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Datko type characterizations for exponential instability in average of cocycles

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Abstract. In this paper, we consider the problem of exponential instability behavior of random dynamical systems described by cocycles in Banach spaces. We prove some continuous and discrete versions of Datko type theorem for the exponential instability in average of cocycles. In addition, two characterizations of the exponential instability in average in terms of Lyapunov functions are given.

1. Introduction

It is well known that the topic of (non)uniform exponential behaviors of nonautonomous differential equations in Banach spaces has been intensively studied, both in the deterministic case and in the stochastic case. During the last couple of decades, various results concerning this subject have witnessed considerable development. One of the most important and fundamental results in the exponential stability theory of dynamical systems was given by Datko [8] in 1972.

Theorem 1.1 (Datko's Theorem). *A uniformly exponentially bounded evolution family $\mathscr{U} = \{U(t,s)\}_{t \geq s \geq 0}$ on a Banach space X is uniformly exponentially stable if and only if there exists $p \geq 1$ such that*

$$\sup_{s \geq 0} \int_s^{\infty} \|U(t,s)x\|^p dt < \infty,$$

for all $x \in X$.

In fact, the Theorem 1.1 was initially proved by Datko [7] in 1970 for C_0-semigroups acting on Hilbert spaces and $p = 2$. Later, using different

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techniques, Pazy in [23] proved the case $p \geq 1$ on arbitrary Banach spaces. Also, we note that a discrete-time version of Datko's theorem was first established by Zabczyk in [32], for the particular case of the discrete semigroups on Banach spaces.

After the seminal work of Datko, with the help of Datko-Pazy theorem, Perron's methods, discrete-time methods, Banach function spaces, Lyapunov functions, Lyapunov norms and other ways, there has been a great number of papers devoted to this area (see [1, 2, 3, 4, 5, 10, 11, 13, 14, 18, 19, 20, 21, 22, 24, 26, 27, 28]) and the references therein). For instance, in [27] and [28] Preda and his collaborators obtained Datko type conditions for the existence of the nonuniform exponential stability in the sense of Barreira and Valls [4] for linear skew-product semiflows and evolution families respectively. In [11] Dragičević gave some Datko type characterizations of strong nonuniform exponential behavior: stability, instability and dichotomy, for both discrete and continuous time cases. Recently, the work [11] motivated Lupa and Popescu [19] to describe the nonuniform exponential stability of evolution families, in terms of a class of admissible Banach spaces. Furthermore, we note that the techniques were gradually improved and expanded in the following directions: from uniform behavior to nonuniform behavior (see [1, 5, 10, 11, 18, 27, 28]), from exponential behavior to polynomial behavior (see [6, 15, 16]), from Datko type theorems to Barbashin type theorems (see [12, 25, 31]), from the deterministic case to the stochastic case (see [9, 17, 29, 30]).

Since some stochastic differential equations arising in nature or engineering involve the discussion of asymptotic properties of cocycles, the study of exponential behaviors of cocycles has attracted the attention of many researchers. In particular, Stoica [29] gave a Perron type characterization for uniform exponential dichotomy in mean square of stochastic cocycles in Hilbert spaces. Later, Barreira, Dragičević and Valls [2] characterized the exponential dichotomy in average of discrete-time cocycles in terms of an admissibility property (see [3] for related results in the case of continuous time). Moreover, a notable contribution in this direction was made by Dragičević in [9] who obtained an interesting Datko type theorem for exponential stability in average of cocycles.

Theorem 1.2 (see [9]). *Let Φ be a cocycle, which is exponentially bounded in average. Then it is exponentially stable in average if and only if there exist $C, p > 0$ such that*

$$\left(\int_s^\infty \left(\int_\Omega \|\Phi_\omega(\tau, s)z(\omega)\| d\mu(\omega)\right)^p d\tau\right)^{1/p} \leq C \int_\Omega \|z(\omega)\| d\mu(\omega),$$

for all $(s, z) \in \mathbb{R}_+ \times \mathscr{F}$.

Inspired by [9], it is natural to study the exponential instability in average of cocycles. The main purpose of this paper is to give some continuous and discrete versions of Datko type theorem for the exponential instability in average of cocycles in Banach spaces. In addition, we use our Datko type results to obtain necessary and sufficient conditions for the existence of exponential instability in average in terms of proper Lyapunov functions. Variants for exponential instability in average of some well-known results in stability theory (Datko [8], Dragičević [9], Zabczyk [32], Preda et al. [27]) are obtained.

