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On a conjecture of Li and Yang

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Abstract. We study the uniqueness problem of an entire function f when it shares two small functions with its derivative $f^{(k)}$ $(k \geq 1)$. This confirms the conjecture posed by Li and Yang [5].

1. Introduction and main result

Let $\mathscr{M}(\mathbb{C})$ be the family of all non-constant functions which are meromorphic in $\mathbb{C}$, whereas $\mathscr{E}(\mathbb{C})$ denotes the family of all non-constant entire functions in $\mathbb{C}$. On the other hand, we denote by $\mathscr{M}_T(\mathbb{C})$ and $\mathscr{E}_T(\mathbb{C})$ the family of all transcendental meromorphic functions in $\mathbb{C}$ and the family of all transcendental entire functions in $\mathbb{C}$ respectively. In this paper, for $f \in \mathscr{M}(\mathbb{C})$, we shall use the standard notations of Nevanlinna's value distribution theory such as $T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \dots$ (see e.g., [4, 10]). We adopt the standard notation $S(r,f)$ for any quantity satisfying the relation $S(r,f) = o(T(r,f))$ as $r \to \infty$ except possibly a set of finite linear measure. Let $a, f \in \mathscr{M}(\mathbb{C})$. Then a is said to be a small function of f if $T(r,a) = S(r,f)$. Denote by $\mathscr{S}(f)$ the family of all small functions of $f \in \mathscr{M}(\mathbb{C})$. Let $a \in \mathscr{S}(f) \cap \mathscr{S}(g)$. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share a CM and if we do not consider the multiplicities, then we say that f and g share a IM.

Rubel and Yang [9] were the first to study entire functions that share values with their derivatives. In 1977, they proved if $f \in \mathscr{E}(\mathbb{C})$ shares two finite distinct values CM with f', then $f \equiv f'$. This result has been generalized from sharing values CM to IM by Mues and Steinmetz [8] and in the case when both shared values are non-zero independently by Gundersen [3]. Since then the subject of sharing values between a meromorphic function and its derivative has been extensively studied by many researchers and a lot of interesting results have been obtained (see [10]).

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In 1991, G. Frank [2] proposed the following conjecture: If $f \in \mathscr{E}(\mathbb{C})$ shares two finite values IM with its k-th derivative $(k \geq 1)$, then $f \equiv f^{(k)}$.

In 2000, Li and Yang [5] fully resolved **Frank's conjecture** in the following form.

Theorem 1.1 ([5]). *Let $f \in \mathscr{E}(\mathbb{C})$ and $a_1, a_2 \in \mathbb{C}$ be distinct. If f and $f^{(k)}$ $(k \geq 1)$ share a_1 and a_2 IM, then $f \equiv f^{(k)}$.*

At the end of the paper, Li and Yang [5] gave rise to the following conjecture:

Conjecture 1.2. *Theorem* 1.1 *still holds when a_1 and a_2 are two arbitrary distinct small functions of f.*

To the knowledge of authors, **Conjecture 1.2** is not still confirmed. In this paper, we settle **Conjecture 1.2** at the cost of considering the fact that $a_1'' \not\equiv a_2''$. We now state our main result as follows.

Theorem 1.3. *Let $f \in \mathscr{E}(\mathbb{C})$ and $a_1, a_2 \in \mathscr{S}(f)$ be non-constant such that $a_1, a_2 \not\equiv \infty$ and $a_1'' \not\equiv a_2''$. If f and $f^{(k)}$ $(k \geq 1)$ share a_1 and a_2 IM, then $f \equiv f^{(k)}$.*

Remark 1.4. *The following example asserts that condition "$a_1, a_2 \not\equiv \infty$" is sharp in Theorem* 1.3.

Example 1.1. *Let $f(z) = c + e^{ce^z}$ and $a_1(z) = \frac{c^2}{c - e^{-z}}$, where $c \in \mathbb{C} \setminus \{0\}$. If $a_2 = \infty$, then f and f' certainly share a_2 CM. On the other hand, we see that f and f' also share a_1 CM, but $f \not\equiv f'$.*

First of all, we generalize the definition of IM to IM*. Let $f, g \in \mathscr{M}(\mathbb{C})$ and $a \in \mathscr{S}(f) \cap \mathscr{S}(g)$. Denote by $\overline{N}_0(r, a)$ the counting function of all common zeroes of $f - a$ and $g - a$ ignoring multiplicities. If $\overline{N}(r, a; f) + \overline{N}(r, a; g) - 2\overline{N}_0(r, a) = S(r, f) + S(r, g)$, then we say f and g share a IM*. One can easily prove that Theorem 1.3 is still valid if condition "IM" is replaced by "IM*".

Remark 1.5. *We can easily see that Theorem* 1.3 *is still valid for any $f \in \mathscr{M}(\mathbb{C})$ satisfying $N(r, f) = S(r, f)$.*

Remark 1.6. *The following examples assert that Theorems* A *and* 1.3 *do not hold for any $f \in \mathscr{M}(\mathbb{C})$ satisfying $N(r, f) \neq S(r, f)$.*

Example 1.2. *Let* $f(z)=\frac{4}{1-3e^{-2z}}$. *Clearly* $N(r,f)\neq S(r,f)$. *Note that* $f'(z)=\frac{-24e^{-2}}{(1-3e^{-2z})^2}$ *and so* f *and* f' *share* 0 *CM. On the other hand, we see that* f *and* f' *share* 2 *IM, but* $f\not\equiv f'$.

Example 1.3. *Let* $a(z)=-\frac{1}{3}e^{-2z}+ce^{-z}$, $b(z)=-\frac{1}{3}e^{-2z}-ce^{-z}$ *and* $h(z)=e^{2ce^{z}}$, *where* $c\in\mathbb{R}\backslash\{0\}$. *Define* $f(z)=b(z)+\frac{b(z)-a(z)}{h(z)-1}$. *Let* $a_1(z)=b'(z)$ *and* $a_2(z)=a'(z)$. *Clearly,* $a_1,a_2\in\mathscr{S}(f)$ *and* $N(r,f)\neq S(r,f)$. *Also, we deduce that* $f'(z)-a_1(z)=e^{2z}(f(z)-a_1(z))(f(z)-b(z))$ *and* $f'(z)-a_2(z)=e^{2z}(f(z)-a_2(z))(f(z)-a(z))$. *Clearly,* f *and* f' *share* a_1 *and* a_2 *IM, but* $f\not\equiv f'$.

Example 1.4 ([6]). *Let* $a,b\in\mathbb{C}$ *such that* $a-b=\sqrt{2}i$ *and* w *be a non-constant solution of the Riccati differential equation* $w'=(w-a_1)(w-a_2)$. *Let* $f(z)=(w(z)-a)(w(z)-b)-\frac{1}{3}$. *Then* $w,f\in\mathscr{M}_T(\mathbb{C})$ *and* $w'\neq 0$. *It is easy to verify that* $f''=6w'f$ *and* $f''+\frac{1}{6}=6\big(f+\frac{1}{6}\big)^2$. *Clearly* f *and* f'' *share* 0 *CM and* $-\frac{1}{6}$ *IM, but* $f\not\equiv f''$.

After considering Theorem 1.3, we ask The following open question:

Open problem. Is it possible to establish Theorem 1.3 without the hypothesis $a_1''\not\equiv a_2''$?

2. Auxiliary lemmas

Lemma 2.1 ([1]). *Let* $f\in\mathscr{M}_T(\mathbb{C})$ *such that* $f^nP(f)=Q(f)$, *where* $P(f)$ *and* $Q(f)$ *are differential polynomials in* f *with functions of small proximity related to* f *as the coefficients and the degree of* $Q(f)$ *is at most* n. *Then* $m(r,P)=S(r,f)$.

Lemma 2.2 ([11]). *If* $f,g\in\mathscr{M}(\mathbb{C})$, *then*

$$N\left(r,\frac{f}{g}\right)-N\left(r,\frac{g}{f}\right)=N(r,f)+N(r,0;g)-N(r,g)-N(r,0;f).$$

Lemma 2.3. *Let* $f\in\mathscr{E}(\mathbb{C})$ *and* $a_1,a_2\in\mathscr{S}(f)$ *such that* $a_1,a_2\not\equiv 0,\infty$ *and* $a_1\not\equiv a_2$. *Suppose*

$$\Delta(f)=\begin{vmatrix} f-a_1 & a_1-a_2\\ f'-a_1' & a_1'-a_2'\end{vmatrix}=\begin{vmatrix} f-a_2 & a_1-a_2\\ f'-a_2' & a_1'-a_2'\end{vmatrix}.$$

Then

(1) $\Delta(f)\not\equiv 0$,

(2) $m\Big(r,\frac{\Delta(f)}{f-a_i}\Big)=S(r,f)$ $(i=1,2)$,

(3) $m\left(r,\frac{\Delta(f)}{(f-a_1)(f-a_2)}\right)=S(r,f)$,

(4) $m\left(r,\frac{\Delta(f)(f-\beta)}{(f-a_1)(f-a_2)}\right)=S(r,f)$, *where* $\beta\in\mathscr{S}(f)$,

(5) $m\left(r,\frac{\Delta(f)(f-f^{(k)})}{(f-a_1)(f-a_2)}\right)=S(r,f)$.

Proof. (1) Suppose to the contrary that $\Delta(f)\equiv 0$. Then clearly we have $\frac{f'-a_1'}{f-a_1}\equiv\frac{a_1'-a_2'}{a_1-a_2}$. On integration, we have $f\equiv a_1+a_0(a_1-a_2)$, where $a_0\in\mathbb{C}\backslash\{0\}$. This shows that $f\in\mathscr{S}(f)$, which is a contradiction. Hence $\Delta(f)\not\equiv 0$.

