v|^{-1})\end{aligned}$$

can be obtained by an appropriate change of variables under the conditions $-\gamma_1(2-\nu_1) < -1$ and $-\gamma_0(2-\nu_2) - 1 < -1$, i.e. $\nu_1 < 2 - 1/\gamma_1$ and $\nu_2 < 2$. Since $2 - 1/\gamma_1 > 1$ and

$$\min_{0\le\nu_1\le1}(-\nu_1/(2-\nu_1)) = -1,$$

we have

$$\int_{-\infty}^{\infty}\|\nabla\tilde{V}^{s}_{|v|,t}\|_{L^\infty}\|x\Phi_0\|dt = O(|v|^{-1}) \tag{2.15}$$

by the optimization argument. In the same way,

$$\begin{aligned}&\int_{-\infty}^{\infty}\|\nabla\tilde{V}^{s}_{|v|,t}\|_{L^\infty}|t|\,\|p\Phi_0\|dt\\ &\quad\le C_3'\int_{-\infty}^{\infty}(1+|v|^{\nu_3/(2-\nu_3)}|t|)^{-\gamma_1(2-\nu_3)}|t|dt\\ &\qquad + C_4'|v|^{\nu_4/(2-\nu_4)-1}\int_{-\infty}^{\infty}(1+|v|^{\nu_4/(2-\nu_4)}|t|)^{-\gamma_0(2-\nu_4)-1}|t|dt\end{aligned}$$

$$= O(|v|^{-2v_3/(2-v_3)}) + O(|v|^{v_4/(2-v_4)-1-2v_4/(2-v_4)})$$
$$= O(|v|^{-2v_3/(2-v_3)}) + O(|v|^{-1-v_4/(2-v_4)})$$

can be obtained by an appropriate change of variables under the conditions $-\gamma_1(2-v_3)+1<-1$ and $-\gamma_0(2-v_4)-1+1<-1$, i.e. $v_3<2-2/\gamma_1$ and $v_4<2-1/\gamma_0$. Since $0<2-2/\gamma_1\le 2-2/(1+\gamma_0)<2/3$, $0<2-1/\gamma_0\le 1$,

$$\inf_{0\le v_3<2-2/\gamma_1}(-2v_3/(2-v_3)) = -2(\gamma_1-1),$$
$$\inf_{0\le v_4<2-1/\gamma_0}(-1-v_4/(2-v_4)) = -2\gamma_0,$$

and $-2\gamma_0\le -2(\gamma_1-1)$, we have

$$\int_{-\infty}^{\infty}\|\nabla\tilde{V}^{\mathrm{s}}_{|v|,t}\|_{L^\infty}|t|\,\|p\Phi_0\|dt = O(|v|^{-2(\gamma_1-1)+\varepsilon}) \tag{2.16}$$

with $\varepsilon>0$, by the optimization argument. (2.15) and (2.16) yield

$$\int_{-\infty}^{\infty}\|\{\tilde{V}^{\mathrm{s}}_{|v|,t}(x)-\tilde{V}^{\mathrm{s}}_{|v|,t}(0)\}e^{-itK_0}\Phi_0\|dt = O(|v|^{-1}) + O(|v|^{-2(\gamma_1-1)+\varepsilon}). \tag{2.17}$$

Therefore, (2.7) can be shown by (2.11) and (2.17).

Remark 2.1. Taking $(\lambda_1|v|\langle t\rangle)^{-1}x$ in place of $(\lambda_1|v|\,|t|)^{-1}x$ as the argument of g in the definition of $\tilde{V}^{\mathrm{s}}_{|v|,t}(x)$ yields the regularity of $(\nabla\tilde{V}^{\mathrm{s}}_{|v|,t})(x)$ also at $t=0$. Hence, we do not have to divide the integral region $\boldsymbol{R}$ of t into any v-dependent neighborhood of 0 and its complement (see below) as in [17], [3], [1], [12] and so on. This fact makes the optimization argument in the above proof rather simple.

Here we will review the argument in the proof of Proposition 2.4 of [12] corresponding to our Lemma 2.4: Let $0<\sigma_1<1$. Taking account of

$$\int_{-\infty}^{\infty} I\,dt = \int_{|t|<|v|^{-\sigma_1}} I\,dt + \int_{|t|\ge|v|^{-\sigma_1}} I\,dt$$

with I in the proof of our Lemma 2.4, we have only to estimate the two terms of the right-hand side separately, as mentioned above. I can be estimated as $I\le\hat{I}_1+\hat{I}_2+\hat{I}_3$, where $\hat{I}_j$'s are defined by replacing $\overline{V}^{\mathrm{s}}_{|v|,t}(x)$ in the definition of I_j's by $\hat{V}^{\mathrm{s}}_{v,t}(x) = V^{\mathrm{s}}(x+vt+e_1t^2/2) - V^{\mathrm{s}}(vt+e_1t^2/2)$:

$$\hat{I}_1 = \|\hat{V}^{\mathrm{s}}_{v,t}(x)F(|x|\ge 3\lambda_1|v|\,|t|)e^{-itK_0}f(p)F(|x|\le\lambda_1|v|\,|t|)\Phi_0\|,$$
$$\hat{I}_2 = \|\hat{V}^{\mathrm{s}}_{v,t}(x)F(|x|\ge 3\lambda_1|v|\,|t|)e^{-itK_0}f(p)F(|x|>\lambda_1|v|\,|t|)\Phi_0\|,$$
$$\hat{I}_3 = \|\hat{V}^{\mathrm{s}}_{v,t}(x)F(|x|<3\lambda_1|v|\,|t|)e^{-itK_0}f(p)\Phi_0\|.$$

$\hat{I}_1 + \hat{I}_2 \le C(1 + \lambda_1 |v|\,|t|)^{-2}$ yields

$$\int_{|t| < |v|^{-\sigma_1}} (\hat{I}_1 + \hat{I}_2) dt = O(|v|^{-1}).$$

On the other hand, $\hat{I}_3 \le C(1 + \lambda_1 |v|\,|t|)^{-\gamma_1}(1 + |t|)$ yields

$$\int_{|t| < |v|^{-\sigma_1}} \hat{I}_3 \, dt = O(|v|^{-1}) + o(|v|^{-1}) = O(|v|^{-1}).$$

Let $0 < \sigma_2 < 1$. I can be also estimated as $I \le \hat{I}_4 + \hat{I}_5 + \hat{I}_6$, where

$$\hat{I}_4 = \| \hat{V}^{\mathrm{s}}_{v,t}(x) F(|x| \ge 3|v|^{\sigma_2}|t|) e^{-itK_0} f(p) F(|x| \le |v|^{\sigma_2}|t|) \Phi_0 \|,$$

$$\hat{I}_5 = \| \hat{V}^{\mathrm{s}}_{v,t}(x) F(|x| \ge 3|v|^{\sigma_2}|t|) e^{-itK_0} f(p) F(|x| > |v|^{\sigma_2}|t|) \Phi_0 \|,$$

$$\hat{I}_6 = \| \hat{V}^{\mathrm{s}}_{v,t}(x) F(|x| < 3|v|^{\sigma_2}|t|) e^{-itK_0} f(p) \Phi_0 \|.$$

$\hat{I}_4 + \hat{I}_5 \le C(1 + |v|^{\sigma_2}|t|)^{-2}$ yields

$$\int_{|t| \ge |v|^{-\sigma_1}} (\hat{I}_4 + \hat{I}_5) dt = O(|v|^{\sigma_1 - 2\sigma_2}).$$

Here we suppose that $|v|$ is so large that $4\sqrt{2}|v|^{\sigma_2 - 1} \le (1 - \delta)/4$ holds. Then $\hat{I}_6 \le C(1 + c_1|v|\,|t|)^{-\gamma_1}(1 + |t|)$ and $\hat{I}_6 \le C(1 + c_2|t|^2)^{-\gamma_1}(1 + |t|)$ hold with c_1 and c_2 in the proof of our Lemma 2.4. These estimates can be obtained rigorously. In [12], Ishida stated that the Hölder inequality yields the estimate (2.44) in the proof of Proposition 2.4 of [12]

$$\int_{|t| \ge |v|^{-\sigma_1}} \hat{I}_6 \, dt \le \left(\int_{|t| \ge |v|^{-\sigma_1}} \hat{I}_6^{q_1} \, dt \right)^{1/q_1} \left(\int_{|t| \ge |v|^{-\sigma_1}} \hat{I}_6^{q_2} \, dt \right)^{1/q_2}$$

with $q_1 > 1$ such that $1/q_1 + 1/q_2 = 1$ (see also (3.46) in the proof of Proposition 3.8 of [12]). However, an appropriate application of the Hölder inequality yields a trivial estimate

$$\int_{|t| \ge |v|^{-\sigma_1}} \hat{I}_6 \, dt \le \left(\int_{|t| \ge |v|^{-\sigma_1}} \hat{I}_6 \, dt \right)^{1/q_1} \left(\int_{|t| \ge |v|^{-\sigma_1}} \hat{I}_6 \, dt \right)^{1/q_2}$$

only, because $\hat{I}_6 = \hat{I}_6^{1/q_1} \times \hat{I}_6^{1/q_2}$. Therefore we think that the results in Propositions 2.4 and 3.8 of [12] have not been obtained rigorously yet. Following our argument, we will use the estimate

$$\hat{I}_6 \le C_1(1 + c_1^{\nu_1} c_2^{1-\nu_1} |v|^{\nu_1} |t|^{2-\nu_1})^{-\gamma_1} + C_2(1 + c_1^{\nu_2} c_2^{1-\nu_2} |v|^{\nu_2} |t|^{2-\nu_2})^{-\gamma_1} |t|$$

for $0 \le \nu_1, \nu_2 \le 1$. As for ν_1, $-\gamma_1(2 - \nu_1) \le -\gamma_1 < -1$ holds; while, as for ν_2, we need that ν_2 satisfies $-\gamma_1(2 - \nu_2) + 1 < -1$, that is, $\nu_2 < 2 - 2/\gamma_1$. For the sake of simplicity, we assume $2 - 2/\gamma_1 \le 1$, that is, $\gamma_1 \le 2$. Then we have the estimate

$$\int_{|t| \ge |v|^{-\sigma_1}} \hat{I}_6 \, dt = O(|v|^{-\nu_1\gamma_1 - \sigma_1\{1-(2-\nu_1)\gamma_1\}}) + O(|v|^{-\nu_2\gamma_1 - \sigma_1\{2-(2-\nu_2)\gamma_1\}})$$

$$= O(|v|^{-\sigma_1(1-2\gamma_1)-(1+\sigma_1)\nu_1\gamma_1}) + O(|v|^{-\sigma_1(2-2\gamma_1)-(1+\sigma_1)\nu_2\gamma_1}).$$

Taking $\nu_1 = 1$ and $\nu_2 = 2 - 2/\gamma_1$, we obtain a finer estimate

$$\int_{|t| \ge |v|^{-\sigma_1}} \hat{I}_6 \, dt = O(|v|^{-\sigma_1(1-2\gamma_1)-(1+\sigma_1)\gamma_1}) + O(|v|^{-\sigma_1(2-2\gamma_1)-(1+\sigma_1)(2-2/\gamma_1)\gamma_1+\varepsilon})$$

$$= O(|v|^{-\gamma_1-\sigma_1(1-\gamma_1)}) + O(|v|^{-(2\gamma_1-2)+\varepsilon}).$$

Taking σ_2 such that $\sigma_1 - 2\sigma_2 = -\gamma_1 - \sigma_1(1-\gamma_1)$, that is, $\sigma_2 = \{\gamma_1 + \sigma_1(2-\gamma_1)\}/2$,

$$\int_{|t| \ge |v|^{-\sigma_1}} I \, dt = O(|v|^{-\gamma_1-\sigma_1(1-\gamma_1)}) + O(|v|^{-2(\gamma_1-1)+\varepsilon})$$

can be obtained. Since one can take σ_1 such that $-\gamma_1 - \sigma_1(1-\gamma_1) \le -(2\gamma_1 - 2) + \varepsilon$, that is, $\sigma_1 \le (2 + \varepsilon - \gamma_1)/(\gamma_1 - 1)$, we finally obtain

$$\int_{|t| \ge |v|^{-\sigma_1}} I \, dt = O(|v|^{-2(\gamma_1-1)+\varepsilon}),$$

which yields (2.17) in the proof of our Lemma 2.4.