2. Preliminaries

In this section, we give some notations and definitions that will be used in the sequel. We denote by $\mathbb{N}$ the set of natural numbers, by $\mathbb{N}^*$ the set of positive integers, by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}_+ = [0, +\infty)$ and by $\Delta = \{(t,s) \in \mathbb{R}_+^2 : t \geq s\}$. For any real number y, the largest integer less than or equal to y is denoted by $[y]$. Let $\Omega = (\Omega, \mathscr{B}, \mu)$ be a probability space, X a Banach space, $\mathscr{L}(X)$ the set of all invertible bounded linear operators acting on X. Let us consider

$$\mathscr{F} = \left\{ z : \Omega \to X : z \text{ is Bochner measurable and } \int_\Omega \|z(\omega)\| d\mu(\omega) < \infty \right\},$$

which is a Banach space endowed with the norm

$$\|z\|_1 := \int_\Omega \|z(\omega)\| d\mu(\omega).$$

Definition 2.1 (see [9]). A measurable map $\varphi : \mathbb{R}_+ \times \Omega \to \Omega$ is called a semiflow on Ω if the following conditions hold:

(i) $\varphi(0, \omega) = \omega$ for all $\omega \in \Omega$;

(ii) $\varphi(t+s, \omega) = \varphi(t, \varphi(s, \omega))$ for all $(t, s, \omega) \in \mathbb{R}_+^2 \times \Omega$.

Definition 2.2 (see [9]). Let φ be a semiflow on Ω. A strongly measurable map $\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(X)$ (i.e., $(t, \omega) \longmapsto \Phi(t, \omega)x$ is Bochner measurable for each $x \in X$) is called a cocycle over φ if the following conditions are satisfied:

(i) $\Phi(0, \omega) = \mathrm{Id}$ (where Id is the identity operator on X) for all $\omega \in \Omega$;

(ii) $\Phi(t+s, \omega) = \Phi(t, \varphi(s, \omega))\Phi(s, \omega)$ for all $(t, s, \omega) \in \mathbb{R}_+^2 \times \Omega$.

In what follows, we denote by

$$\Phi_\omega(t, s) = \Phi(t, \omega)\Phi(s, \omega)^{-1}, \qquad \forall (t, s, \omega) \in \mathbb{R}_+^2 \times \Omega.$$

It is easy to see that $\Phi_\omega(t, r) = \Phi_\omega(t, s)\Phi_\omega(s, r)$, for all $t \geq s \geq r \geq 0$ and $\omega \in \Omega$.

Definition 2.3. A cocycle $\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(X)$ is said to be exponentially bounded in average if there exist $K > 0$ and $\alpha > 0$ such that

$$\int_{\Omega} \|\Phi_\omega(t,s)z(\omega)\| d\mu(\omega) \le K e^{\alpha(t-s)} \int_{\Omega} \|z(\omega)\| d\mu(\omega) \tag{2.1}$$

for all $(t,s,z) \in \Delta \times \mathscr{F}$.

Throughout the paper, for given a cocycle Φ over a semiflow φ, we shall always assume that Φ is exponentially bounded in average.

Definition 2.4. A cocycle $\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(X)$ is exponentially unstable in average if there are $N > 0$ and $v > 0$ such that

$$\int_{\Omega} \|\Phi_\omega(t,s)z(\omega)\| d\mu(\omega) \ge N e^{v(t-s)} \int_{\Omega} \|z(\omega)\| d\mu(\omega), \tag{2.2}$$

for all $(t,s,z) \in \Delta \times \mathscr{F}$.

Remark 2.5. *A cocycle Φ is exponentially unstable in average if and only if there exist $N > 0$ and $v > 0$ such that*

$$\int_{\Omega} \|\Phi_\omega(t,t_0)z(\omega)\| d\mu(\omega) \ge N e^{v(t-s)} \int_{\Omega} \|\Phi_\omega(s,t_0)z(\omega)\| d\mu(\omega),$$

for all $t \ge s \ge t_0 \ge 0$ and $z \in \mathscr{F}$.

3. Exponential instability in average

In this section, we present the main results of this paper. First, the continuous version of Datko type theorem for exponential instability in average of cocycles is established. Next, we prove two discrete characterizations of Datko type by using the continuous version. Finally, we show how exponential instability in average can be characterized in terms of Lyapunov functions.

The following Theorem 3.1 is a crucial result in this paper, which is an integral characterization of Datko type.

