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Existence of BV flow via elliptic regularization

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ABSTRACT. We investigate a mean curvature flow obtained via elliptic regularization, and prove that it is not only a Brakke flow, but additionally a generalized BV flow proposed by Stuvard and Tonegawa. In particular, we show that the change in volume of the evolving phase can be expressed in terms of the generalized mean curvature of the Brakke flow.

1. Introduction

Arising as the natural L^2-gradient flow of the area functional, the mean curvature flow (henceforth referred to as MCF) is arguably one of the most fundamental geometric flows. The unknown of MCF is a one-parameter family $\{M_t\}_{t\geq 0}$ of surfaces in the Euclidean space (or more generally some Riemannian manifold) such that the normal velocity vector V of M_t equals its mean curvature vector h at each point for every time, i.e.,

$$V = h \qquad \text{on } M_t. \tag{1.1}$$

When given a compact smooth surface M_0, a unique smooth solution exists for a finite time until singularities such as shrinkage and neck pinching occur. Numerous frameworks of generalized solutions of MCF that allow singularities have been proposed and studied: we mention, among others, the Brakke flow [2], level set flow [3, 6], BV flow [13, 11, 12], L^2 flow [15, 1], generalized BV flow [19]. These weak solutions have been investigated by numerous researchers in the last 40 years or so from varying viewpoints.

The aim of the present paper is to show that the flow arising from elliptic regularization [9] is a generalized BV flow in addition to being a Brakke flow. A generalized BV flow consists of a pair of phase function and Brakke flow, in many ways reminiscent to Ilmanen's enhanced motion described in [9], but the one which relates the volume change of phase and the Brakke flow in an

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explicit manner. When an $(n-1)$-dimension smooth MCF $\{M_t\}_{t\geq 0}$ is the boundary of open sets $\{E_t\}_{t\geq 0}$, the following equality holds naturally for all $\phi \in C_c^1(\mathbb{R}^n \times \mathbb{R}^+)$:

$$\frac{d}{dt}\int_{E_t} \phi(x,t)dx = \int_{E_t} \partial_t \phi(x,t)dx + \int_{M_t} \phi(x,t)h(x,t) \cdot \nu_{M_t}(x)d\mathscr{H}^{n-1}(x). \quad (1.2)$$

Here, $h(\cdot,t)$ is the mean curvature of M_t and ν_{M_t} is the unit outer normal vector of M_t. The notion of BV flow utilizes this equality to characterize the motion law, roughly speaking: a family of sets of finite perimeter $\{E_t\}_{t\geq 0}$ is a BV flow if the reduced boundary $M_t := \partial^* E_t$ has generalized mean curvature vector $h(\cdot,t)$ satisfying (1.2). The underlying assumption of the BV flow is that the generalized mean curvature vector is derived from $\partial^* E_t$. The generalized BV flow is proposed by Stuvard and Tonegawa [19] so that the flow can allow possible integer multiplicities (≥ 2) for the underlying Brakke flow while still keeping the equality (1.2) for sets of finite perimeter. When the higher multiplicity portion of the Brakke flow has null measure, the generalized BV flow corresponds to a BV flow. Additionally having this equality (1.2) has a certain conceptual advantage for the Brakke flow in that some non-uniqueness and stability issues of Brakke flow can be resolved: Fischer et al. [7] showed that a BV flow with smooth initial datum necessarily coincides with the smooth MCF until the time when some singularity appears. There are some conditional existence results for BV flows such as [13, 11, 12] under an assumption that the approximate solutions converge to the limit without loss of surface energy. In the present paper, we prove without any condition that the solution arising from elliptic regularization is a generalized BV solution with possible higher integer multiplicities on the side of Brakke flow.

We next briefly mention closely related works on the MCF using elliptic regularization. Ilmanen's elliptic regularization [9] gives a Brakke flow by first minimizing weighted area functional with a parameter. The solution of this minimization is a translative soliton and smooth (except for a small set of singularity of codimension 7), and by letting the parameter converge to 0, one obtains a Brakke flow as a limit of the smooth flows. One advantage of this method is that one can apply White's local regularity theorem for this Brakke flow arising from the elliptic regularization [23]. Elliptic regularization has also been studied for constructing MCFs with the Neumann boundary conditions by Edelen [4] and Dirichlet boundary conditions by White [25]. In addition, by Schulze and White [16], this method is utilized to construct a MCF with a triple junction by setting up an appropriate minimizing problem within the class of flat chains.

The key element of the present paper is an estimate of L^2 boundedness of approximate velocity for elliptic regularization. The idea is to find the convergence of the velocity vector representing the motion of phase boundaries using the concept of measure function pairs by Hutchinson [8]. The existence of velocity leads to absolute continuity of phase boundary measure in space-time with respect to the weight measure of the Brakke flow. More precisely, if $\{\mu_t\}$ is the Brakke flow, and $\{E_t\}$ is the family of sets of finite perimeter driven by μ_t, we may obtain $d|\partial S| \ll d\mu_t dt$, where $S = \{(x,t) \mid x \in E_t\}$ is the space-time track of $\{E_t\}$, and $|\partial S|$ is the total variation measure of the characteristic function of S in space-time. Once this is done, we may recover the formula (1.2) using a suitable version of co-area formula from geometric measure theory.

The paper is organized as follows. In Section 2, we set our notation and explain the main result. In Section 3, we review an outline of elliptic regularization. In Section 4, we construct the approximate velocity and show that the existence of velocity leads to a generalized BV flow, and then we prove that the limit of the translative soliton is indeed a generalized BV flow.

2. Preliminaries and main results

2.1. Basic notation. We shall use the same notation for the most part adopted in [19, Section 2]. In particular, the ambient space we will be working in is the Euclidean space $\mathbb{R}^n$ or its open subset U, and $\mathbb{R}^+$ will denote the interval $[0, \infty)$. The coordinates (x,t) are set in the product space $\mathbb{R}^n \times \mathbb{R}$, and t will be thought of and referred to as "time". We will denote $\mathbf{p}$ and $\mathbf{q}$ the projections of $\mathbb{R}^n \times \mathbb{R}$ onto its factor, so that $\mathbf{p}(x,t) = x$ and $\mathbf{q}(x,t) = t$. If $A \subset \mathbb{R}^n$ is (Borel) measurable, $\mathscr{L}^n(A)$ will denote the Lebesgue measure of A, whereas $\mathscr{H}^k(A)$ denotes the k-dimensional Hausdorff measure of A. When $x \in \mathbb{R}^n$ and $r > 0$, $B_r(x)$ denotes the closed ball centered at x with radius r. More generally, if k is an integer, then $B_r^k(x)$ will denote closed balls in $\mathbb{R}^k$. The symbols ∇, ∇', Δ, and ∇^2 denote the spatial gradient and the full gradient in $\mathbb{R}^n \times \mathbb{R}$, Laplacian, and Hessian, respectively. The symbol ∂_t will denote the time derivative.

A positive Radon measure μ on $\mathbb{R}^n$ (or "space-time" $\mathbb{R}^{n+1}$) is always also regarded as a positive linear functional on the space $C_c^0(\mathbb{R}^n)$ of continuous and compactly supported functions, with pairing denoted $\mu(\phi)$ for $\phi \in C_c^0(\mathbb{R}^n)$. The restriction of μ to a Borel set A is denoted $\mu \llcorner_A$, so that $(\mu \llcorner_A)(E) := \mu(A \cap E)$ for any $E \subset \mathbb{R}^n$. The support of μ is denoted $\operatorname{supp} \mu$, and it is the closed set defined by

$$\operatorname{supp} \mu := \{x \in \mathbb{R}^n \mid \mu(B_r(x)) > 0 \text{ for all } r > 0\}.$$

For $1 \le p \le \infty$, the space of p-integrable functions with respect to μ is denoted $L^p(\mu)$. If $\mu = \mathscr{L}^n$, $L^p(\mathscr{L}^n)$ is simply written $L^p(\mathbb{R}^n)$. For a signed or vector-valued measure μ, $|\mu|$ denotes its total variation. For two Radon measures μ and $\bar{\mu}$, when the measure $\bar{\mu}$ is absolutely continuous with respect to μ, we write $\bar{\mu} \ll \mu$. We say that a function $f \in L^1(\mathbb{R}^n)$ has a bounded variation, written $f \in BV(\mathbb{R}^n)$, if

$$\sup\left\{ \int_{\mathbb{R}^n} f \operatorname{div} X \, dx \,\middle|\, X \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ \|X\|_{C^0} \le 1 \right\} < \infty.$$

If $f \in BV(\mathbb{R}^n)$, then there exists an $\mathbb{R}^n$-valued Radon measure (which we will call the measure derivative of f denoted by ∇f) satisfying

$$\int_{\mathbb{R}^n} f \operatorname{div} X \, dx = -\int_{\mathbb{R}^n} X \cdot d\nabla f \qquad \text{for all } X \in C_c^1(\mathbb{R}^n;\mathbb{R}^n).$$

For a set $E \subset \mathbb{R}^n$, χ_E is the characteristic function of E, defined by $\chi_E = 1$ if $x \in E$ and $\chi_E = 0$ otherwise. We say that E has a finite perimeter if $\chi_E \in BV(\mathbb{R}^n)$. When E is a set of finite perimeter, then the measure derivative $\nabla\chi_E$ is the associated Gauss-Green measure, and its total variation $|\nabla\chi_E|$ is the perimeter measure; by De Giorgi's structure theorem, $|\nabla\chi_E| = \mathscr{H}^{n-1} \llcorner_{\partial^* E}$, where $\partial^* E$ is the reduced boundary of E, and $\nabla\chi_E = -\nu_E|\nabla\chi_E| = -\nu_E \mathscr{H}^{n-1} \llcorner_{\partial^* E}$, where ν_E is the outer pointing unit normal vector field to $\partial^* E$.

