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Symmetries of coefficients of three-term relations for the hypergeometric series

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ABSTRACT. Any three hypergeometric series whose respective parameters, a, b and c, differ by integers satisfy a linear relation with coefficients that are rational functions of a, b, c and the variable x. These relations are called three-term relations. This paper gives explicit formulas describing symmetries of the coefficients of three-term relations.

1. Introduction

The hypergeometric series is defined by

$$F(a,b,c;x) = F\left(\begin{matrix} a,b \\ c \end{matrix};x\right) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} x^n.$$

Here, $(\alpha)_n$ denotes $\Gamma(\alpha+n)/\Gamma(\alpha)$, which equals $\alpha(\alpha+1)\cdots(\alpha+n-1)$ for any positive integer n. It is assumed that c is such that the denominator factor $(c)_n$ is never zero.

As mentioned in [1, Section 2.5, p. 94], it is known that for any triples of integers (k,l,m) and (k',l',m'), the three hypergeometric series

$$F\left(\begin{matrix} a+k, b+l \\ c+m \end{matrix};x\right), \qquad F\left(\begin{matrix} a+k', b+l' \\ c+m' \end{matrix};x\right), \qquad F\left(\begin{matrix} a,b \\ c \end{matrix};x\right)$$

satisfy a linear relation with coefficients that are rational functions of a, b, c and x. We call such a relation a three-term relation. Gauss obtained the three-term relations in the cases

$$(k,l,m),(k',l',m') \in \{(1,0,0),(-1,0,0),(0,1,0),(0,-1,0),(0,0,1),(0,0,-1)\},$$

where $(k,l,m) \neq (k',l',m')$; thus, there are $\binom{6}{2} = 15$ pairs of (k,l,m) and (k',l',m'). See [5, Chapter 4, p. 71] for the 15 three-term relations obtained by Gauss.

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We consider the three-term relations of the following form:

$$F\begin{pmatrix} a+k, b+l \\ c+m \end{pmatrix}; x\Big) = Q \cdot F\begin{pmatrix} a+1, b+1 \\ c+1 \end{pmatrix}; x\Big) + R \cdot F\begin{pmatrix} a, b \\ c \end{pmatrix}; x\Big). \tag{1}$$

Note that the pair (Q, R) of rational functions of a, b, c and x is uniquely determined by (k, l, m) (cf. [4, Chapter 6, Section 23]). See [2, Section 3] for the explicit expressions for Q and R with sums of products of two hypergeometric series. Ebisu [3, Section 2.3] noticed that the coefficient Q in (1) has 48 symmetries, and using these symmetries, he gave many special values of the hypergeometric series. As examples of Q's symmetries, Ebisu presented two explicit formulas [3, (2.7), (2.11)]:

$$\begin{aligned} Q\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x\right) &= \frac{ab}{(c-a)(c-b)}(1-x)^{m-k-l} \\ &\quad \times Q\left(\begin{matrix} m-k, m-l \\ m \end{matrix}; \begin{matrix} c-a, c-b \\ c \end{matrix}; x\right), \end{aligned} \tag{2}$$

$$Q\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x\right) = \frac{b}{b-c}(1-x)^{2-k} Q\left(\begin{matrix} k, m-l \\ m \end{matrix}; \begin{matrix} a, c-b \\ c \end{matrix}; \frac{x}{x-1}\right), \tag{3}$$

where we write Q in (1) as $Q\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x\right)$. The 48 symmetries of Q are directly related to the Kummer's 24 solutions. Including the trivial permutation of the local exponents at infinity, the hypergeometric differential equation has 48 symmetries (which correspond to the permutations of the 3 singular points, and to the permutations of the local exponents at them), and Q inherit those symmetries.

On the other hand, Vidūnas considered the three-term relations of the form

$$F\begin{pmatrix} a+k, b+l \\ c+m \end{pmatrix}; x\Big) = \tilde{Q} \cdot F\begin{pmatrix} a+1, b \\ c \end{pmatrix}; x\Big) + \tilde{R} \cdot F\begin{pmatrix} a, b \\ c \end{pmatrix}; x\Big),$$

and gave an explicit formula describing $\tilde{Q}$'s symmetry [6, (11)]. We will see that Q also has the same symmetry which corresponds to reversal of the three-term relations.