Theorem 3.1. *The cocycle Φ is exponentially unstable in average if and only if there exist $C > 0$ and $p > 0$ such that*

$$\left(\int_s^{\infty} \left(\int_{\Omega} \|\Phi_\omega(\tau,s)z(\omega)\| d\mu(\omega) \right)^{-p} d\tau \right)^{1/p} \le C \left(\int_{\Omega} \|z(\omega)\| d\mu(\omega) \right)^{-1}, \tag{3.1}$$

for all $(s,z) \in \mathbb{R}_+ \times (\mathscr{F} \setminus \{0\})$.

Proof. Necessity. If the cocycle Φ is exponentially unstable in average, then by Definition 2.4, there are $N > 0$ and $v > 0$ such that (2.2) holds. Take now any $p > 0$. By (2.2), we have

$$\begin{aligned}
&\int_s^\infty \left(\int_\Omega \|\Phi_\omega(\tau, s)z(\omega)\| d\mu(\omega)\right)^{-p} d\tau \\
&\quad \le \int_s^\infty \left(Ne^{v(\tau-s)} \int_\Omega \|z(\omega)\| d\mu(\omega)\right)^{-p} d\tau \\
&\quad = N^{-p}\left(\int_\Omega \|z(\omega)\| d\mu(\omega)\right)^{-p} \int_s^\infty e^{-vp(\tau-s)}\, d\tau \\
&\quad = N^{-p}\left(\int_\Omega \|z(\omega)\| d\mu(\omega)\right)^{-p} \int_0^\infty e^{-vp\tau}\, d\tau \\
&\quad = N^{-p}(vp)^{-1}\left(\int_\Omega \|z(\omega)\| d\mu(\omega)\right)^{-p}
\end{aligned}$$

for all $(s, z) \in \mathbb{R}_+ \times (\mathscr{F}\backslash\{0\})$. Hence, (3.1) holds with $C = N^{-1}(vp)^{-1/p}$.

Sufficiency. Let $(t, s, z) \in \Delta \times (\mathscr{F}\backslash\{0\})$ and $\tau \in [t, t+1]$. We note that it follows from (2.1) that

$$\begin{aligned}
\left(\int_\Omega \|\Phi_\omega(\tau, s)z(\omega)\| d\mu(\omega)\right)^p &= \left(\int_\Omega \|\Phi_\omega(\tau, t)\Phi_\omega(t, s)z(\omega)\| d\mu(\omega)\right)^p \\
&\le \left(Ke^{\alpha(\tau-t)} \int_\Omega \|\Phi_\omega(t, s)z(\omega)\| d\mu(\omega)\right)^p \\
&\le K^p e^{\alpha p}\left(\int_\Omega \|\Phi_\omega(t, s)z(\omega)\| d\mu(\omega)\right)^p,
\end{aligned}$$

and thus

$$\left(\int_\Omega \|\Phi_\omega(t, s)z(\omega)\| d\mu(\omega)\right)^{-p} \le K^p e^{\alpha p}\left(\int_\Omega \|\Phi_\omega(\tau, s)z(\omega)\| d\mu(\omega)\right)^{-p}.$$

By integrating on $[t, t+1]$ and using (3.1), we have

$$\begin{aligned}
\left(\int_\Omega \|\Phi_\omega(t, s)z(\omega)\| d\mu(\omega)\right)^{-p} &\le \int_t^{t+1} K^p e^{\alpha p}\left(\int_\Omega \|\Phi_\omega(\tau, s)z(\omega)\| d\mu(\omega)\right)^{-p} d\tau \\
&\le K^p e^{\alpha p} \int_s^\infty \left(\int_\Omega \|\Phi_\omega(\tau, s)z(\omega)\| d\mu(\omega)\right)^{-p} d\tau \\
&\le C^p K^p e^{\alpha p}\left(\int_\Omega \|z(\omega)\| d\mu(\omega)\right)^{-p},
\end{aligned}$$

which implies that

$$\int_{\Omega} \|\Phi_{\omega}(t,s)z(\omega)\| d\mu(\omega) \geq C^{-1}K^{-1}e^{-\alpha} \int_{\Omega} \|z(\omega)\| d\mu(\omega), \tag{3.2}$$

for all $(t,s,z) \in \Delta \times \mathscr{F}$.