As in [3], we introduce auxiliary wave operators

$$\Omega_{G,v}^{\mathrm{s},\pm} = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH} U_{G,v}^{\mathrm{s}}(t),$$

where $U_{G,v}^{\mathrm{s}}(t) = e^{-itH_0} M_{G,v}^{\mathrm{s}}(t)$ and $M_{G,v}^{\mathrm{s}}(t) = e^{-i\int_0^t V^{\mathrm{s}}(vs+e_1 s^2/2)\,ds}$. We know that

$$\Omega_{G,v}^{\mathrm{s},\pm} = W^{\pm} I_{G,v}^{\mathrm{s},\pm}, \qquad I_{G,v}^{\mathrm{s},\pm} = \operatorname*{s-lim}_{t\to\pm\infty} M_{G,v}^{\mathrm{s}}(t)$$

exist, by virtue of (2.5). As emphasized in [3], the v-dependent Graf-type modifier $M_{G,v}^{\mathrm{s}}(t)$ commutes with any operators. This fact will be used frequently. Then the following can be obtained as in [3], so we omit the proof:

LEMMA 2.5. *Let v and Φ_v be as in Theorem 2.1, and $\varepsilon > 0$. Then*

$$\sup_{t\in\boldsymbol{R}} \|(e^{-itH}\Omega_{G,v}^{\mathrm{s},-} - U_{G,v}^{\mathrm{s}}(t))\Phi_v\| = O(|v|^{\max\{-1,-2(\gamma_1-1)+\varepsilon\}}) \tag{2.18}$$

holds as $|v| \to \infty$ for $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$ and $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}(1/2, 1)$.

Now we will show Theorem 2.1:

Proof (Proof of Theorem 2.1). Since the proof is quite similar to the one of Theorem 2.1 of [3], we give its sketch only.

Suppose that $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$ and $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}(1/2, 5/4)$. We first note that S is represented as

$$S = (W^+)^* W^- = I_{G,v}^{\mathrm{s}} (\Omega_{G,v}^{\mathrm{s},+})^* \Omega_{G,v}^{\mathrm{s},-},$$

$$I_{G,v}^{\mathrm{s}} = I_{G,v}^{\mathrm{s},+} (I_{G,v}^{\mathrm{s},-})^* = e^{-i\int_{-\infty}^{\infty} V^{\mathrm{s}}(vs + e_1 s^2/2) ds}.$$

Noting $[S, p_j] = [S - I_{G,v}^{\mathrm{s}}, p_j - v_j]$, $(p_j - v_j)\Phi_v = (p_j \Phi_0)_v$ and

$$\begin{aligned} i(S - I_{G,v}^{\mathrm{s}})\Phi_v &= I_{G,v}^{\mathrm{s}} i(\Omega_{G,v}^{\mathrm{s},+} - \Omega_{G,v}^{\mathrm{s},-})^* \Omega_{G,v}^{\mathrm{s},-} \Phi_v \\ &= I_{G,v}^{\mathrm{s}} \int_{-\infty}^{\infty} U_{G,v}^{\mathrm{s}}(t)^* V_t e^{-itH} \Omega_{G,v}^{\mathrm{s},-} \Phi_v \, dt \end{aligned}$$

with $V_t = V^{\mathrm{vs}}(x) + V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt + e_1 t^2/2)$, we have

$$|v|(i[S, p_j]\Phi_v, \Psi_v) = I_{G,v}^{\mathrm{s}} \{I(v) + R(v)\}$$

with

$$\begin{aligned} I(v) = |v| \int_{-\infty}^{\infty} &[(V_t U_{G,v}^{\mathrm{s}}(t)(p_j \Phi_0)_v, U_{G,v}^{\mathrm{s}}(t)\Psi_v) \\ &- (V_t U_{G,v}^{\mathrm{s}}(t)\Phi_v, U_{G,v}^{\mathrm{s}}(t)(p_j \Psi_0)_v)] dt, \\ R(v) = |v| \int_{-\infty}^{\infty} &[((e^{-itH}\Omega_{G,v}^{\mathrm{s},-} - U_{G,v}^{\mathrm{s}}(t))(p_j \Phi_0)_v, V_t U_{G,v}^{\mathrm{s}}(t)\Psi_v) \\ &- ((e^{-itH}\Omega_{G,v}^{\mathrm{s},-} - U_{G,v}^{\mathrm{s}}(t))\Phi_v, V_t U_{G,v}^{\mathrm{s}}(t)(p_j \Psi_0)_v)] dt. \end{aligned}$$

By Lemmas 2.3, 2.4 and 2.5, we have

$$|R(v)| = O(|v|^{1+2\max\{-1, -2(\gamma_1 - 1)+\varepsilon\}}) = O(|v|^{\max\{-1, 5-4\gamma_1+2\varepsilon\}}). \tag{2.19}$$

Then we need the condition $5 - 4\gamma_1 < 0$ in order to get $\lim_{|v|\to\infty} R(v) = 0$, because one can take $\varepsilon > 0$ so small that $5 - 4\gamma_1 + 2\varepsilon < 0$. This is equivalent to $\gamma_1 > 5/4$.

The rest of the proof is the same as in [17] and [3]. So we omit it.

By virtue of Theorem 2.1 and the Plancherel formula associated with the Radon transform (see Helgason [10]), Theorem 1.1 can be shown in the same way as in the proof of Theorem 1.2 of [17] (see also [8]). Thus we omit its proof.

3. The case where $V^{s} \in \mathscr{V}^{s}(1/2, 5/4)$ and $V^{1} \neq 0$

The main purpose of this section is showing the following reconstruction formula, which yields Theorem 1.2. For the sake of brevity, we put

$$U_D(t) = e^{-itH_0} M_D(t), \qquad M_D(t) = e^{-i\int_0^t V^1(ps+e_1 s^2/2)\,ds}. \tag{3.1}$$

THEOREM 3.1. *Let $\hat{v} \in \boldsymbol{R}^n$ be given such that $|\hat{v}| = 1$ and $|\hat{v} \cdot e_1| < 1$. Put $v = |v|\hat{v}$. Let $\eta > 0$ be given, and $\Phi_0, \Psi_0 \in L^2(\boldsymbol{R}^n)$ be such that $\hat{\Phi}_0, \hat{\Psi}_0 \in C_0^{\infty}(\boldsymbol{R}^n)$ with $\operatorname{supp} \hat{\Phi}_0, \operatorname{supp} \hat{\Psi}_0 \subset \{\xi \in \boldsymbol{R}^n \mid |\xi| < \eta\}$. Put $\Phi_v = e^{iv\cdot x}\Phi_0$ and $\Psi_v = e^{iv\cdot x}\Psi_0$. Let $V^{vs} \in \mathscr{V}^{vs}$, $V^{s} \in \mathscr{V}^{s}(1/2, 5/4)$, and $V^1 \in \mathscr{V}_D^1(3/8)$. Then the following holds:*

$$\begin{aligned}
&\lim_{|v|\to\infty} \left\{ |v|(i[S_D, p_j]\Phi_v, \Psi_v) - \int_{-\infty}^{\infty} i((\partial_j V^1)(x) U_D(t)\Phi_v, U_D(t)\Psi_v)\,dt \right\} \\
&\quad = \int_{-\infty}^{\infty} [(V^{vs}(x + \hat{v}\tau) p_j \Phi_0, \Psi_0) - (V^{vs}(x + \hat{v}\tau)\Phi_0, p_j \Psi_0) \\
&\qquad + i((\partial_j V^{s})(x + \hat{v}\tau)\Phi_0, \Psi_0)]\,d\tau
\end{aligned} \tag{3.2}$$

for $1 \le j \le n$.

We first need the following lemma:

LEMMA 3.2. *Let v and Φ_v be as in Theorem 3.1, and $V^1 \in \mathscr{V}_D^1(1/4)$ with $\gamma_D < 1/2$. Then, for $0 \le \nu_1, \nu_2, \nu_3 \le 1$, there exists a positive constant C such that*

$$\begin{aligned}
&\|\langle x\rangle^2 M_D(t)\Phi_v\| = \|\langle x\rangle^2 M_{D,v}(t)\Phi_0\| \\
&\qquad \le C(1 + |v|^{-(2\gamma_D+1)\nu_1}|t|^{4-(2\gamma_D+1)(2-\nu_1)} + |v|^{-(\gamma_D+1)\nu_2}|t|^{3-(\gamma_D+1)(2-\nu_2)} \\
&\qquad\qquad + |v|^{-(\gamma_D+1/2)\nu_3}|t|^{2-(\gamma_D+1/2)(2-\nu_3)})
\end{aligned} \tag{3.3}$$

holds as $|v| \to \infty$, where $M_{D,v}(t) = e^{-iv\cdot x} M_D(t) e^{iv\cdot x} = e^{-i\int_0^t V^1(ps+vs+e_1 s^2/2)\,ds}$.

PROOF. First of all, we introduce

$$\tilde{V}^1_{|v|,t}(x) = V^1(x + vt + e_1 t^2/2) g((\lambda_1 |v| \langle t\rangle)^{-1} x),$$

by mimicking the definition of $\tilde{V}^{s}_{|v|,t}(x)$ in the proof of Lemma 2.4. Since $\operatorname{supp} \hat{\Phi}_0 \subset \{\xi \in \boldsymbol{R}^n \mid |\xi| < \eta\}$,

$$M_{D,v}(t)\Phi_0 = e^{-i\int_0^t \tilde{V}^1_{|v|,s}(ps)\,ds}\Phi_0 \tag{3.4}$$

holds for $|v| \ge \eta/(3\lambda_1)$. Since $x = i\nabla_p$,

$$e^{i\int_0^t \tilde{V}^1_{|v|,s}(ps)\,ds} x e^{-i\int_0^t \tilde{V}^1_{|v|,s}(ps)\,ds} = x + \int_0^t s(\nabla \tilde{V}^1_{|v|,s})(ps)\,ds \tag{3.5}$$

holds. This yields

$$
\begin{aligned}
&e^{i\int_0^t \tilde{V}^1_{|v|,s}(ps)ds}\langle x\rangle^2 e^{-i\int_0^t \tilde{V}^1_{|v|,s}(ps)ds}\\
&\quad = 1 + \left(x + \int_0^t s(\nabla \tilde{V}^1_{|v|,s})(ps)ds\right)^2\\
&\quad = \langle x\rangle^2 + i\left(\int_0^t s^2(\Delta \tilde{V}^1_{|v|,s})(ps)ds\right) + 2\left(\int_0^t s(\nabla \tilde{V}^1_{|v|,s})(ps)ds\right)\cdot x\\
&\qquad + \left(\int_0^t s(\nabla \tilde{V}^1_{|v|,s})(ps)ds\right)^2.
\end{aligned}
$$

Now we will estimate $\|\nabla \tilde{V}^1_{|v|,t}\|_{L^\infty}$ and $\|\Delta \tilde{V}^1_{|v|,t}\|_{L^\infty}$. In the same way as in the proof of Lemma 2.4, for $0 \le v_1, v_2, v_3, v_4, v_5 \le 1$ and $|v| \ge 1$,

$$
\begin{aligned}
\|\nabla \tilde{V}^1_{|v|,t}\|_{L^\infty} &\le C_1(1 + |v|^{v_1/(2-v_1)}|t|)^{-(\gamma_D+1/2)(2-v_1)}\\
&\quad + C_2|v|^{v_2/(2-v_2)-1}(1 + |v|^{v_2/(2-v_2)}|t|)^{-\gamma_D(2-v_2)-1}
\end{aligned} \tag{3.6}
$$

and

$$
\begin{aligned}
\|\Delta \tilde{V}^1_{|v|,t}\|_{L^\infty} &\le C_3(1 + |v|^{v_3/(2-v_3)}|t|)^{-(\gamma_D+1)(2-v_3)}\\
&\quad + C_4|v|^{v_4/(2-v_4)-1}(1 + |v|^{v_4/(2-v_4)}|t|)^{-(\gamma_D+1/2)(2-v_4)-1}\\
&\quad + C_5|v|^{2v_5/(2-v_5)-2}(1 + |v|^{v_5/(2-v_5)}|t|)^{-\gamma_D(2-v_5)-2}
\end{aligned} \tag{3.7}
$$

can be obtained. By noting $1 - (\gamma_D + 1/2)(2 - v_1) \ge -2\gamma_D > -1$ and $1 - \gamma_D(2 - v_2) - 1 \ge -2\gamma_D > -1$,

$$
\begin{aligned}
&\left\|\int_0^t s(\nabla \tilde{V}^1_{|v|,s})(ps)ds\right\|_{\mathscr{B}(L^2)}\\
&\quad \le C_1 \int_0^{|t|} s(|v|^{v_1/(2-v_1)}s)^{-(\gamma_D+1/2)(2-v_1)}ds\\
&\qquad + C_2|v|^{v_2/(2-v_2)-1}\int_0^{|t|} s(|v|^{v_2/(2-v_2)}s)^{-\gamma_D(2-v_2)-1}ds\\
&\quad = C_1'|v|^{-(\gamma_D+1/2)v_1}|t|^{2-(\gamma_D+1/2)(2-v_1)} + C_2'|v|^{-\gamma_D v_2-1}|t|^{1-\gamma_D(2-v_2)}\\
&\quad = C_1'|v|^{1-2\gamma_D}(|v|^{-1}|t|)^{2-(\gamma_D+1/2)(2-v_1)} + C_2'|v|^{-2\gamma_D}(|v|^{-1}|t|)^{1-\gamma_D(2-v_2)}
\end{aligned}
$$