A subset $M \subset \mathbb{R}^n$ is countably k-rectifiable if it admits a covering

$$M \subset Z \cup \bigcup_{i \in \mathbb{N}} f_i(\mathbb{R}^k)$$

where $\mathscr{H}^k(Z) = 0$ and $f_i : \mathbb{R}^k \to \mathbb{R}^n$ is Lipschitz. If M is countably k-rectifiable, $\mathscr{H}^k$-measurable and $\mathscr{H}^k(M) < \infty$, M has a measure-theoretic tangent plane called approximate tangent plane for $\mathscr{H}^k$-a.e. $x \in M$ ([17, Theorem 11.6]), denoted by $T_x M$. We may simply refer to it as the tangent plane at $x \in M$ without fear of confusion. A Radon measure μ is said to be k-rectifiable if there are a countably k-rectifiable, $\mathscr{H}^k$-measurable set M and a positive function $\theta \in L^1(\mathscr{H}^k \llcorner_M)$ such that $\mu = \theta\mathscr{H}^k \llcorner_M$. This function θ is called multiplicity of μ. The approximate tangent plane of M in this case (which exists μ-a.e.) is denoted by $T_x\mu$. When θ is an integer for μ-a.e., μ is said to be integral. The first variation $\delta\mu : C_c^1(\mathbb{R}^n;\mathbb{R}^n) \to \mathbb{R}$ of a rectifiable Radon measure μ is defined by

$$\delta\mu(X) = \int_{\mathbb{R}^n} \operatorname{div}_{T_x\mu} X \, d\mu,$$

where $P_{T_x\mu}$ is the orthogonal projection from $\mathbb{R}^n$ to $T_x\mu$, and $\operatorname{div}_{T_x\mu} X = \operatorname{tr}(P_{T_x\mu}\nabla X)$. For an open set $U \subset \mathbb{R}^n$, the total variation $|\delta\mu|(U)$ of μ is defined by

$$|\delta\mu|(U) = \sup\{\delta\mu(X) \mid X \in C_c^1(U;\mathbb{R}^n),\ \|X\|_{C^0} \le 1\}.$$

If the total variation $|\delta\mu|(\tilde{U})$ is finite for any bounded subset $\tilde{U}$ of U, then $\delta\mu$ is called locally bounded, and we can regard $|\delta\mu|$ as a measure. If $|\delta\mu| \ll \mu$, then the Radon–Nikodým derivative (times -1) is called the generalized mean curvature vector h of μ, and we have

$$\delta\mu(X) = -\int_{\mathbb{R}^n} X \cdot h \, d\mu \qquad \text{for all } X \in C_c^1(\mathbb{R}^n;\mathbb{R}^n).$$

If μ is integral, then h and $T_x\mu$ are orthogonal for μ-a.e. by Brakke's perpendicularity theorem [2, Chapter 5].

2.2. Weak notions of mean curvature flow and main result. In this subsection, we introduce some weak solutions to the MCF. We briefly define and comment upon the three of interest in the present paper: We begin with the notion of Brakke flow introduced by Brakke [2].

Definition 2.1. A family of Radon measures $\{\mu_t\}_{t\in\mathbb{R}^+}$ in $\mathbb{R}^n$ is *an $(n-1)$-dimensional Brakke flow* if the following four conditions are satisfied:

(1): For a.e. $t \in \mathbb{R}^+$, μ_t is integral and $\delta\mu_t$ is locally bounded and absolutely continuous with respect to μ_t (thus the generalized mean curvature exists for a.e. t, denoted by h).

(2): For all $s > 0$ and all compact set $K \subset \mathbb{R}^n$, $\sup_{t\in[0,s]} \mu_t(K) < \infty$.

(3): The generalized mean curvature h satisfies $h \in L^2(d\mu_t dt)$.

(4): For all $0 \le t_1 < t_2 < \infty$ and all test functions $\phi \in C_c^1(\mathbb{R}^n \times \mathbb{R}^+;\mathbb{R}^+)$,

$$\mu_{t_2}(\phi(\cdot,t_2)) - \mu_{t_1}(\phi(\cdot,t_1)) \le \int_{t_1}^{t_2}\int_{\mathbb{R}^n} (\nabla\phi(x,t) - \phi(x,t)h(x,t)) \cdot h(x,t) + \partial_t\phi(x,t)d\mu_t(x)dt. \tag{2.1}$$

The inequality (2.1) is motivated by the following identity,

$$\int_{M_t} \phi(x,t)d\mathscr{H}^{n-1}\Big|_{t=t_1}^{t_2} = \int_{t_1}^{t_2}\int_{M_t} (\nabla\phi - \phi h)\cdot V + \partial_t\phi \, d\mathscr{H}^{n-1}dt, \tag{2.2}$$

where M_t is an $(n-1)$-dimensional smooth surface, h is the mean curvature vector, and V is the normal velocity vector of M_t. In particular, if $\{M_t\}_{t\in[0,T)}$ is a smooth MCF (hence $V = h$), setting $\mu_t := \mathscr{H}^{n-1}\llcorner_{M_t}$ defines a Brakke flow for which (2.1) is satisfied with the equality. Conversely, if $\mu_t = \mathscr{H}^{n-1}\llcorner_{M_t}$ with smooth M_t satisfies (2.1), then one can prove that $\{M_t\}_{t\in[0,T)}$ is a classical solution to the MCF. The notion of Brakke flow is equivalently (and originally in [2]) formulated in the framework of varifolds, but we use the above formulation using Radon measures, mainly for convenience.

The following definition of L^2 flow (modified slightly for our purpose) was given by Mugnai and Röger [15].

DEFINITION 2.2 (L^2 flow). A family of Radon measures $\{\mu_t\}_{t\in\mathbb{R}^+}$ in $\mathbb{R}^n$ is *an* $(n-1)$*-dimensional* L^2 *flow* if it satisfies (1)–(2) in Definition 2.1 as well as the following:

(a): The generalized mean curvature $h(\cdot,t)$ (which exists for a.e. $t\in\mathbb{R}^+$ by (1)) satisfies $h(\cdot,t)\in L^2(\mu_t;\mathbb{R}^n)$, and $d\mu := d\mu_t dt$ is a Radon measure on $\mathbb{R}^n\times\mathbb{R}^+$.

(b): There exists a vector field $V\in L^2(\mu;\mathbb{R}^n)$ and a constant $C=C(\mu_t)>0$ such that

(b′1): $V(x,t)\perp T_x\mu_t$ for μ-a.e. $(x,t)\in\mathbb{R}^n\times\mathbb{R}^+$,

(b′2): For every test functions $\phi\in C_c^1(\mathbb{R}^n\times(0,\infty))$, it holds

$$\left|\int_0^\infty\int_{\mathbb{R}^n}\partial_t\phi(x,t)+\nabla\phi(x,t)\cdot V(x,t)d\mu_t(x)dt\right|\le C\|\phi\|_{C^0}. \quad (2.3)$$

The vector field V satisfying (2.3) is called the generalized velocity vector in the sense of L^2 flow. This definition interprets equality (2.2) as a functional expression of the area change.

Finally, we introduce the concept of generalized BV flow suggested by Stuvard and Tonegawa [19].

DEFINITION 2.3 (Generalized BV flow). Let $\{\mu_t\}_{t\in\mathbb{R}^+}$ and $\{E_t\}_{t\in\mathbb{R}^+}$ be families of Radon measures and sets of finite perimeter, respectively. The pair $(\{\mu_t\}_{t\in\mathbb{R}^+},\{E_t\}_{t\in\mathbb{R}^+})$ is *a generalized BV flow* if all of the following hold:

(i): $\{\mu_t\}_{t\in\mathbb{R}^+}$ is a Brakke flow.

(ii): For all $t\in\mathbb{R}^+$, $|\nabla\chi_{E_t}|\le\mu_t$.

(iii): For all $0\le t_1<t_2<\infty$ and all test functions $\phi\in C_c^1(\mathbb{R}^n\times\mathbb{R}^+)$,

$$\int_{E_t}\phi(x,t)dx\Big|_{t=t_1}^{t_2}=\int_{t_1}^{t_2}\int_{E_t}\partial_t\phi(x,t)dxdt + \int_{t_1}^{t_2}\int_{\partial^*E_t}\phi(x,t)h(x,t)\cdot\nu_{E_t}(x)d\mathscr{H}^{n-1}(x)dt. \quad (2.4)$$

If μ_t and E_t satisfy the above definition, we say "$V=h$" in the sense of generalized BV flow. This definition expresses that the interface ∂^*E_t is driven by the mean curvature of μ_t. If $\mu_t=|\nabla\chi_{E_t}|$ for a.e. t, the characterization (2.4) coincides with the notion of BV flow considered by Luckhaus–Sturzenhecker in [13] since the mean curvature of ∂^*E_t is naturally defined to be $h(\cdot,t)$ in this case. On the other hand, while the original BV flow is characterized only by (2.4), here μ_t is additionally a Brakke flow to which one can apply White's local regularity theorem [23] (see [10, 21, 18] for more general regularity theorems for Brakke flow).

REMARK 2.4. The relationship between the Brakke flow $\{\mu_t\}_{t\in\mathbb{R}^+}$ and the sets of finite perimeter $\{E_t\}_{t\in\mathbb{R}^+}$ in the generalized BV flow appears unclear from the definition.

(1): If the initial datum E_0 satisfies the following assumption

$$\sup_{0<r<r_0,\, x\in\mathbb{R}^n} \frac{|\nabla\chi_{E_0}|(B_r(x))}{\omega_{n-1}r^{n-1}} \le 1 + o(r_0),$$

where $\omega_{n-1} = \mathscr{L}^{n-1}(B_1^{n-1})$, then there exists $T > 0$ such that $\mu_t = |\nabla\chi_{E_t}|$ for a.e. $t \in [0, T]$. One can see this fact by examining the time variation of the surface density using Huisken's monotonicity formula and demonstrating that a rapid increase in density does not occur for a short time interval (for more details, see [20, Proposition 8.6] for example).

(2): In general, the perimeter measure of the phase function $|\nabla\chi_{E_t}|$ and the Brakke flow μ_t may be different Radon measures. This discrepancy arises from the fact that convergence of the surface measures is lower semi-continuous when the MCF is constructed by approximation. It is not known that this discrepancy does not occur in the elliptic regularization.

REMARK 2.5. Note that one can prove that Brakke flow is an L^2 flow of $V = h$ in general, but the opposite implication may not hold in general. The following is a simple counterexample. Define

$$E_t = \begin{cases} \varnothing & (0 \le t < 1) \\ \{x \in \mathbb{R}^n \mid |x|^2 \le 1 - 2(n-1)(t-1)\} & \left(1 \le t \le 1 + \frac{1}{2(n-1)}\right), \end{cases}$$

and consider $\mu_t := \mathscr{H}^{n-1}\llcorner_{\partial E_t}$. One can show that it defines an L^2 flow with $V = h$ but it is not a Brakke flow.

The following claim is the essence of the main result of the present paper.

THEOREM 2.6. *Suppose that $E_0 \subset \mathbb{R}^n$ is a set of finite perimeter and consider the initial value problem of MCF starting from $\partial^* E_0$. Then, the elliptic regularization of Ilmanen* [9] *produces a generalized BV flow of $V = h$. Namely, in addition to a Brakke flow $\{\mu_t\}_{t\in\mathbb{R}^+}$ with $\mu_0 = \mathscr{H}^{n-1}\llcorner_{\partial^* E_0}$ (whose existence was proved in* [9]*), there exists a family of sets of finite perimeter $\{E_t\}_{t\in\mathbb{R}^+}$ such that $(\{\mu_t\}_{t\in\mathbb{R}^+}, \{E_t\}_{t\in\mathbb{R}^+})$ is a generalized BV flow.*

In the present paper, we discuss the case where the flow is considered in $\mathbb{R}^n$. Since the analysis is local in nature, the same characterization holds equally (with appropriate modifications) for flows in [4, 16, 25].

3. Review on the existence of Brakke flow via elliptic regularization

In this section, we briefly review Ilmanen's proof on the existence of Brakke flow in [9] for the convenience of the reader (see also a lecture note on MCF by White [24]).