Taking now $t \geq s \geq 0$, $\tau \in [s,t]$ and $z \in \mathscr{F} \backslash \{0\}$. By (3.2), we deduce that

$$\begin{aligned} \int_{\Omega} \|\Phi_{\omega}(t,s)z(\omega)\| d\mu(\omega) &= \int_{\Omega} \|\Phi_{\omega}(t,\tau)\Phi_{\omega}(\tau,s)z(\omega)\| d\mu(\omega) \\ &\geq C^{-1}K^{-1}e^{-\alpha} \int_{\Omega} \|\Phi_{\omega}(\tau,s)z(\omega)\| d\mu(\omega), \end{aligned}$$

which yields that

$$\left(\int_{\Omega} \|\Phi_{\omega}(t,s)z(\omega)\| d\mu(\omega)\right)^{-p} \leq C^p K^p e^{\alpha p} \left(\int_{\Omega} \|\Phi_{\omega}(\tau,s)z(\omega)\| d\mu(\omega)\right)^{-p}.$$

Integrating now on $[s,t]$ and using (3.1), we get

$$\begin{aligned} &(t-s)^{1/p} \left(\int_{\Omega} \|\Phi_{\omega}(t,s)z(\omega)\| d\mu(\omega)\right)^{-1} \\ &\leq \left(\int_s^t C^p K^p e^{\alpha p} \left(\int_{\Omega} \|\Phi_{\omega}(\tau,s)z(\omega)\| d\mu(\omega)\right)^{-p} d\tau\right)^{1/p} \\ &\leq CKe^{\alpha} \left(\int_s^{\infty} \left(\int_{\Omega} \|\Phi_{\omega}(\tau,s)z(\omega)\| d\mu(\omega)\right)^{-p} d\tau\right)^{1/p} \\ &\leq C^2 K e^{\alpha} \left(\int_{\Omega} \|z(\omega)\| d\mu(\omega)\right)^{-1}. \end{aligned}$$

Therefore

$$C \int_{\Omega} \|\Phi_{\omega}(t,s)z(\omega)\| d\mu(\omega) \geq C^{-1}K^{-1}e^{-\alpha}(t-s)^{1/p} \int_{\Omega} \|z(\omega)\| d\mu(\omega), \tag{3.3}$$

for all $(t,s,z) \in \Delta \times \mathscr{F}$.

From (3.2) and (3.3) we observe that

$$\int_{\Omega} \|\Phi_{\omega}(t,s)z(\omega)\| d\mu(\omega) \geq \frac{1+(t-s)^{1/p}}{(1+C)CKe^{\alpha}} \int_{\Omega} \|z(\omega)\| d\mu(\omega), \tag{3.4}$$

for all $(t,s,z) \in \Delta \times \mathscr{F}$.

It follows from (3.4) that there exists $\xi \in \mathbb{N}^*$ such that

$$\int_\Omega \|\Phi_\omega(t,s)z(\omega)\| d\mu(\omega) \geq e \int_\Omega \|z(\omega)\| d\mu(\omega)$$
$$\text{for all } t - s \geq \xi \text{ and } z \in \mathscr{F}. \tag{3.5}$$

For each $(t,s) \in \Delta$, there exist $n \in \mathbb{N}$ and $r \in [0, \xi)$ such that $t - s = n\xi + r$. By (3.5) and (3.2), we obtain that

$$\begin{aligned}
\int_\Omega \|\Phi_\omega(t,s)z(\omega)\| d\mu(\omega) &= \int_\Omega \|\Phi_\omega(n\xi + s + r, s)z(\omega)\| d\mu(\omega) \\
&\geq e \int_\Omega \|\Phi_\omega((n-1)\xi + s + r, s)z(\omega)\| d\mu(\omega) \\
&\geq \cdots \\
&\geq e^n \int_\Omega \|\Phi_\omega(s + r, s)z(\omega)\| d\mu(\omega) \\
&\geq C^{-1}K^{-1}e^{-\alpha}e^n \int_\Omega \|z(\omega)\| d\mu(\omega) \\
&= C^{-1}K^{-1}e^{-(\alpha+1)}e^{n+1} \int_\Omega \|z(\omega)\| d\mu(\omega) \\
&\geq C^{-1}K^{-1}e^{-(\alpha+1)}e^{(t-s)/\xi} \int_\Omega \|z(\omega)\| d\mu(\omega),
\end{aligned}$$

and thus (2.2) holds with $N = C^{-1}K^{-1}e^{-(\alpha+1)}$ and $v = 1/\xi$. The proof is complete.

We now give below a discrete version of the Datko type theorem by making use of Theorem 3.1.