can be obtained. Taking account of

$$[1-2\gamma_D, 3/2-\gamma_D] = \{2-(\gamma_D+1/2)(2-\nu_1) \,|\, 0 \le \nu_1 \le 1\}$$
$$\supset [1-2\gamma_D, 1-\gamma_D] = \{1-\gamma_D(2-\nu_2) \,|\, 0 \le \nu_2 \le 1\},$$

we obtain the estimate

$$\left\| \int_0^t s(\nabla \tilde{V}^1_{|v|,s})(ps)ds \right\|_{\mathscr{B}(L^2)} \le C_1''|v|^{-(\gamma_D+1/2)\nu_1}|t|^{2-(\gamma_D+1/2)(2-\nu_1)} \tag{3.8}$$

for $0 \le \nu_1 \le 1$. By noting $2-(\gamma_D+1)(2-\nu_3) \ge -2\gamma_D > -1$, $2-(\gamma_D+1/2)\cdot(2-\nu_4)-1 \ge -2\gamma_D > -1$ and $2-\gamma_D(2-\nu_5)-2 \ge -2\gamma_D > -1$,

$$\begin{aligned}
&\left\| \int_0^t s^2(\Delta \tilde{V}^1_{|v|,s})(ps)ds \right\|_{\mathscr{B}(L^2)} \\
&\le C_3'|v|^{-(\gamma_D+1)\nu_3}|t|^{3-(\gamma_D+1)(2-\nu_3)} + C_4'|v|^{-(\gamma_D+1/2)\nu_4-1}|t|^{2-(\gamma_D+1/2)(2-\nu_4)} \\
&\quad + C_5'|v|^{-\gamma_D\nu_5-2}|t|^{1-\gamma_D(2-\nu_5)} \\
&= C_3'|v|^{1-2\gamma_D}(|v|^{-1}|t|)^{3-(\gamma_D+1)(2-\nu_3)} + C_4'|v|^{-2\gamma_D}(|v|^{-1}|t|)^{2-(\gamma_D+1/2)(2-\nu_4)} \\
&\quad + C_5'|v|^{-1-2\gamma_D}(|v|^{-1}|t|)^{1-\gamma_D(2-\nu_5)}
\end{aligned}$$

can be obtained. Taking account of

$$[1-2\gamma_D, 2-\gamma_D] = \{3-(\gamma_D+1)(2-\nu_3) \,|\, 0 \le \nu_3 \le 1\}$$
$$\supset [1-2\gamma_D, 3/2-\gamma_D] = \{2-(\gamma_D+1/2)(2-\nu_4) \,|\, 0 \le \nu_4 \le 1\}$$
$$\supset [1-2\gamma_D, 1-\gamma_D] = \{1-\gamma_D(2-\nu_5) \,|\, 0 \le \nu_5 \le 1\},$$

we obtain the estimate

$$\left\| \int_0^t s^2(\Delta \tilde{V}^1_{|v|,s})(ps)ds \right\|_{\mathscr{B}(L^2)} \le C_3''|v|^{-(\gamma_D+1)\nu_3}|t|^{3-(\gamma_D+1)(2-\nu_3)} \tag{3.9}$$

for $0 \le \nu_3 \le 1$.

Based on the above observations, the lemma can be proved.

Then the following lemma can be shown as in [3].

LEMMA 3.3. *Let v and Φ_v be as in Theorem* 3.1, *and* $V^1 \in \mathscr{V}^1_D(1/4)$. *Then*

$$\int_{-\infty}^{\infty} \|V^{\mathrm{vs}}(x)U_D(t)\Phi_v\|dt = O(|v|^{-1}) \tag{3.10}$$

holds as $|v| \to \infty$ *for* $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$.

PROOF. For the sake of brevity, we put $I = \|V^{\mathrm{vs}}(x)U_D(t)\Phi_v\|$. For simplicity, we suppose $\gamma_D < 1/2$. Since by virtue of the Avron-Herbst formula (2.8), I can be written as

$$\begin{aligned} I &= \|V^{\mathrm{vs}}(x + vt + e_1t^2/2)e^{-itK_0}M_{D,v}(t)\Phi_0\| \\ &= \|V^{\mathrm{vs}}(x + vt + e_1t^2/2)(1 + K_0)^{-1}e^{-itK_0}f(p)M_{D,v}(t)(1 + K_0)\Phi_0\|, \end{aligned}$$

we estimate this in the same way as in the proof of Lemma 2.3, which is omitted in this paper (cf. the proof of Lemma 2.4).

$$\|V^{\mathrm{vs}}(x)U_D(t)\Phi_v\| \le I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \|V^{\mathrm{vs}}(x + vt + e_1t^2/2)(1 + K_0)^{-1} \\ &\quad \times F(|x| \ge 3\lambda_1|v|\,|t|)e^{-itK_0}f(p)F(|x| \le \lambda_1|v|\,|t|)M_{D,v}(t)(1 + K_0)\Phi_0\|, \\ I_2 &= \|V^{\mathrm{vs}}(x + vt + e_1t^2/2)(1 + K_0)^{-1} \\ &\quad \times F(|x| \ge 3\lambda_1|v|\,|t|)e^{-itK_0}f(p)F(|x| > \lambda_1|v|\,|t|)M_{D,v}(t)(1 + K_0)\Phi_0\| \\ I_3 &= \|V^{\mathrm{vs}}(x + vt + e_1t^2/2)(1 + K_0)^{-1} \\ &\quad \times F(|x| < 3\lambda_1|v|\,|t|)e^{-itK_0}f(p)M_{D,v}(t)(1 + K_0)\Phi_0\|. \end{aligned}$$

As for I_1, by virtue of Proposition 2.2 and $\|V^{\mathrm{vs}}(x + vt + e_1t^2/2)\cdot(1 + K_0)^{-1}\|_{\mathscr{B}(L^2)} = \|V^{\mathrm{vs}}(1 + K_0)^{-1}\|_{\mathscr{B}(L^2)}$,

$$I_1 \le C(1 + \lambda_1|v|\,|t|)^{-2}$$

holds for $|v| > \eta/\lambda_1$, which yields

$$\int_{-\infty}^{\infty} I_1\,dt = O(|v|^{-1}).$$

As for I_3, by that (2.12) holds for $|t| \ge 1$ and $|x| < 3\lambda_1|v|\,|t| < 4\lambda_1|v|\langle t\rangle$, and (1.8),

$$\int_{-\infty}^{\infty} I_3\,dt = O(|v|^{-1})$$

can be obtained. As for I_2, by virtue of Lemma 3.2,

$$\begin{aligned} I_2 &\le C(1 + \lambda_1|v|\,|t|)^{-2}(1 + |v|^{-(2\gamma_D+1)v_1}|t|^{4-(2\gamma_D+1)(2-v_1)} \\ &\quad + |v|^{-(\gamma_D+1)v_2}|t|^{3-(\gamma_D+1)(2-v_2)} + |v|^{-(\gamma_D+1/2)v_3}|t|^{2-(\gamma_D+1/2)(2-v_3)}) \end{aligned}$$

holds for $0 \le v_1, v_2, v_3 \le 1$ (cf. I_2 in the proof of Lemma 2.3). Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} I_2\, dt &= O(|v|^{-1}) + O(|v|^{-(2\gamma_D+1)v_1-\{5-(2\gamma_D+1)(2-v_1)\}}) \\ &\quad + O(|v|^{-(\gamma_D+1)v_2-\{4-(\gamma_D+1)(2-v_2)\}}) \\ &\quad + O(|v|^{-(\gamma_D+1/2)v_3-\{3-(\gamma_D+1/2)(2-v_3)\}}) \\ &= O(|v|^{-1}) + O(|v|^{-3+4\gamma_D-2(2\gamma_D+1)v_1}) + O(|v|^{-2+2\gamma_D-2(\gamma_D+1)v_2}) \\ &\quad + O(|v|^{-2+2\gamma_D-2(\gamma_D+1/2)v_3})\end{aligned}$$

can be obtained by an appropriate change of variables under the conditions $-2+4-(2\gamma_D+1)(2-v_1) < -1$, $-2+3-(\gamma_D+1)(2-v_2) < -1$ and $-2+2-(\gamma_D+1/2)(2-v_3) < -1$, i.e. $v_1 < 2-3/(2\gamma_D+1)$, $v_2 < 2-2/(\gamma_D+1)$ and $v_3 < 2-1/(\gamma_D+1/2)$. Since $0 < 2-3/(2\gamma_D+1) < 1/2$, $2/5 < 2-2/(\gamma_D+1) < 2/3$, $2/3 < 2-1/(\gamma_D+1/2) < 1$, and

$$\inf_{0\le v_1<2-3/(2\gamma_D+1)} (-3+4\gamma_D-2(2\gamma_D+1)v_1) = -1-4\gamma_D,$$

$$\inf_{0\le v_2<2-2/(\gamma_D+1)} (-2+2\gamma_D-2(\gamma_D+1)v_2) = -2-2\gamma_D,$$

$$\inf_{0\le v_3<2-1/(\gamma_D+1/2)} (-2+2\gamma_D-2(\gamma_D+1/2)v_3) = -2-2\gamma_D,$$

we obtain

$$\int_{-\infty}^{\infty} I_2\, dt = O(|v|^{-1}).$$

Based on the above observations, the lemma can be proved.

The following lemma can be also proved in the same way as in the proof of Lemmas 2.4 and 3.3.

LEMMA 3.4. *Let* v *and* Φ_v *be as in Theorem* 3.1, $\varepsilon > 0$, *and* $V^1 \in \mathscr{V}_D^1(1/4)$. *Then*

$$\int_{-\infty}^{\infty} \|\{V^s(x) - V^s(vt+e_1t^2/2)\}U_D(t)\Phi_v\|dt = O(|v|^{\max\{-1,-2(\gamma_1-1)+\varepsilon\}}) \tag{3.11}$$

holds as $|v| \to \infty$ *for* $V^s \in \mathscr{V}^s(1/2,1)$.

PROOF. Since the proof is quite similar to the proof of Lemma 2.4, we sketch it: For the sake of brevity, we put $I = \|\{V^s(x) -$

$V^s(vt + e_1 t^2/2)\} U_D(t)\Phi_v\|$. For simplicity, we suppose $\gamma_D < 1/2$. As in the proof of Lemma 2.4, we estimate I as

$$\begin{aligned} I &= \|\{V^s(x + vt + e_1 t^2/2) - V^s(vt + e_1 t^2/2)\} e^{-itK_0} M_{D,v}(t)\Phi_0\| \\ &\le \|\bar{V}^s_{|v|,t}(x) e^{-itK_0} f(p) M_{D,v}(t)\Phi_0\| \\ &\quad + \|\{\tilde{V}^s_{|v|,t}(x) - \tilde{V}^s_{|v|,t}(0)\} e^{-itK_0} M_{D,v}(t)\Phi_0\|. \end{aligned} \tag{3.12}$$

Here we used $M_{D,v}(t)\Phi_0 = f(p) M_{D,v}(t)\Phi_0$. As for the first term of the inequality (3.12), we estimate it as

$$\|\bar{V}^s_{|v|,t}(x) e^{-itK_0} f(p) M_{D,v}(t)\Phi_0\| \le I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \|\bar{V}^s_{|v|,t}(x) F(|x| \ge 3\lambda_1 |v|\,|t|) e^{-itK_0} f(p) F(|x| \le \lambda_1 |v|\,|t|) M_{D,v}(t)\Phi_0\|, \\ I_2 &= \|\bar{V}^s_{|v|,t}(x) F(|x| \ge 3\lambda_1 |v|\,|t|) e^{-itK_0} f(p) F(|x| > \lambda_1 |v|\,|t|) M_{D,v}(t)\Phi_0\|, \\ I_3 &= \|\bar{V}^s_{|v|,t}(x) F(|x| < 3\lambda_1 |v|\,|t|) e^{-itK_0} f(p) M_{D,v}(t)\Phi_0\|. \end{aligned}$$

As for I_1, by virtue of Proposition 2.2,

$$I_1 \le C(1 + \lambda_1 |v|\,|t|)^{-2}$$

holds for $|v| > \eta/\lambda_1$. As for I_2, by virtue of Lemma 3.2,

$$\begin{aligned} I_2 \le C(1 + \lambda_1 |v|\,|t|)^{-2} (1 &+ |v|^{-(2\gamma_D+1)v_1} |t|^{4-(2\gamma_D+1)(2-v_1)} \\ &+ |v|^{-(\gamma_D+1)v_2} |t|^{3-(\gamma_D+1)(2-v_2)} + |v|^{-(\gamma_D+1/2)v_3} |t|^{2-(\gamma_D+1/2)(2-v_3)}) \end{aligned}$$

holds. As for I_3, by the definition of $\tilde{V}^s_{|v|,t}(x)$, $I_3 = 0$ holds. Combining these estimates with the results in the proof of Lemma 3.3, we obtain

$$\int_{-\infty}^{\infty} \|\bar{V}^s_{|v|,t}(x) e^{-itK_0} f(p) M_{D,v}(t)\Phi_0\| dt = O(|v|^{-1}). \tag{3.13}$$