3.1. Translative functional and Euler-Lagrange equation. The method of construction in [9] uses the framework of rectifiable current and for general codimensional case. Since we are concerned only with the hypersurface case, we work with surfaces realized as boundaries of sets of finite perimeter. In the following, the time variable is temporarily denoted as z. Let $\varepsilon > 0$ be fixed. The symbol $\mathbf{e}_{n+1}$ will denote the standard basis pointing the time direction, i.e., $\mathbf{e}_{n+1} = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$. We define the following functional for a set of finite perimeter $S \subset \mathbb{R}^{n+1}$:

$$I^\varepsilon(S) := \frac{1}{\varepsilon}\int_{\mathbb{R}^{n+1}} \exp\left(-\frac{z}{\varepsilon}\right) d|\nabla' \chi_S|(x, z). \tag{3.1}$$

We will say that a stationary point S^ε of I^ε is a translative soliton of the MCF with velocity $-(1/\varepsilon)\mathbf{e}_{n+1}$. The name "translative soliton" is based on the fact that the translation of S^ε in the time direction $S^\varepsilon - (t/\varepsilon)\mathbf{e}_{n+1}$ results in a MCF, as in the grim reaper type MCF. We consider a set of finite perimeter $E_0 \subset \mathbb{R}^n$ as an initial datum and find a stationary point S^ε for I^ε by minimization. The existence of a minimizer follows from the compactness theorem of set of finite perimeter (see [9, Section 3.2]).

Lemma 3.1. *Let $E_0 \subset \mathbb{R}^n$ be a set of finite perimeter. Then there exists a set of finite perimeter $S^\varepsilon \subset \mathbb{R}^n \times \mathbb{R}$ such that*

(1): *$S^\varepsilon \subset \mathbb{R}^n \times \mathbb{R}^+$ and $\{z = 0\} \cap \partial^* S^\varepsilon = E_0$,*

(2): *$I^\varepsilon(S^\varepsilon) \le \mathscr{H}^{n-1}(\partial^* E_0)$,*

(3): *S^ε is a minimizer for I^ε among the sets $\tilde{S} \subset \mathbb{R}^n \times \mathbb{R}^+$ with $\{z = 0\} \cap \partial^* \tilde{S} = E_0$.*

Definition 3.2. We define $S_z = \{x \in \mathbb{R}^n \mid (x, z) \in S\}$, that is, S_z is the horizontal slice of S at height z.

The well-known regularity theory of geometric measure theory shows that the support of $|\nabla' \chi_{S^\varepsilon}|$ in $\{z > 0\}$ is a smooth hypersurface except for a closed set of codimension 7 ([17]). For such S^ε, we calculate the first variation of I^ε and obtain the following equations (see [9, Section 2.6]).

Lemma 3.3. *Let S^ε be as in Lemma 3.1 and let h^ε denote the mean curvature of $\partial^* S^\varepsilon$ (computed as a submanifold in $\mathbb{R}^n \times (0,\infty)$). Then we have all of the following for $|\nabla' \chi_{S^\varepsilon}|$-a.e. in $\{z > 0\}$:*

(1): $\varepsilon h^\varepsilon + P_{T_{(x,z)}(\partial^* S^\varepsilon)^\perp}(\mathbf{e}_{n+1}) = 0,$

(2): $|h^\varepsilon| \le 1/\varepsilon,$

(3): $P_{T_{(x,z)}(\partial^* S^\varepsilon)^\perp}(h^\varepsilon) = h^\varepsilon,$

(4): $\varepsilon^2 |h^\varepsilon|^2 + |P_{T_{(x,z)}(\partial^* S^\varepsilon)}(\mathbf{e}_{n+1})|^2 = 1.$

By considering the integration of this equation in the time direction, we obtain the following (see [9, Section 4.5]).

Lemma 3.4. *For every $0 \le z_1 < z_2$,*

$$\int_{\partial^* S^\varepsilon_{z_2}} |P_{T_{(x,z_2)}(\partial^* S^\varepsilon)}(\mathbf{e}_{n+1})| \, d\mathscr{H}^{n-1}(x) + \int_{\partial^* S^\varepsilon \cap (\mathbb{R}^n \times (z_1, z_2))} \varepsilon |h^\varepsilon|^2 \, d\mathscr{H}^n(x,z)$$
$$= \int_{\partial^* S^\varepsilon_{z_1}} |P_{T_{(x,z_1)}(\partial^* S^\varepsilon)}(\mathbf{e}_{n+1})| \, d\mathscr{H}^{n-1}(x). \tag{3.2}$$

In particular, for any $\varepsilon > 0$,

$$\max\left\{ \sup_{z>0} \int_{\partial^* S^\varepsilon_z} |P_{T_{(x,z)}(\partial^* S^\varepsilon)}(\mathbf{e}_{n+1})| \, d\mathscr{H}^{n-1}(x), \right.$$
$$\left. \int_{\partial^* S^\varepsilon \cap (\mathbb{R}^n \times (0,\infty))} \varepsilon |h^\varepsilon|^2 \, d\mathscr{H}^n(x,z) \right\} \le \mathscr{H}^{n-1}(\partial^* E_0). \tag{3.3}$$

Now consider $E^\varepsilon = \kappa_\varepsilon(S^\varepsilon)$ in which S^ε is shrunk by the map $\kappa_\varepsilon(x,z) = (x, \varepsilon z)$ in the z direction. Since κ_ε is the contraction map by ε to z, the determinant of Jacobian matrix of κ_ε on $\partial^* S^\varepsilon$ is $(|P_{T_{(x,z)}(\partial^* S^\varepsilon)}(\nabla' \mathbf{p}(x,z))|^2 + \varepsilon^2 |P_{T_{(x,z)}(\partial^* S^\varepsilon)^\perp}(\mathbf{e}_{n+1})|^2)^{1/2}$. Therefore, by Lemma 3.4, we have the following for the mass of $\partial^* S^\varepsilon$ and $\partial^* E^\varepsilon$ (see [9, Section 5.1 and 5.3]).

Lemma 3.5. *For any open interval $A = (a,b) \subset \mathbb{R}^+$, we obtain*

$$|\nabla' \chi_{S^\varepsilon}|(\mathbb{R}^n \times A) \le (\mathscr{L}^1(A) + \varepsilon) \mathscr{H}^{n-1}(\partial^* E_0), \tag{3.4}$$

$$|\nabla' \chi_{E^\varepsilon}|(\mathbb{R}^n \times A) \le (\mathscr{L}^1(A) + \varepsilon^2 + (\mathscr{L}^1(A) + \varepsilon^2)^{1/2}) \mathscr{H}^{n-1}(\partial^* E_0). \tag{3.5}$$

In particular, the result holds for any $\mathscr{L}^1$-measurable set $A \subset \mathbb{R}^+$ by approximation.

For this S^ε, we define the following notation:

$$\sigma_{-t/\varepsilon}(x,z) := \left(x, z - \frac{t}{\varepsilon}\right), \qquad S^\varepsilon(t) := \sigma_{-t/\varepsilon}(S^\varepsilon), \qquad \mu_t^\varepsilon := |\nabla' \chi_{S^\varepsilon(t)}|. \tag{3.6}$$

This μ_t^ε is a Brakke flow on the (x,z) components of $W^\varepsilon := \{(x,z,t) \in (\mathbb{R}^n \times \mathbb{R}) \times [0,\infty) \mid z > -t/\varepsilon\}$. Since the Brakke flow μ_t^ε satisfies $\mu_t^\varepsilon(\mathbb{R}^n \times (z_1,z_2)) \le ((z_2 - z_1) + \varepsilon)\mathscr{H}^{n-1}(\partial^* E_0)$ by (3.4), we can apply the compactness theorem for Brakke flow to μ_t^ε [9, Section 7.1]. Thus taking a further subsequence from μ_t^ε, there exists a Brakke flow $\{\bar{\mu}_t\}_{t\ge 0}$ on the (x,z) components of $W := (\mathbb{R}^n \times \mathbb{R} \times (0,\infty)) \cup (\mathbb{R}^n \times (0,\infty) \times \{0\})$ such that μ_t^ε converges to $\bar{\mu}_t$ as Radon measure. Since $\bar{\mu}_t$ is invariant to translations in the z direction, we have the following from the product lemma [9, Lemma 8.5]. See [9, Section 8] for the details of the above discussion.

Lemma 3.6. *Let $\theta \in C_c^2(\mathbb{R};\mathbb{R}^+)$ with $\int_{\mathbb{R}} \theta(z)dz = 1$ and* $\operatorname{supp}\theta \subset (0,\infty)$ *be fixed. We define a Radon measure μ_t on $\phi \in C_c^0(\mathbb{R}^n;\mathbb{R}^+)$ by*

$$\mu_t(\phi) := \bar{\mu}_t(\theta\phi),$$

then μ_t is independent of θ and the following hold:

(1): *$\bar{\mu}_t = \mu_t \otimes \mathscr{L}^1$ except for countable $t \ge 0$,*

(2): *$\{\mu_t\}_{t\ge 0}$ is a Brakke flow on $\mathbb{R}^n$.*

When applying the compactness theorem of Brakke flow, we take a further subsequence using the compactness of set of finite perimeter: there exists a set of finite perimeter $E \subset \mathbb{R}^n \times \mathbb{R}^+$ such that

$$\chi_{E^\varepsilon} \to \chi_E \qquad \text{in } L^1_{loc}(\mathbb{R}^n \times \mathbb{R}^+),$$

$$|\nabla' \chi_E|(\phi) \le \liminf_{\varepsilon \to +0} |\nabla' \chi_{E^\varepsilon}|(\phi) \qquad \text{for all } \phi \in C_c^0(\mathbb{R}^n \times \mathbb{R}^+;\mathbb{R}^+).$$

Since $S^\varepsilon(t)$ is the translation of S^ε by $-t/\varepsilon$ in the z direction, and E^ε is the contraction by ε in the z direction as defined above, one can check the following:

$$S^\varepsilon(t)_z = S^\varepsilon_{z+t/\varepsilon} = E^\varepsilon_{t+\varepsilon z}, \tag{3.7}$$

where $S^\varepsilon(t)_z$, $S^\varepsilon_{z+t/\varepsilon}$ and $E^\varepsilon_{t+\varepsilon z}$ are the horizontal slices as in Definition 3.2.

4. A generalized BV flow: Proof of Theorem 2.6

The key to the proof is to construct an approximate velocity and obtain a suitable L^2 estimate for a convergence of the velocity.

4.1. Existence of measure-theoretic velocities. In this subsection, even after taking a subsequence, we use the same notation ε for simplicity. In establishing (2.3), the main tool is the co-area formula applied to S^ε. It is a formula describing the rate of change of the volume of E_t, and the obstacle to obtain this is the possible presence of a portion where $\partial^* S^\varepsilon$ is "close to being horizontal", that is, we want to make sure that the domain $\{|P_{T_{(x,z)}(\partial^* S^\varepsilon)}(\mathbf{e}_{n+1})| \le 1 - \varepsilon^{1/2}\}$ vanishes when passing to the limit $\varepsilon \to +0$. According to Lemma 3.3(4), if $|P_{T_{(x,z)}(\partial^* S^\varepsilon)}(\mathbf{e}_{n+1})| \le 1 - \varepsilon^{1/2}$, we have $1/\varepsilon^{3/2} \le |h^\varepsilon|^2 (\le 1/\varepsilon^2)$. Using this fact and the L^2 boundedness of h^ε (Lemma 3.4), we can prove that the domain $\{|P_{T_{(x,z)}(\partial^* S^\varepsilon)}(\mathbf{e}_{n+1})| \le 1 - \varepsilon^{1/2}\}$ is vanishing as $\varepsilon \to +0$.