In this paper, by conjoining the 48 symmetries of Q inherited from the hypergeometric differential equation and a new symmetry corresponding to reversal of the three-term relations, we show that Q has 96 symmetries. In addition, by giving a relation between Q and R, we show that R also has 96 symmetries.

To avoid ambiguity, we first define the notion of a symmetry of Q and R. For the parameters a, b, c and the variable x, let S_{abc}, S_x and S be the sets defined by

$$S_{abc} := \{n_0 + n_1 a + n_2 b + n_3 c \mid n_i \in \mathbb{Z}\},$$

$$S_x := \left\{x, \frac{x}{x-1}, 1-x, \frac{x-1}{x}, \frac{1}{x}, \frac{1}{1-x}\right\},$$

$$S := \left\{ \left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} \alpha_1, \alpha_2 \\ \alpha_3 \end{matrix}; \beta \right) \middle| \, k, l, m \in \mathbb{Z}, \, \alpha_i \in S_{abc}, \, \beta \in S_x \right\},$$

and let T be the set of all rational functions of a, b, c and x. Also, let $\mathrm{Map}(S, T)$ denote the set of all functions $P : S \to T$. Then, Q and R can be regarded as elements of $\mathrm{Map}(S, T)$; namely,

$$Q = Q\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right), \qquad R = R\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right) \in \mathrm{Map}(S, T).$$

For a group G acting on S, we define the action of G on $\mathrm{Map}(S, T)$ by $(\sigma P)(z) := P(\sigma^{-1} z)$, where $\sigma \in G$, $P \in \mathrm{Map}(S, T)$ and $z \in S$. The definition of the notion of a symmetry of any $P \in \mathrm{Map}(S, T)$ is as follows:

Definition 1. Suppose that a group G acts on S, and take any $\sigma \in G$ and $P \in \mathrm{Map}(S, T)$. If for any $k, l, m \in \mathbb{Z}$, there exist $\alpha_1, \ldots, \alpha_n \in S_{abc}$ and $i_1, \ldots, i_n, j_1, j_2, j_3 \in \mathbb{Z}$ satisfying

$$(\sigma P)\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right) = (\alpha_1)_{i_1} \cdots (\alpha_n)_{i_n} (-1)^{j_1} x^{j_2} (1-x)^{j_3} P\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right),$$

then we say that P has a symmetry under σ. If P has a symmetry under an arbitrary $\sigma \in G$, then we say that P has symmetries under the action of G.

The following theorem provides Q's symmetries.

Theorem 1. *The coefficient Q of* (1) *has* 96 *symmetries. In fact, the following hold:*

$$Q\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right) = \frac{(c+1)_m (c)_m (-1)^{m-k-l-1} x^{-m} (1-x)^{m-k-l}}{(a+1)_k (b+1)_l (c-a)_{m-k} (c-b)_{m-l}} \times Q\left(\begin{matrix} -k, -l \\ -m \end{matrix}; \begin{matrix} a+k, b+l \\ c+m \end{matrix}; x \right), \tag{4}$$

$$Q\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right) = \frac{a}{a-c} (1-x)^{2-l} Q\left(\begin{matrix} m-k, l \\ m \end{matrix}; \begin{matrix} c-a, b \\ c \end{matrix}; \frac{x}{x-1} \right), \tag{5}$$

$$Q\begin{pmatrix} k,l \\ m \end{pmatrix}; \begin{matrix} a,b \\ c \end{matrix}; x\Big) = \frac{(c+1)_{m-1}(c-a-b-1)_{m+1-k-l}}{(c-a)_{m-k}(c-b)_{m-l}} \times Q\left(\begin{matrix} k,l \\ k+l-m \end{matrix}; \begin{matrix} a,b \\ a+b+1-c \end{matrix}; 1-x\right), \tag{6}$$

$$Q\left(\begin{matrix} k,l \\ m \end{matrix}; \begin{matrix} a,b \\ c \end{matrix}; x\right) = Q\left(\begin{matrix} l,k \\ m \end{matrix}; \begin{matrix} b,a \\ c \end{matrix}; x\right). \tag{7}$$

In addition, combining these formulas, we are able to obtain the other 92 *explicit formulas describing Q's symmetries.*

The symmetry (4) is counterpart of the symmetry given in [6, (11)]. The symmetry (5) immediately follows from the symmetries (2) and (3).