(2) Note that for $i=1,2$, we have $\Delta(f)=(a_1'-a_2')(f-a_i)-(a_1-a_2)(f'-a_i')$, i.e., $\frac{\Delta(f)}{f-a_i}=a_1'-a_2'-(a_1-a_2)\frac{f'-a_i'}{f-a_i}$. Consequently

$$\begin{aligned} m\left(r,\frac{\Delta(f)}{f-a_i}\right)&\leq m(r,a_1'-a_2')+m(r,a_1-a_2)+m\left(r,\frac{f'-a_i'}{f-a_i}\right)+\log 2\\ &=S(r,f), \end{aligned}$$

for $i=1,2$ and so (2) holds.

(3) We see that $\frac{\Delta(f)}{(f-a_1)(f-a_2)}=\frac{1}{a_1-a_2}\left[\frac{\Delta(f)}{f-a_1}-\frac{\Delta(f)}{f-a_2}\right]$. Now (3) follows directly from (2).

(4) We see that $\frac{\Delta(f)(f-\beta)}{(f-a_1)(f-a_2)}=\frac{\Delta(f)}{f-a_1}+\frac{(a_2-\beta)\Delta(f)}{(f-a_1)(f-a_2)}$ and so

$$\begin{aligned} m\left(r,\frac{\Delta(f)(f-\beta)}{(f-a_1)(f-a_2)}\right)&\leq m\left(r,\frac{\Delta(f)}{f-a_1}\right)+m\left(r,\frac{\Delta(f)}{(f-a_1)(f-a_2)}\right)\\ &\quad+m(r,a_2-\beta). \end{aligned}$$

Now (4) follows directly from (2) and (3).

(5) We see that $m\left(r,\frac{\Delta(f)(f-f^{(k)})}{(f-a_1)(f-a_2)}\right)\leq m\left(r,\frac{\Delta(f)f}{(f-a_1)(f-a_2)}\right)+m\left(r,1-\frac{f^{(k)}}{f}\right)$. Now (5) follows directly from (4).

Lemma 2.4 ([7]). *Let* $f\in\mathscr{M}(\mathbb{C})$ *and* $R(f)=\frac{P(f)}{Q(f)}$, *where* $P(f)=\sum_{k=0}^{p}a_kf^k$ *and* $Q(f)=\sum_{j=0}^{q}b_jf^j$ *are two mutually prime polynomials in* f. *If* $a_k,b_j\in\mathscr{S}(f)$ *such that* $a_p\not\equiv 0$ *and* $b_q\not\equiv 0$, *then* $T(r,R(f))=\max\{p,q\}T(r,f)+S(r,f)$.

Lemma 2.5 ([4]). *Let* $f\in\mathscr{M}(\mathbb{C})$ *and* $a_1,a_2\in\mathscr{S}(f)$. *Then* $T(r,f)\leq\overline{N}(r,f)+\sum_{i=1}^{2}\overline{N}(r,a_i;f)+S(r,f)$.

Henceforth for $f\in\mathscr{E}(\mathbb{C})$, we define the following auxiliary functions

$$\phi=\frac{\Delta(f)(f-f^{(k)})}{(f-a_1)(f-a_2)},\tag{2.1}$$

$$\psi = \frac{\Delta(f^{(k)})(f - f^{(k)})}{(f^{(k)} - a_1)(f^{(k)} - a_2)}, \tag{2.2}$$

$$H_{nm} = n\phi - m\psi, \tag{2.3}$$

where $m, n \in \mathbb{N}$ and

$$H = \frac{\Delta(f)}{(f - a_1)(f - a_2)} - \frac{\Delta(f^{(k)})}{(f^{(k)} - a_1)(f^{(k)} - a_2)}. \tag{2.4}$$

Differentiating twice, we obtain from (2.1) that

$$\begin{aligned}
&((a_1'' - a_2'')(f - a_1) - (a_1 - a_2)(f'' - a_1''))(f - f^{(k)}) \\
&\quad + ((a_1' - a_2')(f - a_1) - (a_1 - a_2)(f' - a_1'))(f' - f^{(k+1)}) \\
&\quad = \phi'(f - a_1)(f - a_2) + \phi(f' - a_1')(f - a_2) + \phi(f - a_1)(f' - a_2')
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
&((a_1''' - a_2''')(f - a_1) + (a_1'' - a_2'')(f' - a_1') - (a_1' - a_2')(f'' - a_1'') \\
&\quad - (a_1 - a_2)(f''' - a_1'''))(f - f^{(k)}) \\
&\quad + 2((a_1'' - a_2'')(f - a_1) - (a_1 - a_2)(f'' - a_1''))(f' - f^{(k+1)}) \\
&\quad + ((a_1' - a_2')(f - a_1) - (a_1 - a_2)(f' - a_1'))(f'' - f^{(k+2)}) \\
&= \phi''(f - a_1)(f - a_2) + 2\phi'(f' - a_1')(f - a_2) + 2\phi'(f - a_1)(f' - a_2') \\
&\quad + \phi(f'' - a_1'')(f - a_2) + 2\phi(f' - a_1')(f' - a_2') \\
&\quad + \phi(f - a_1)(f'' - a_2'').
\end{aligned} \tag{2.6}$$

Definition 2.6. Let $k, m, n \in \mathbb{N}$ and $a_1, a_2 \in \mathscr{S}(f)$. Denote by $S_{(m,n)}(a_1)$ the set of those points $z \in \mathbb{C}$ such that z is an a_1-point of f of order m and an a_1-point of $f^{(k)}$ of order n. The set $S_{(m,n)}(a_2)$ can be defined similarly. Let $\overline{N}_{(m,n)}(r, a_1; f)$ denote the reduced counting function of f with respect to the set $S_{(m,n)}(a_1)$. Similarly $\overline{N}_{(m,n)}(r, a_2; f)$ denotes the reduced counting function of f with respect to the set $S_{(m,n)}(a_2)$.

Let $z_{p,q} \in S_{(p,q)}(a_1)$ such that $\phi(z_{p,q}) \neq 0, \infty$ and $a_1(z_{p,q}) - a_2(z_{p,q}) \neq 0, \infty$. Then in some neighbourhood of $z_{p,q}$, we get by Taylor's expansion

$$\begin{cases}
f(z) - a_1(z) = b_p(z - z_{p,q})^p + b_{p+1}(z - z_{p,q})^{p+1} + \cdots \quad (b_p \neq 0), \\
f^{(k)}(z) - a_1(z) = c_q(z - z_{p,q})^q + c_{q+1}(z - z_{p,q})^{q+1} + \cdots \quad (c_q \neq 0), \\
\phi(z) = d_0 + d_1(z - z_{p,q}) + d_2(z - z_{p,q})^2 + \cdots \quad (d_0 \neq 0).
\end{cases} \tag{2.7}$$

Clearly

$$\begin{cases} f'(z)-a_1'(z)=pb_p(z-z_{p,q})^{p-1}+(p+1)b_{p+1}(z-z_{p,q})^{p}+\cdots, \\ f''(z)-a_1''(z)=p(p-1)b_p(z-z_{p,q})^{p-2} \\ \qquad\qquad +p(p+1)b_{p+1}(z-z_{p,q})^{p-1}+\cdots, \\ f^{(k+1)}(z)-a_1'(z)=qc_q(z-z_{p,q})^{q-1}+(q+1)c_{q+1}(z-z_{p,q})^{q}+\cdots, \\ f^{(k+2)}(z)-a_1''(z)=q(q-1)c_q(z-z_{p,q})^{q-2} \\ \qquad\qquad +(q+1)qc_{q+1}(z-z_{p,q})^{q-1}+\cdots \\ \phi'(z)=d_1+2d_2(z-z_{p,q})+3d_2(z-z_{p,q})^2\ldots, \\ \phi''(z)=2d_2+6d_2(z-z_{p,q})\ldots. \end{cases} \tag{2.8}$$

Now from (2.7) and (2.8), we see that $z_{p,q}$ is a zero of $\Delta(f(z))$ of multiplicity $p-1$ and

$$f(z)-f^{(k)}(z)=\begin{cases} b_p(z-z_{p,q})^p+\cdots, & \text{if } p<q \\ -c_q(z-z_{p,q})^q-\cdots, & \text{if } p>q \\ (b_p-c_p)(z-z_{p,q})^p+\cdots, & \text{if } p=q. \end{cases} \tag{2.9}$$

If $z_{p,1}\in S_{(p,1)}(a_1)$ $(p\geq 2)$ such that $\phi(z_{p,1})\neq 0,\infty$ and $a_1(z_{p,1})-a_2(z_{p,1})\neq 0,\infty$, then from (2.5), (2.7)–(2.9), one can easily conclude that

$$d_0=pc_1, \qquad \text{i.e., } f^{(k+1)}(z_{p,1})-a_1'(z_{p,1})=c_1=\frac{\phi(z_{p,1})}{p}. \tag{2.10}$$

Let $\hat{z}_{p,q}\in S_{(p,q)}(a_2)$ such that $\phi(\hat{z}_{p,q})\neq 0,\infty$ and $a_1(\hat{z}_{p,q})-a_2(\hat{z}_{p,q})\neq 0,\infty$.