Lemma 4.1. *For* S^ε *of Section* 3.1, *we define*

$$\Sigma^{\varepsilon,k} := \left\{ (x,z) \in \partial^* S^\varepsilon \,\middle|\, |P_{T_{(x,z)}(\partial^* S^\varepsilon)}(\mathbf{e}_{n+1})|^2 \le 1 - \frac{1}{k} \right\}.$$

Then if $1 < k \le \varepsilon^{-1/2}$, *we have* $\lim_{\varepsilon \to +0} |\nabla' \chi_{S^\varepsilon}|(\Sigma^{\varepsilon,k}) = 0$.

Proof. Let $1 < k \le \varepsilon^{-1/2}$ be fixed. From Lemma 3.3(1) and

$$1 = |P_{T_{(x,z)}(\partial^* S^\varepsilon)}(\mathbf{e}_{n+1})|^2 + |P_{T_{(x,z)}(\partial^* S^\varepsilon)^\perp}(\mathbf{e}_{n+1})|^2,$$

we have

$$|\nabla' \chi_{S^\varepsilon}|(\Sigma^{\varepsilon,k}) = |\nabla' \chi_{S^\varepsilon}|\left(\left\{ (x,z) \in \partial^* S^\varepsilon \,\middle|\, \frac{1}{k\varepsilon^2} \le |h^\varepsilon|^2 \right\} \right).$$

Thus, by using Markov's inequality and the L^2 boundedness of h^ε (3.4), we compute

$$\begin{aligned} |\nabla' \chi_{S^\varepsilon}|(\Sigma^{\varepsilon,k}) &= |\nabla' \chi_{S^\varepsilon}|(\{1 \le k\varepsilon^2 |h^\varepsilon|^2\}) \\ &\le k\varepsilon \int_{\mathbb{R}^n \times \mathbb{R}} \varepsilon |h^\varepsilon|^2 d|\nabla' \chi_{S^\varepsilon}| \le \varepsilon^{1/2} \mathscr{H}^{n-1}(\partial^* E_0). \end{aligned} \tag{4.1}$$

By taking $\varepsilon \to +0$, we obtain $\lim_{\varepsilon \to +0} |\nabla' \chi_{S^\varepsilon}|(\Sigma^{\varepsilon,k}) = 0$. □

The following two lemmas relate μ_t and E_t so that these fit in the framework of generalized BV flow.

Lemma 4.2. *Taking a further subsequence if necessary, we have* $d|\nabla \chi_{E_t^\varepsilon}|dt \rightharpoonup d\mu_t dt$ *as Radon measure.*

Remark 4.3. By Remark 2.4(2), $d|\nabla \chi_{E_t}|dt$ and $d\mu_t dt$ may not coincide in general.

PROOF. By (3.5) and the co-area formula (see, for example, [5, Theorem 3.10] or [14, Theorem 13.1]), we obtain

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^n} d|\nabla\chi_{E_t^\varepsilon}|dt \le |\nabla'\chi_{E^\varepsilon}|(\mathbb{R}^n\times(t_1,t_2))$$
$$\le ((t_2-t_1)+\varepsilon^2+((t_2-t_1)+\varepsilon^2)^{1/2})\mathscr{H}^{n-1}(\partial^* E_0) \qquad (4.2)$$

for all $0\le t_1<t_2<\infty$. Thus, by the compactness theorem of Radon measure, $d|\nabla\chi_{E_t^\varepsilon}|dt$ converges to some Radon measure.

Next, we fix $\phi\in C_c^0(\mathbb{R}^n\times\mathbb{R}^+)$ and $z\ge 0$. Parallel translating with respect to time, for sufficiently small $\varepsilon>0$, we have

$$\left|\int_0^\infty\int_{\mathbb{R}^n}\phi\, d|\nabla\chi_{E_t^\varepsilon}|dt-\int_0^\infty\int_{\mathbb{R}^n}\phi\, d|\nabla\chi_{E_{t+\varepsilon z}^\varepsilon}|dt\right|$$
$$\le\int_0^\infty\int_{\mathbb{R}^n}|\phi(x,t-\varepsilon z)-\phi(x,t)|d|\nabla\chi_{E_t^\varepsilon}|dt$$
$$\le\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}^+}|\phi(x,t-\varepsilon z)-\phi(x,t)|\,|\nabla'\chi_{E^\varepsilon}|(K), \qquad (4.3)$$

where K is a sufficiently large compact set for ϕ and we used the co-area formula. Therefore, by letting $\varepsilon\to+0$ in the above, we can deduce

$$\lim_{\varepsilon\to+0}\int_0^\infty\int_{\mathbb{R}^n}\phi\, d|\nabla\chi_{E_t^\varepsilon}|dt=\lim_{\varepsilon\to+0}\int_0^\infty\int_{\mathbb{R}^n}\phi\, d|\nabla\chi_{E_{t+\varepsilon z}^\varepsilon}|dt. \qquad (4.4)$$

Finally, we prove $d|\nabla\chi_{E_t^\varepsilon}|dt\rightharpoonup d\mu_t dt$ as Radon measure. Let $\phi\in C_c^0(\mathbb{R}^n\times\mathbb{R}^+)$, and let $\theta\in C_c^2(\mathbb{R};\mathbb{R}^+)$ with $\int_{\mathbb{R}}\theta(z)dz=1$ and $\operatorname{supp}\theta\subset(0,\infty)$ be arbitrary. To use the co-area formula for $\partial^* S^\varepsilon(t)$, we translate $\int_0^\infty\int_{\mathbb{R}^n}\phi\, d\mu_t dt$ as

$$\int_0^\infty\int_{\mathbb{R}^n}\phi\, d\mu_t dt$$
$$=\lim_{\varepsilon\to+0}\int_0^\infty\int_{\mathbb{R}^{n+1}}\theta(z)\phi(x,t)d\mu_t^\varepsilon(x,z)dt$$
$$=\lim_{\varepsilon\to+0}\left\{\int_0^\infty\int_{\mathbb{R}^{n+1}}\theta(z)\phi(x,t)(1-|P_{T_{(x,z)}(\partial^* S^\varepsilon(t))}(\mathbf{e}_{n+1})|)d|\nabla'\chi_{S^\varepsilon(t)}|(x,z)dt\right.$$
$$\left.+\int_0^\infty\int_{\mathbb{R}^{n+1}}\theta(z)\phi(x,t)|P_{T_{(x,z)}(\partial^* S^\varepsilon(t))}(\mathbf{e}_{n+1})|d|\nabla'\chi_{S^\varepsilon(t)}|(x,z)dt\right\}. \qquad (4.5)$$

Since $|P_{T_{(x,z)}(\partial^* S^\varepsilon)}(\mathbf{e}_{n+1})|^2\approx 1$ from Lemma 4.1, setting A_ε and B_ε by

$$A_\varepsilon := \int_0^\infty \int_{\mathbb{R}^{n+1}} \theta(z)\phi(x,t)(1-|P_{T_{(x,z)}(\partial^* S^\varepsilon(t))}(\mathbf{e}_{n+1})|)d|\nabla' \chi_{S^\varepsilon(t)}|(x,z)dt,$$

$$B_\varepsilon := \int_0^\infty \int_{\mathbb{R}^{n+1}} \theta(z)\phi(x,t)|P_{T_{(x,z)}(\partial^* S^\varepsilon(t))}(\mathbf{e}_{n+1})|d|\nabla' \chi_{S^\varepsilon(t)}|(x,z)dt,$$

we can predict $A_\varepsilon \to 0$ and $B_\varepsilon \approx \int_0^\infty \int_{\mathbb{R}^n} \phi \, d|\nabla \chi_{E_t^\varepsilon}|dt$.

For B_ε, by the co-area formula and (3.7), we obtain

$$\begin{aligned}\lim_{\varepsilon\to+0} B_\varepsilon &= \lim_{\varepsilon\to+0}\int_0^\infty \int_{\mathbb{R}^{n+1}} \theta\phi \, d|\nabla \chi_{S^\varepsilon_{t/\varepsilon+z}}|dzdt \\ &= \lim_{\varepsilon\to+0}\int_0^\infty \int_{\mathbb{R}^{n+1}} \theta\phi \, d|\nabla \chi_{E^\varepsilon_{t+\varepsilon z}}|dzdt \\ &= \lim_{\varepsilon\to+0}\int_0^\infty \int_{\mathbb{R}^{n}} \phi \, d|\nabla \chi_{E^\varepsilon_{t}}|dt. \end{aligned} \tag{4.6}$$

Here, we also used Fubini's theorem to change the order of integration with respect to z and t, (4.4) and $\int_{\mathbb{R}} \theta(z)dz = 1$ for the second line to the third line.

Now we consider A_ε. For $\Sigma^{\varepsilon,k}$ of Lemma 4.1, $1 - |P_{T_{(x,z)}(\partial^* S^\varepsilon)}(\mathbf{e}_{n+1})|^2 < 1/k$ is satisfied for all $(x,z) \in \partial^* S^\varepsilon \backslash \Sigma^{\varepsilon,k}$. Hence, by using the co-area formula, the mass boundedness of $\partial^* S_z$ (3.3) and $\int_{\mathbb{R}} \theta(z)dz = 1$, we calculate as