Theorem 3.2. *The cocycle Φ is exponentially unstable in average if and only if there exist $\tilde{C}, p > 0$ such that*

$$\left(\sum_{n=[s]+1}^{\infty} \left(\int_\Omega \|\Phi_\omega(n,s)z(\omega)\| d\mu(\omega) \right)^{-p} \right)^{1/p} \leq \tilde{C} \left(\int_\Omega \|z(\omega)\| d\mu(\omega) \right)^{-1}, \tag{3.6}$$

for all $(s,z) \in \mathbb{R}_+ \times (\mathscr{F} \setminus \{0\})$.

Proof. Necessity. We suppose that the cocycle Φ is exponentially unstable in average. Using (2.2) we deduce that for any $p > 0$,

$$\begin{aligned}
&\sum_{n=[s]+1}^{\infty}\left(\int_{\Omega}\|\Phi_{\omega}(n,s)z(\omega)\|d\mu(\omega)\right)^{-p}\\
&\quad\le \sum_{n=[s]+1}^{\infty}\left(Ne^{v(n-s)}\int_{\Omega}\|z(\omega)\|d\mu(\omega)\right)^{-p}\\
&\quad= N^{-p}\left(\int_{\Omega}\|z(\omega)\|d\mu(\omega)\right)^{-p}\sum_{n=[s]+1}^{\infty}e^{-vp(n-s)}\\
&\quad= \frac{e^{vp(s-[s]-1)}}{1-e^{-vp}}N^{-p}\left(\int_{\Omega}\|z(\omega)\|d\mu(\omega)\right)^{-p}\\
&\quad\le \frac{N^{-p}}{1-e^{-vp}}\left(\int_{\Omega}\|z(\omega)\|d\mu(\omega)\right)^{-p},
\end{aligned}$$

and we conclude that (3.6) holds with $\tilde{C} = N^{-1}(1-e^{-vp})^{-1/p}$.

Sufficiency. Let $(t,s,z) \in \Delta \times (\mathscr{F}\backslash\{0\})$. If $s \le t \le [s]+1$, then by (2.1), we have

$$\begin{aligned}
\int_{\Omega}\|\Phi_{\omega}([s]+1,s)z(\omega)\|d\mu(\omega) &= \int_{\Omega}\|\Phi_{\omega}([s]+1,t)\Phi_{\omega}(t,s)z(\omega)\|d\mu(\omega)\\
&\le Ke^{\alpha}\int_{\Omega}\|\Phi_{\omega}(t,s)z(\omega)\|d\mu(\omega). \qquad (3.7)
\end{aligned}$$

From (3.6) and (3.7) we obtain that

$$\begin{aligned}
\left(\int_{\Omega}\|\Phi_{\omega}(t,s)z(\omega)\|d\mu(\omega)\right)^{-p} &\le K^{p}e^{\alpha p}\left(\int_{\Omega}\|\Phi_{\omega}([s]+1,s)z(\omega)\|d\mu(\omega)\right)^{-p}\\
&\le \tilde{C}^{p}K^{p}e^{\alpha p}\left(\int_{\Omega}\|z(\omega)\|d\mu(\omega)\right)^{-p}.
\end{aligned}$$

Integrating now on $[s,[s]+1]$ we deduce that

$$\begin{aligned}
&\int_{s}^{[s]+1}\left(\int_{\Omega}\|\Phi_{\omega}(t,s)z(\omega)\|d\mu(\omega)\right)^{-p}dt\\
&\quad\le \tilde{C}^{p}K^{p}e^{\alpha p}\int_{s}^{[s]+1}\left(\int_{\Omega}\|z(\omega)\|d\mu(\omega)\right)^{-p}dt\\
&\quad\le \tilde{C}^{p}K^{p}e^{\alpha p}\left(\int_{\Omega}\|z(\omega)\|d\mu(\omega)\right)^{-p}. \qquad (3.8)
\end{aligned}$$

If $t \ge [s]+1$, we denote by $i = [t]$. Then, it follows from (2.1) that

$$\int_\Omega \|\Phi_\omega(i+1,s)z(\omega)\|d\mu(\omega) = \int_\Omega \|\Phi_\omega(i+1,t)\Phi_\omega(t,s)z(\omega)\|d\mu(\omega)$$
$$\leq Ke^\alpha \int_\Omega \|\Phi_\omega(t,s)z(\omega)\|d\mu(\omega),$$

and thus

$$\int_i^{i+1}\left(\int_\Omega \|\Phi_\omega(t,s)z(\omega)\|d\mu(\omega)\right)^{-p}dt$$
$$\leq K^p e^{\alpha p}\left(\int_\Omega \|\Phi_\omega(i+1,s)z(\omega)\|d\mu(\omega)\right)^{-p}. \tag{3.9}$$