Then in some neighbourhood of $\hat{z}_{p,q}$, we get by Taylor's expansion

$$\begin{cases} f(z)-a_2(z)=\hat{b}_p(z-\hat{z}_{p,q})^p+\hat{b}_{p+1}(z-\hat{z}_{p,q})^{p+1}+\cdots \quad (\hat{b}_p\neq 0), \\ f^{(k)}(z)-a_2(z)=\hat{c}_q(z-\hat{z}_{p,q})^q+\hat{c}_{q+1}(z-\hat{z}_{p,q})^{q+1}+\cdots \quad (\hat{c}_q\neq 0), \\ \phi(z)=\hat{d}_0+\hat{d}_1(z-\hat{z}_{p,q})+\hat{d}_2(z-\hat{z}_{p,q})^2+\cdots \quad (\hat{d}_0\neq 0). \end{cases} \tag{2.11}$$

Similarly if $\hat{z}_{p,1}\in S_{(p,1)}(a_2)$ $(p\geq 2)$ such that $\phi(\hat{z}_{p,1})\neq 0,\infty$ and $a_1(\hat{z}_{p,1})-a_2(\hat{z}_{p,1})\neq 0,\infty$, then we immediately get

$$f^{(k+1)}(\hat{z}_{p,1})-a_2'(\hat{z}_{p,1})=\hat{c}_1=-\frac{\phi(\hat{z}_{p,1})}{p}. \tag{2.12}$$

Lemma 2.7. *Let $f\in\mathscr{E}(\mathbb{C})$ and $a_1,a_2\in\mathscr{S}(f)$ such that $a_1,a_2\not\equiv\infty$ and $a_1\not\equiv a_2$. If f and $f^{(k)}$ share a_1, a_2 IM and $T(r,f)=T(r,f^{(k)})+S(r,f)$, then $f\equiv f^{(k)}$.*

Proof. Suppose to the contrary that $f\not\equiv f^{(k)}$. Since f and $f^{(k)}$ share a_1 and a_2 IM, we have

$$\begin{aligned}\overline{N}(r,a_1;f)+\overline{N}(r,a_2;f) &\leq N(r,0;f-f^{(k)})+S(r,f)\\ &\leq T(r,f-f^{(k)})+S(r,f)\\ &\leq m(r,f-f^{(k)})+S(r,f)\\ &\leq m(r,f)+m\left(r,1-\frac{f^{(k)}}{f}\right)+S(r,f)\\ &\leq T(r,f)+S(r,f). \end{aligned} \tag{2.13}$$

Also using Lemma 2.5, we have $T(r,f)\leq\overline{N}(r,a_1;f)+\overline{N}(r,a_2;f)+S(r,f)$. Therefore we conclude that

$$T(r,f)=\overline{N}(r,a_1;f)+\overline{N}(r,a_2;f)+S(r,f). \tag{2.14}$$

Again from Lemma 2.3, we see that $\Delta(f)\not\equiv 0$ and so $\phi\not\equiv 0$. If possible suppose that $\Delta(f^{(k)})\equiv 0$. Then clearly we have

$$\frac{f^{(k+1)}-a_1'}{f^{(k)}-a_1}\equiv\frac{a_1'-a_2'}{a_1-a_2}.$$

On integration, we have $f^{(k)}\equiv a_1+a_0(a_1-a_2)$, where $a_0\in\mathbb{C}\setminus\{0\}$. This shows that $f^{(k)}\in\mathscr{S}(f)$. Since $T(r,f)=T(r,f^{(k)})+S(r,f)$, it follows that $f\in\mathscr{S}(f)$ which is a contradiction. Therefore $\Delta(f^{(k)})\not\equiv 0$ and so $\psi\not\equiv 0$. Let $z_{p,q}\in S_{(p,q)}(a_1)$ such that $a_1(z_{p,q}), a_1(z_{p,q})-a_2(z_{p,q})\neq 0,\infty$ and $a_1'(z_{p,q})-a_2'(z_{p,q})\neq 0$. Then from (2.1) and (2.9), we conclude that $z_{p,q}$ is a zero of $\phi(z)$ of multiplicity $t-1$, where $t\geq\min\{p,q\}\geq 1$ and so ϕ is holomorphic at $z_{p,q}$.

Let $\hat{z}_{p,q}\in S_{(p,q)}(a_2)$ such that $a_1(\hat{z}_{p,q}), a_1(\hat{z}_{p,q})-a_2(\hat{z}_{p,q})\neq 0,\infty$ and $a_1'(\hat{z}_{p,q})-a_2'(\hat{z}_{p,q})\neq 0$. Then, in the same way as above, one can easily prove that ϕ is also holomorphic at $\hat{z}_{p,q}$. As a result we have $N(r,\phi)=S(r,f)$. Also from Lemma 2.3, we get $m(r,\phi)=S(r,f)$ and so $\phi\in\mathscr{S}(f)$.

Denote by $\overline{N}(r,a_1;f,f^{(k)}\,|\geq 2)$ the reduced counting function of common multiple 0-points of $f-a_1$ and $f^{(k)}-a_1$. Since $\phi\in\mathscr{S}(f)$, it follows that

$$\overline{N}(r,a_1;f,f^{(k)}\,|\geq 2)\leq N(r,0;\phi)\leq S(r,f). \tag{2.15}$$

Similarly we have

$$\overline{N}(r,a_2;f,f^{(k)}\,|\geq 2)=S(r,f). \tag{2.16}$$

Denote by $N(r,0;f-f^{(k)}\,|\,f\neq a_1,a_2)$ the counting function of those 0-points of $f-f^{(k)}$ which are neither the 0-points of $f-a_1$ nor the 0-points of $f-a_2$. We denote by $\overline{N}_{(s+1}(r,0;f-f^{(k)}\,|\,f=a_1,a_2)$ the reduced counting function of those 0-points of $f - f^{(k)}$ with multiplicity greater than s which are the 0-points of both $f - a_1$ and $f - a_2$.

Now from (2.1), we can easily deduce that

$$\overline{N}_{(2}(r, 0; f - f^{(k)} \mid f = a_1, a_2) + N(r, 0; \Delta(f)(f - f^{(k)}) \mid f \neq a_1, a_2)$$
$$= S(r, f). \tag{2.17}$$

Let $a_3 = a_1 + l(a_1 - a_2)$, where "$l \in \mathbb{N}$", and let

$$F = \frac{f - a_1}{a_2 - a_1}. \tag{2.18}$$

Clearly $a_3 \not\equiv a_1, a_2$ and $T(r, F) = T(r, f) + S(r, f)$. Now using the second fundamental theorem and (2.14), we have

$$\begin{aligned}
2T(r, f) &= 2T(r, F) + S(r, f) \\
&\leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}(r, 1; F) + \overline{N}(r, -l; F) + S(r, f) \\
&\leq \overline{N}(r, a_1; f) + \overline{N}(r, a_2; f) + \overline{N}(r, a_3; f) + S(r, f) \\
&\leq 2T(r, f) - m(r, a_3; f) + S(r, f),
\end{aligned}$$

i.e., $m(r, a_3; f) = S(r, f)$. Also from (2.1), we see that

$$\frac{1}{f} = \frac{\Delta(f)}{\phi(f - a_1)(f - a_2)} \left(1 - \frac{f^{(k)}}{f}\right)$$

and so using Lemma 2.3, we get $m(r, 0; f) = S(r, f)$. Therefore we have

$$m(r, 0; f) = S(r, f) \qquad \text{and} \qquad m(r, a_3; f) = S(r, f), \tag{2.19}$$

where $a_3 = a_1 + l(a_1 - a_2)$, and $l \in \mathbb{N}$. Let $G = \frac{f^{(k)} - a_1}{a_2 - a_1}$. Clearly $T(r, G) = T(r, f^{(k)}) + S(r, f)$. Note that f and $f^{(k)}$ share a_1, a_2 IM and $T(r, f) = T(r, f^{(k)}) + S(r, f)$. Now from (2.14) and using the second fundamental theorem, we have

$$\begin{aligned}
2T(r, f^{(k)}) &= 2T(r, G) + S(r, f) \\
&\leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, 1; G) + \overline{N}(r, -l; G) + S(r, f) \\
&\leq \overline{N}(r, a_1; f^{(k)}) + \overline{N}(r, a_2; f^{(k)}) + \overline{N}(r, a_3; f^{(k)}) + S(r, f) \\
&\leq \overline{N}(r, a_1; f) + \overline{N}(r, a_2; f) + T(r, f^{(k)}) - m(r, a_3; f^{(k)}) + S(r, f) \\
&= T(r, f) + T(r, f^{(k)}) - m(r, a_3; f^{(k)}) + S(r, f)
\end{aligned}$$

$$= 2T(r, f^{(k)}) - m(r, a_3; f^{(k)}) + S(r, f),$$

$$\text{i.e., } m(r, a_3; f^{(k)}) = S(r, f). \tag{2.20}$$

Again from (2.19) and (2.20), we have

$$m\left(r, \frac{f^{(k)} - a_3}{f - a_3}\right) = m\left(r, \frac{f^{(k)} - a_3^{(k)} + a_3^{(k)} - a_3}{f - a_3}\right) \le m(r, a_3; f) + S(r, f)$$

$$= S(r, f),$$

$$\text{i.e., } m\left(r, \frac{f^{(k)} - a_3}{f - a_3}\right) = S(r, f). \tag{2.21}$$

Since $T(r, f) = T(r, f^{(k)}) + S(r, f)$, from Lemma 2.2, (2.19) and (2.21), we get

$$\begin{aligned}
m\left(r, \frac{f - a_3}{f^{(k)} - a_3}\right) &= T\left(r, \frac{f - a_3}{f^{(k)} - a_3}\right) - N\left(r, \frac{f - a_3}{f^{(k)} - a_3}\right) \\
&= T\left(r, \frac{f^{(k)} - a_3}{f - a_3}\right) - N\left(r, \frac{f - a_3}{f^{(k)} - a_3}\right) + O(1) \\
&= N\left(r, \frac{f^{(k)} - a_3}{f - a_3}\right) + m\left(r, \frac{f^{(k)} - a_3}{f - a_3}\right) \\
&\quad - N\left(r, \frac{f - a_3}{f^{(k)} - a_3}\right) + O(1) \\
&= N\left(r, \frac{f^{(k)} - a_3}{f - a_3}\right) - N\left(r, \frac{f - a_3}{f^{(k)} - a_3}\right) + S(r, f) \\
&= N(r, a_3; f) - N(r, a_3; f^{(k)}) + S(r, f) \\
&= N(r, a_3; f) + m(r, a_3; f) \\
&\quad - \{N(r, a_3; f^{(k)}) + m(r, a_3; f^{(k)})\} + S(r, f) \\
&= T(r, f) - T(r, f^{(k)}) + S(r, f) = S(r, f),
\end{aligned}$$

$$\text{i.e., } m\left(r, \frac{f - a_3}{f^{(k)} - a_3}\right) = S(r, f). \tag{2.22}$$

Since f and $f^{(k)}$ share a_1, a_2 IM, we can easily see that $N(r, \psi) = S(r, f)$. Note that

$$\psi = \frac{\Delta(f^{(k)})(f^{(k)} - a_3)}{(f^{(k)} - a_1)(f^{(k)} - a_2)} \left(\frac{f - a_3}{f^{(k)} - a_3} - 1\right)$$

and so using Lemma 2.3 and (2.22), we have

$$\begin{aligned}m(r,\psi) &\le m\left(r,\frac{\Delta(f^{(k)})(f^{(k)}-a_3)}{(f^{(k)}-a_1)(f^{(k)}-a_2)}\right)+m\left(r,\frac{f-a_3}{f^{(k)}-a_3}\right)+O(1)\\ &\le S(r,f),\end{aligned}$$

i.e., $m(r,\psi)=S(r,f)$. Consequently, $\psi\in\mathcal{S}(f)$. We now consider the following two cases.