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^{n+1}} |\theta\phi|\chi_{\sigma_{-t/\varepsilon}(\partial^* S^\varepsilon \backslash \Sigma^{\varepsilon,k})}(1-|P_{T_{(x,z)}(\partial^* S^\varepsilon(t))}(\mathbf{e}_{n+1})|^2)d|\nabla' \chi_{S^\varepsilon(t)}|dt \\ &\leq \int_0^\infty \int_{\mathbb{R}^{n+1}} |\theta\phi|\chi_{\sigma_{-t/\varepsilon}(\partial^* S^\varepsilon \backslash \Sigma^{\varepsilon,k})}\frac{1}{k}\frac{|P_{T_{(x,z)}(\partial^* S^\varepsilon(t))}(\mathbf{e}_{n+1})|^2}{1-\frac{1}{k}} \, d|\nabla' \chi_{S^\varepsilon(t)}|dt \\ &\leq \frac{\mathscr{L}^1(K)\sup|\phi|}{k-1}\int_{\mathbb{R}}\int_{\mathbb{R}^n} \theta|P_{T_{(x,z)}(\partial^* S^\varepsilon)}(\mathbf{e}_{n+1})|d|\nabla \chi_{S_z^\varepsilon}|dz \\ &\leq \frac{\mathscr{L}^1(K)\sup|\phi|}{k-1}\mathscr{H}^{n-1}(\partial^* E_0),\end{aligned}$$

where K is a sufficiently large bounded interval for ϕ. Therefore, by using Lemma 4.1 for $k = \varepsilon^{-1/2}$, we obtain

$$\begin{aligned}|A_\varepsilon| &\leq \int_0^\infty \int_{\mathbb{R}^{n+1}} |\theta\phi|(\chi_{\sigma_{-t/\varepsilon}(\Sigma^{\varepsilon,\varepsilon^{-1/2}})} + \chi_{\sigma_{-t/\varepsilon}(\partial^* S^\varepsilon \backslash \Sigma^{\varepsilon,\varepsilon^{-1/2}})}) \\ &\qquad \times (1-|P_{T_{(x,z)}(\partial^* S^\varepsilon(t))}(\mathbf{e}_{n+1})|^2)d|\nabla' \chi_{S^\varepsilon(t)}|dt \\ &\leq \int_0^\infty \int_{\mathbb{R}^{n+1}} |\theta\phi|\chi_{\sigma_{-t/\varepsilon}(\Sigma^{\varepsilon,\varepsilon^{-1/2}})}d|\nabla' \chi_{S^\varepsilon(t)}|dt\end{aligned}$$

$$+\int_0^\infty \int_{\mathbb{R}^{n+1}} |\theta\phi|\chi_{\sigma_{-t/\varepsilon}(\partial^* S^\varepsilon \setminus \Sigma^{\varepsilon,\varepsilon^{-1/2}})}$$

$$\times (1 - |P_{T_{(x,z)}(\partial^* S^\varepsilon(t))}(\mathbf{e}_{n+1})|^2) d|\nabla' \chi_{S^\varepsilon(t)}| dt$$

$$\leq C(\theta,\phi,E_0)\left(\int_{\mathbb{R}^{n+1}} \chi_{\Sigma^{\varepsilon,\varepsilon^{-1/2}}}\, d|\nabla'\chi_{S^\varepsilon}| + \frac{1}{\varepsilon^{-1/2}-1}\right)$$

$$\leq C(\theta,\phi,E_0)(|\nabla'\chi_{S^\varepsilon}|(\Sigma^{\varepsilon,\varepsilon^{-1/2}}) + \varepsilon^{1/2}), \tag{4.7}$$

where $C(\theta,\phi,E_0)$ is a constant that depends only on θ, ϕ and the initial value E_0. By Lemma 4.1, we obtain $\lim_{\varepsilon\to+0} A_\varepsilon = 0$. Thanks to (4.5)–(4.7), we have $d|\nabla\chi_{E_t^\varepsilon}|dt \rightharpoonup d\mu_t dt$ as Radon measure. □

Lemma 4.4. *Taking a further subsequence if necessary, we have* $|\nabla\chi_{E_t}|(\phi) \leq \mu_t(\phi)$ *for all* $\phi \in C_c^0(\mathbb{R}^n;\mathbb{R}^+)$ *and for a.e.* $t \geq 0$.

Proof. From Section 3.1, $\chi_{E^\varepsilon} \to \chi_E$ in $L^1_{loc}(\mathbb{R}^n \times \mathbb{R}^+)$. Let $z > 0$ be arbitrary. Since the parallel translation is continuous in L^1, we have $\chi_{E^\varepsilon+\varepsilon z\mathbf{e}_{n+1}} \to \chi_E$ in $L^1_{loc}(\mathbb{R}^n\times\mathbb{R}^+)$. Taking a further subsequence from $\{E^\varepsilon + \varepsilon z\mathbf{e}_{n+1}\}$, $\chi_{E^\varepsilon}(x, t+\varepsilon z)$ converges to $\chi_E(x,t)$ for $\mathscr{L}^{n+1}$-a.e. $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$, and by Fubini, $\chi_{E^\varepsilon_{t+\varepsilon z}} \to \chi_{E_t}$ in $L^1_{loc}(\mathbb{R}^n)$ for a.e. $t \geq 0$. Thus we obtain $|\nabla\chi_{E_t}|(\phi) \leq \liminf_{\varepsilon\to+0} |\nabla\chi_{E^\varepsilon_{t+\varepsilon z}}|(\phi)$ for all $\phi \in C_c^0(\mathbb{R}^n;\mathbb{R}^+)$ and a.e. $t \geq 0$ by the lower semi-continuity of variation measure. Let $\phi \in C_c^0(\mathbb{R}^n;\mathbb{R}^+)$ and $\theta \in C_c^2(\mathbb{R};\mathbb{R}^+)$ with $\int_{\mathbb{R}} \theta(z)dz = 1$, $\operatorname{supp}\theta \subset (0,\infty)$ be arbitrary. Then we obtain

$$\mu_t(\phi) = \bar{\mu}_t(\theta\phi) = \lim_{\varepsilon\to+0} \int_{\mathbb{R}^{n+1}} \theta\phi\, d|\nabla'\chi_{S^\varepsilon(t)}|$$

$$\geq \liminf_{\varepsilon\to+0} \int_{\mathbb{R}^{n+1}} \theta\phi |P_{T_{(x,z)}(\partial^* S(t))}(\mathbf{e}_{n+1})| d|\nabla'\chi_{S^\varepsilon(t)}|$$

$$= \liminf_{\varepsilon\to+0} \int_{\mathbb{R}} \theta \int_{\mathbb{R}^n} \phi\, d|\nabla\chi_{S^\varepsilon(t)_z}| dz$$

$$= \liminf_{\varepsilon\to+0} \int_{\mathbb{R}} \theta \int_{\mathbb{R}^n} \phi\, d|\nabla\chi_{S^\varepsilon_{z+t/\varepsilon}}| dz$$

$$= \liminf_{\varepsilon\to+0} \int_{\mathbb{R}} \theta \int_{\mathbb{R}^n} \phi\, d|\nabla\chi_{E^\varepsilon_{t+\varepsilon z}}| dz$$

$$\geq \int_{\mathbb{R}} \theta \liminf_{\varepsilon\to+0} |\nabla\chi_{E^\varepsilon_{t+\varepsilon z}}|(\phi) dz \geq |\nabla\chi_{E_t}|(\phi) \tag{4.8}$$

where we used the co-area formula, Fatou's Lemma, the lower semi-continuity of $|\nabla\chi_{E_t}|$, $\int_{\mathbb{R}} \theta(z)dz = 1$ and (3.7). □

Finally, we construct an approximate velocity to show that $d|\nabla'\chi_E| \ll d\mu_t dt$. If the co-area formula is available, we obtain

$$\int_E \partial_t \phi \, dxdt = \int_{\partial^* E} \phi \mathbf{q}(\nu_E) d\mathscr{H}^n = \int_0^\infty \int_{\partial^* E_t} \phi \frac{\mathbf{q}(\nu_E)}{|\mathbf{p}(\nu_E)|} \, d\mathscr{H}^{n-1} dt \tag{4.9}$$

for all $\phi \in C_c^1(\mathbb{R}^n \times (0, \infty))$. Since we have that the domain $\{|\mathbf{p}(\nu_{S^\varepsilon})| \approx 0\}$ where the co-area formula is not applicable goes to measure 0 from Lemma 4.1 and $|\mathbf{p}(\nu_{S^\varepsilon})| = |P_{T_{(x,z)}(\partial^* S)}(\mathbf{e}_{n+1})|$, we may construct the approximate velocity based on (4.9).

Proposition 4.5. *Taking a further subsequence if necessary, there exists $V \in L^2(d\mu_t dt)$ such that*

$$\int_E \partial_t \phi \, dxdt = -\int_0^\infty \int_{\mathbb{R}^n} \phi V \, d\mu_t dt, \tag{4.10}$$

for all $\phi \in C_c^1(\mathbb{R}^n \times (0, \infty))$.

Proof. To use Lemma 4.1, we assume that $\varepsilon > 0$ is sufficiently small. Let $\phi \in C_c^1(\mathbb{R}^n \times (0, \infty))$ be arbitrary. We define the approximate velocity of E^ε by

$$\begin{aligned} V_\varepsilon(x,t) &:= -\chi_{\kappa_\varepsilon(\partial^* S^\varepsilon \setminus \Sigma^{\varepsilon,2})} \frac{\mathbf{q}(\nu_{E^\varepsilon})}{|\mathbf{p}(\nu_{E^\varepsilon})|}(x,t) \\ &= \begin{cases} -\frac{\mathbf{q}(\nu_{E^\varepsilon})}{|\mathbf{p}(\nu_{E^\varepsilon})|}(x,t) & ((x,t) \in \kappa_\varepsilon(\partial^* S^\varepsilon \setminus \Sigma^{\varepsilon,2})) \\ 0 & ((x,t) \in \kappa_\varepsilon(\Sigma^{\varepsilon,2})). \end{cases} \end{aligned}$$

(Note that $\mathbf{p}$ and $\mathbf{q}$ are the projections of $\mathbb{R}^n \times \mathbb{R}$ onto its factor, so that $\mathbf{p}(x,t) = x$ and $\mathbf{q}(x,t) = t$.) Since the map κ_ε shrinks z-variable by ε, the following holds as the relationship between the unit normal vectors of S^ε and E^ε:

$$\frac{\mathbf{q}(\nu_{S^\varepsilon})}{|\mathbf{p}(\nu_{S^\varepsilon})|}(x,z) = \varepsilon \frac{\mathbf{q}(\nu_{E^\varepsilon})}{|\mathbf{p}(\nu_{E^\varepsilon})|}(x,t), \qquad t = \varepsilon z. \tag{4.11}$$

Furthermore, since the area element of κ_ε is ε and $t = \varepsilon z$, we have

$$\int_{E^\varepsilon} \partial_t \phi \, dxdt = \int_{S^\varepsilon} \partial_z \phi \, dxdz. \tag{4.12}$$

A simple geometric argument shows $|\mathbf{p}(\nu_{S^\varepsilon})| = |P_{T_{(x,z)}(\partial^* S^\varepsilon)}(\mathbf{e}_{n+1})|$, and by the definition of $\Sigma^{\varepsilon,2}$, we may deduce $1/2 < |\mathbf{p}(\nu_{S^\varepsilon})|^2$ on $\partial^* S^\varepsilon \setminus \Sigma^{\varepsilon,2}$. Thus, by the co-area formula and (4.12), we obtain

$$
\begin{aligned}
\int_{E^\varepsilon} \partial_t \phi \, dxdt &= \int_{S^\varepsilon} \partial_z \phi \, dxdz \\
&= \int_{\mathbb{R}^n \times (0,\infty)} (\chi_{\Sigma^{\varepsilon,2}} + \chi_{\partial^* S^\varepsilon \setminus \Sigma^{\varepsilon,2}}) \phi \mathbf{q}(\nu_{S^\varepsilon}) d|\nabla' \chi_{S^\varepsilon}| \\
&= \int_{\mathbb{R}^n \times (0,\infty)} \chi_{\Sigma^{\varepsilon,2}} \phi \mathbf{q}(\nu_{S^\varepsilon}) d|\nabla' \chi_{S^\varepsilon}| \\
&\quad + \int_0^\infty \int_{\mathbb{R}^n} \phi \chi_{\partial^* S^\varepsilon \setminus \Sigma^{\varepsilon,2}} \frac{\mathbf{q}(\nu_{S^\varepsilon})}{|\mathbf{p}(\nu_{S^\varepsilon})|} \, d|\nabla \chi_{S^\varepsilon_z}| dz \\
&= \int_{\mathbb{R}^n \times (0,\infty)} \chi_{\Sigma^{\varepsilon,2}} \phi \mathbf{q}(\nu_{S^\varepsilon}) d|\nabla' \chi_{S^\varepsilon}| - \int_0^\infty \int_{\mathbb{R}^n} \phi V_\varepsilon \, d|\nabla \chi_{E^\varepsilon_t}| dt. \qquad (4.13)
\end{aligned}
$$