From (3.9) and (3.6) we get that

$$\int_{[s]+1}^\infty\left(\int_\Omega \|\Phi_\omega(t,s)z(\omega)\|d\mu(\omega)\right)^{-p}dt$$
$$= \sum_{i=[s]+1}^\infty \int_i^{i+1}\left(\int_\Omega \|\Phi_\omega(t,s)z(\omega)\|d\mu(\omega)\right)^{-p}dt$$
$$\leq K^p e^{\alpha p}\sum_{i=[s]+1}^\infty\left(\int_\Omega \|\Phi_\omega(i+1,s)z(\omega)\|d\mu(\omega)\right)^{-p}$$
$$= K^p e^{\alpha p}\sum_{i=[s]+2}^\infty\left(\int_\Omega \|\Phi_\omega(i,s)z(\omega)\|d\mu(\omega)\right)^{-p}$$
$$\leq \tilde{C}^p K^p e^{\alpha p}\left(\int_\Omega \|z(\omega)\|d\mu(\omega)\right)^{-p}. \tag{3.10}$$

Combining (3.8) with (3.10), we have that

$$\int_s^\infty\left(\int_\Omega \|\Phi_\omega(t,s)z(\omega)\|d\mu(\omega)\right)^{-p}dt \leq 2\tilde{C}^p K^p e^{\alpha p}\left(\int_\Omega \|z(\omega)\|d\mu(\omega)\right)^{-p}.$$

Therefore, (3.1) holds with $C = 2^{1/p}\tilde{C}Ke^\alpha$. Now by Theorem 3.1, we conclude that Φ is exponentially unstable in average.

Another discrete characterization of the exponential instability in average is given by:

Theorem 3.3. *Assume that the cocycle Φ has exponential decay in average, that is, there exist $\tilde{K} > 0$ and $\tilde{\alpha} > 0$ such that*

$$\int_\Omega \|\Phi_\omega(t,s)z(\omega)\|d\mu(\omega) \geq \tilde{K}e^{-\tilde{\alpha}(t-s)}\int_\Omega \|z(\omega)\|d\mu(\omega) \tag{3.11}$$

for all $(t, s, z) \in \Delta \times \mathscr{F}$. *Then it is exponentially unstable in average if and only if there exist* $\tilde{C}, p > 0$ *such that*

$$\left(\sum_{n=0}^{\infty}\left(\int_{\Omega}\|\Phi_{\omega}(s+n, s)z(\omega)\|d\mu(\omega)\right)^{-p}\right)^{1/p} \leq \tilde{C}\left(\int_{\Omega}\|z(\omega)\|d\mu(\omega)\right)^{-1}, \quad (3.12)$$

for all $(s, z) \in \mathbb{R}_{+} \times (\mathscr{F} \backslash \{0\})$.

Proof. Necessity. It is a simple verification that (3.12) holds for any $p > 0$ and $\tilde{C} = N^{-1}(1 - e^{-vp})^{-1/p}$, where N, v are given by Definition 2.4.

Sufficiency. Let $(s, z) \in \mathbb{R}_{+} \times (\mathscr{F} \backslash \{0\})$. Using (3.11) and (3.12) we obtain that

$$\begin{aligned}
&\int_{s}^{\infty}\left(\int_{\Omega}\|\Phi_{\omega}(\tau, s)z(\omega)\|d\mu(\omega)\right)^{-p}d\tau \\
&\quad = \sum_{n=0}^{\infty}\int_{s+n}^{s+n+1}\left(\int_{\Omega}\|\Phi_{\omega}(\tau, s)z(\omega)\|d\mu(\omega)\right)^{-p}d\tau \\
&\quad = \sum_{n=0}^{\infty}\int_{s+n}^{s+n+1}\left(\int_{\Omega}\|\Phi_{\omega}(\tau, s+n)\Phi_{\omega}(s+n, s)z(\omega)\|d\mu(\omega)\right)^{-p}d\tau \\
&\quad \leq \sum_{n=0}^{\infty}\int_{s+n}^{s+n+1}\tilde{K}^{-p}e^{\tilde{\alpha}p(\tau-s-n)}\left(\int_{\Omega}\|\Phi_{\omega}(s+n, s)z(\omega)\|d\mu(\omega)\right)^{-p}d\tau \\
&\quad \leq \sum_{n=0}^{\infty}\tilde{K}^{-p}e^{\tilde{\alpha}p}\left(\int_{\Omega}\|\Phi_{\omega}(s+n, s)z(\omega)\|d\mu(\omega)\right)^{-p} \\
&\quad \leq \tilde{C}^{p}\tilde{K}^{-p}e^{\tilde{\alpha}p}\left(\int_{\Omega}\|z(\omega)\|d\mu(\omega)\right)^{-p}.
\end{aligned}$$