Case 1. Suppose that $H_{nm}\equiv 0$. Then from (2.1) and (2.2), we have

$$n\left(\frac{f'-a_1'}{f-a_1}-\frac{f'-a_2'}{f-a_2}\right)\equiv m\left(\frac{f^{(k+1)}-a_1'}{f^{(k)}-a_1}-\frac{f^{(k+1)}-a_2'}{f^{(k)}-a_2}\right).$$

On integration, we have

$$\left(\frac{f-a_1}{f-a_2}\right)^n\equiv c_1\left(\frac{f^{(k)}-a_1}{f^{(k)}-a_2}\right)^m, \tag{2.23}$$

where $c_1\in\mathbb{C}\setminus\{0\}$. First we suppose that $n\ne m$. Then using Lemma 2.4, we get from (2.23) that $nT(r,f)=mT(r,f^{(k)})+S(r,f)$, which contradicts the fact that $T(r,f)=T(r,f^{(k)})+S(r,f)$. Next we suppose that $n=m$. Then from (2.23), we have

$$\frac{f-a_1}{f-a_2}\equiv c_2\frac{f^{(k)}-a_1}{f^{(k)}-a_2}, \tag{2.24}$$

where $c_2\in\mathbb{C}\setminus\{0\}$. If $c_2=1$, then from (2.24), we have $f\equiv f^{(k)}$, which is a contradiction. Hence $c_2\ne 1$. Now from (2.24), we get

$$\frac{1-c_2}{c_2}\,\frac{f-a_4}{f-a_2}\equiv\frac{a_2-a_1}{f^{(k)}-a_2}, \tag{2.25}$$

where $a_4=\frac{a_1-a_2c_2}{1-c_2}$ such that $a_4\not\equiv a_1,a_2$. Since $f\in\mathscr{E}$ and f, $f^{(k)}$ share a_2 IM, from (2.25), we conclude that $N(r,a_4;f)=S(r,f)$. Also we see that

$$\frac{f-a_1}{a_2-a_1}+\frac{c_2}{1-c_2}=\frac{f-a_4}{a_2-a_1}.$$

Now using the second fundamental theorem, we get from (2.14) and (2.18) that

$$\begin{aligned}2T(r,f)&=2T(r,F)+S(r,f)\\ &\le \overline{N}(r,0,F)+\overline{N}(r,1;F)+\overline{N}\left(r,-\frac{c_2}{1-c_2};F\right)+S(r,f)\\ &=\overline{N}(r,a_1;f)+\overline{N}(r,a_2;f)+S(r,f)\\ &=T(r,f)+S(r,f),\end{aligned}$$

which is a contradiction.

Case 2. Suppose that $H_{nm} \not\equiv 0$ for all $m, n \in \mathbb{N}$. Let $z_{m,n} \in S_{(m,n)}(a_1) \cup S_{(m,n)}(a_2)$ such that $a_1(z_{m,n}), a_2(z_{m,n}) \neq 0, \infty$ and $a_1(z_{m,n}) - a_2(z_{m,n}) \neq 0$. Now from (2.1) and (2.2), we see that

$$H_{nm} = (f - f^{(k)})\left[\left(n\frac{f' - a_2'}{f - a_2} - m\frac{f^{(k+1)} - a_2'}{f^{(k)} - a_2}\right) - \left(n\frac{f' - a_1'}{f - a_1} - m\frac{f^{(k+1)} - a_1'}{f^{(k)} - a_1}\right)\right],$$

and so $H_{nm}(z_{m,n}) = 0$. Therefore using the first fundamental theorem, we also have,

$$\begin{aligned}
&\overline{N}_{(m,n)}(r, a_1; f) + \overline{N}_{(m,n)}(r, a_2; f) \\
&\quad \leq N(r, 0; H_{nm}) + \sum_{i=1}^{2} N(r, 0; a_i) + \sum_{i=1}^{2} N(r, \infty; a_i) \\
&\qquad + N(r, 0; a_1 - a_2) + S(r, f) \\
&\quad \leq T(r, H_{nm}) + S(r, f) \leq T(r, \phi) + T(r, \psi) + S(r, f) = S(r, f). \qquad (2.26)
\end{aligned}$$

Finally from (2.14) and (2.26), we have

$$\begin{aligned}
T(r, f) &= \overline{N}(r, a_1; f) + \overline{N}(r, a_2; f) + S(r, f) \\
&= \sum_{m,n} (\overline{N}_{(m,n)}(r, a_1; f) + \overline{N}_{(m,n)}(r, a_2; f)) + S(r, f) \\
&= \sum_{m+n \geq 5} (\overline{N}_{(m,n)}(r, a_1; f) + \overline{N}_{(m,n)}(r, a_2; f)) + S(r, f) \\
&\leq \frac{1}{5}(N(r, a_1; f) + N(r, a_1; f^{(k)}) + N(r, a_2; f) + N(r, a_2; f^{(k)})) + S(r, f) \\
&\leq \frac{1}{5}(2T(r, f) + 2T(r, f^{(k)})) + S(r, f) = \frac{4}{5}T(r, f) + S(r, f),
\end{aligned}$$

which is impossible here. Hence $f \equiv f^{(k)}$.

3. Proof of the theorem

Proof (Proof of Theorem 1.3). If possible suppose that f is a non-constant polynomial. Since a small function of a polynomial is a constant, it follows that $a_1, a_2 \in \mathbb{C}$. This contradicts the fact that $a_1' - a_2' \notin \mathbb{C}$. Hence $f \in \mathscr{E}_T(\mathbb{C})$. Now we divide the following two cases.

Case 1. Suppose that $\phi \not\equiv 0$, where ϕ is defined by (2.1). Clearly $f \not\equiv f^{(k)}$. Now from the proof of Lemma 2.7, we see that $\phi \in \mathscr{S}(f)$. Also from (2.17), we have

$$\overline{N}_{(2}(r,0;f-f^{(k)}\,|\,f=a_1,a_2)+N(r,0;\Delta(f)(f-f^{(k)})\,|\,f\neq a_1,a_2)$$
$$=S(r,f). \tag{3.1}$$

On the other hand, from (2.15) and (2.16), we have

$$\overline{N}(r,a_1;f,f^{(k)}\,|\geq 2)+\overline{N}(r,a_2;f,f^{(k)}\,|\geq 2)=S(r,f). \tag{3.2}$$

Let ψ be defined by (2.2). Since $\Delta(f^{(k)})\not\equiv 0$, it follows that $\psi\not\equiv 0$. Now rewriting (2.1), we get

$$f'=\frac{\alpha_{1,2}f^2+\alpha_{1,1}f+\alpha_{1,0}+Q_1}{f-f^{(k)}}, \tag{3.3}$$

where $\alpha_{1,2}=\frac{a_1'-a_2'-\phi}{a_1-a_2}$, $\alpha_{1,1}=a_1'-a_1\frac{a_1'-a_2'}{a_1-a_2}+\frac{(a_1+a_2)\phi}{a_1-a_2}$, $\alpha_{1,0}=-\frac{\phi a_1a_2}{a_1-a_2}$ and $Q_1=-\frac{a_1'-a_2'}{a_1-a_2}ff^{(k)}-\left(a_1'-a_1\frac{a_1'-a_2'}{a_1-a_2}\right)f^{(k)}$. Now we divide the following two sub-cases.

Sub-case 1.1. Suppose that $\phi\not\equiv a_1'-a_2'$. Certainly $\alpha_{1,2}\not\equiv 0$. Now by induction and using (3.3) repeatedly, we obtain the following

$$f^{(k)}=\frac{\sum_{j=0}^{2k}\alpha_{k,j}f^j+Q_k}{(f-f^{(k)})^{2k-1}}, \tag{3.4}$$

where

$$Q_k=\sum_{\substack{l<2k\\ l+j_1+j_2+\cdots+j_k\leq 2k}}\beta_{l,j_1,j_2,\dots,j_k}f^l(f^{(k)})^{j_1}(f^{(k+1)})^{j_2}\dots(f^{(2k-1)})^{j_k}. \tag{3.5}$$

Here $\alpha_{k,j},\beta_{l,j_1,j_2,\dots,j_k}\in\mathscr{S}(f)$ and $\psi_i:=\alpha_{i,2i}$ satisfies the recurrence formula

$$\psi_1=\alpha_{1,2},\qquad \psi_{i+1}=\psi_i'+\psi_1\psi_i,\quad i=1,2,\dots,k-1. \tag{3.6}$$

From the recurrence formula (3.6) for ψ_i, we can easily derive the expression

$$\psi_k=\psi_1^k+Q(\psi_1), \tag{3.7}$$

where $Q(\psi_1)$ is a differential polynomial in ψ_1 with a degree less than or equal to $k-1$.

Now we divide the following two sub-cases.