From Lemma 4.1, we see that

$$\lim_{\varepsilon \to +0} \int_{\mathbb{R}^n \times (0,\infty)} \chi_{\Sigma^{\varepsilon,2}} \phi \mathbf{q}(\nu_{S^\varepsilon}) d|\nabla' \chi_{S^\varepsilon}| = 0. \qquad (4.14)$$

Next, we prove the following with respect to the second term:

$$\int_0^\infty \int_{\mathbb{R}^n} |V_\varepsilon|^2 d|\nabla \chi_{E^\varepsilon_t}| dt \le C < \infty, \qquad (4.15)$$

where C is a constant that depends only on $\mathcal{H}^{n-1}(\partial^* E_0)$. By (4.11), we see that

$$
\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^n} |V_\varepsilon|^2 d|\nabla \chi_{E^\varepsilon_t}| dt &= \int_0^\infty \int_{\mathbb{R}^n} \left(\frac{1}{\varepsilon} \chi_{\partial^* S^\varepsilon \setminus \Sigma^{\varepsilon,2}} \frac{\mathbf{q}(\nu_{S^\varepsilon})}{|\mathbf{p}(\nu_{S^\varepsilon})|} \right)^2 \varepsilon \, d|\nabla \chi_{S^\varepsilon_z}| dz \\
&= \frac{1}{\varepsilon} \int_0^\infty \int_{\mathbb{R}^n} \left(\chi_{\partial^* S^\varepsilon \setminus \Sigma^{\varepsilon,2}} \frac{\mathbf{q}(\nu_{S^\varepsilon})}{|\mathbf{p}(\nu_{S^\varepsilon})|} \right)^2 d|\nabla \chi_{S^\varepsilon_z}| dz.
\end{aligned}
$$

From $1/2 < |\mathbf{p}(\nu_{S^\varepsilon})|^2$ on $\partial^* S^\varepsilon \setminus \Sigma^{\varepsilon,2}$ and Lemma 3.3(1), we obtain

$$\left(\chi_{\partial^* S^\varepsilon \setminus \Sigma^{\varepsilon,2}} \frac{\mathbf{q}(\nu_{S^\varepsilon})}{|\mathbf{p}(\nu_{S^\varepsilon})|} \right)^2 \le 2\varepsilon^2 |h^\varepsilon|^2.$$

Thus, by (3.3), we have

$$
\begin{aligned}
&\frac{1}{\varepsilon} \int_0^\infty \int_{\mathbb{R}^n} \left(\chi_{\partial^* S^\varepsilon \setminus \Sigma^{\varepsilon,2}} \frac{\mathbf{q}(\nu_{S^\varepsilon})}{|\mathbf{p}(\nu_{S^\varepsilon})|} \right)^2 d|\nabla \chi_{S^\varepsilon_z}| dz \\
&\quad \le 2 \int_0^\infty \int_{\mathbb{R}^n} \varepsilon |h^\varepsilon|^2 d|\nabla \chi_{S^\varepsilon_z}| dz \le 2\mathcal{H}^{n-1}(\partial^* E_0),
\end{aligned}
$$

and thus (4.15) is proved. As Lemma 4.2 and (4.15) are valid, we can apply Theorem A.3 to $(d|\nabla\chi_{E_t^\varepsilon}|dt, V_\varepsilon)$. Therefore, taking a further subsequence if necessary, we obtain a function $V \in L^2(d\mu_t dt)$ such that

$$\lim_{\varepsilon\to+0}\int_0^\infty\int_{\mathbb{R}^n}\phi V_\varepsilon\, d|\nabla\chi_{E_t^\varepsilon}|dt = \int_0^\infty\int_{\mathbb{R}^n}\phi V\, d\mu_t dt \tag{4.16}$$

for all $\phi \in C_c^1(\mathbb{R}^n \times (0,\infty))$. By (4.14), (4.16) and $\chi_{E^\varepsilon} \to \chi_E$ in $L^1_{loc}(\mathbb{R}^n \times \mathbb{R})$, we obtain

$$\begin{aligned}\int_E \partial_t\phi\, dxdt &= \lim_{\varepsilon\to+0}\int_{E^\varepsilon}\partial_t\phi\, dxdt \\ &= \lim_{\varepsilon\to+0}\left(-\int_0^\infty\int_{\mathbb{R}^n}\phi V_\varepsilon\, d|\nabla\chi_{E_t^\varepsilon}|dt\right) = -\int_0^\infty\int_{\mathbb{R}^n}\phi V\, d\mu_t dt,\end{aligned}$$

for all $\phi \in C_c^1(\mathbb{R}^n \times (0,\infty))$. This completes the proof. □

Lemma 4.6. *There exists $G \subset \mathbb{R}^+$ with $\mathscr{L}^1(\mathbb{R}^+\setminus G) = 0$ such that χ_{E_t} is $1/2$-Hölder continuous in $L^1(\mathbb{R}^n)$ norm with respect to t on G.*

Proof. Let $G \subset \mathbb{R}^+$ be a set such that $t \in G$ is a Lebesgue point of function $f_\phi(s) := \int_{E_s}\phi\, dx$ for any $\phi \in C_c^1(\mathbb{R}^n)$. By choosing a countable dense set of functions in $C_c^1(\mathbb{R}^n)$ and using a standard result in measure theory, one can prove that such G is a full-measure set in $\mathbb{R}^+$. Let t_1, t_2 $(t_1 < t_2)$ be arbitrary points in G, and consider a smooth approximation η of $\chi_{[t_1,t_2]}$. Use $\phi(x)\eta(t)$ in (4.10) and let $\eta \to \chi_{[t_1,t_2]}$ to obtain

$$-\int_{E_{t_2}}\phi\, dx + \int_{E_{t_1}}\phi\, dx = -\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\phi V\, d\mu_t dt.$$

By approximation, we may replace ϕ by $\chi_{E_{t_1}}$ and obtain

$$\begin{aligned}|\mathscr{L}^n(E_{t_1}) - \mathscr{L}^n(E_{t_2}\cap E_{t_1})| &\le \left(\int_{t_1}^{t_2}\mu_t(E_{t_1})dt\right)^{1/2}\|V\|_{L^2(d\mu_t dt)} \\ &\le (t_2-t_1)^{1/2}\mathscr{H}^{n-1}(\partial^* E_0)^{1/2}\|V\|_{L^2(d\mu_t dt)},\end{aligned} \tag{4.17}$$

where we also used $\mu_t(\mathbb{R}^n) \le \mu_0(\mathbb{R}^n) = \mathscr{H}^{n-1}(\partial^* E_0)$. This inequality is due to the energy decreasing property of Brakke flow which follows from (2.1). The left-hand side of (4.17) is $\mathscr{L}^n(E_{t_1}\setminus E_{t_2})$. One can obtain the similar estimate for $\mathscr{L}^n(E_{t_2}\setminus E_{t_1})$ by considering $\chi_{E_{t_2}}$. This proves the claim. □

Remark 4.7. By Lemma 4.6, if necessary, we may re-define E so that χ_{E_t} is $1/2$-Hölder continuous in $L^1(\mathbb{R}^n)$ on $\mathbb{R}^+$. We also point out that, one can re-define the Brakke flow $\{\mu_t\}_{t\in\mathbb{R}^+}$ so that it is left-continuous at all $t\in\mathbb{R}^+$. This is because, for any $\phi\in C_c^2(\mathbb{R}^n)$, $\mu_t(\phi)-C(\phi)t$ is a monotone decreasing function of t for a suitable $C>0$, and is discontinuous on a countable set at most (see for example [22, Proposition 3.3]). At these discontinuous points, one may re-define μ_t (by approaching from the left) so that it is left-continuous while keeping (2.1). Now the claim of Lemma 4.4 is for a.e. $t\geq 0$, while Definition 2.3(ii) is for all $t\geq 0$. Let $\tilde{G}$ be the set of points where the conclusion of Lemma 4.4 holds, which is a full-measure set of $\mathbb{R}^+$. For any $t\notin\tilde{G}$, we may choose a sequence $\{t_i\}\subset\tilde{G}$ approaching from left to t. Since $\chi_{E_{t_i}}\to\chi_{E_t}$ in $L^1(\mathbb{R}^n)$, we have for any $\phi\in C_c^0(\mathbb{R}^n;\mathbb{R}^+)$

$$|\nabla\chi_{E_t}|(\phi)\leq \liminf_{i\to\infty}|\nabla\chi_{E_{t_i}}|(\phi)\leq \liminf_{i\to\infty}\mu_{t_i}(\phi)=\mu_t(\phi).$$

Here the first inequality is due to the lower semi-continuous property, the second is due to $t_i\in\tilde{G}$, and the last is the left-continuity of μ_t. Thus we have the desired property Definition 2.3(ii).

Even if a family of perimeter measures $\{|\nabla\chi_{E_t}|\}_{t\geq 0}$ is a Brakke flow, the pair $(\{|\nabla\chi_{E_t}|\}_{t\geq 0},\{E_t\}_{t\geq 0})$ may not be a genenralized BV flow. For example, define

$$E_t=\begin{cases}\{x\in\mathbb{R}^n\mid |x|^2\leq 1-2(n-1)t\} & \left(0\leq t<\frac{1}{4(n-1)}\right),\\ \emptyset & \left(\frac{1}{4(n-1)}\leq t\right),\end{cases}$$

this is a simple counterexample, that is, the formula (2.4) fails at $t=1/(4(n-1))$. We can expect such a phenomenon where the formula (2.4) does not hold to occur due to a discontinuity to time direction in the measure-theoretic sense. The existence of the velocity ensures that the discontinuity does not occur.

PROPOSITION 4.8. *For* $\{\mu_t\}_{t\geq 0}$ *and* $\{E_t\}_{t\geq 0}$ *of Section* 3.1, *we have* $d|\nabla'\chi_E|\ll d\mu_t dt$.