Hence (3.1) holds with $C = \tilde{C}K^{-1}e^{\tilde{\alpha}}$. By virtue of Theorem 3.1, we conclude that Φ is exponentially unstable in average.

Remark 3.4. *Theorems* 3.1, 3.2 *and* 3.3 *are certain versions of the classical exponential stability results due to Datko* [8], *Dragičević* [9], *and Zabczyk* [32], *for exponential instability in average of cocycles.*

In the following, we give two characterizations for the exponential instability in average of cocycles by using Lyapunov functions.

Theorem 3.5. *The cocycle* Φ *is exponentially unstable in average if and only if there exist* $p > 0$ *and* $W : \mathbb{R}_{+} \times (\mathscr{F} \backslash \{0\}) \to \mathbb{R}_{+}$ *such that:*

(i) *there exists* $L > 0$ *such that* $W(t, z) \leq L(\int_{\Omega}\|z(\omega)\|d\mu(\omega))^{-p}$, *for all* $(t, z) \in \mathbb{R}_{+} \times (\mathscr{F} \backslash \{0\})$*;*

(ii) $W(t, \Phi_\omega(t,s)z(\omega)) + \int_s^t (\int_\Omega \|\Phi_\omega(\tau,s)z(\omega)\| d\mu(\omega))^{-p} d\tau = W(s, z(\omega))$, *for all* $(t,s,z) \in \Delta \times (\mathscr{F}\backslash\{0\})$, *where* $\Phi_\omega(t,s)z(\omega) = \Phi(t,\omega)\Phi(s,\omega)^{-1}z(\omega)$.

Proof. Necessity. If the cocycle Φ is exponentially unstable in average, then by Definition 2.4, there are $N > 0$ and $v > 0$ such that (2.2) holds. Take an arbitrary $p > 0$. We define $W : \mathbb{R}_+ \times (\mathscr{F}\backslash\{0\}) \to \mathbb{R}_+$ by the formula

$$W(t,z) = \int_t^\infty \left(\int_\Omega \|\Phi_\omega(\tau,t)z(\omega)\| d\mu(\omega) \right)^{-p} d\tau. \tag{3.13}$$

It follows from (2.2) that

$$\begin{aligned} W(t,z) &\le \int_t^\infty \left(N e^{v(\tau-t)} \int_\Omega \|z(\omega)\| d\mu(\omega) \right)^{-p} d\tau \\ &= N^{-p} \left(\int_\Omega \|z(\omega)\| d\mu(\omega) \right)^{-p} \int_t^\infty e^{-vp(\tau-t)}\, d\tau \\ &= N^{-p}(vp)^{-1} \left(\int_\Omega \|z(\omega)\| d\mu(\omega) \right)^{-p}, \end{aligned}$$

which shows that the first assertion holds with $L = N^{-p}(vp)^{-1}$. Moreover, by (3.13), it is clear that

$$\begin{aligned} W(t, \Phi_\omega(t,s)z(\omega)) &= \int_t^\infty \left(\int_\Omega \|\Phi_\omega(\tau,t)\Phi_\omega(t,s)z(\omega)\| d\mu(\omega) \right)^{-p} d\tau \\ &= \int_t^\infty \left(\int_\Omega \|\Phi_\omega(\tau,s)z(\omega)\| d\mu(\omega) \right)^{-p} d\tau \\ &= W(s, z(\omega)) - \int_s^t \left(\int_\Omega \|\Phi_\omega(\tau,s)z(\omega)\| d\mu(\omega) \right)^{-p} d\tau. \end{aligned}$$