Sub-case 1.1.1. Suppose that $\psi_k=\alpha_{k,2k}\not\equiv 0$. Then using (2.19) and the lemma of logarithmic derivative, we get from (3.5) that

$$m\left(r,\frac{Q_k}{f^{2k-1}f^{(k)}}\right)=S(r,f). \tag{3.8}$$

Again from (3.4), we have

$$\sum_{j=0}^{2k} \alpha_{k,j} f^j = f^{(k)}(f - f^{(k)})^{2k-1} - Q_k. \tag{3.9}$$

Now from Lemma 2.4, (3.8) and (3.9), we obtain that

$$\begin{aligned} 2kT(r,f) + S(r,f) &= T\left(r, \sum_{j=0}^{2k} \alpha_{k,j} f^j\right) \\ &\le m\left(r, \left(1 - \frac{f^{(k)}}{f}\right)^{2k-1} - \frac{Q_k}{f^{2k-1}f^{(k)}}\right) \\ &\quad + m(r, f^{2k-1}f^{(k)}) + S(r,f) \\ &\le m\left(r, \left(1 - \frac{f^{(k)}}{f}\right)^{2k-1}\right) + m\left(r, \frac{Q_k}{f^{2k-1}f^{(k)}}\right) \\ &\quad + m(r, f^{2k-1}) + m(r, f^{(k)}) + S(r,f) \\ &\le (2k-1)T(r,f) + T(r, f^{(k)}) + S(r,f), \end{aligned}$$

i.e., $T(r,f) \le T(r,f^{(k)}) + S(r,f)$. Since $f \in \mathscr{E}_T(\mathbb{C})$, we have $T(r,f^{(k)}) \le T(r,f) + S(r,f)$. Therefore $T(r,f) = T(r,f^{(k)}) + S(r,f)$. Consequently, from Lemma 2.7, one can see that $f \equiv f^{(k)}$, which is impossible here.

Sub-case 1.1.2. Suppose that $\psi_k = \alpha_{k,2k} \equiv 0$. Then from (3.6), we have $\psi'_{k-1} + \psi_1\psi_{k-1} \equiv 0$. On integration, we have $\psi_{k-1}(z) = c_0 e^{\xi(z)}$, where $\xi(z) = -\int_0^z \psi_1(z)\mathrm{d}z \in \mathscr{M}(\mathbb{C})$ and $c_0 \in \mathbb{C}\backslash\{0\}$. We know that if $\xi(z)$ has a pole at the point z_0, then z_0 is an essential singularity of $e^{\xi(z)}$. Since $\psi_{k-1} \in \mathscr{M}(\mathbb{C})$, it follows that $\xi \in \mathscr{E}(\mathbb{C})$ and so $\psi_1 \in \mathscr{E}(\mathbb{C})$. On the other hand, we see that if ψ_1 is a polynomial, then ψ_{i+1} is also a polynomial for $i = 1, 2, \ldots, k-1$. In that case we arrive at a contradiction. Hence $\psi_1 \in \mathscr{E}_T(\mathbb{C})$. Now from (3.7), we see that $\psi_1^k = -Q(\psi_1)$. Using Lemma 2.1, we evaluate that $m(r,\psi_1) = S(r,\psi_1)$. Since $N(r,\psi_1) = 0$, it follows that $T(r,\psi_1) = S(r,\psi_1)$, which is impossible.

Sub-case 1.2. Suppose that $\phi = a_1' - a_2'$. Now from (2.1), we have

$$\begin{aligned} f - f^{(k)} &= \frac{(a_1' - a_2')(f - a_1)(f - a_2)}{(a_1' - a_2')(f - a_1) - (a_1 - a_2)(f' - a_1')} \\ &= \frac{(a_1' - a_2')(f - a_1)(f - a_2)}{\Delta(f)}. \end{aligned} \tag{3.10}$$

Since $(a_1'-a_2')(f-a_1)-(a_1-a_2)(f'-a_1')=(a_1'-a_2')(f-a_2)-(a_1-a_2)\cdot(f'-a_2')$, from (3.10), we have

$$f^{(k)}-a_1=-\frac{(a_1-a_2)(f-a_1)(f'-a_2')}{\Delta(f)} \qquad \text{and}$$
$$f^{(k)}-a_2=-\frac{(a_1-a_2)(f-a_2)(f'-a_1')}{\Delta(f)}. \tag{3.11}$$

Also from (2.5), we have

$$\begin{aligned}
&(a_1''-a_2'')(f-a_1)^2-(a_1''-a_2'')(f-a_1)(f^{(k)}-a_1)\\
&\quad -(a_1-a_2)(f-a_1)(f''-a_1'')+(a_1-a_2)(f''-a_1'')(f^{(k)}-a_1)\\
&\quad +(a_1'-a_2')(f-a_1)(f'-a_1')-(a_1'-a_2')(f-a_1)(f^{(k+1)}-a_1')\\
&\quad -(a_1-a_2)(f'-a_1')^2+(a_1-a_2)(f'-a_1')(f^{(k+1)}-a_1')\\
&=(a_1''-a_2'')(f-a_1)(f-a_2)+(a_1'-a_2')(f'-a_1')(f-a_2)\\
&\quad +(a_1'-a_2')(f-a_1)(f'-a_2'). \qquad (3.12)
\end{aligned}$$

Let $z_{p,1}\in S_{(p,1)}(a_1)$ $(p\geq 2)$ such that $a_1(z_{p,1})-a_2(z_{p,1})\neq 0,\infty$ and $a_1'(z_{p,1})-a_2'(z_{p,1})\neq 0$. Clearly, from (2.10), we have

$$f^{(k+1)}(z_{p,1})-a_1'(z_{p,1})=c_1=\frac{a_1'(z_{p,1})-a_2'(z_{p,1})}{p}.$$

In some neighbourhood of $z_{p,1}$, it is easy to calculate, from (2.7), (2.8) and (3.12) that

$$\begin{aligned}
&(b_pc_2p^2(p+1)(a_1(z_{p,1})-a_2(z_{p,1}))-b_p(a_1'(z_{p,1})-a_2'(z_{p,1}))^2\\
&\quad +b_{p+1}(p+1)^2(a_1(z_{p,1})-a_2(z_{p,1}))(a_1'(z_{p,1})-a_2'(z_{p,1}))\\
&\quad -pb_p((a_1''(z_{p,1})-a_2''(z_{p,1}))(a_1(z_{p,1})-a_2(z_{p,1}))+(a_1'(z_{p,1})-a_2'(z_{p,1}))^2)\\
&\quad -b_{p+1}p(p+1)(a_1(z_{p,1})-a_2(z_{p,1}))(a_1'(z_{p,1})-a_2'(z_{p,1})))(z-z_{p,1})^p\\
&\quad +A_{p+1}(z-z_{p,1})^{p+1}+\cdots\equiv 0 \qquad (A_{p+1}\in\mathbb{C}),
\end{aligned}$$

which shows that

$$\begin{aligned}
&(b_pc_2p^2(p+1)(a_1(z_{p,1})-a_2(z_{p,1}))-b_p(a_1'(z_{p,1})-a_2'(z_{p,1}))^2\\
&\quad +b_{p+1}(p+1)^2(a_1(z_{p,1})-a_2(z_{p,1}))(a_1'(z_{p,1})-a_2'(z_{p,1}))
\end{aligned}$$

$$- pb_p((a_1''(z_{p,1}) - a_2''(z_{p,1}))(a_1(z_{p,1}) - a_2(z_{p,1})) + (a_1'(z_{p,1}) - a_2'(z_{p,1}))^2)$$
$$- b_{p+1}p(p+1)(a_1(z_{p,1}) - a_2(z_{p,1}))(a_1'(z_{p,1}) - a_2'(z_{p,1})) = 0. \tag{3.13}$$

On the other hand, from (2.6), we have

$$\begin{aligned}
&(a_1''' - a_2''')(f - a_1)^2 - (a_1''' - a_2''')(f - a_1)(f^{(k)} - a_1) \\
&\quad + (a_1'' - a_2'')(f - a_1)(f' - a_1') - (a_1'' - a_2'')(f' - a_1')(f^{(k)} - a_1) \\
&\quad - (a_1' - a_2')(f - a_1)(f'' - a_1'') + (a_1' - a_2')(f'' - a_1'')(f^{(k)} - a_1) \\
&\quad - (a_1 - a_2)(f - a_1)(f''' - a_1''') + (a_1 - a_2)(f''' - a_1''')(f^{(k)} - a_1) \\
&\quad + 2(a_1'' - a_2'')(f - a_1)(f' - a_1') - 2(a_1'' - a_2'')(f - a_1)(f^{(k+1)} - a_1') \\
&\quad - 2(a_1 - a_2)(f' - a_1')(f'' - a_1'') + (a_1' - a_2')(f - a_1)(f'' - a_1'') \\
&\quad + 2(a_1 - a_2)(f'' - a_1'')(f^{(k+1)} - a_1') - (a_1' - a_2')(f - a_1)(f^{(k+2)} - a_1'') \\
&\quad - (a_1 - a_2)(f' - a_1')(f'' - a_1'') + (a_1 - a_2)(f' - a_1')(f^{(k+2)} - a_1'') \\
&= (a_1''' - a_2''')(f - a_1)(f - a_2) + 2(a_1'' - a_2'')(f' - a_1')(f - a_2) \\
&\quad + 2(a_1'' - a_2'')(f - a_1)(f' - a_2') + (a_1' - a_2')(f'' - a_1'')(f - a_2) \\
&\quad + 2(a_1' - a_2')(f' - a_1')(f' - a_2') + (a_1' - a_2')(f - a_1)(f'' - a_2'').
\end{aligned} \tag{3.14}$$