PROOF. From Proposition 4.5,

$$\nabla'\chi_E(x,t)=(\nabla\chi_{E_t}\,dt,V(x,t)d\mu_t dt)$$

is satisfied in the sense of vector-valued measure. Thus, from Lemma 4.4, we obtain $d|\nabla'\chi_E|\ll d\mu_t dt$. □

REMARK 4.9. Setting $d\mu=d|\nabla\chi_{E_t}|dt$ and $E=\{(x,t)\mid t>0,\ x\in E_t\}$, where E_t is as in the above counterexample, then one can obtain that $d|\nabla'\chi_E|\ll d\mu_t dt$

is not satisfied by the following calculation:

$$(\mathrm{supp}\,\mu) = \bigcup_{0<t\le 1/(4(n-1))} \{|x|^2 = 1 - 2(n-1)t\} \times \{t\},$$

$$(\mathrm{supp}|\nabla'\chi_E|) = \bigcup_{0<t\le 1/(4(n-1))} \{|x|^2 = 1 - 2(n-1)t\} \times \{t\}$$

$$\cup \{x \in \mathbb{R}^n \mid |x|^2 \le 1/2\} \times \{1/(4(n-1))\}.$$

4.2. Basic properties of L^2 flow and set of finite perimeter. In this subsection, we state the properties of L^2 flow and set of (locally) finite perimeter. The proof of Theorem 2.6 will follow from those properties. The arguments in this subsection are mostly contained in [15, 19] and we include this for the convenience of the reader.

Proposition 4.10. *Let $\{\mu_t\}_{t\in\mathbb{R}^+}$ and E be as in Section 3.1 and let $d\mu = d\mu_t dt$. Then $\mu\llcorner_{\partial^* E}$ is a rectifiable Radon measure and we have the following for $\mathscr{H}^n$-a.e. $(x,t) \in \partial^* E \cap \{t > 0\}$:*

(1): *the tangent space $T_{(x,t)}\mu$ exists, and $T_{(x,t)}\mu = T_{(x,t)}(\partial^* E)$,*

(2): $\begin{pmatrix} h(x,t) \\ 1 \end{pmatrix} \in T_{(x,t)}\mu$,

(3): *$x \in \partial^* E_t$, and $T_x\mu_t = T_x(\partial^* E_t)$,*

(4): *$\mathbf{p}(\nu_E(x,t)) \neq 0$, and $\nu_{E_t}(x) = |\mathbf{p}(\nu_E(x,t))|^{-1}\mathbf{p}(\nu_E(x,t))$,*

(5): *$T_x(\partial^* E_t) \times \{0\}$ is linear subspace of $T_{(x,t)}\mu$.*

The crucial step of the proof of Theorem 2.6 is to prove the above proposition, for which the L^2 flow property of μ_t plays a pivotal role, and this proposition is proved in detail by [19, Lemma 4.7]. In this paper, we will give a brief outline of the proof of Proposition 4.10.

First, the following are simple propositions of L^2 flow by [15, Proposition 3.3] and [19, Theorem 4.3].

Proposition 4.11. *Let $\{\mu_t\}_{t\in\mathbb{R}^+}$ and V be an L^2 flow in Definition 2.2, and let μ be the space-time measure $d\mu = d\mu_t dt$. Then,*

$$\begin{pmatrix} V(x,t) \\ 1 \end{pmatrix} \in T_{(x,t)}\mu \tag{4.18}$$

at μ-a.e. $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$ wherever the tangent space $T_{(x,t)}\mu$ exists.

Proposition 4.12. *The Brakke flow $\{\mu_t\}_{t\in\mathbb{R}^+}$ in Definition 2.1 with $\mu_0(\mathbb{R}^n) < \infty$ is an L^2 flow with the velocity $V = h$ in Definition 2.2. Namely,* *there exists* $C = C(\mu_t) > 0$ *such that*

$$\left| \int_0^\infty \int_{\mathbb{R}^n} \partial_t \phi(x,t) + \nabla\phi(x,t) \cdot h(x,t) d\mu_t(x) dt \right| \le C \|\phi\|_{C^0}, \tag{4.19}$$

for all $\phi \in C_c^1(\mathbb{R}^n \times (0,\infty))$.

Next, before the proof of Proposition 4.10, we will need some consequences of Huisken's monotonicity formula for MCF. Now we briefly state the consequences necessary to prove the main result. See [22, Section 3.2] for discussion below in detail. First, we set some notation. For $(y,s) \in \mathbb{R}^n \times \mathbb{R}^+$, we define the backward heat kernel $\rho_{(y,s)}$ by

$$\rho_{(y,s)}(x,t) := \frac{1}{(4\pi(s-t))^{(n-1)/2}} \exp\left(-\frac{|x-y|^2}{4(s-t)}\right),$$

for all $0 \le t < s$ and $x \in \mathbb{R}^n$, as well as the truncated kernel

$$\hat\rho^r_{(y,s)}(x,t) := \eta\left(\frac{x-y}{r}\right)\rho_{(y,s)}(x,t),$$

where $r > 0$ and $\eta \in C_c^\infty(B_2(0);\mathbb{R}^+)$ is a suitable cut-off function such that $\eta \equiv 1$ on $B_1(0)$, $0 \le \eta \le 1$, $|\nabla\eta| \le 2$ and $\|\nabla^2\eta\| \le 4$. The following is a variant of Huisken's monotonicity formula for MCF (for example, see [22, Section 3.2] in detail).

LEMMA 4.13. *Let* $\{\mu_t\}_{t\in\mathbb{R}^+}$ *is a Brakke flow in Definition* 2.1. *Then there exists* $c(n) > 0$ *with the following property. For every* $0 \le t_1 < t_2 < s < \infty$, $y \in \mathbb{R}^n$ *and* $r > 0$, *it holds that*

$$\mu_t(\hat\rho^r_{(y,s)}(x,t))\Big|_{t=t_1}^{t_2} \le c(n)\frac{t_2-t_1}{r^2} \sup_{t\in[t_1,t_2]} \frac{\mu_t(B_{2r})}{r^{n-1}}. \tag{4.20}$$

As a consequence, Lemma 4.13 and a local mass bound (Definition 2.1(2)) indicate the following, which provides the upper bound of mass density ratio ([22, Proposition 3.5]).

LEMMA 4.14. *Let* $\{\mu_t\}_{t\in\mathbb{R}^+}$ *is a Brakke flow in Definition* 2.1, *and let* $d\mu = d\mu_t dt$. *For any* $\delta > 0$, $x_0 \in \mathbb{R}^n$ *and* $R > 0$, *there exists* $c(\delta,n,R) > 0$ *with the following property. For any* $t \in [\delta,\infty)$ *and* $B_r(y) \subset B_R(x_0)$, *we have*

$$\frac{\mu_t(B_r(y))}{r^{n-1}} \le c(\delta,n,R) \sup_{s\in[0,t]} \mu_s(B_{3R}(x_0)).$$

In particular, $\Theta^{*n}(\mu,(x,t)) < \infty$ *for all* $(x,t) \in \mathbb{R}^n \times (0,\infty)$ *holds.*

For the proof of Proposition 4.10, we need the following general facts on sets of finite perimeter ([14, Theorem 18.11]).

Lemma 4.15. *If $E \subset \mathbb{R}^n \times \mathbb{R}$ is a set of locally finite perimeter, then the horizontal section $E_t = \{x \in \mathbb{R}^n \mid (x,t) \in E\}$ as in Definition 3.2 is a set of locally finite perimeter in $\mathbb{R}^n$ for a.e. $t \in \mathbb{R}$, and the following properties hold:*

(1): $\mathscr{H}^{n-1}(\partial^* E_t \Delta (\partial^* E)_t) = 0$,

(2): $\mathbf{p}(\nu_E(x,t)) \neq 0$ *for* $\mathscr{H}^{n-1}$*-a.e.* $x \in (\partial^* E)_t$,

(3): $\nabla \chi_{E_t} = |\mathbf{p}(\nu_E(x,t))|^{-1}\mathbf{p}(\nu_E(x,t))\mathscr{H}^{n-1}\llcorner_{(\partial^* E)_t}$.

Proof (Proof of Proposition 4.10). First of all, we will prove that $\mu\llcorner_{\partial^* E}$ is a rectifiable Radon measure. It is not difficult to see that $\mu \ll \mathscr{H}^n$. Indeed, let $A \subset \mathbb{R}^n \times \mathbb{R}$ be a set with $\mathscr{H}^n(A) = 0$, and let the set $D_k := \{(x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \mid \Theta^{*n}(\mu,(x,t)) \leq k\}$ for each $k \in \mathbb{N}$. By [17, Theorem 3.2], we have

$$\mu(A \cap D_k) \leq 2^n k \mathscr{H}^n(A \cap D_k) = 0$$

for all $k \in \mathbb{N}$. Furthermore, by the upper bound of mass density ratio (Lemma 4.14), we see that $\mu(A \setminus \bigcup_{k=1}^{\infty} D_k) = 0$. Thus we obtain $\mu(A) = 0$, that is, $\mu \ll \mathscr{H}^n$ holds. Since $\mu \ll \mathscr{H}^n$, $|\nabla' \chi_E| = \mathscr{H}^n\llcorner_{\partial^* E}$ and Proposition 4.8, we see that

$$\mu\llcorner_{\partial^* E} \ll |\nabla' \chi_E|, \qquad |\nabla' \chi_E| \ll \mu\llcorner_{\partial^* E}.$$

By Radon–Nikodým theorem, there exists a function $f = (d\mu\llcorner_{\partial^* E})/d|\nabla' \chi_E|$ with $0 < f < \infty$ for $|\nabla' \chi_E|$-a.e., $f \in L^1_{loc}(|\nabla' \chi_E|)$ and $\mu\llcorner_{\partial^* E} = f|\nabla' \chi_E| = f\mathscr{H}^n\llcorner_{\partial^* E}$. This shows that $\mu\llcorner_{\partial^* E}$ is a rectifiable Radon measure and the tangent space $T_{(x,t)}(\mu\llcorner_{\partial^* E})$ with multiplicity f exists for $\mathscr{H}^n$-a.e. $(x,t) \in \partial^* E \cap \{t > 0\}$. For the next step, we prove that $T_{(x,t)}\mu = T_{(x,t)}(\partial^* E)$ for $\mathscr{H}^n$-a.e. $(x,t) \in \partial^* E \cap \{t > 0\}$. Now, by [17, Theorem 3.5], we see that

$$\limsup_{r \to +0} \frac{\mu(B_r^{n+1}(x,t)) \setminus \partial^* E)}{r^n} = 0 \qquad \text{for } \mathscr{H}^n\text{-a.e. } (x,t) \in \partial^* E \cap \{t > 0\}.$$

Let then $\phi \in C_c^0(B_1^{n+1}(0))$ be arbitrary, we have

$$\lim_{r \to +0} \left| \int_{\mathbb{R}^n \times (0,\infty) \setminus \partial^* E} \frac{1}{r^n} \phi\left(\frac{1}{r}(y - x, s - t)\right) d\mu(y,s) \right|$$

$$\leq \|\phi\|_{C^0} \limsup_{r \to +0} \frac{\mu(B_r^{n+1}(x,t)) \setminus \partial^* E)}{r^n} = 0$$

for $\mathscr{H}^n$-a.e. $(x,t) \in \partial^* E \cap \{t > 0\}$. Thus, by $f \in L^1_{loc}(|\nabla' \chi_E|)$, we obtain at each Lebesgue point of f

$$\lim_{r\to+0}\int_{\mathbb{R}^n\times(0,\infty)}\frac{1}{r^n}\phi\left(\frac{1}{r}(y-x,s-t)\right)d\mu(y,s)$$
$$=\lim_{r\to+0}\int_{\partial^*E}\frac{1}{r^n}\phi\left(\frac{1}{r}(y-x,s-t)\right)\frac{d\mu}{d|\nabla'\chi_E|}(y,s)d|\nabla'\chi_E|(y,s)$$
$$=f(x,t)\int_{T_{(x,t)}(\partial^*E)}\phi(y,s)d\mathscr{H}^n(y,s)$$

for all $\phi\in C_c^0(\mathbb{R}^n\times\mathbb{R})$ and $\mathscr{H}^n$-a.e. $(x,t)\in\partial^*E\cap\{t>0\}$. This completes the proof of $T_{(x,t)}\mu=T_{(x,t)}(\partial^*E)$.