Sufficiency. We note that

$$\int_s^t \left(\int_\Omega \|\Phi_\omega(\tau,s)z(\omega)\| d\mu(\omega) \right)^{-p} d\tau \le W(s, z(\omega)) \le L \left(\int_\Omega \|z(\omega)\| d\mu(\omega) \right)^{-p},$$

for all $(t,s,z) \in \Delta \times (\mathscr{F}\backslash\{0\})$. Now, taking the limit for $t \to \infty$ we obtain

$$\left(\int_s^\infty \left(\int_\Omega \|\Phi_\omega(\tau,s)z(\omega)\| d\mu(\omega) \right)^{-p} d\tau \right)^{1/p} \le L^{1/p} \left(\int_\Omega \|z(\omega)\| d\mu(\omega) \right)^{-1},$$

for all $(s,z) \in \mathbb{R}_+ \times (\mathscr{F}\backslash\{0\})$. By Theorem 3.1, we conclude that Φ is exponentially unstable in average.

Theorem 3.6. *The cocycle* Φ *is exponentially unstable in average if and only if there exist* $p > 0$ *and* $W : \mathbb{R}_+ \times (\mathscr{F}\backslash\{0\}) \to \mathbb{R}_+$ *such that:*

(i) *there exists* $L > 0$ *such that* $W(s,z) \le L(\int_\Omega \|z(\omega)\| d\mu(\omega))^{-p}$, *for all* $(s,z) \in \mathbb{R}_+ \times (\mathscr{F}\backslash\{0\})$;

(ii) $W(s+n, \Phi_\omega(s+n,s)z(\omega)) + \sum_{i=[s]+1}^{[s]+n}(\int_\Omega \|\Phi_\omega(i,s)z(\omega)\| d\mu(\omega))^{-p} = W(s, z(\omega))$, *for all* $(n,s,z) \in \mathbb{N}^* \times \mathbb{R}_+ \times (\mathscr{F}\backslash\{0\})$.

Proof. Necessity. Take an arbitrary $p > 0$. We define

$$W : \mathbb{R}_+ \times (\mathscr{F}\backslash\{0\}) \to \mathbb{R}_+,$$

$$W(s,z) = \sum_{i=[s]+1}^{\infty} \left(\int_\Omega \|\Phi_\omega(i,s)z(\omega)\| d\mu(\omega) \right)^{-p}. \tag{3.14}$$

Using (2.2), it can easily be verified that the first assertion holds. In addition, it follows from (3.14) that

$$\begin{aligned} &W(s+n, \Phi_\omega(s+n,s)z(\omega)) \\ &= \sum_{i=[s]+n+1}^{\infty} \left(\int_\Omega \|\Phi_\omega(i,s+n)\Phi_\omega(s+n,s)z(\omega)\| d\mu(\omega) \right)^{-p} \\ &= \sum_{i=[s]+n+1}^{\infty} \left(\int_\Omega \|\Phi_\omega(i,s)z(\omega)\| d\mu(\omega) \right)^{-p} \\ &= W(s,z(\omega)) - \sum_{i=[s]+1}^{[s]+n} \left(\int_\Omega \|\Phi_\omega(i,s)z(\omega)\| d\mu(\omega) \right)^{-p}, \end{aligned}$$

for all $(n,s,z) \in \mathbb{N}^* \times \mathbb{R}_+ \times (\mathscr{F}\backslash\{0\})$.

Sufficiency. Since

$$\sum_{i=[s]+1}^{[s]+n} \left(\int_\Omega \|\Phi_\omega(i,s)z(\omega)\| d\mu(\omega) \right)^{-p} \le W(s,z(\omega)) \le L\left(\int_\Omega \|z(\omega)\| d\mu(\omega) \right)^{-p}$$

for all $(n,s,z) \in \mathbb{N}^* \times \mathbb{R}_+ \times (\mathscr{F}\backslash\{0\})$, it follows that

$$\left(\sum_{i=[s]+1}^{\infty} \left(\int_\Omega \|\Phi_\omega(i,s)z(\omega)\| d\mu(\omega) \right)^{-p} \right)^{1/p} \le L^{1/p} \left(\int_\Omega \|z(\omega)\| d\mu(\omega) \right)^{-1},$$

for all $(s,z) \in \mathbb{R}_+ \times (\mathscr{F}\backslash\{0\})$. By Theorem 3.2, we conclude that Φ is exponentially unstable in average.

Remark 3.7. *Theorems* 3.5 *and* 3.6 *are variants for the case of exponential instability in average of a well-known result due to Preda et al. (see Theorem* 3.3 *in* [27]*) for (non)uniform exponential stability.*

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