The following lemma is counterpart of the relation given in [6, (8)].

LEMMA 1. *The coefficients of* (1) *satisfy the following relation:*

$$R\left(\begin{matrix} k,l \\ m \end{matrix}; \begin{matrix} a,b \\ c \end{matrix}; x\right) = \frac{c(c+1)}{(a+1)(b+1)x(1-x)} Q\left(\begin{matrix} k-1,l-1 \\ m-1 \end{matrix}; \begin{matrix} a+1,b+1 \\ c+1 \end{matrix}; x\right).$$

The following theorem is derived from Theorem 1 and Lemma 1.

THEOREM 2. *The coefficient R of* (1) *has* 96 *symmetries. In fact, the following hold:*

$$R\left(\begin{matrix} k,l \\ m \end{matrix}; \begin{matrix} a,b \\ c \end{matrix}; x\right) = \frac{(c+1)_{m-1}(c)_{m-1}(-1)^{m-k-l}x^{1-m}(1-x)^{m+1-k-l}}{(a+1)_{k-1}(b+1)_{l-1}(c-a)_{m-k}(c-b)_{m-l}} \times R\left(\begin{matrix} 2-k,2-l \\ 2-m \end{matrix}; \begin{matrix} a+k-1,b+l-1 \\ c+m-1 \end{matrix}; x\right),$$

$$R\left(\begin{matrix} k,l \\ m \end{matrix}; \begin{matrix} a,b \\ c \end{matrix}; x\right) = (1-x)^{-l} R\left(\begin{matrix} m+1-k,l \\ m \end{matrix}; \begin{matrix} c-a-1,b \\ c \end{matrix}; \frac{x}{x-1}\right),$$

$$R\left(\begin{matrix} k,l \\ m \end{matrix}; \begin{matrix} a,b \\ c \end{matrix}; x\right) = \frac{(c)_m(c-a-b)_{m-k-l}}{(c-a)_{m-k}(c-b)_{m-l}} R\left(\begin{matrix} k,l \\ k+l-m \end{matrix}; \begin{matrix} a,b \\ a+b+1-c \end{matrix}; 1-x\right),$$

$$R\left(\begin{matrix} k,l \\ m \end{matrix}; \begin{matrix} a,b \\ c \end{matrix}; x\right) = R\left(\begin{matrix} l,k \\ m \end{matrix}; \begin{matrix} b,a \\ c \end{matrix}; x\right).$$

In addition, combining these formulas, we are able to obtain the other 92 *explicit formulas describing R's symmetries.*

2. Proof of Theorem 1

We prove Theorem 1.

Let G be the group generated by the following four mappings so that G acts on S:

$$\sigma_0 : \begin{pmatrix} k,l \\ m \end{pmatrix}; \begin{matrix} a,b \\ c \end{matrix}; x \Big) \mapsto \begin{pmatrix} -k,-l \\ -m \end{pmatrix}; \begin{matrix} a+k, b+l \\ c+m \end{matrix}; x \Big),$$

$$\sigma_0 : \left(\begin{matrix} k,l \\ m \end{matrix}; \begin{matrix} a,b \\ c \end{matrix}; x \right) \mapsto \left(\begin{matrix} -k,-l \\ -m \end{matrix}; \begin{matrix} a+k, b+l \\ c+m \end{matrix}; x \right),$$

$$\sigma_1 : \left(\begin{matrix} k,l \\ m \end{matrix}; \begin{matrix} a,b \\ c \end{matrix}; x \right) \mapsto \left(\begin{matrix} m-k,l \\ m \end{matrix}; \begin{matrix} c-a, b \\ c \end{matrix}; \frac{x}{x-1} \right),$$

$$\sigma_2 : \left(\begin{matrix} k,l \\ m \end{matrix}; \begin{matrix} a,b \\ c \end{matrix}; x \right) \mapsto \left(\begin{matrix} k,l \\ k+l-m \end{matrix}; \begin{matrix} a, b \\ a+b+1-c \end{matrix}; 1-x \right),$$