In some neighbourhood of $z_{p,1}$, it is easy to calculate, from (2.7), (2.8) and (3.14) that

$$\begin{aligned}
&(b_p c_2 p^2(p+1)(a_1(z_{p,1}) - a_2(z_{p,1})) + b_p(p-1)(a_1'(z_{p,1}) - a_2'(z_{p,1}))^2 \\
&\quad + b_{p+1}(p+1)^2(a_1(z_{p,1}) - a_2(z_{p,1}))(a_1'(z_{p,1}) - a_2'(z_{p,1})) \\
&\quad - 2pb_p((a_1''(z_{p,1}) - a_2''(z_{p,1}))(a_1(z_{p,1}) - a_2(z_{p,1})) + (a_1'(z_{p,1}) - a_2'(z_{p,1}))^2) \\
&\quad - b_{p+1}p(p+1)(a_1(z_{p,1}) - a_2(z_{p,1}))(a_1'(z_{p,1}) - a_2'(z_{p,1})))(z - z_{p,1})^{p-1} \\
&\quad + B_p(z - z_{p,1})^p + \cdots \equiv 0 \qquad (B_p \in \mathbb{C}),
\end{aligned}$$

which shows that

$$\begin{aligned}
&b_p c_2 p^2(p+1)(a_1(z_{p,1}) - a_2(z_{p,1})) + b_p(p-1)(a_1'(z_{p,1}) - a_2'(z_{p,1}))^2 \\
&\quad + b_{p+1}(p+1)^2(a_1(z_{p,1}) - a_2(z_{p,1}))(a_1'(z_{p,1}) - a_2'(z_{p,1})) \\
&\quad - 2pb_p((a_1''(z_{p,1}) - a_2''(z_{p,1}))(a_1(z_{p,1}) - a_2(z_{p,1})) + (a_1'(z_{p,1}) - a_2'(z_{p,1}))^2) \\
&\quad - b_{p+1}p(p+1)(a_1(z_{p,1}) - a_2(z_{p,1}))(a_1'(z_{p,1}) - a_2'(z_{p,1})) = 0.
\end{aligned} \tag{3.15}$$

Now from (3.13) and (3.15), we have

$$b_p p(a_1(z_{p,1}) - a_2(z_{p,1}))(a_1''(z_{p,1}) - a_2''(z_{p,1})) = 0. \tag{3.16}$$

Since $b_p \neq 0$ and $a_1(z_{p,1}) - a_2(z_{p,1}) \neq 0, \infty$, from (3.16) we have $a_1''(z_{p,1}) - a_2''(z_{p,1}) = 0$.

Now since $a_1' - a_2' \notin \mathbb{C}$, it follows that

$$\sum_{p\geq 2} \overline{N}_{(p,1)}(r, a_1; f) \leq N(r, 0; a_1'' - a_2'') \leq S(r, f). \tag{3.17}$$

Let $\hat{z}_{p,1} \in S_{(p,1)}(a_2)$ $(p \geq 2)$ such that $a_1(\hat{z}_{p,1}) - a_2(\hat{z}_{p,1}) \neq 0, \infty$ and $a_1'(\hat{z}_{p,1}) - a_2'(\hat{z}_{p,1}) \neq 0$. Clearly from (2.12), we have

$$f^{(k+1)}(\hat{z}_{p,1}) - a_2'(\hat{z}_{p,1}) = \hat{c}_1 = -\frac{a_1'(\hat{z}_{p,1}) - a_2'(\hat{z}_{p,1})}{p}.$$

Now proceeding in the same way as done above and using (2.11) instead of (2.7), one can easily deduce that $a_1''(\hat{z}_{p,1}) - a_2''(\hat{z}_{p,1}) = 0$ and so

$$\sum_{p\geq 2} \overline{N}_{(p,1)}(r, a_2; f) \leq N(r, 0; a_1'' - a_2'') \leq S(r, f). \tag{3.18}$$

Therefore from (3.2), (3.17) and (3.18), we see that

$$\begin{aligned} \overline{N}_{(2}(r, a_1; f) + \overline{N}_{(2}(r, a_2; f) &\leq \sum_{p\geq 2} (\overline{N}_{(p,1)}(r, a_1; f) + \overline{N}_{(p,1)}(r, a_2; f)) \\ &= S(r, f). \end{aligned} \tag{3.19}$$

Let $l \in \mathbb{N}$ and $a \in \mathbb{C} \cup \{\infty\}$. We use $N_{(l+1}(r, a; f)$ to denote the counting function of a-points of f with multiplicity greater than l. Similarly, $\overline{N}_{(l+1}(r, a; f)$ is its reduced function. Now we divide the following three sub-cases.

Sub-case 1.2.1. Suppose that

$$\overline{N}_{(2}(r, a_1; f^{(k)}) = S(r, f) \qquad \text{and} \qquad \overline{N}_{(2}(r, a_2; f^{(k)}) = S(r, f).$$

Then from (3.2), one can easily obtain that

$$\sum_{q\geq 2} (\overline{N}_{(1,q)}(r, a_1; f) + \overline{N}_{(1,q)}(r, a_2; f)) = S(r, f). \tag{3.20}$$

Now from (2.14), (3.19) and (3.20), we deduce that

$$\begin{aligned} T(r,f) &= \overline{N}(r,a_1;f) + \overline{N}(r,a_2;f) + S(r,f) \\ &= \sum_{i=1}^{2} \overline{N}_{(1,1)}(r,a_i;f) + \sum_{p\geq 2}(\overline{N}_{(p,1)}(r,a_1;f) + \overline{N}_{(p,1)}(r,a_2;f)) \\ &\quad + \sum_{q\geq 2}(\overline{N}_{(1,q)}(r,a_1;f) + \overline{N}_{(1,q)}(r,a_2;f)) + S(r,f) \\ &= \overline{N}_{(1,1)}(r,a_1;f) + \overline{N}_{(1,1)}(r,a_2;f) + S(r,f). \end{aligned} \tag{3.21}$$

Let $\alpha \in \mathscr{S}(f)$ be arbitrary. Now using the first fundamental theorem, we get from (2.2), Lemmas 2.2 and 2.3 that

$$\begin{aligned} m(r,\psi) &= m\left(r, \frac{\Delta(f^{(k)})(f^{(k)}-\alpha)}{(f^{(k)}-a_1)(f^{(k)}-a_2)}\left(\frac{f-\alpha}{f^{(k)}-\alpha}-1\right)\right) \\ &\leq m\left(r, \frac{\Delta(f^{(k)})(f^{(k)}-\alpha)}{(f^{(k)}-a_1)(f^{(k)}-a_2)}\right) + m\left(r, \frac{f-\alpha}{f^{(k)}-\alpha}-1\right) \\ &\leq m\left(r, \frac{f-\alpha}{f^{(k)}-\alpha}\right) + S(r,f) \\ &= T\left(r, \frac{f^{(k)}-\alpha}{f-\alpha}\right) - N\left(r, \frac{f-\alpha}{f^{(k)}-\alpha}\right) + S(r,f) \\ &= m\left(r, \frac{f^{(k)}-\alpha}{f-\alpha}\right) + N\left(r, \frac{f^{(k)}-\alpha}{f-\alpha}\right) - N\left(r, \frac{f-\alpha}{f^{(k)}-\alpha}\right) + S(r,f) \\ &\leq m\left(r, \frac{f^{(k)}-\alpha^{(k)}}{f-\alpha}\right) + m\left(r, \frac{\alpha^{(k)}-\alpha}{f-\alpha}\right) \\ &\quad + N(r,\alpha;f) - N(r,\alpha;f^{(k)}) + S(r,f) \\ &\leq m(r,\alpha;f) + N(r,\alpha;f) - N(r,\alpha;f^{(k)}) + S(r,f) \\ &= T(r,f) - N(r,\alpha;f^{(k)}) + S(r,f). \end{aligned}$$

Under the given conditions, we have $N(r,\psi) = S(r,f)$. Consequently we have

$$T(r,\psi) \leq T(r,f) - N(r,\alpha;f^{(k)}) + S(r,f). \tag{3.22}$$

Now we consider the following two sub-cases.

Sub-case 1.2.1.1. Suppose that $H_{11} \equiv 0$. Then immediately we have $T(r,f) = T(r,f^{(k)}) + S(r,f)$ and so by Lemma 2.5, we get $f \equiv f^{(k)}$, which is impossible.

Sub-case 1.2.1.2. Suppose that $H_{11} \not\equiv 0$. Let $z_{1,1} \in S_{(1,1)}(a_1) \cup S_{(1,1)}(a_2)$. Then it is easy to obtain that $H_{11}(z_{1,1}) = 0$ and so we conclude that

$$\sum_{i=1}^{2} \overline{N}_{(1,1)}(r, a_i; f) \leq N(r, 0; H_{11}) + S(r, f)$$

$$\leq T(r, H_{11}) + S(r, f) \leq T(r, \psi) + S(r, f). \tag{3.23}$$

Then from (3.21), (3.22) and (3.23), we have

$$T(r, f) \leq T(r, \psi) + S(r, f) \leq T(r, f) - N(r, \alpha; f^{(k)}) + S(r, f),$$

$$\text{i.e., } N(r, \alpha; f^{(k)}) = S(r, f),$$

where $\alpha \in \mathscr{S}(f)$ is arbitrary. In particular we have $\overline{N}(r, a_1; f^{(k)}) + \overline{N}(r, a_2; f^{(k)}) = S(r, f)$. Since f and $f^{(k)}$ share a_1 and a_2 IM, we have $\overline{N}(r, a_1; f) + \overline{N}(r, a_2; f) = S(r, f)$ and so $\overline{N}_{(1,1)}(r, a_1; f) + \overline{N}_{(1,1)}(r, a_2; f) = S(r, f)$. Therefore from (3.21), we arrive at a contradiction.