As for the second term of the inequality (3.12), in the same way as in the proof of Lemma 2.4, we see that

$$\begin{aligned} &\|\{\tilde{V}^s_{|v|,t}(x) - \tilde{V}^s_{|v|,t}(0)\} e^{-itK_0} M_{D,v}(t)\Phi_0\| \\ &\quad \le \|\nabla \tilde{V}^s_{|v|,t}\|_{L^\infty} \|(x + pt) M_{D,v}(t)\Phi_0\| \\ &\quad \le \|\nabla \tilde{V}^s_{|v|,t}\|_{L^\infty} \left(\|x\Phi_0\| + \left\| \left(\int_0^t s(\nabla \tilde{V}^1_{|v|,s})(ps) ds \right) \Phi_0 \right\| + |t|\,\|p\Phi_0\| \right) \end{aligned}$$

holds. Here we used (3.4) and (3.5). As for the estimates of

$$\int_{-\infty}^{\infty} \|\nabla \tilde{V}^{\mathrm{s}}_{|v|,t}\|_{L^\infty} \|x\Phi_0\| dt, \qquad \int_{-\infty}^{\infty} \|\nabla \tilde{V}^{\mathrm{s}}_{|v|,t}\|_{L^\infty} |t| \, \|p\Phi_0\| dt,$$

we already obtained (2.15) and (2.16) in the proof of Lemma 2.4. Hence, we have only to estimate

$$\int_{-\infty}^{\infty} \|\nabla \tilde{V}^{\mathrm{s}}_{|v|,t}\|_{L^\infty} \left\| \left(\int_0^t s(\nabla \tilde{V}^{1}_{|v|,s})(ps)ds \right) \Phi_0 \right\| dt.$$

To this end, we consider

$$\begin{aligned}
\tilde{I}_1 &= \int_{-\infty}^{\infty} (1 + |v|^{v_1/(2-v_1)}|t|)^{-\gamma_1(2-v_1)} |v|^{-(\gamma_D+1/2)v_3} |t|^{2-(\gamma_D+1/2)(2-v_3)} dt \\
&= O(|v|^{-(\gamma_D+1/2)v_3 - \{3-(\gamma_D+1/2)(2-v_3)\}v_1/(2-v_1)}), \\
\tilde{I}_2 &= \int_{-\infty}^{\infty} |v|^{v_2/(2-v_2)-1} (1 + |v|^{v_2/(2-v_2)}|t|)^{-\gamma_0(2-v_2)-1} \\
&\quad \times |v|^{-(\gamma_D+1/2)v_4} |t|^{2-(\gamma_D+1/2)(2-v_4)} dt \\
&= O(|v|^{v_2/(2-v_2)-1-(\gamma_D+1/2)v_4 - \{3-(\gamma_D+1/2)(2-v_4)\}v_2/(2-v_2)}) \\
&= O(|v|^{-1-(\gamma_D+1/2)v_4 - \{2-(\gamma_D+1/2)(2-v_4)\}v_2/(2-v_2)}),
\end{aligned}$$

which can be obtained by an appropriate change of variables under the conditions $-\gamma_1(2 - v_1) + 2 - (\gamma_D + 1/2)(2 - v_3) < -1$ and $-\gamma_0(2 - v_2) - 1 + 2 - (\gamma_D + 1/2)(2 - v_4) < -1$, i.e. $\gamma_1 v_1 + (\gamma_D + 1/2)v_3 < 2\{(\gamma_1 - 1) + \gamma_D\}$ and $\gamma_0 v_2 + (\gamma_D + 1/2)v_4 < 2(\gamma_0 + \gamma_D) - 1$. Noting

$$\gamma_1 + (\gamma_D + 1/2) - 2\{(\gamma_1 - 1) + \gamma_D\} = 5/2 - \gamma_1 - \gamma_D > 3/2 - \gamma_0 - 1/2 \geq 0$$

by $\gamma_0 \leq 1$, $\gamma_1 \leq 1 + \gamma_0$ and $\gamma_D < 1/2$, we see

$$\{\gamma_1 v_1 + (\gamma_D + 1/2)v_3 \,|\, 0 \leq v_1, \ v_3 \leq 1\} \supset [0, 2\{(\gamma_1 - 1) + \gamma_D\}].$$

We also note that for v_1 and v_3 such that $\gamma_1 v_1 + (\gamma_D + 1/2)v_3 = 2\{(\gamma_1 - 1) + \gamma_D\}$,

$$\begin{aligned}
-(\gamma_D &+ 1/2)v_3 - \{3 - (\gamma_D + 1/2)(2 - v_3)\}v_1/(2 - v_1) \\
&= -(\gamma_D + 1/2)v_3 - \{2 - 2\gamma_D + (\gamma_D + 1/2)v_3\} \\
&\quad \times \frac{2\{(\gamma_1 - 1) + \gamma_D\} - (\gamma_D + 1/2)v_3}{2\gamma_1 - 2\{(\gamma_1 - 1) + \gamma_D\} + (\gamma_D + 1/2)v_3} \\
&= -2\{(\gamma_1 - 1) + \gamma_D\}
\end{aligned}$$

holds. This yields

$$\inf_{v_1, v_3}(-(\gamma_D+1/2)v_3-\{3-(\gamma_D+1/2)(2-v_3)\}v_1/(2-v_1))$$
$$=-2\{(\gamma_1-1)+\gamma_D\}. \tag{3.14}$$

Noting

$$\gamma_0+(\gamma_D+1/2)-\{2(\gamma_0+\gamma_D)-1\}=3/2-\gamma_0-\gamma_D>0$$

by $\gamma_0\le 1$ and $\gamma_D<1/2$, we see

$$\{\gamma_0 v_2+(\gamma_D+1/2)v_4 \,|\, 0\le v_2,\ v_4\le 1\}\supset[0,2(\gamma_0+\gamma_D)-1].$$

We also note that for v_2 and v_4 such that $\gamma_0 v_2+(\gamma_D+1/2)v_4=2(\gamma_0+\gamma_D)-1$,

$$\begin{aligned}&-1-(\gamma_D+1/2)v_4-\{2-(\gamma_D+1/2)(2-v_4)\}v_2/(2-v_2)\\&=-1-(\gamma_D+1/2)v_4-\{1-2\gamma_D+(\gamma_D+1/2)v_4\}\\&\quad\times\frac{\{2(\gamma_0+\gamma_D)-1\}-(\gamma_D+1/2)v_4}{2\gamma_0-\{2(\gamma_0+\gamma_D)-1\}+(\gamma_D+1/2)v_4}\\&=-1-\{2(\gamma_0+\gamma_D)-1\}=-2(\gamma_0+\gamma_D)\end{aligned}$$

holds. This yields

$$\inf_{v_2, v_4}(-1-(\gamma_D+1/2)v_4-\{2-(\gamma_D+1/2)(2-v_4)\}v_2/(2-v_2))$$
$$=-2(\gamma_0+\gamma_D). \tag{3.15}$$

(3.14), (3.15) and $-2(\gamma_0+\gamma_D)\le-2\{(\gamma_1-1)+\gamma_D\}$ imply

$$\int_{-\infty}^{\infty}\|\nabla\tilde{V}^{s}_{|v|,t}\|_{L^\infty}\left\|\left(\int_0^t s(\nabla\tilde{V}^{1}_{|v|,s})(ps)ds\right)\Phi_0\right\|dt=O(|v|^{-2\{(\gamma_1-1)+\gamma_D\}+\varepsilon}) \tag{3.16}$$

with $\varepsilon>0$.

By (3.13), (2.15), (2.16) and (3.16), we obtain (3.11).

The following lemma is the key in this section.

Lemma 3.5. *Let v and Φ_v be as in Theorem 3.1, $\varepsilon>0$, and $V^1\in\mathscr{V}^1_D(1/4)$. Then*

$$\int_{-\infty}^{\infty}\|\{V^1(x)-V^1(pt-e_1t^2/2)\}U_D(t)\Phi_v\|dt=O(|v|^{-(4\gamma_D-1)+\varepsilon}) \tag{3.17}$$

holds as $|v|\to\infty$.

Proof. For the sake of brevity, we put

$$I = \|\{V^1(x) - V^1(pt - e_1t^2/2)\}U_D(t)\Phi_v\|.$$

For simplicity, we suppose $\gamma_D < 1/2$. We first note that by virtue of the Avron-Herbst formula (2.8),

$$\begin{aligned} e^{-itH_0} i\frac{d}{dt}(M_D(t)) &= e^{-itH_0}V^1(pt + e_1t^2/2)M_D(t) \\ &= V^1((p - e_1t)t + e_1t^2/2)e^{-itH_0}M_D(t) \\ &= V^1(pt - e_1t^2/2)U_D(t) \end{aligned} \tag{3.18}$$

holds. In the same way as in the proof of Lemma 3.4, I can be written as

$$\begin{aligned} I &= \|\{V^1(x + e_1t^2/2) - V^1(pt + e_1t^2/2)\}e^{-itK_0}M_D(t)\Phi_v\| \\ &= \|\{V^1(x + vt + e_1t^2/2) - V^1(pt + vt + e_1t^2/2)\}e^{-itK_0}M_{D,v}(t)\Phi_0\|. \end{aligned}$$

Now we will deal with this by using $\tilde{V}^1_{|v|,t}(x)$ which is introduced in the proof of Lemma 3.2, and mimicking the argument in the proof of Lemma 3.4. Hence, we estimate it as

$$\begin{aligned} I &= \|\{V^1(x + vt + e_1t^2/2) - \tilde{V}^1_{|v|,t}(pt)\}e^{-itK_0}M_{D,v}(t)\Phi_0\| \\ &\le \|\overline{V}^1_{|v|,t}(x)e^{-itK_0}f(p)M_{D,v}(t)\Phi_0\| \\ &\quad + \|\{\tilde{V}^1_{|v|,t}(x) - \tilde{V}^1_{|v|,t}(pt)\}e^{-itK_0}M_{D,v}(t)\Phi_0\| \end{aligned} \tag{3.19}$$

for $|v| \ge \eta/(3\lambda_1)$, where $\overline{V}^1_{|v|,t}(x) = V^1(x + vt + e_1t^2/2) - \tilde{V}^1_{|v|,t}(x)$. As for the first term of the inequality (3.19), in the same way as in the proof of Lemma 3.4,

$$\int_{-\infty}^{\infty} \|\overline{V}^1_{|v|,t}(x)e^{-itK_0}f(p)M_{D,v}(t)\Phi_0\|dt = O(|v|^{-1}) \tag{3.20}$$

can be obtained, because $I_3 = 0$ also in the long-range case. As for the second term of the inequality (3.19), we first note that by virtue of the Baker-Campbell-Hausdorff formula, $\tilde{V}^1_{|v|,t}(x) - \tilde{V}^1_{|v|,t}(pt)$ can be written as

$$\begin{aligned} \tilde{V}^1_{|v|,t}(x) - \tilde{V}^1_{|v|,t}(pt) &= \left(\int_0^1 (\nabla\tilde{V}^1_{|v|,t})(\theta x + (1-\theta)pt)d\theta\right)\cdot(x - pt) \\ &\quad + \frac{i}{2}\int_0^1 t(\Delta\tilde{V}^1_{|v|,t})(\theta x + (1-\theta)pt)d\theta. \end{aligned} \tag{3.21}$$