$$\sigma_3 : \left(\begin{matrix} k,l \\ m \end{matrix}; \begin{matrix} a,b \\ c \end{matrix}; x \right) \mapsto \left(\begin{matrix} l,k \\ m \end{matrix}; \begin{matrix} b,a \\ c \end{matrix}; x \right),$$

where a group operation is defined as the composition of elements in G. Let $\sigma_4 := \sigma_1\sigma_3\sigma_1\sigma_3$ and $\sigma_5 := \sigma_2\sigma_4\sigma_2\sigma_4\sigma_3$ to make them become

$$\sigma_4 : \left(\begin{matrix} k,l \\ m \end{matrix}; \begin{matrix} a,b \\ c \end{matrix}; x \right) \mapsto \left(\begin{matrix} m-k,m-l \\ m \end{matrix}; \begin{matrix} c-a, c-b \\ c \end{matrix}; x \right),$$

$$\sigma_5 : \left(\begin{matrix} k,l \\ m \end{matrix}; \begin{matrix} a,b \\ c \end{matrix}; x \right) \mapsto \left(\begin{matrix} -k,-l \\ -m \end{matrix}; \begin{matrix} 1-a, 1-b \\ 2-c \end{matrix}; x \right).$$

Then, we obtain the following lemma.

Lemma 2. *The structure of G is identified as*

$$G \cong \langle\sigma_0\rangle \times (\langle\sigma_1,\sigma_2\rangle \ltimes (\langle\sigma_3\rangle \times \langle\sigma_4\rangle \times \langle\sigma_5\rangle)) \cong \mathbb{Z}/2\mathbb{Z} \times (S_3 \ltimes (\mathbb{Z}/2\mathbb{Z})^3),$$

where S_3 is the symmetric group of degree 3; thus, the order of G equals $2 \cdot 3! \cdot 2^3 = 96$.

Proof. First, $G \cong \langle\sigma_0\rangle \times \langle\sigma_1,\sigma_2,\sigma_3\rangle$ holds. Next, since $\langle\sigma_3,\sigma_4,\sigma_5\rangle$ is normal in $\langle\sigma_1,\sigma_2,\sigma_3\rangle$ and satisfies $\langle\sigma_1,\sigma_2\rangle \cap \langle\sigma_3,\sigma_4,\sigma_5\rangle = \{\mathrm{id}\}$, where id denotes the identity element of G, it holds that $\langle\sigma_1,\sigma_2,\sigma_3\rangle = \langle\sigma_1,\sigma_2\rangle \ltimes \langle\sigma_3,\sigma_4,\sigma_5\rangle$. Finally, from $\sigma_i^2 = \mathrm{id}$ $(0 \le i \le 5)$, $\sigma_1\sigma_2 = \sigma_2\sigma_1\sigma_2\sigma_1$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ $(3 \le i, j \le 5)$, the proof of the lemma is complete. □

The formulas (4)–(7) describe symmetries of Q under $\sigma_0, \dots, \sigma_3$, respectively. Hence, we can complete the proof of Theorem 1 by proving (4)–(7). As stated in the introduction, (5) is obtained by combining (2) and (3). Also, (7) immediately follows from the fact that $F(\alpha,\beta,\gamma;x)$ is symmetric with respect to the exchange of α and β. Below, we prove (4) and (6). From the uniqueness of analytic continuation, it is sufficient to prove it for $|x| < 1/2$. Thus, we assume $|x| < 1/2$. We introduce two expressions for Q given in [2]. Let f_i $(i = 1,2,5,6)$ be the functions defined by

$$f_1\begin{pmatrix} a,b \\ c \end{pmatrix}; x\Bigr) := f\begin{pmatrix} a,b \\ c \end{pmatrix}; x\Bigr),$$

$$f_1\left(\begin{matrix} a,b \\ c \end{matrix}; x\right) := f\left(\begin{matrix} a,b \\ c \end{matrix}; x\right),$$

$$f_2\left(\begin{matrix} a,b \\ c \end{matrix}; x\right) := f\left(\begin{matrix} a,b \\ a+b+1-c \end{matrix}; 1-x\right),$$