Sub-case 1.2.2. Suppose that

$$\text{either } \overline{N}_{(2}(r, a_1; f^{(k)}) = S(r, f) \text{ and } \overline{N}_{(2}(r, a_2; f^{(k)}) \neq S(r, f)$$

$$\text{or } \overline{N}_{(2}(r, a_1; f^{(k)}) \neq S(r, f) \text{ and } \overline{N}_{(2}(r, a_2; f^{(k)}) = S(r, f).$$

Without loss of generality we may assume that $\overline{N}_{(2}(r, a_1; f^{(k)}) = S(r, f)$ and $\overline{N}_{(2}(r, a_2; f^{(k)}) \neq S(r, f)$.

Now from (3.2), one can easily show that

$$\sum_{q \geq 2} \overline{N}_{(1,q)}(r, a_1; f) = S(r, f). \tag{3.24}$$

Consequently from (3.19) and (3.24), we deduce that

$$\overline{N}(r, a_1; f) = \sum_{p,q} \overline{N}_{(p,q)}(r, a_1; f)$$

$$= \overline{N}_{(1,1)}(r, a_1; f) + \sum_{p \geq 2} \overline{N}_{(p,1)}(r, a_1; f) + \sum_{q \geq 2} \overline{N}_{(1,q)}(r, a_1; f)$$

$$= \overline{N}_{(1,1)}(r, a_1; f) + S(r, f). \tag{3.25}$$

Let

$$\varphi_1 = \frac{f - f^{(k)}}{(f - a_1)(f - a_2)}. \tag{3.26}$$

Clearly $\varphi_1 \not\equiv 0$. Since $\phi = a_1' - a_2'$, from (2.1) and (3.26), we have

$$(a_1' - a_2')\left(f - a_1 - \frac{1}{\varphi_1}\right) = (a_1 - a_2)(f' - a_1'). \tag{3.27}$$

Let $\hat{z}_{1,q} \in S_{(1,q)}(a_2)$ $(q \geq 2)$ such that $a_1(\hat{z}_{1,q}) - a_2(\hat{z}_{1,q}) \neq 0, \infty$, $a_1'(\hat{z}_{1,q}) - a_2'(\hat{z}_{1,q}) \neq 0$ and $\varphi_1(\hat{z}_{1,q}) \neq 0, \infty$. Clearly $\Delta(f(\hat{z}_{1,q})) \neq 0$ and so from (3.11), we conclude that $\hat{z}_{1,q}$ is a zero of $f' - a_1'$ of multiplicity $q-1$. On the other hand, from (3.27), we conclude that $\hat{z}_{1,q}$ is a zero of $f - a_1 - \frac{1}{\varphi_1}$, i.e., $f(\hat{z}_{1,q}) = a_1(\hat{z}_{1,q}) + \frac{1}{\varphi_1(\hat{z}_{1,q})}$. Also since f and $f^{(k)}$ share a_2 IM, it follows that $f(\hat{z}_{1,q}) = a_2(\hat{z}_{1,q})$. Consequently we have $\varphi_1(\hat{z}_{1,q}) = \frac{1}{a_2(\hat{z}_{1,q}) - a_1(\hat{z}_{1,q})}$.

We claim that $\varphi_1 \notin \mathbb{C}$. If possible suppose that $\varphi_1 \in \mathbb{C}\backslash\{0\}$. Then we have $\varphi_1 = \frac{1}{a_2(\hat{z}_{1,q}) - a_1(\hat{z}_{1,q})}$. If $\varphi_1 \not\equiv \frac{1}{a_2 - a_1}$, then

$$\sum_{q \geq 2} \overline{N}_{(1,q)}(r, a_2; f) \leq N\left(r, 0; \varphi_1 - \frac{1}{a_2 - a_1}\right) \leq S(r, f). \tag{3.28}$$

Also from (3.2), we have $\overline{N}(r, a_2; f, f^{(k)} \,|\geq 2) = S(r, f)$ and so from (3.28), we conclude that $\overline{N}_{(2}(r, a_2; f^{(k)}) = S(r, f)$, which is impossible. Hence $\varphi_1 \equiv \frac{1}{a_2 - a_1}$. This shows that $a_2 - a_1 \in \mathbb{C}$, which is again impossible. Hence $\varphi_1 \notin \mathbb{C}$. Let $\phi_1 = \frac{\varphi_1'}{\varphi_1}$. Clearly $\phi_1 \not\equiv 0$.

Let $z_{1,q} \in S_{(1,q)}(a_1)$ such that $a_1(z_{1,q}) - a_2(z_{1,q}) \neq 0, \infty$. Then $z_{1,q}$ is a zero of $f - f^{(k)}$. Consequently $\varphi_1(z_{1,q}) \neq \infty$. Similarly if $\hat{z}_{1,q} \in S_{(1,q)}(a_2)$ such that $a_1(\hat{z}_{1,q}) - a_2(\hat{z}_{1,q}) \neq 0, \infty$, then $\varphi_1(\hat{z}_{1,q}) \neq \infty$. Now from (3.1), (3.2) and (3.19), we conclude that $\overline{N}(r, 0; \varphi_1) + N(r, \varphi_1) = S(r, f)$. Then $\phi_1 \in \mathscr{S}(f)$. Now by logarithmic differentiation, we get from (3.26) that

$$\phi_1 = \frac{f' - f^{(k+1)}}{f - f^{(k)}} - \frac{f' - a_1'}{f - a_1} - \frac{f' - a_2'}{f - a_2}, \qquad \text{i.e.,}$$

$$\begin{aligned}
&\phi_1(f - a_1)^2(f - a_2) - \phi_1(f - a_1)(f - a_2)(f^{(k)} - a_1) \\
&\quad = -(f - a_1)(f - a_2)(f^{(k+1)} - a_1') + (f' - a_1')(f - a_2)(f^{(k)} - a_1) \\
&\qquad + (f - a_1)(f' - a_2')(f^{(k)} - a_1) - (f - a_1)^2(f' - a_2')
\end{aligned} \tag{3.29}$$

or

$$\begin{aligned}
&\phi_1(f - a_1)(f - a_2)^2 - \phi_1(f - a_1)(f - a_2)(f^{(k)} - a_2) \\
&\quad = -(f - a_1)(f - a_2)(f^{(k+1)} - a_2') + (f' - a_1')(f - a_2)(f^{(k)} - a_2) \\
&\qquad + (f - a_1)(f' - a_2')(f^{(k)} - a_2) - (f' - a_1')(f - a_2)^2.
\end{aligned} \tag{3.30}$$

Let $\hat{z}_{1,q} \in S_{(1,q)}(a_2)$ $(q \geq 3)$ such that $a_1(\hat{z}_{1,q}) - a_2(\hat{z}_{1,q}) \neq 0, \infty$ and $a_1'(\hat{z}_{1,q}) - a_2'(\hat{z}_{1,q}) \neq 0$. Clearly $\Delta(f(\hat{z}_{1,q})) \neq 0$ and so from (3.11), we conclude that $\hat{z}_{1,q}$ is a zero of $f' - a_1'$ of multiplicity $q-1$. Then from (3.30), it is easy to calculate $\phi_1(\hat{z}_{1,q}) = 0$. Since $\phi_1 \not\equiv 0$, it follows that

$$\sum_{q \geq 3} \overline{N}_{(1,q)}(r, a_2; f) \leq N(r, 0; \phi_1) \leq S(r, f). \tag{3.31}$$

Now from (3.2) and (3.31), we can easily conclude that

$$\overline{N}_{(3}(r, a_2; f^{(k)}) = S(r, f). \tag{3.32}$$

Now we consider the following two sub-cases.

Sub-case 1.2.2.1. Suppose that $H \equiv 0$. Then on integration, we have

$$\frac{f - a_1}{f - a_2} \equiv c_2 \frac{f^{(k)} - a_1}{f^{(k)} - a_2},$$

where $c_2 \in \mathbb{C}\backslash\{0\}$. Now by Lemma 2.4, we conclude that $T(r, f) = T(r, f^{(k)}) + S(r, f)$ and so by Lemma 2.5, we have $f \equiv f^{(k)}$, which is impossible.

Sub-case 1.2.2.2. Suppose that $H \not\equiv 0$. It is easy to obtain that $m(r, H) = S(r, f)$ and $N(r, H) = \overline{N}_{(1,2)}(r, a_2; f)$. Therefore

$$T(r, H) = m(r, H) + N(r, H) = \overline{N}_{(1,2)}(r, a_2; f) + S(r, f). \tag{3.33}$$

Now from (2.13) and (2.14), we have $T(r, f) = m(r, f - f^{(k)}) + S(r, f)$. Therefore from (2.1), (2.2), (3.22) and (3.33), we have

$$\begin{aligned} T(r, f) &= m\left(r, \frac{H(f - f^{(k)})}{H}\right) + S(r, f) \\ &= m\left(r, \frac{\phi - \psi}{H}\right) + S(r, f) \\ &= T\left(r, \frac{H}{\phi - \psi}\right) + S(r, f) \\ &\leq T(r, \psi) + T(r, H) + S(r, f) \\ &\leq T(r, f) + \overline{N}_{(1,2)}(r, a_2; f) - N(r, \alpha; f^{(k)}) + S(r, f) \end{aligned}$$

and so

$$N(r, \alpha; f^{(k)}) \leq \overline{N}_{(1,2)}(r, a_2; f) + S(r, f). \tag{3.34}$$

Suppose that $\alpha = a_2$. Then from (3.34), we have

$$\overline{N}(r, a_2; f^{(k)}) \le \overline{N}_{(1,2)}(r, a_2; f) + S(r, f). \tag{3.35}$$

Since f and $f^{(k)}$ share a_2 IM, from (3.2) and (3.19), it follows that

$$\overline{N}(r, a_2; f^{(k)}) + S(r, f) = \overline{N}(r, a_2; f) + S(r, f) = \sum_{i=1}^{2} \overline{N}_{(1,i)}(r, a_2; f) + S(r, f),$$

and so from (3.35), we conclude that $\overline{N}_{(1,1)}(r, a_2; f) = S(r, f)$. Let

$$G_1 = \frac{f^{(k+1)} - a_1'}{f^{(k)} - a_1} - \frac{f' - a_1'}{f - a_1} - \frac{a_1' - a_2'}{a_1 - a_2}. \tag{3.36}$$

If $G_1 \equiv 0$, then on integration we have $f^{(k)} - a_1 = c_3(a_1 - a_2)(f - a_1)$, where $c_3 \in \mathbb{C}\backslash\{0\}$, and so by Lemma 2.4, we get $T(r, f) = T(r, f^{(k)}) + S(r, f)$. Now by Lemma 2.7, we conclude that $f \equiv f^{(k)}$, which is impossible here. Hence $G_1 \not\equiv 0$. Also from (3.25), it is easy to prove that $N(r, G_1) = S(r, f)$. Since $m(r, G_1) = S(r, f)$, it follows that $G_1 \in \mathscr{S}(f)$.