By Proposition 4.11, Proposition 4.12, and the above argument, (1) and (2) are proved. Next, we prove (3) and (4). By Lemma 4.15, we have the following for a.e. $t>0$ and $\mathscr{H}^{n-1}$-a.e. $x\in(\partial^*E)_t$:

$$\mathscr{H}^{n-1}(\partial^*E_t\Delta(\partial^*E)_t)=0, \tag{4.21}$$

$$\mathbf{p}(\nu_E(x,t))\neq 0, \tag{4.22}$$

$$\nu_{E_t}(x)=\frac{\mathbf{p}(\nu_E(x,t))}{|\mathbf{p}(\nu_E(x,t))|}. \tag{4.23}$$

Let $A:=\{t>0\,|\,(4.21)\text{ fails}\}$ and for every $t>0$ set $A_t:=\{x\in(\partial^*E)_t\,|\,x\notin\partial^*E_t$ or (4.22)–(4.23) fail$\}$, so that $\mathscr{L}^1(A)=0$ and $\mathscr{H}^{n-1}(A_t)=0$ for every $t\in(0,\infty)\backslash A$. Consider then the characteristic function $\chi(x,t):=\chi_{A_t}(x)$ on $\mathbb{R}^n\times(0,\infty)$, since $\mathscr{L}^1(A)=0$ and $\mathscr{H}^{n-1}(A_t)=0$ for every $t\in(0,\infty)\backslash A$, we have

$$\int_{\partial^*E}\chi(x,t)|\nabla^{\partial^*E}(\mathbf{q}(x,t))|d\mathscr{H}^n(x,t)=\int_0^\infty\int_{(\partial^*E)_t}\chi(x,t)d\mathscr{H}^{n-1}dt$$
$$=\int_0^\infty\mathscr{H}^{n-1}(A_t)dt=\int_A\mathscr{H}^{n-1}(A_t)dt=0,$$

where we used the co-area formula in the first line, and where ∇^{∂^*E} is the gradient on the tangent plane of ∂^*E, that is,

$$\nabla^{\partial^*E}\mathbf{q}(x,t)=P_{T_{(x,t)}(\partial^*E)}(\nabla\mathbf{q}(x,t)).$$

Here, combining (1) and (2), we see that

$$\begin{pmatrix}h(x,t)\\1\end{pmatrix}\in T_{(x,t)}(\partial^*E)\qquad\text{at }\mathscr{H}^n\text{-a.e. }(x,t)\in\partial^*E\cap\{t>0\},$$

which implies $|\nabla^{\partial^*E}(\mathbf{q}(x,t))|>0$ for $\mathscr{H}^n$-a.e. $(x,t)\in\partial^*E\cap\{t>0\}$. Hence, it must be $\chi(x,t)=0$ for $\mathscr{H}^n$-a.e. $(x,t)\in\partial^*E\cap\{t>0\}$, thus the first part of

(3) and (4) are proved. For the proof of the identity $T_x\mu_t = T_x(\partial^* E_t)$, by repeating the argument of the first half of (1) and Lemma 4.14, we have $\mu_t \ll \mathscr{H}^{n-1}$ for all $t > 0$. Thus it is obtained by Lemma 4.4 and repeating the argument of (1) at fixed t.

Finally, we prove (5). Taking a point $(x,t) \in \partial^* E$ as satisfying (1)–(4) of this Proposition, we can calculate as

$$^t(z,0)\cdot \nu_E(x,t) = z\cdot \mathbf{p}(\nu_E(x,t)) = |\mathbf{p}(\nu_E(x,t))|(z\cdot \nu_{E_t}(x)) = 0$$

for all $z \in T_x(\partial^* E_t)$. This completes the proof of (5). □

4.3. Boundaries move by mean curvature. In this subsection, we prove Theorem 2.6 by rephrasing the velocity V in Proposition 4.5 as the mean curvature and by using geometric measure theory. The argument for this rephrasing corresponds to the proof of the equality (2.4). Since Definition 2.3(ii) is treated in Remark 4.7, we can deduce that $(\{\mu_t\}_{t\in\mathbb{R}^+}, \{E_t\}_{t\in\mathbb{R}^+})$ as in Section 3.1 is a generalized BV flow.

Proof (Proof of Theorem 2.6). We fix a test function $\phi \in C_c^1(\mathbb{R}^n \times (0,\infty))$ arbitrarily. Then, by using Gauss-Green's theorem for set of finite perimeter, we have

$$\int_{\mathbb{R}^n\times(0,\infty)} \partial_t\phi\chi_E \,dxdt = \int_{\partial^* E} \phi\mathbf{q}(\nu_E)\,d\mathscr{H}^n. \tag{4.24}$$

Let G be the set satisfying Proposition 4.10(1)–(5). Then for all $(x,t)\in G$, we have

$$T_{(x,t)}\mu = (T_x(\partial^* E_t)\times\{0\}) \oplus \operatorname{span}\begin{pmatrix} h(x,t) \\ 1 \end{pmatrix}$$
$$\text{(by Proposition 4.10(2)).} \tag{4.25}$$

By $h(x,t)\perp T_x\mu_t$, (4.25), Proposition 4.10(1) and (4), we have

$$\nu_E(x,t) = \frac{1}{\sqrt{1+|h(x,t)|^2}}\begin{pmatrix} \nu_{E_t}(x) \\ -h(x,t)\cdot\nu_{E_t}(x) \end{pmatrix}. \tag{4.26}$$

By (4.26) and $h(x,t)\perp T_x\mu_t$ again, for all $(x,t)\in G$, we can calculate the $i\times(n+1)$ component of the matrix $I_{n+1} - \nu_E\otimes\nu_E$ for $i = 1,\ldots,n+1$ as

$$(I_{n+1} - \nu_E\otimes\nu_E)_{i,(n+1)}(x,t) = \begin{cases} \frac{-(\nu_{E_t}(x))_i(h(x,t)\cdot\nu_{E_t}(x))}{1+|h(x,t)|^2} & (i = 1,\ldots,n), \\ \frac{1}{1+|h(x,t)|^2} & (i = n+1), \end{cases}$$

where I_{n+1} is the $(n+1)$-identity matrix and $(\nu_{E_t})_i$ is the i-th component of ν_{E_t}. According to this calculation, $\nabla \mathbf{q} = \mathbf{e}_{n+1}$ and $T_{(x,t)}\mu = T_{(x,t)}(\partial^* E)$ on G, we obtain that the co-area factor of the projection $\mathbf{q}$ satisfies

$$|\nabla^{\partial^* E}\mathbf{q}(x,t)| = |(I_{n+1} - \nu_E \otimes \nu_E)(\nabla \mathbf{q}(x,t))| = \frac{1}{\sqrt{1+|h(x,t)|^2}}. \tag{4.27}$$

Due to (4.24)–(4.27) and the co-area formula, we compute as

$$\begin{aligned}\int_{\mathbb{R}^n\times(0,\infty)} \partial_t\phi\chi_E \, dxdt &= -\int_G \phi h\cdot \nu_{E_t}\frac{1}{\sqrt{1+|h|^2}}\, d\mathscr{H}^n \\ &= -\int_{\partial^* E} \phi h\cdot \nu_{E_t}|\nabla^{\partial^* E}\mathbf{q}(x,t)| d\mathscr{H}^n \\ &= -\int_0^\infty \int_{\partial^* E\cap\{\mathbf{q}=t\}} \phi h\cdot\nu_{E_t}\, d\mathscr{H}^{n-1}dt \\ &= -\int_0^\infty \int_{\partial^* E_t} \phi h\cdot\nu_{E_t}\, d\mathscr{H}^{n-1}dt, \end{aligned} \tag{4.28}$$

where we used $\mathscr{H}^n(\partial^* E\setminus G) = 0$. By the same cut-off argument of Lemma 4.6 for (4.28) and Remark 4.7, we deduce

$$\begin{aligned}&\int_{E_{t_2}} \phi(x,t_2)dx - \int_{E_{t_1}} \phi(x,t_1)dx \\ &\quad = \int_{t_1}^{t_2}\int_{E_t} \partial_t\phi\, dxdt + \int_{t_1}^{t_2}\int_{\partial^* E_t} \phi h\cdot\nu_{E_t}\, d\mathscr{H}^{n-1}dt \end{aligned} \tag{4.29}$$

for all $0 < t_1 < t_2 < \infty$. Use the continuity of Remark 4.7, we obtain the above equality for all $0 \le t_1 < t_2 < \infty$ and all $\phi \in C_c^1(\mathbb{R}^n \times \mathbb{R}^+)$. This completes the proof. □

Appendix A. Measure-function pairs

Here, we recall the notion of measure-function pairs introduced by Hutchinson in [8].

DEFINITION A.1. Let $E \subset \mathbb{R}^n$ be an open set and let μ be a Radon measure on E. Suppose $f \in L^1(\mu;\mathbb{R}^d)$. Then we say that (μ, f) is an $\mathbb{R}^d$-valued measure-function pair over E.

We define the notion of convergence for a sequence of $\mathbb{R}^d$-valued measure-function pairs over E.

Definition A.2. Let $\{(\mu_i, f_i)\}_{i=1}^{\infty}$ and (μ, f) be $\mathbb{R}^d$-valued measure-function pairs over E. Suppose

$$\mu_i \rightharpoonup \mu$$

as Radon measure on E. Then we call (μ_i, f_i) converges to (μ, f) in the weak sense if

$$\int_E f_i \cdot \phi \, d\mu_i \to \int_E f \cdot \phi \, d\mu$$

for all $\phi \in C_c^0(E; \mathbb{R}^d)$.

We present a less general version of [8, Theorem 4.4.2] to the extent that it can be used in this paper.

Theorem A.3. *Suppose that $\mathbb{R}^d$-valued measure-function pairs $\{(\mu_i, f_i)\}_{i=1}^{\infty}$ satisfy*

$$\sup_i \int_E |f_i|^2 d\mu_i < \infty.$$

Then the following hold:

(1): *There exist a subsequence $\{(\mu_{i_j}, f_{i_j})\}_{j=1}^{\infty}$ and an $\mathbb{R}^d$-valued measure-function pair (μ, f) such that (μ_{i_j}, f_{i_j}) converges to (μ, f) as measure-function pair.*

(2): *If (μ_{i_j}, f_{i_j}) converges to (μ, f), then*

$$\int_E |f|^2 d\mu \leq \liminf_{j\to\infty} \int_E |f_{i_j}|^2 d\mu_{i_j} < \infty.$$

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