$$f_5\left(\begin{matrix} a,b \\ c \end{matrix}; x\right) := x^{1-c} f\left(\begin{matrix} a+1-c, b+1-c \\ 2-c \end{matrix}; x\right),$$

$$f_6\left(\begin{matrix} a,b \\ c \end{matrix}; x\right) := (1-x)^{c-a-b} f\left(\begin{matrix} c-a, c-b \\ c+1-a-b \end{matrix}; 1-x\right),$$

where $f\left(\begin{matrix} a,b \\ c \end{matrix}; x\right) := \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(c)} F\left(\begin{matrix} a,b \\ c \end{matrix}; x\right)$. Then, Q can be expressed as

$$Q = \frac{ab(c)_m}{c(a)_k(b)_l} q,$$

where

$$\begin{aligned} q &:= q\left(\begin{matrix} k,l \\ m \end{matrix}; \begin{matrix} a,b \\ c \end{matrix}; x\right) \\ &= \frac{f_5\left(\begin{matrix} a,b \\ c \end{matrix}; x\right) f_1\left(\begin{matrix} a+k, b+l \\ c+m \end{matrix}; x\right) - f_1\left(\begin{matrix} a,b \\ c \end{matrix}; x\right) f_5\left(\begin{matrix} a+k, b+l \\ c+m \end{matrix}; x\right)}{f_5\left(\begin{matrix} a,b \\ c \end{matrix}; x\right) f_1\left(\begin{matrix} a+1, b+1 \\ c+1 \end{matrix}; x\right) - f_1\left(\begin{matrix} a,b \\ c \end{matrix}; x\right) f_5\left(\begin{matrix} a+1, b+1 \\ c+1 \end{matrix}; x\right)}, \quad (8) \\ &= \frac{(-1)^{m+1-k-l}}{(c-a)_{m-k}(c-b)_{m-l}} \\ &\quad \times \frac{f_6\left(\begin{matrix} a,b \\ c \end{matrix}; x\right) f_2\left(\begin{matrix} a+k, b+l \\ c+m \end{matrix}; x\right) - f_2\left(\begin{matrix} a,b \\ c \end{matrix}; x\right) f_6\left(\begin{matrix} a+k, b+l \\ c+m \end{matrix}; x\right)}{f_6\left(\begin{matrix} a,b \\ c \end{matrix}; x\right) f_2\left(\begin{matrix} a+1, b+1 \\ c+1 \end{matrix}; x\right) - f_2\left(\begin{matrix} a,b \\ c \end{matrix}; x\right) f_6\left(\begin{matrix} a+1, b+1 \\ c+1 \end{matrix}; x\right)}. \quad (9) \end{aligned}$$

These expressions (8) and (9) follow immediately from [2, p. 260, (3.5)] and [2, p. 264, the expression above Theorem 3.8], respectively. First, using (8), we prove (4). Applying σ_0 to (8), we have

$$(\sigma_0 q)\left(\begin{matrix} k,l \\ m \end{matrix}; \begin{matrix} a,b \\ c \end{matrix}; x\right) = -\frac{W(a,b,c;x)}{W(a+k, b+l, c+m; x)} q\left(\begin{matrix} k,l \\ m \end{matrix}; \begin{matrix} a,b \\ c \end{matrix}; x\right),$$

where $W(a,b,c;x)$ denotes the denominator of (8); namely,

$$W(a,b,c;x) := f_5\begin{pmatrix} a,b \\ c \end{pmatrix}; x\Big) f_1\begin{pmatrix} a+1,b+1 \\ c+1 \end{pmatrix}; x\Big) - f_1\begin{pmatrix} a,b \\ c \end{pmatrix}; x\Big) f_5\begin{pmatrix} a+1,b+1 \\ c+1 \end{pmatrix}; x\Big).$$

From the formula [2, p. 262, Lemma 3.6]

$$W(a,b,c;x) = -\frac{\Gamma(a)\Gamma(b)\Gamma(a+1-c)\Gamma(b+1-c)}{\Gamma(c)\Gamma(1-c)} x^{-c}(1-x)^{c-a-b-1},$$

we obtain

$$\frac{W(a,b,c;x)}{W(a+k,b+l,c+m;x)} = \frac{(-1)^{k+l-m}(c-a)_{m-k}(c-b)_{m-l}}{(a)_k(b)_l} x^m(1-x)^{k+l-m}.$$