By virtue of this, we see that

$$
\begin{aligned}
&\|\{\tilde{V}^1_{|v|,t}(x) - \tilde{V}^1_{|v|,t}(pt)\}e^{-itK_0}M_{D,v}(t)\Phi_0\| \\
&\quad \le \left\| \int_0^1 (\nabla \tilde{V}^1_{|v|,t})(\theta x + (1-\theta)pt)d\theta \right\|_{\mathscr{B}(L^2)} \|(x-pt)e^{-itK_0}M_{D,v}(t)\Phi_0\| \\
&\qquad + \left\| \frac{i}{2}\int_0^1 t(\Delta \tilde{V}^1_{|v|,t})(\theta x + (1-\theta)pt)d\theta \right\|_{\mathscr{B}(L^2)} \|e^{-itK_0}M_{D,v}(t)\Phi_0\| \\
&\quad \le \|\nabla \tilde{V}^1_{|v|,t}\|_{L^\infty} \|xM_{D,v}(t)\Phi_0\| + \frac{1}{2}|t|\,\|\Delta \tilde{V}^1_{|v|,t}\|_{L^\infty}\|\Phi_0\| \\
&\quad \le \|\nabla \tilde{V}^1_{|v|,t}\|_{L^\infty}\left(\|x\Phi_0\| + \left\| \left(\int_0^t s(\nabla \tilde{V}^1_{|v|,s})(ps)ds \right)\Phi_0 \right\| \right) \\
&\qquad + \frac{1}{2}|t|\,\|\Delta \tilde{V}^1_{|v|,t}\|_{L^\infty}\|\Phi_0\|
\end{aligned}
$$

holds. Here we used $e^{itK_0}(x-pt)e^{-itK_0} = x$, (3.4) and (3.5). In order to estimate

$$\int_{-\infty}^{\infty} \|\nabla \tilde{V}^1_{|v|,t}\|_{L^\infty}\|x\Phi_0\|dt,$$

we consider

$$
\begin{aligned}
\tilde{I}_{1,1} &= \int_{-\infty}^{\infty} (1+|v|^{v_1/(2-v_1)}|t|)^{-(\gamma_D+1/2)(2-v_1)}dt = O(|v|^{-v_1/(2-v_1)}), \\
\tilde{I}_{1,2} &= \int_{-\infty}^{\infty} |v|^{v_2/(2-v_2)-1}(1+|v|^{v_2/(2-v_2)}|t|)^{-\gamma_D(2-v_2)-1}dt \\
&= O(|v|^{v_2/(2-v_2)-1-v_2/(2-v_2)}) = O(|v|^{-1})
\end{aligned}
$$

by (3.6), which can be obtained by an appropriate change of variables under the conditions $-(\gamma_D+1/2)(2-v_1) < -1$ and $-\gamma_D(2-v_2)-1 < -1$, i.e. $v_1 < 2-1/(\gamma_D+1/2)$ and $v_2 < 2$. Since $2/3 < 2-1/(\gamma_D+1/2) < 1$,

$$\inf_{0 \le v_1 < 2-1/(\gamma_D+1/2)} (-v_1/(2-v_1)) = -2\gamma_D,$$

and $-2\gamma_D > -1$, we obtain

$$\int_{-\infty}^{\infty} \|\nabla \tilde{V}^1_{|v|,t}\|_{L^\infty}\|x\Phi_0\|dt = O(|v|^{-2\gamma_D+\varepsilon}) \tag{3.22}$$

with $\varepsilon > 0$. In order to estimate

$$\int_{-\infty}^{\infty} \|\nabla \tilde{V}^1_{|v|,t}\|_{L^\infty}\left\| \left(\int_0^t s(\nabla \tilde{V}^1_{|v|,s})(ps)ds \right)\Phi_0 \right\| dt,$$

we consider

$$\begin{aligned}
\tilde{I}_{2,1} &= \int_{-\infty}^{\infty} (1+|v|^{v_1/(2-v_1)}|t|)^{-(\gamma_D+1/2)(2-v_1)}|v|^{-(\gamma_D+1/2)v_3}|t|^{2-(\gamma_D+1/2)(2-v_3)}dt \\
&= O(|v|^{-(\gamma_D+1/2)v_3-\{3-(\gamma_D+1/2)(2-v_3)\}v_1/(2-v_1)}), \\
\tilde{I}_{2,2} &= \int_{-\infty}^{\infty} |v|^{v_2/(2-v_2)-1}(1+|v|^{v_2/(2-v_2)}|t|)^{-\gamma_D(2-v_2)-1} \\
&\quad \times |v|^{-(\gamma_D+1/2)v_4}|t|^{2-(\gamma_D+1/2)(2-v_4)}dt \\
&= O(|v|^{v_2/(2-v_2)-1-(\gamma_D+1/2)v_4-\{3-(\gamma_D+1/2)(2-v_4)\}v_2/(2-v_2)}) \\
&= O(|v|^{-1-(\gamma_D+1/2)v_4-\{2-(\gamma_D+1/2)(2-v_4)\}v_2/(2-v_2)})
\end{aligned}$$

by (3.6) and (3.8), which can be obtained under the conditions $-(\gamma_D+1/2)\cdot(2-v_1)+2-(\gamma_D+1/2)(2-v_3)<-1$ and $-\gamma_D(2-v_2)-1+2-(\gamma_D+1/2)\cdot(2-v_4)<-1$, i.e. $(\gamma_D+1/2)(v_1+v_3)<4\gamma_D-1$ and $\gamma_D v_2+(\gamma_D+1/2)v_4<4\gamma_D-1$. Noting

$$2(\gamma_D+1/2)-(4\gamma_D-1)=2-2\gamma_D>1>0$$

by $\gamma_D<1/2$, we see

$$\{(\gamma_D+1/2)(v_1+v_3)\,|\,0\le v_1,\ v_3\le 1\}\supset[0,4\gamma_D-1].$$

We also note that for v_1 and v_3 such that $(\gamma_D+1/2)(v_1+v_3)=4\gamma_D-1$,

$$\begin{aligned}
&-(\gamma_D+1/2)v_3-\{3-(\gamma_D+1/2)(2-v_3)\}v_1/(2-v_1) \\
&\quad = -(\gamma_D+1/2)v_3-\{2-2\gamma_D+(\gamma_D+1/2)v_3\} \\
&\qquad \times \frac{(4\gamma_D-1)-(\gamma_D+1/2)v_3}{2(\gamma_D+1/2)-(4\gamma_D-1)+(\gamma_D+1/2)v_3} \\
&\quad = -(4\gamma_D-1)
\end{aligned}$$

holds. This yields

$$\begin{aligned}
&\inf_{v_1,v_3}(-(\gamma_D+1/2)v_3-\{3-(\gamma_D+1/2)(2-v_3)\}v_1/(2-v_1)) \\
&\quad = -(4\gamma_D-1). \qquad (3.23)
\end{aligned}$$

Noting

$$\gamma_D+(\gamma_D+1/2)-(4\gamma_D-1)=3/2-2\gamma_D>1/2>0$$

by $\gamma_D < 1/2$, we see

$$\{\gamma_D v_2 + (\gamma_D + 1/2)v_4 \,|\, 0 \le v_2,\ v_4 \le 1\} \supset [0, 4\gamma_D - 1].$$

We also note that for v_2 and v_4 such that $\gamma_D v_2 + (\gamma_D + 1/2)v_4 = 4\gamma_D - 1$,

$$\begin{aligned} &-1 - (\gamma_D + 1/2)v_4 - \{2 - (\gamma_D + 1/2)(2 - v_4)\}v_2/(2 - v_2) \\ &\quad = -1 - (\gamma_D + 1/2)v_4 - \{1 - 2\gamma_D + (\gamma_D + 1/2)v_4\} \\ &\qquad \times \frac{(4\gamma_D - 1) - (\gamma_D + 1/2)v_4}{2\gamma_D - (4\gamma_D - 1) + (\gamma_D + 1/2)v_4} \\ &\quad = -1 - (4\gamma_D - 1) = -4\gamma_D \end{aligned}$$

holds. This yields

$$\begin{aligned} &\inf_{v_2, v_4}(-1 - (\gamma_D + 1/2)v_4 - \{2 - (\gamma_D + 1/2)(2 - v_4)\}v_2/(2 - v_2)) \\ &\quad = -4\gamma_D. \end{aligned} \tag{3.24}$$

(3.23) and (3.24) imply

$$\int_{-\infty}^{\infty} \|\nabla \tilde{V}^1_{|v|,t}\|_{L^\infty} \left\| \left(\int_0^t s(\nabla \tilde{V}^1_{|v|,s})(ps)ds \right) \Phi_0 \right\| dt = O(|v|^{-(4\gamma_D - 1) + \varepsilon}) \tag{3.25}$$

with $\varepsilon > 0$. In order to estimate

$$\int_{-\infty}^{\infty} |t| \, \|\Delta \tilde{V}^1_{|v|,t}\|_{L^\infty} \|\Phi_0\| dt,$$

we consider

$$\begin{aligned} \tilde{I}_{3,1} &= \int_{-\infty}^{\infty} (1 + |v|^{v_1/(2-v_1)}|t|)^{-(\gamma_D+1)(2-v_1)}|t|dt = O(|v|^{-2v_1/(2-v_1)}), \\ \tilde{I}_{3,2} &= \int_{-\infty}^{\infty} |v|^{v_2/(2-v_2)-1}(1 + |v|^{v_2/(2-v_2)}|t|)^{-(\gamma_D+1/2)(2-v_2)-1}|t|dt \\ &= O(|v|^{v_2/(2-v_2)-1-2v_2/(2-v_2)}) = O(|v|^{-1-v_2/(2-v_2)}), \\ \tilde{I}_{3,3} &= \int_{-\infty}^{\infty} |v|^{2v_3/(2-v_3)-2}(1 + |v|^{v_3/(2-v_3)}|t|)^{-\gamma_D(2-v_3)-2}|t|dt \\ &= O(|v|^{2v_3/(2-v_3)-2-2v_3/(2-v_3)}) = O(|v|^{-2}) \end{aligned}$$

by (3.7), which can be obtained by an appropriate change of variables under the conditions $-(\gamma_D + 1)(2 - v_1) + 1 < -1$, $-(\gamma_D + 1/2)(2 - v_2) - 1 + 1 < -1$

and $-\gamma_D(2-\nu_3)-2+1<-1$, i.e. $\nu_1<2-2/(\gamma_D+1)$, $\nu_2<2-1/(\gamma_D+1/2)$ and $\nu_3<2$. Since $2/5<2-2/(\gamma_D+1)<2/3$, $2/3<2-1/(\gamma_D+1/2)<1$,

$$\inf_{0\le \nu_1<2-2/(\gamma_D+1)}(-2\nu_1/(2-\nu_1))=-2\gamma_D,$$

$$\inf_{0\le \nu_2<2-1/(\gamma_D+1/2)}(-1-\nu_2/(2-\nu_2))=-1-2\gamma_D,$$

and $-2<-1-2\gamma_D<-2\gamma_D$, we obtain

$$\int_{-\infty}^{\infty}|t|\,\|\Delta\tilde{V}^1_{|v|,t}\|_{L^\infty}\|\Phi_0\|dt=O(|v|^{-2\gamma_D+\varepsilon}) \tag{3.26}$$

with $\varepsilon>0$. By (3.20), (3.22), (3.25) and (3.26), we finally obtain (3.17) because of $-1<-2\gamma_D<-(4\gamma_D-1)$.

In the same way as in [3], we introduce auxiliary wave operators

$$\Omega^{s,\pm}_{D,G,v}=\operatorname*{s-lim}_{t\to\pm\infty}e^{itH}U^{s}_{D,G,v}(t),$$

where $U^{s}_{D,G,v}(t)=U_D(t)M^{s}_{G,v}(t)$ and $M^{s}_{G,v}(t)=e^{-i\int_0^t V^s(vs+e_1s^2/2)ds}$ as in §2. Then we see that

$$\Omega^{s,\pm}_{D,G,v}=W^{\pm}_D I^{s,\pm}_{G,v},\qquad I^{s,\pm}_{G,v}=\operatorname*{s-lim}_{t\to\pm\infty}M^{s}_{G,v}(t)$$

exist. Therefore, by Lemmas 3.3, 3.4 and 3.5, the following lemma can be obtained as Lemma 2.5. Thus we omit the proof.

LEMMA 3.6. *Let v and Φ_v be as in Theorem 3.1, and $\varepsilon>0$. Then*

$$\sup_{t\in\boldsymbol{R}}\|(e^{-itH}\Omega^{s,-}_{D,G,v}-U^{s}_{D,G,v}(t))\Phi_v\|=O(|v|^{\max\{-1,-2(\gamma_1-1)+\varepsilon,-(4\gamma_D-1)+\varepsilon\}}) \tag{3.27}$$

holds as $|v|\to\infty$ for $V^{vs}\in\mathscr{V}^{vs}$, $V^s\in\mathscr{V}^s(1/2,1)$, and $V^1\in\mathscr{V}^1_D(1/4)$.

Now we will show Theorem 3.1:

PROOF (Proof of Theorem 3.1). Since the proof is quite similar to the one of Theorem 2.1, we give its sketch only.