Let $\hat{z}_{1,2} \in S_{(1,2)}(a_2)$ such that $a_1(\hat{z}_{1,2}) - a_2(\hat{z}_{1,2}) \ne 0, \infty$, $a_1'(\hat{z}_{1,2}) - a_2'(\hat{z}_{1,2}) \ne 0$ and $\phi(\hat{z}_{1,2}) \ne 0, \infty$. Then $f^{(k)}(\hat{z}_{1,2}) = a_2(\hat{z}_{1,2})$ and $f^{(k+1)}(\hat{z}_{1,2}) = a_2'(\hat{z}_{1,2})$. Also from (3.11), one can easily conclude that $\hat{z}_{1,2}$ is a simple zero of $f' - a_1'$, i.e., $f'(\hat{z}_{1,2}) = a_1'(\hat{z}_{1,2})$.

Now from (3.36), we conclude that $G_1(\hat{z}_{1,2}) = 0$ and so $\overline{N}_{(1,2)}(r, a_2; f) \le N(r, 0; G_1) + S(r, f) \le S(r, f)$. Consequently, $\overline{N}_{(2}(r, a_2; f) = S(r, f)$, which is impossible.

Sub-case 1.2.3. Suppose that

$$\overline{N}_{(2}(r, a_1; f^{(k)}) \ne S(r, f) \qquad \text{and} \qquad \overline{N}_{(2}(r, a_2; f^{(k)}) \ne S(r, f).$$

Let $z_{1,q} \in S_{(1,q)}(a_1)$ $(q \ge 3)$ such that $a_1(z_{1,q}) - a_2(z_{1,q}) \ne 0, \infty$ and $a_1'(z_{1,q}) - a_2'(z_{1,q}) \ne 0$. Clearly, $\Delta(f(z_{1,q})) \ne 0$ and so from (3.11), we conclude that $z_{1,q}$ is a zero of $f' - a_2'$ of multiplicity $q - 1$. Now from (3.29), we calculate that $\phi_1(z_{1,q}) = 0$. Since $\phi_1 \not\equiv 0$, we get

$$\sum_{q \ge 3} \overline{N}_{(1,q)}(r, a_1; f) \le N(r, 0; \phi_1) \le S(r, f). \tag{3.37}$$

Now, from (3.2) and (3.37), we can easily conclude that

$$\overline{N}_{(3}(r, a_1; f^{(k)}) = S(r, f). \tag{3.38}$$

Also, from (3.32), we have

$$\overline{N}_{(3}(r, a_2; f^{(k)}) = S(r, f). \tag{3.39}$$

In this case from, (2.14), we have

$$T(r,f) = \sum_{i=1}^{2}(\overline{N}_{(1,1)}(r,a_i;f) + \overline{N}_{(1,2)}(r,a_i;f)) + S(r,f). \tag{3.40}$$

From Sub-case 1.2.1.1, we conclude that $H_{11} \not\equiv 0$. Let $z_{1,1} \in S_{(1,1)}(a_1) \cup S_{(1,1)}(a_2)$. Then from (3.22) and (3.23), we see that

$$\sum_{i=1}^{2} \overline{N}_{(1,1)}(r,a_i;f) \le T(r,f) - N(r,\alpha;f^{(k)}) + S(r,f),$$

where $\alpha \in \mathscr{S}(f)$ is arbitrary, and so from (3.40), we have

$$N(r,\alpha;f^{(k)}) \le \sum_{i=1}^{2} \overline{N}_{(1,2)}(r,a_i;f) + S(r,f),$$

$$\text{i.e., } \overline{N}(r,\alpha;f^{(k)}) \le \sum_{i=1}^{2} \overline{N}_{(1,2)}(r,a_i;f) + S(r,f). \tag{3.41}$$

Suppose $\alpha = a_1$. Since

$$\overline{N}(r,a_1;f^{(k)}) = \overline{N}(r,a_1;f) = \overline{N}_{(1,1)}(r,a_1;f) + \overline{N}_{(1,2)}(r,a_1;f) + S(r,f),$$

from (3.41), we conclude that $\overline{N}_{(1,1)}(r,a_1;f) \le N_{(1,2)}(r,a_2;f) + S(r,f)$. Again if we take $\alpha = a_2$, then from (3.41), we can easily deduce that

$$\overline{N}_{(1,1)}(r,a_2;f) \le N_{(1,2)}(r,a_1;f) + S(r,f).$$

Consequently from (3.40), we have

$$T(r,f) \le 2\sum_{i=1}^{2} \overline{N}_{(1,2)}(r,a_i;f) + S(r,f). \tag{3.42}$$

Now we divide the following two sub-cases.

Sub-case 1.2.3.1. Suppose that $H_{21} \equiv 0$. Then on integration, we have

$$\left(\frac{f-a_1}{f-a_2}\right)^2 = c_1 \frac{f^{(k)}-a_1}{f^{(k)}-a_2},$$

where $c_1 \in \mathbb{C}\setminus\{0\}$. Now using Lemma 2.4, we deduce that $2T(r,f) = T(r,f^{(k)}) + S(r,f)$. Since $f \in \mathscr{E}_T(\mathbb{C})$, it follows that $T(r,f^{(k)}) \le T(r,f) + S(r,f)$. Consequently we have $T(r,f) = S(r,f)$, which is impossible.

Sub-case 1.2.3.2. Suppose that $H_{21} \not\equiv 0$. Let $z_{1,2} \in S_{(1,2)}(a_1) \cup S_{(1,2)}(a_2)$. Then it is easy to obtain that $H_{21}(z_{1,2}) = 0$ and so we conclude that

$$\sum_{i=1}^{2} \overline{N}_{(1,2)}(r, a_i; f) \le N(r, 0; H_{21}) + S(r, f)$$
$$\le T(r, H_{21}) + S(r, f)$$
$$\le T(r, \psi) + S(r, f). \tag{3.43}$$

Now from (3.22) and (3.43), we see that

$$\sum_{i=1}^{2} \overline{N}_{(1,2)}(r, a_i; f) \le T(r, f) - N(r, \alpha; f^{(k)}) + S(r, f), \tag{3.44}$$

where $\alpha \in \mathscr{S}(f)$ is arbitrary. In particular from (3.44), we have

$$\sum_{i=1}^{2} \overline{N}_{(1,2)}(r, a_i; f) \le T(r, f) - N(r, a_1; f^{(k)}) + S(r, f) \tag{3.45}$$

and $$\sum_{i=1}^{2} \overline{N}_{(1,2)}(r, a_i; f) \le T(r, f) - N(r, a_2; f^{(k)}) + S(r, f). \tag{3.46}$$

Adding (3.45) and (3.46), we have

$$2\sum_{i=1}^{2} \overline{N}_{(1,2)}(r, a_i; f) \le 2T(r, f) - N(r, a_1; f^{(k)}) - N(r, a_2; f^{(k)}) + S(r, f). \tag{3.47}$$

Now using (3.42), from (3.47) we get

$$N(r, a_1; f^{(k)}) + N(r, a_2; f^{(k)}) \le T(r, f) + S(r, f). \tag{3.48}$$

Again, from (3.42) and (3.48), we conclude that

$$N(r, a_1; f^{(k)}) + N(r, a_2; f^{(k)}) \le 2\sum_{i=1}^{2} \overline{N}_{(1,2)}(r, a_i; f) + S(r, f). \tag{3.49}$$

Note that

$$\sum_{1=1}^{2} (\overline{N}_{(1,1)}(r, a_i; f) + 2\overline{N}_{(1,2)}(r, a_i; f)) \le N(r, a_1; f^{(k)}) + N(r, a_2; f^{(k)}) + S(r, f),$$

and so from (3.49), we conclude that

$$\sum_{1=1}^{2} \overline{N}_{(1,1)}(r, a_i; f) = S(r, f). \tag{3.50}$$

Now, from (3.40) and (3.50), we deduce that

$$\begin{aligned} T(r,f) &= \overline{N}(r,a_1;f) + \overline{N}(r,a_2;f) + S(r,f) \\ &= \overline{N}_{(1,2)}(r,a_1;f) + \overline{N}_{(1,2)}(r,a_2;f) + S(r,f) \\ &\le \overline{N}_{(2}(r,a_1;f^{(k)}) + \overline{N}_{(2}(r,a_2;f^{(k)}) + S(r,f) \\ &\le \frac{1}{2}(N(r,a_1;f^{(k)}) + N(r,a_2;f^{(k)})) + S(r,f) \\ &\le T(r,f^{(k)}) + S(r,f). \end{aligned} \tag{3.51}$$

Since $f \in \mathscr{E}_T(\mathbb{C})$, it follows that $T(r,f^{(k)}) \le T(r,f) + S(r,f)$ and so from (3.51), we conclude that $T(r,f) = T(r,f^{(k)}) + S(r,f)$. Consequently from Lemma 2.7, one can conclude that $f \equiv f^{(k)}$, which is impossible here.

Case 2. Suppose that $\phi \equiv 0$. Since $\Delta(f) \not\equiv 0$, it follows that $f \equiv f^{(k)}$. This completes the proof.

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