Therefore, it follows that

$$(\sigma_0 q)\left(\begin{matrix} k,l \\ m \end{matrix}; \begin{matrix} a,b \\ c \end{matrix}; x\right) = \frac{(-1)^{k+l-m-1}(c-a)_{m-k}(c-b)_{m-l}}{(a)_k(b)_l} x^m(1-x)^{k+l-m} \times q\left(\begin{matrix} k,l \\ m \end{matrix}; \begin{matrix} a,b \\ c \end{matrix}; x\right). \tag{10}$$

Multiplying both sides of (10) by

$$\frac{(a+k)(b+l)(c+m)_{-m}}{(c+m)(a+k)_{-k}(b+l)_{-l}}$$

completes the proof of (4). Next, using (8) and (9), we prove (6). When we apply σ_2 to (8), the numerator becomes

$$f_5\left(\begin{matrix} a,b \\ a+b+1-c \end{matrix}; 1-x\right) f_1\left(\begin{matrix} a+k,b+l \\ a+b+1-c+k+l-m \end{matrix}; 1-x\right) - f_1\left(\begin{matrix} a,b \\ a+b+1-c \end{matrix}; 1-x\right) f_5\left(\begin{matrix} a+k,b+l \\ a+b+1-c+k+l-m \end{matrix}; 1-x\right), \tag{11}$$

and the denominator becomes

$$f_5\left(\begin{matrix} a,b \\ a+b+1-c \end{matrix}; 1-x\right) f_1\left(\begin{matrix} a+1,b+1 \\ a+b+2-c \end{matrix}; 1-x\right) - f_1\left(\begin{matrix} a,b \\ a+b+1-c \end{matrix}; 1-x\right) f_5\left(\begin{matrix} a+1,b+1 \\ a+b+2-c \end{matrix}; 1-x\right). \tag{12}$$

From the definitions of f_i $(i = 1, 2, 5, 6)$, we can rewrite (11) and (12) as

$$f_6\begin{pmatrix} a, b \\ c \end{pmatrix}; x\Bigr) f_2\begin{pmatrix} a+k, b+l \\ c+m \end{pmatrix}; x\Bigr) - f_2\begin{pmatrix} a, b \\ c \end{pmatrix}; x\Bigr) f_6\begin{pmatrix} a+k, b+l \\ c+m \end{pmatrix}; x\Bigr), \tag{13}$$

$$f_6\begin{pmatrix} a, b \\ c \end{pmatrix}; x\Bigr) f_2\begin{pmatrix} a+1, b+1 \\ c+1 \end{pmatrix}; x\Bigr) - f_2\begin{pmatrix} a, b \\ c \end{pmatrix}; x\Bigr) f_6\begin{pmatrix} a+1, b+1 \\ c+1 \end{pmatrix}; x\Bigr), \tag{14}$$

respectively. Comparing (13)/(14) with (9), we obtain

$$(\sigma_2 q)\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x\right) = (-1)^{k+l-m-1}(c-a)_{m-k}(c-b)_{m-l}\, q\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x\right). \tag{15}$$

Multiplying both sides of (15) by

$$\frac{ab(a+b+1-c)_{k+l-m}}{(a+b+1-c)(a)_k(b)_l}$$

completes the proof of (6).

3. Proof of Lemma 1

We prove Lemma 1.

Replacing (k, l, m, a, b, c) by $(k-1, l-1, m-1, a+1, b+1, c+1)$ in (1), we have

$$F\left(\begin{matrix} a+k, b+l \\ c+m \end{matrix}; x\right) = Q' \cdot F\left(\begin{matrix} a+2, b+2 \\ c+2 \end{matrix}; x\right) + R' \cdot F\left(\begin{matrix} a+1, b+1 \\ c+1 \end{matrix}; x\right), \tag{16}$$

where

$$Q' := Q\left(\begin{matrix} k-1, l-1 \\ m-1 \end{matrix}; \begin{matrix} a+1, b+1 \\ c+1 \end{matrix}; x\right),$$

$$R' := R\left(\begin{matrix} k-1, l-1 \\ m-1 \end{matrix}; \begin{matrix} a+1, b+1 \\ c+1 \end{matrix}; x\right).$$