Suppose that $V^{vs}\in\mathscr{V}^{vs}$, $V^s\in\mathscr{V}^s(1/2,5/4)$ and $V^1\in\mathscr{V}^1_D(3/8)$. We first note that S_D is represented as

$$S_D=(W^+_D)^*W^-_D=I^{s}_{G,v}(\Omega^{s,+}_{D,G,v})^*\Omega^{s,-}_{D,G,v},$$

$$I^{s}_{G,v}=I^{s,+}_{G,v}(I^{s,-}_{G,v})^*=e^{-i\int_{-\infty}^{\infty}V^s(vs+e_1s^2/2)ds}.$$

Noting $[S_D, p_j] = [S_D - I^{\rm s}_{G,v}, p_j - v_j]$, $(p_j - v_j)\Phi_v = (p_j\Phi_0)_v$ and

$$i(S_D - I^{\rm s}_{G,v})\Phi_v = I^{\rm s}_{G,v} i(\Omega^{{\rm s},+}_{D,G,v} - \Omega^{{\rm s},-}_{D,G,v})^*\Omega^{{\rm s},-}_{D,G,v}\Phi_v$$
$$= I^{\rm s}_{G,v}\int_{-\infty}^{\infty} U^{\rm s}_{D,G,v}(t)^* V_t^D e^{-itH}\Omega^{{\rm s},-}_{D,G,v}\Phi_v\, dt$$

with

$$V_t^D = V^{\rm vs}(x) + V^{\rm s}(x) - V^{\rm s}(vt + e_1t^2/2) + V^1(x) - V^1(pt - e_1t^2/2),$$

we have

$$|v|(i[S_D, p_j]\Phi_v, \Psi_v) = I^{\rm s}_{G,v}\{I^D(v) + R^D(v)\}$$

with

$$I^D(v) = |v|\int_{-\infty}^{\infty}[(V_t^D U^{\rm s}_{D,G,v}(t)(p_j\Phi_0)_v, U^{\rm s}_{D,G,v}(t)\Psi_v)$$
$$- (V_t^D U^{\rm s}_{D,G,v}(t)\Phi_v, U^{\rm s}_{D,G,v}(t)(p_j\Psi_0)_v)]dt,$$
$$R^D(v) = |v|\int_{-\infty}^{\infty}[((e^{-itH}\Omega^{{\rm s},-}_{D,G,v} - U^{\rm s}_{D,G,v}(t))(p_j\Phi_0)_v, V_t^D U^{\rm s}_{D,G,v}(t)\Psi_v)$$
$$- ((e^{-itH}\Omega^{{\rm s},-}_{D,G,v} - U^{\rm s}_{D,G,v}(t))\Phi_v, V_t^D U^{\rm s}_{D,G,v}(t)(p_j\Psi_0)_v)]dt.$$

By Lemmas 3.3, 3.4, 3.5 and 3.6, we have

$$|R^D(v)| = O(|v|^{1+2\max\{-1,-2(\gamma_1-1)+\varepsilon,-(4\gamma_D-1)+\varepsilon\}})$$
$$= O(|v|^{\max\{-1,5-4\gamma_1+2\varepsilon,3-8\gamma_D+2\varepsilon\}}). \tag{3.28}$$

In the same way as in the proof of Theorem 2.1, we need the conditions $5 - 4\gamma_1 < 0$ and $3 - 8\gamma_D < 0$, which are equivalent to $\gamma_1 > 5/4$ and $\gamma_D > 3/8$, in order to get $\lim_{|v|\to\infty} R^D(v) = 0$.

The rest of the proof is the same as in [17] and [3]. So we omit it.

By virtue of Theorem 3.1, Theorem 1.2 can be shown in the same way as in the proof of Theorem 1.1. Thus we omit its proof.

4. The case where $V^{\rm s} \in \tilde{\mathscr{V}}^{\rm s}(1/2, 1, 5/4)$

Throughout this section, we suppose $V^{\rm s} \in \tilde{\mathscr{V}}^{\rm s}(1/2, 1, 1)$. Then the following reconstruction formulas, which are Theorems 2.1 and 3.1 with replacing $V^{\rm s} \in \mathscr{V}^{\rm s}(1/2, 5/4)$ by $V^{\rm s} \in \tilde{\mathscr{V}}^{\rm s}(1/2, 1, 5/4)$, can be obtained:

THEOREM 4.1. *Let the notation in this theorem be the same as in Theorem* 2.1. *Let* $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$, $V^{\mathrm{s}} \in \tilde{\mathscr{V}}^{\mathrm{s}}(1/2, 1, 5/4)$. *Then* (2.1) *holds for* $1 \le j \le n$.

THEOREM 4.2. *Let the notation in this theorem be the same as in Theorem* 3.1. *Let* $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$, $V^{\mathrm{s}} \in \tilde{\mathscr{V}}^{\mathrm{s}}(1/2, 1, 5/4)$, $V^{\mathrm{l}} \in \mathscr{V}_D^{\mathrm{l}}(3/8)$. *Then* (3.2) *holds for* $1 \le j \le n$.

In order to prove Theorems 4.1 and 4.2, we will improve a series of lemmas in §2 and §3, for $V^{\mathrm{s}} \in \tilde{\mathscr{V}}^{\mathrm{s}}(1/2, 1, 1)$. To this end, we will introduce

$$U_D^{\mathrm{s}}(t) = e^{-itH_0} M_D^{\mathrm{s}}(t), \qquad M_D^{\mathrm{s}}(t) = e^{-i\int_0^t V^{\mathrm{s}}(ps + e_1 s^2/2)\,ds}. \tag{4.1}$$

In [14], instead of $M_D^{\mathrm{s}}(t)$, $e^{-i\int_0^t V^{\mathrm{s}}(p_\perp s + e_1 s^2/2)\,ds}$ was used, as mentioned in §1. The Dollard-type modifier $M_D^{\mathrm{s}}(t)$ seems more appropriate for the problem considered in this paper than $e^{-i\int_0^t V^{\mathrm{s}}(p_\perp s + e_1 s^2/2)\,ds}$. We first give the following lemma:

LEMMA 4.3. *Let* v *and* Φ_v *be as in Theorem* 4.1, *and* $V^{\mathrm{s}} \in \tilde{\mathscr{V}}^{\mathrm{s}}(1/2, 1, 1)$. *If* $\gamma_2 > 3/2$, *then there exists a positive constant* C *such that*

$$\|\langle x\rangle^2 M_D^{\mathrm{s}}(t)\Phi_v\| = \|\langle x\rangle^2 M_{D,v}^{\mathrm{s}}(t)\Phi_0\| \le C \tag{4.2}$$

holds as $|v| \to \infty$, *where* $M_{D,v}^{\mathrm{s}}(t) = e^{-iv\cdot x} M_D^{\mathrm{s}}(t) e^{iv\cdot x} = e^{-i\int_0^t V^{\mathrm{s}}(ps + vs + e_1 s^2/2)\,ds}$. *On the other hand, if* $\gamma_2 \le 3/2$, *then, for* $0 \le v_1 \le 1$, *there exists a positive constant* C *such that*

$$\|\langle x\rangle^2 M_D^{\mathrm{s}}(t)\Phi_v\| = \|\langle x\rangle^2 M_{D,v}^{\mathrm{s}}(t)\Phi_0\| \le C(1 + |v|^{-\gamma_2 v_1}|t|^{3-\gamma_2(2-v_1)}) \tag{4.3}$$

holds as $|v| \to \infty$, *where only when* $\gamma_2 = 3/2$, *we assume* $v_1 \ne 0$ *additionally.*

PROOF. Since the proof is quite similar to the one of Lemma 3.2, we will sketch it.

We first note that since $\operatorname{supp} \hat{\Phi}_0 \subset \{\xi \in \boldsymbol{R}^n \mid |\xi| < \eta\}$,

$$M_{D,v}^{\mathrm{s}}(t)\Phi_0 = e^{-i\int_0^t \tilde{V}_{|v|,s}^{\mathrm{s}}(ps)\,ds}\Phi_0 \tag{4.4}$$

holds for $|v| \ge \eta/(3\lambda_1)$. Hence, as in the proof of Lemma 3.2,

$$\begin{aligned}
&\|\langle x\rangle^2 M_{D,v}^{\mathrm{s}}(t)\Phi_0\| \\
&\quad \le \|\langle x\rangle^2 \Phi_0\| + \left\| \left(\int_0^t s^2 (\Delta \tilde{V}_{|v|,s}^{\mathrm{s}})(ps)\,ds \right) \Phi_0 \right\| \\
&\qquad + 2\left\| \left(\int_0^t s (\nabla \tilde{V}_{|v|,s}^{\mathrm{s}})(ps)\,ds \right) \cdot x\Phi_0 \right\| + \left\| \left(\int_0^t s (\nabla \tilde{V}_{|v|,s}^{\mathrm{s}})(ps)\,ds \right)^2 \Phi_0 \right\|
\end{aligned}$$

can be obtained. Now we will estimate $\|\nabla \tilde{V}_{|v|,t}^{\mathrm{s}}\|_{L^\infty}$ and $\|\Delta \tilde{V}_{|v|,t}^{\mathrm{s}}\|_{L^\infty}$. In the same way as in the proof of Lemma 3.2, for $0 \le v_1, v_2, v_3, v_4, v_5 \le 1$

and $|v| \geq 1$,

$$\|\nabla \tilde{V}^{s}_{|v|,t}\|_{L^\infty} \leq C'_1(1 + |v|^{\nu_1/(2-\nu_1)}|t|)^{-\gamma_1(2-\nu_1)} + C'_2|v|^{\nu_2/(2-\nu_2)-1}(1 + |v|^{\nu_2/(2-\nu_2)}|t|)^{-\gamma_0(2-\nu_2)-1} \tag{4.5}$$

and

$$\|\Delta \tilde{V}^{s}_{|v|,t}\|_{L^\infty} \leq C'_3(1 + |v|^{\nu_3/(2-\nu_3)}|t|)^{-\gamma_2(2-\nu_3)} + C'_4|v|^{\nu_4/(2-\nu_4)-1}(1 + |v|^{\nu_4/(2-\nu_4)}|t|)^{-\gamma_1(2-\nu_4)-1} + C'_5|v|^{2\nu_5/(2-\nu_5)-2}(1 + |v|^{\nu_5/(2-\nu_5)}|t|)^{-\gamma_0(2-\nu_5)-2} \tag{4.6}$$

can be obtained (cf. (3.6) and (3.7)). Since $-2\gamma_1 + 1 < -1$ and $-2\gamma_0 - 1 + 1 < -1$ by assumption, the estimate

$$\left\| \int_0^t s(\nabla \tilde{V}^{s}_{|v|,s})(ps)ds \right\|_{\mathscr{B}(L^2)} \leq C \tag{4.7}$$

can be obtained immediately by (4.5) with $\nu_1 = \nu_2 = 0$. Since $-2\gamma_1 - 1 + 2 < -1$ and $-2\gamma_0 - 2 + 2 < -1$ by assumption, we see that $|t|^2 \times$ (the second and third terms of the right-hand side of (4.6) with $\nu_4 = \nu_5 = 0$) are integrable in $\boldsymbol{R}$. Hence, we have only to watch

$$\tilde{I} = C'_3 \int_0^{|t|} s^2(1 + |v|^{\nu_3/(2-\nu_3)}s)^{-\gamma_2(2-\nu_3)}ds.$$

If $\gamma_2 > 3/2$, then $-2\gamma_2 + 2 < -1$ holds, which implies that there exists $C > 0$ independent of t such that $\tilde{I} \leq C$ holds, by taking $\nu_3 = 0$. On the other hand, if $\gamma_2 \leq 3/2$, then

$$\tilde{I} \leq C''_3|v|^{-\gamma_2\nu_3}|t|^{3-\gamma_2(2-\nu_3)}$$

can be obtained easily, where only when $\gamma_2 = 3/2$, we assume $\nu_3 \neq 0$ additionally.

Based on the above observations, the lemma can be proved.

By virtue of Lemma 4.3, the following lemma can be obtained in the same way as in the proof of Lemma 3.3.

Lemma 4.4. *Let v and Φ_v be as in Theorem 4.1, and $V^{s} \in \tilde{\mathscr{V}}^{s}(1/2, 1, 1)$. Then*

$$\int_{-\infty}^{\infty} \|V^{vs}(x)M^{s}_D(t)\Phi_v\|dt = O(|v|^{-1}) \tag{4.8}$$

holds as $|v| \to \infty$ for $V^{vs} \in \mathscr{V}^{vs}$.

PROOF. We have only to mention some changes compared to the proof of Lemma 3.3: We consider the case where $\gamma_2 < 3/2$ only. The estimate I_2 in the proof of Lemma 3.3 has to be replaced by

$$I_2 \leq C(1+\lambda_1|v|\,|t|)^{-2}(1+|v|^{-\gamma_2 v_1}|t|^{3-\gamma_2(2-v_1)})$$

for $0 \leq v_1 \leq 1$. Therefore

$$\int_{-\infty}^{\infty} I_2\, dt = O(|v|^{-1}) + O(|v|^{-\gamma_2 v_1 - \{4-\gamma_2(2-v_1)\}})$$

can be obtained by an appropriate change of variables under the condition $-2+3-\gamma_2(2-v_1) < -1$, i.e. $v_1 < 2-2/\gamma_2$. Since $0 < 2-2/\gamma_2 < 2/3$ and

$$\inf_{0 \leq v_1 < 2-2/\gamma_2}(-\gamma_2 v_1 - \{4-\gamma_2(2-v_1)\}) = -2\gamma_2,$$

we obtain

$$\int_{-\infty}^{\infty} I_2\, dt = O(|v|^{-1}).$$

Based on the above observations, the lemma can be proved.

The following lemma can be obtained as in the proof of Lemma 3.5:

LEMMA 4.5. *Let* v *and* Φ_v *be as in Theorem* 4.1, $\varepsilon > 0$, *and* $V^{\mathrm{s}} \in \tilde{\mathscr{V}}^{\mathrm{s}}(1/2, 1, 1)$. *Then*

$$\int_{-\infty}^{\infty} \|\{V^{\mathrm{s}}(x) - V^{\mathrm{s}}(pt - e_1 t^2/2)\} U_D^{\mathrm{s}}(t)\Phi_v\| dt = O(|v|^{\max\{-1, -2(\gamma_2-1)+\varepsilon\}}) \tag{4.9}$$

holds as $|v| \to \infty$.

PROOF. We have only to mention some changes compared to the proof of Lemma 3.5: We consider the case where $\gamma_2 < 3/2$ only. For the sake of brevity, we put $I = \|\{V^{\mathrm{s}}(x) - V^{\mathrm{s}}(pt - e_1 t^2/2)\} U_D^{\mathrm{s}}(t)\Phi_v\|$. I can be estimated as

$$\begin{aligned} I &= \|\{V^{\mathrm{s}}(x + vt + e_1 t^2/2) - V^{\mathrm{s}}(pt + vt + e_1 t^2/2)\} e^{-itK_0} M_{D,v}^{\mathrm{s}}(t)\Phi_0\| \\ &\leq \|\bar{V}_{|v|,t}^{\mathrm{s}}(x) e^{-itK_0} f(p) M_{D,v}^{\mathrm{s}}(t)\Phi_0\| \\ &\quad + \|\{\tilde{V}_{|v|,t}^{\mathrm{s}}(x) - \tilde{V}_{|v|,t}^{\mathrm{s}}(pt)\} e^{-itK_0} M_{D,v}^{\mathrm{s}}(t)\Phi_0\| \end{aligned} \tag{4.10}$$

for $|v| \geq \eta/(3\lambda_1)$. As for the first term of the inequality (4.10), the estimate

$$\int_{-\infty}^{\infty} \|\bar{V}_{|v|,t}^{\mathrm{s}}(x) e^{-itK_0} f(p) M_{D,v}^{\mathrm{s}}(t)\Phi_0\| dt = O(|v|^{-1}) \tag{4.11}$$

can be obtained as in the proof of Lemma 3.5. On the other hand, as for the second term of the inequality (4.10), we see that

$$\begin{aligned}
&\|\{\tilde{V}^{\mathrm{s}}_{|v|,t}(x) - \tilde{V}^{\mathrm{s}}_{|v|,t}(pt)\}e^{-itK_0}M^{\mathrm{s}}_{D,v}(t)\Phi_0\| \\
&\quad \le \|\nabla\tilde{V}^{\mathrm{s}}_{|v|,t}\|_{L^\infty}\left(\|x\Phi_0\| + \left\|\left(\int_0^t s(\nabla\tilde{V}^{\mathrm{s}}_{|v|,s})(ps)ds\right)\Phi_0\right\|\right) \\
&\qquad + \frac{1}{2}|t|\,\|\Delta\tilde{V}^{\mathrm{s}}_{|v|,t}\|_{L^\infty}\|\Phi_0\| \\
&\quad \le C_1\|\nabla\tilde{V}^{\mathrm{s}}_{|v|,t}\|_{L^\infty} + C_2|t|\,\|\Delta\tilde{V}^{\mathrm{s}}_{|v|,t}\|_{L^\infty}
\end{aligned}$$

holds, by virtue of the Baker-Campbell-Hausdorff formula. Here we used (4.7). The estimate

$$\int_{-\infty}^{\infty} C_1\|\nabla\tilde{V}^{\mathrm{s}}_{|v|,t}\|_{L^\infty}dt = O(|v|^{-1}) \tag{4.12}$$

can be obtained in the same way as in the proof of Lemma 2.4 (cf. (2.15)). In order to estimate

$$\int_{-\infty}^{\infty} C_2|t|\,\|\Delta\tilde{V}^{\mathrm{s}}_{|v|,t}\|_{L^\infty}dt,$$

we consider

$$\begin{aligned}
\tilde{I}_{2,1} &= \int_{-\infty}^{\infty}(1+|v|^{v_1/(2-v_1)}|t|)^{-\gamma_2(2-v_1)}|t|dt = O(|v|^{-2v_1/(2-v_1)}), \\
\tilde{I}_{2,2} &= \int_{-\infty}^{\infty}|v|^{v_2/(2-v_2)-1}(1+|v|^{v_2/(2-v_2)}|t|)^{-\gamma_1(2-v_2)-1}|t|dt \\
&= O(|v|^{v_2/(2-v_2)-1-2v_2/(2-v_2)}) = O(|v|^{-1-v_2/(2-v_2)}), \\
\tilde{I}_{2,3} &= \int_{-\infty}^{\infty}|v|^{2v_3/(2-v_3)-2}(1+|v|^{v_3/(2-v_3)}|t|)^{-\gamma_0(2-v_3)-2}|t|dt \\
&= O(|v|^{2v_3/(2-v_3)-2-2v_3/(2-v_3)}) = O(|v|^{-2})
\end{aligned}$$

by (4.6), which can be obtained by an appropriate change of variables under the conditions $-\gamma_2(2-v_1)+1 < -1$, $-\gamma_1(2-v_2)-1+1 < -1$ and $-\gamma_0(2-v_3)-2+1 < -1$, i.e. $v_1 < 2-2/\gamma_2$, $v_2 < 2-1/\gamma_1$ and $v_3 < 2$. Since $0 < 2-2/\gamma_2 < 2/3$, $2-1/\gamma_1 > 1$,

$$\inf_{0\le v_1<2-2/\gamma_2}(-2v_1/(2-v_1)) = -2(\gamma_2-1),$$

and $-2(\gamma_2 - 1) > -1$, we have

$$\int_{-\infty}^{\infty} |t| \, \|\Delta \tilde{V}^{\mathrm{s}}_{|v|,t}\|_{L^\infty} \|\Phi_0\| dt = O(|v|^{-2(\gamma_2-1)+\varepsilon}) \tag{4.13}$$

with $\varepsilon > 0$. By (4.11), (4.12) and (4.13), (4.9) can be obtained.

By modifying the argument in [14], we introduce auxiliary wave operators

$$\Omega_D^{\mathrm{s},\pm} = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH} U_D^{\mathrm{s}}(t)$$

with $H = H_0 + V^{\mathrm{vs}} + V^{\mathrm{s}}$. Then we see that

$$\Omega_D^{\mathrm{s},\pm} = W^{\pm} I_D^{\mathrm{s},\pm}, \qquad I_D^{\mathrm{s},\pm} = \operatorname*{s-lim}_{t\to\pm\infty} M_D^{\mathrm{s}}(t) = e^{-i\int_0^{\pm\infty} V^{\mathrm{s}}(ps+e_1s^2/2)ds}$$

exist. Therefore, by Lemmas 4.4 and 4.5, the following lemma can be obtained as an improvement of Lemma 2.5. Thus we omit the proof.

Lemma 4.6. *Let v and Φ_v be as in Theorem* 4.1 *and* $\varepsilon > 0$. *Then*

$$\sup_{t\in\boldsymbol{R}} \|(e^{-itH}\Omega_D^{\mathrm{s},-} - U_D^{\mathrm{s}}(t))\Phi_v\| = O(|v|^{\max\{-1,-2(\gamma_2-1)+\varepsilon\}}) \tag{4.14}$$

holds as $|v| \to \infty$ *for* $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$ *and* $V^{\mathrm{s}} \in \tilde{\mathscr{V}}^{\mathrm{s}}(1/2, 1, 1)$.

Now we will show Theorem 4.1:

Proof (Proof of Theorem 4.1). Since the proof is quite similar to the one of Theorem 2.1, we give its sketch only.

Suppose that $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$ and $V^{\mathrm{s}} \in \tilde{\mathscr{V}}^{\mathrm{s}}(1/2, 1, 5/4)$. We first note that S is represented as

$$S = (W^+)^* W^- = I_D^{\mathrm{s},+}(\Omega_D^{\mathrm{s},+})^* \Omega_D^{\mathrm{s},-}(I_D^{\mathrm{s},-})^*.$$

Unlike $I_{G,v}^{\mathrm{s},\pm}$, $I_D^{\mathrm{s},\pm}$ do not commute with $\Omega_D^{\mathrm{s},\pm}$. Putting

$$I_D^{\mathrm{s}} = I_D^{\mathrm{s},+}(I_D^{\mathrm{s},-})^* = e^{-i\int_{-\infty}^{\infty} V^{\mathrm{s}}(ps+e_1s^2/2)ds}$$

and noting $[S, p_j] = [S - I_D^{\mathrm{s}}, p_j - v_j]$, $(p_j - v_j)\Phi_v = (p_j\Phi_0)_v$ and

$$\begin{aligned} i(S - I_D^{\mathrm{s}})\Phi_v &= I_D^{\mathrm{s},+} i(\Omega_D^{\mathrm{s},+} - \Omega_D^{\mathrm{s},-})^* \Omega_D^{\mathrm{s},-}(I_D^{\mathrm{s},-})^*\Phi_v \\ &= I_D^{\mathrm{s},+} \int_{-\infty}^{\infty} U_D^{\mathrm{s}}(t)^* V_{t,D} e^{-itH} \Omega_D^{\mathrm{s},-}((I_{D,v}^{\mathrm{s},-})^*\Phi_0)_v dt \end{aligned}$$

with

$$V_{t,D} = V^{\mathrm{vs}}(x) + V^{\mathrm{s}}(x) - V^{\mathrm{s}}(pt - e_1t^2/2),$$

$$(I_{D,v}^{\mathrm{s},\pm})^* = e^{-iv\cdot x}(I_D^{\mathrm{s},\pm})^* e^{iv\cdot x} = e^{i\int_0^{\pm\infty} V^{\mathrm{s}}(ps+vs+e_1s^2/2)ds},$$

we have

$$|v|(i[S, p_j]\Phi_v, \Psi_v) = I_D(v) + R_D(v)$$

with

$$I_D(v) = |v| \int_{-\infty}^{\infty} [(V_{t,D} U_D^{\mathrm{s}}(t)(p_j(I_{D,v}^{\mathrm{s},-})^* \Phi_0)_v, U_D^{\mathrm{s}}(t)((I_{D,v}^{\mathrm{s},+})^* \Psi_0)_v)$$
$$- (V_{t,D} U_D^{\mathrm{s}}(t)((I_{D,v}^{\mathrm{s},-})^* \Phi_0)_v, U_D^{\mathrm{s}}(t)(p_j(I_{D,v}^{\mathrm{s},+})^* \Psi_0)_v]dt,$$

$$R_D(v) = |v| \int_{-\infty}^{\infty} [(e^{-itH}\Omega_D^{\mathrm{s},-} - U_D^{\mathrm{s}}(t))(p_j(I_{D,v}^{\mathrm{s},-})^* \Phi_0)_v, V_{t,D} U_D^{\mathrm{s}}(t)((I_{D,v}^{\mathrm{s},+})^* \Psi_0)_v)$$
$$- (e^{-itH}\Omega_D^{\mathrm{s},-} - U_D^{\mathrm{s}}(t))((I_{D,v}^{\mathrm{s},-})^* \Phi_0)_v, V_{t,D} U_D^{\mathrm{s}}(t)(p_j(I_{D,v}^{\mathrm{s},+})^* \Psi_0)_v]dt.$$

Here we used that p_j commutes with $(I_{D,v}^{\mathrm{s},\pm})^*$. By Lemmas 4.4, 4.5 and 4.6, we have

$$|R_D(v)| = O(|v|^{1+2\max\{-1,-2(\gamma_2-1)+\varepsilon\}}) = O(|v|^{\max\{-1,5-4\gamma_2+2\varepsilon\}}). \tag{4.15}$$

Here we used $\operatorname{supp} \mathscr{F}[(I_{D,v}^{\mathrm{s},-})^* \Phi_0] = \operatorname{supp} \mathscr{F}[\Phi_0]$ and $\operatorname{supp} \mathscr{F}[(I_{D,v}^{\mathrm{s},+})^* \Psi_0] = \operatorname{supp} \mathscr{F}[\Psi_0]$. Then we need the condition $5 - 4\gamma_2 < 0$ in order to get $\lim_{|v|\to\infty} R_D(v) = 0$. Using the Avron-Herbst formula (2.8) and (2.9), $I_D(v)$ is rewritten as

$$I_D(v) = |v| \int_{-\infty}^{\infty} [(\hat{V}_{t,D} e^{-itK_0} M_{D,v}^{\mathrm{s}}(t) p_j (I_{D,v}^{\mathrm{s},-})^* \Phi_0, e^{-itK_0} M_{D,v}^{\mathrm{s}}(t)(I_{D,v}^{\mathrm{s},+})^* \Psi_0)$$
$$- (\hat{V}_{t,D} e^{-itK_0} M_{D,v}^{\mathrm{s}}(t)(I_{D,v}^{\mathrm{s},-})^* \Phi_0, e^{-itK_0} M_{D,v}^{\mathrm{s}}(t) p_j (I_{D,v}^{\mathrm{s},+})^* \Psi_0]dt$$

with

$$\hat{V}_{t,D} = V^{\mathrm{vs}}(x + vt + e_1 t^2/2) + V^{\mathrm{s}}(x + vt + e_1 t^2/2) - V^{\mathrm{s}}(pt + vt + e_1 t^2/2).$$