As is well known, $F(a, b, c; x)$ satisfies

$$\partial F\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \frac{ab}{c} F\left(\begin{matrix} a+1, b+1 \\ c+1 \end{matrix}; x\right),$$

where $\partial := d/dx$, and is a solution of the hypergeometric differential equation $L_{abc}\, y = 0$, where

$$L_{abc} := \partial^2 + \frac{c-(a+b+1)x}{x(1-x)}\partial - \frac{ab}{x(1-x)}.$$

Using these facts, we have

$$
\begin{aligned}
0 &= L_{abc}F\begin{pmatrix} a, b \\ c \end{pmatrix}; x\Bigr) \\
&= \partial^2 F\begin{pmatrix} a, b \\ c \end{pmatrix}; x\Bigr) + \frac{c-(a+b+1)x}{x(1-x)}\partial F\begin{pmatrix} a, b \\ c \end{pmatrix}; x\Bigr) - \frac{ab}{x(1-x)}F\begin{pmatrix} a, b \\ c \end{pmatrix}; x\Bigr) \\
&= \frac{ab(a+1)(b+1)}{c(c+1)}F\begin{pmatrix} a+2, b+2 \\ c+2 \end{pmatrix}; x\Bigr) \\
&\quad + \frac{ab\{c-(a+b+1)x\}}{cx(1-x)}F\begin{pmatrix} a+1, b+1 \\ c+1 \end{pmatrix}; x\Bigr) - \frac{ab}{x(1-x)}F\begin{pmatrix} a, b \\ c \end{pmatrix}; x\Bigr).
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
F\begin{pmatrix} a+2, b+2 \\ c+2 \end{pmatrix}; x\Bigr) &= -\frac{(c+1)\{c-(a+b+1)x\}}{(a+1)(b+1)x(1-x)}F\begin{pmatrix} a+1, b+1 \\ c+1 \end{pmatrix}; x\Bigr) \\
&\quad + \frac{c(c+1)}{(a+1)(b+1)x(1-x)}F\begin{pmatrix} a, b \\ c \end{pmatrix}; x\Bigr).
\end{aligned}
$$

Using this, we rewrite (16) as

$$
\begin{aligned}
F\begin{pmatrix} a+k, b+l \\ c+m \end{pmatrix}; x\Bigr) &= \left\{-\frac{(c+1)\{c-(a+b+1)x\}}{(a+1)(b+1)x(1-x)}Q' + R'\right\}F\begin{pmatrix} a+1, b+1 \\ c+1 \end{pmatrix}; x\Bigr) \\
&\quad + \frac{c(c+1)}{(a+1)(b+1)x(1-x)}Q' \cdot F\begin{pmatrix} a, b \\ c \end{pmatrix}; x\Bigr).
\end{aligned} \tag{17}
$$

Equating the coefficients of $F(a, b, c; x)$ in (1) and (17) completes the proof of Lemma 1.

4. Proof of Theorem 2

Using Theorem 1 and Lemma 1, we prove Theorem 2.

Let $\tilde{G}$ be the group generated by $\tau\sigma_i\tau^{-1}$ $(i = 0, 1, 2, 3)$, where τ is the mapping defined by

$$
\tau : \begin{pmatrix} k, l \\ m \end{pmatrix}; \begin{matrix} a, b \\ c \end{matrix}; x\Bigr) \mapsto \begin{pmatrix} k+1, l+1 \\ m+1 \end{pmatrix}; \begin{matrix} a-1, b-1 \\ c-1 \end{matrix}; x\Bigr).
$$

Then, $\tilde{G}$ is isomorphic to G; thus, the order of $\tilde{G}$ also equals 96. By combining the four formulas in Theorem 1 with the formula in Lemma 1, we can obtain the four formulas in Theorem 2. These four resulting formulas assert that R has symmetries under $\tau\sigma_i\tau^{-1}$ $(i = 0, 1, 2, 3)$, respectively. This completes the proof of Theorem 2.

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