Since

$$[V^{\mathrm{s}}(x + vt + e_1 t^2/2) - V^{\mathrm{s}}(pt + vt + e_1 t^2/2), p_j] = i(\partial_j V^{\mathrm{s}})(x + vt + e_1 t^2/2)$$

and $-\gamma_1 < -1$, $I_D(v)$ is rewritten as

$$I_D(v) = |v| \int_{-\infty}^{\infty} [(V^{\mathrm{vs}}(x + vt + e_1 t^2/2) e^{-itK_0} M_{D,v}^{\mathrm{s}}(t) p_j (I_{D,v}^{\mathrm{s},-})^* \Phi_0,$$
$$e^{-itK_0} M_{D,v}^{\mathrm{s}}(t)(I_{D,v}^{\mathrm{s},+})^* \Psi_0)$$
$$- (V^{\mathrm{vs}}(x + vt + e_1 t^2/2) e^{-itK_0} M_{D,v}^{\mathrm{s}}(t)(I_{D,v}^{\mathrm{s},-})^* \Phi_0,$$
$$e^{-itK_0} M_{D,v}^{\mathrm{s}}(t) p_j (I_{D,v}^{\mathrm{s},+})^* \Psi_0)]dt$$

$$+ |v| \int_{-\infty}^{\infty} (i(\partial_j V^{\mathrm{s}})(x + vt + e_1 t^2/2) e^{-itK_0} M_{D,v}^{\mathrm{s}}(t)(I_{D,v}^{\mathrm{s},-})^* \Phi_0, \\ e^{-itK_0} M_{D,v}^{\mathrm{s}}(t)(I_{D,v}^{\mathrm{s},+})^* \Psi_0) dt$$

$$= \int_{-\infty}^{\infty} l_{D,v}(\tau) d\tau$$

with

$$l_{D,v}(\tau) = (V^{\mathrm{vs}}(x + \hat{v}\tau + e_1(\tau/|v|)^2/2) e^{-i(\tau/|v|)K_0} M_{D,v}^{\mathrm{s}}(\tau/|v|) p_j (I_{D,v}^{\mathrm{s},-})^* \Phi_0, \\ e^{-i(\tau/|v|)K_0} M_{D,v}^{\mathrm{s}}(\tau/|v|)(I_{D,v}^{\mathrm{s},+})^* \Psi_0) \\ - (V^{\mathrm{vs}}(x + \hat{v}\tau + e_1(\tau/|v|)^2/2) e^{-i(\tau/|v|)K_0} M_{D,v}^{\mathrm{s}}(\tau/|v|)(I_{D,v}^{\mathrm{s},-})^* \Phi_0, \\ e^{-i(\tau/|v|)K_0} M_{D,v}^{\mathrm{s}}(\tau/|v|) p_j (I_{D,v}^{\mathrm{s},+})^* \Psi_0) \\ + i((\partial_j V^{\mathrm{s}})(x + \hat{v}\tau + e_1(\tau/|v|)^2/2) e^{-i(\tau/|v|)K_0} M_{D,v}^{\mathrm{s}}(\tau/|v|)(I_{D,v}^{\mathrm{s},-})^* \Phi_0, \\ e^{-i(\tau/|v|)K_0} M_{D,v}^{\mathrm{s}}(\tau/|v|)(I_{D,v}^{\mathrm{s},+})^* \Psi_0).$$

Here we note that

$$\operatorname*{s-lim}_{|v|\to\infty} (I_{D,v}^{\mathrm{s},\pm})^* = \mathrm{Id}. \tag{4.16}$$

In fact, for $|\xi| < \eta$ and $|v| \geq \eta/(4\lambda_1)$,

$$|\xi s + vs + e_1 s^2/2| \geq \max\{c_1|v|\,|s|, c_2|s|^2\}$$

holds by (2.12). This yields that $V^{\mathrm{s}}(\xi s + vs + e_1 s^2/2)$ is integrable on $\boldsymbol{R}$ uniformly in $|v|$ because of $\gamma_0 > 1/2$, and that $\lim_{|v|\to\infty} V^{\mathrm{s}}(\xi s + vs + e_1 s^2/2) = 0$ for any $s \neq 0$. Hence, the Lebesgue dominated convergence theorem yields

$$\lim_{|v|\to\infty} \int_0^{\pm\infty} V^{\mathrm{s}}(\xi s + vs + e_1 s^2/2) ds = 0,$$

which implies (4.16). We also note

$$\operatorname*{s-lim}_{|v|\to\infty} e^{-i(\tau/|v|)K_0} M_{D,v}^{\mathrm{s}}(\tau/|v|) = \mathrm{Id},$$

which can be shown easily. Since in the proof of Lemma 2.3, $|l_{D,v}(\tau)|$ can be estimated as

$$|l_{D,v}(\tau)| \leq C\{\|V^{\mathrm{vs}}(x)(1 + K_0)^{-1} F(|x| \geq \lambda_1|\tau|)\|_{\mathscr{B}(L^2)} \\ + (1 + |\tau|)^{-2} + (1 + |\tau|)^{-\gamma_1}\},$$

whose right-hand side is $|v|$-independent and integrable on $\boldsymbol{R}$. Therefore we obtain

$$\lim_{|v|\to\infty} I_D(v) = \int_{-\infty}^{\infty} [(V^{\mathrm{vs}}(x+\hat{v}\tau)p_j\Phi_0, \Psi_0) - (V^{\mathrm{vs}}(x+\hat{v}\tau)\Phi_0, p_j\Psi_0) + i((\partial_j V^{\mathrm{s}})(x+\hat{v}\tau)\Phi_0, \Psi_0)]d\tau$$

by the Lebesgue dominated convergence theorem. This yields the theorem.

As for Theorem 4.2, we need the following lemmas, which are the versions of Lemmas 3.2, 3.3, 3.4, 3.5 and 3.6 in the case where $V^{\mathrm{s}} \in \tilde{\mathscr{V}}^{\mathrm{s}}(1/2, 1, 1)$. Since their proofs are yielded by the above observations, we omit them: We will introduce

$$U^{\mathrm{s}}_{D,D}(t) = e^{-itH_0} M^{\mathrm{s}}_{D,D}(t), \qquad M^{\mathrm{s}}_{D,D}(t) = M_D(t) M^{\mathrm{s}}_D(t). \tag{4.17}$$

Here we note that $M_D(t)$ does commute with $M^{\mathrm{s}}_D(t)$.

Lemma 4.7. *Let v and Φ_v be as in Theorem 4.2, $V^{\mathrm{s}} \in \tilde{\mathscr{V}}^{\mathrm{s}}(1/2, 1, 1)$, and $V^1 \in \mathscr{V}^1_D(1/4)$ with $\gamma_D < 1/2$. If $\gamma_2 > 3/2$, then, for $0 \le \nu_1, \nu_2, \nu_3 \le 1$, there exists a positive constant C such that*

$$\begin{aligned} \|\langle x\rangle^2 M^{\mathrm{s}}_{D,D}(t)\Phi_v\| &= \|\langle x\rangle^2 M^{\mathrm{s}}_{D,D,v}(t)\Phi_0\| \\ &\le C(1 + |v|^{-(2\gamma_D+1)\nu_1}|t|^{4-(2\gamma_D+1)(2-\nu_1)} \\ &\quad + |v|^{-(\gamma_D+1)\nu_2}|t|^{3-(\gamma_D+1)(2-\nu_2)} \\ &\quad + |v|^{-(\gamma_D+1/2)\nu_3}|t|^{2-(\gamma_D+1/2)(2-\nu_3)}) \end{aligned} \tag{4.18}$$

holds as $|v| \to \infty$, where $M^{\mathrm{s}}_{D,D,v}(t) = e^{-iv\cdot x} M^{\mathrm{s}}_{D,D}(t) e^{iv\cdot x}$. On the other hand, if $\gamma_2 \le 3/2$, then, for $0 \le \nu_1, \nu_2, \nu_3, \nu_4 \le 1$, there exists a positive constant C such that

$$\begin{aligned} \|\langle x\rangle^2 M^{\mathrm{s}}_{D,D}(t)\Phi_v\| &= \|\langle x\rangle^2 M^{\mathrm{s}}_{D,D,v}(t)\Phi_0\| \\ &\le C(1 + |v|^{-(2\gamma_D+1)\nu_1}|t|^{4-(2\gamma_D+1)(2-\nu_1)} \\ &\quad + |v|^{-(\gamma_D+1)\nu_2}|t|^{3-(\gamma_D+1)(2-\nu_2)} \\ &\quad + |v|^{-(\gamma_D+1/2)\nu_3}|t|^{2-(\gamma_D+1/2)(2-\nu_3)} \\ &\quad + |v|^{-\gamma_2\nu_4}|t|^{3-\gamma_2(2-\nu_4)}) \end{aligned} \tag{4.19}$$

holds as $|v| \to \infty$, where only when $\gamma_2 = 3/2$, we assume $\nu_1 \ne 0$ additionally.

LEMMA 4.8. *Let v and Φ_v be as in Theorem 4.2, $V^{\mathrm{s}} \in \tilde{\mathscr{V}}^{\mathrm{s}}(1/2,1,1)$, and $V^1 \in \mathscr{V}_D^1(1/4)$. Then*

$$\int_{-\infty}^{\infty} \|V^{\mathrm{vs}}(x) U_{D,D}^{\mathrm{s}}(t) \Phi_v\| dt = O(|v|^{-1}) \tag{4.20}$$

holds as $|v| \to \infty$ for $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$.

LEMMA 4.9. *Let v and Φ_v be as in Theorem 4.2, $\varepsilon > 0$, $V^{\mathrm{s}} \in \tilde{\mathscr{V}}^{\mathrm{s}}(1/2,1,1)$, and $V^1 \in \mathscr{V}_D^1(1/4)$. Then*

$$\int_{-\infty}^{\infty} \|\{V^{\mathrm{s}}(x) - V^{\mathrm{s}}(pt - e_1 t^2/2)\} U_{D,D}^{\mathrm{s}}(t) \Phi_v\| dt$$
$$= O(|v|^{\max\{-1, -2(\gamma_2 - 1)+\varepsilon, -2\{(\gamma_1 - 1)+\gamma_D\}+\varepsilon\}}) \tag{4.21}$$

holds as $|v| \to \infty$.

LEMMA 4.10. *Let v and Φ_v be as in Theorem 4.2, $\varepsilon > 0$, $V^{\mathrm{s}} \in \tilde{\mathscr{V}}^{\mathrm{s}}(1/2,1,1)$, and $V^1 \in \mathscr{V}_D^1(1/4)$. Then*

$$\int_{-\infty}^{\infty} \|\{V^1(x) - V^1(pt - e_1 t^2/2)\} U_{D,D}^{\mathrm{s}}(t) \Phi_v\| dt = O(|v|^{-(4\gamma_D - 1)+\varepsilon}) \tag{4.22}$$

holds as $|v| \to \infty$.

Here we introduce auxiliary wave operators

$$\Omega_{D,D}^{\mathrm{s},\pm} = \operatorname*{s-lim}_{t \to \pm\infty} e^{itH} U_{D,D}^{\mathrm{s}}(t)$$

with $H = H_0 + V^{\mathrm{vs}} + V^{\mathrm{s}} + V^1$. Then we see that

$$\Omega_{D,D}^{\mathrm{s},\pm} = W_D^{\pm} I_D^{\mathrm{s},\pm}, \qquad I_D^{\mathrm{s},\pm} = \operatorname*{s-lim}_{t \to \pm\infty} M_D^{\mathrm{s}}(t)$$

exist.

LEMMA 4.11. *Let v and Φ_v be as in Theorem 4.2, and $\varepsilon > 0$. Then*

$$\sup_{t \in \boldsymbol{R}} \|(e^{-itH} \Omega_{D,D}^{\mathrm{s},-} - U_{D,D}^{\mathrm{s}}(t)) \Phi_v\|$$
$$= O(|v|^{\max\{-1, -2(\gamma_2 - 1)+\varepsilon, -2\{(\gamma_1 - 1)+\gamma_D\}+\varepsilon, -(4\gamma_D - 1)+\varepsilon\}}) \tag{4.23}$$

holds as $|v| \to \infty$ for $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$, $V^{\mathrm{s}} \in \tilde{\mathscr{V}}^{\mathrm{s}}(1/2,1,1)$, and $V^1 \in \mathscr{V}_D^1(1/4)$.

Based on Lemmas 4.8, 4.9, 4.10 and 4.11, Theorem 4.2 can be shown by the additional conditions

$$\gamma_2 > 5/4, \qquad \gamma_1 + \gamma_D > 5/4, \qquad \gamma_D > 3/8 \tag{4.24}$$

in the same way as in the proof of Theorem 3.1. So we omit its proof. Here we note that $\gamma_1 + \gamma_D > 5/4$ is satsified even when $\gamma_1 > 1$ and $\gamma_D > 1/4$.

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