Hiroshima Math. J., **53** (2023), 171–189
doi:10.32917/h2022005

Asymptotic forms of solutions of half-linear ordinary differential equations with integrable perturbations

Sokea Luey and Hiroyuki Usami

(Received March 17, 2022)
(Revised August 26, 2022)

Abstract. We consider perturbed half-linear ordinary differential equations near ∞. We clarify asymptotic forms of the nontrivial solutions as $t \to \infty$ under p-th power integrability conditions imposed on the perturbations. Generalized Riccati equations associated with the half-linear equations under consideration are employed to prove the main results.

1. Introduction and statements of main results

Let us consider the quasilinear ordinary differential equations of the form

$$(|u'|^{\alpha-1}u')' = \alpha(1 + b(t))|u|^{\alpha-1}u, \qquad t \ge t_0 \ge 0, \tag{1}$$

where $\alpha > 0$ is a constant, and $b(t)$ is a continuous function defined on $[t_0, \infty)$.

The objective of this paper is to determine asymptotic forms of nontrivial solutions of equation (1).

When $\alpha = 1$, that is, when equation (1) is reduced into the linear equation

$$u'' = (1 + b(t))u, \qquad t \ge t_0, \tag{2}$$

such problems have been investigated since the middle of the twentieth century by many mathematicians. Roughly speaking, if $b(t)$ is small near ∞ in some sense, it can be shown that every nontrivial solution of (2) satisfies

$$u(t) \sim ce^t \qquad \text{or} \qquad u(t) \sim ce^{-t} \qquad \text{as } t \to \infty$$

for some constant $c \neq 0$; see, for example, [1, 3] in details. (Here and in the sequel the symbol "$f(t) \sim g(t)$ as $t \to \infty$" means that $\lim_{t\to\infty} f(t)/g(t) = 1$ for functions $f(t)$ and $g(t)$ defined near ∞.)

This work was supported by JSPS KAKENHI Grant number 21K03307.
2020 *Mathematics Subject Classification*. Primary 34E10; Secondary 34C11.
Key words and phrases. Half-linear equation, asymptotic form.

Recently the authors have tried to extend such results to the half-linear equation (1), and obtained the following theorem:

Theorem 1.0 ([5]). *Suppose that*

$$\lim_{t\to\infty} b(t) = 0 \qquad \text{and} \qquad \int^{\infty} |b(t)|dt < \infty.$$

Then, every nontrivial solution u *of* (1) *satisfies*

$$u(t) \sim ce^{t} \qquad \text{or} \qquad u(t) \sim ce^{-t} \qquad \text{as } t \to \infty$$

for some constant $c \neq 0$.

However, when $\alpha = 1$, it is well known that Theorem 1.0 can be proved only under the condition $\int^{\infty} |b(t)|dt < \infty$. Therefore we conjecture that the assumptions of Theorem 1.0 can be weakened. In this paper, we give an affirmative answer to this conjecture.

Let us consider equation (1) under the following conditions:

(A_1) $\alpha > 0$ is a constant;

(A_2) $b(t)$ is a continuous function on $[t_0, \infty)$, $t_0 \geq 0$;

(A_3) $b(t)$ belongs to $L^p[t_0, \infty)$ for some $p > 1$:

$$\|b\|_{L^p[t_0,\infty)} \equiv \left(\int_{t_0}^{\infty} |b(t)|^p dt\right)^{1/p} < \infty. \tag{3}$$

We always assume (A_1)–(A_3) in this paper without further mention. A C^1-function $u(t)$ defined near ∞ is called a solution of (1) if $|u'|^{\alpha-1}u'$ is of class C^1 and (1) holds. It should be noted that every local solution of (1) can be prolonged to a global solution existing on $[t_0, \infty)$; see for example [2, Chapter 1].

We will show as a first step

Theorem 1.1. *Every nontrivial solution of* (1) *is of constant sign near* ∞.

A function $u(t)$ is a solution of (1) if so is $-u(t)$. Therefore by Theorem 1.1 we may treat mainly eventually positive solutions without loss of generality.

Our main results are as follows:

Theorem 1.2. *Every positive solution* u *of* (1) *satisfies exactly one of the following two asymptotic properties:*

(i) $u' \sim u$ *as* $t \to \infty$;

(ii) $u' \sim -u$ *as* $t \to \infty$.

For $p \in (1, 2]$, we can get asymptotic forms of positive solutions:

Theorem 1.3. *Let* $1 < p \leq 2$. *Then every positive solution u of* (1) *has the asymptotic form either*

$$u(t) = c \exp\left(t + \frac{1}{\alpha+1}\int^t b(s)ds + o(1)\right) \qquad \text{as } t \to \infty; \tag{4}$$

or

$$u(t) = c \exp\left(-t - \frac{1}{\alpha+1}\int^t b(s)ds + o(1)\right) \qquad \text{as } t \to \infty, \tag{5}$$

where $c > 0$ *is a constant.*

Remark 1.4. As will be shown in Proposition 2.2, surely there are positive solutions u, respectively, satisfying the properties (i) and (ii) of Theorem 1.2, and (4) and (5).

Example 1.5. Let us consider the equation

$$(|u'|^{\alpha-1}u')' = \alpha\left(1 + \frac{1}{t^\sigma}\right)|u|^{\alpha-1}u, \qquad t \geq 1, \tag{6}$$

where $\sigma > 0$ is a constant.

(i) By Theorem 1.2, every positive solution u of (6) satisfies either $u' \sim u$ or $u' \sim -u$ as $t \to \infty$.

(ii) Let $1/2 < \sigma < 1$. By Theorem 1.3, every positive solution u of (6) satisfies either

$$u(t) = c \exp\left(t + \frac{t^{1-\sigma}}{(\alpha+1)(1-\sigma)} + o(1)\right) \qquad \text{as } t \to \infty$$

or

$$u(t) = c \exp\left(-t - \frac{t^{1-\sigma}}{(\alpha+1)(1-\sigma)} + o(1)\right) \qquad \text{as } t \to \infty,$$

where $c > 0$ is a constant.

(iii) Let $\sigma = 1$. By Theorem 1.3, every positive solution u of (6) satisfies either

$$u(t) = ct^{1/(\alpha+1)} \exp(t + o(1)) \qquad \text{as } t \to \infty$$

or

$$u(t) = ct^{-1/(\alpha+1)} \exp(-t + o(1)) \qquad \text{as } t \to \infty,$$

where $c > 0$ is a constant.

Notice that, if $0 < \sigma \le 1$, Theorem 1.0 is not applicable for equation (6).

This paper is organized as follows. In Section 2 preparatory results are given. Theorem 1.1 is a simple consequence of the results here. In Section 3 the proof of Theorem 1.2 is given, and in Section 4 the proof of Theorem 1.3 is given.

In [4, 6, 7] analogous results to Theorem 1.0–Theorem 1.3 are obtained under somewhat different assumptions.

2. Preparatory results

In this section we give preparatory considerations and results which will be employed later.

To see Theorem 1.1 we employ the following result which is well known for the case $\alpha = 1$.

PROPOSITION 2.1. *If equation* (1) *has a solution of constant sign near* ∞, *then every nontrivial solution of* (1) *is of constant sign near* ∞.

This proposition is a direct consequence of Sturm's comparison theorem for half-linear equations whose proof is found, for example, in [2, Theorem 1.2.3]. We must notice that, actually condition (A_3) is not required to see this proposition. By this proposition, to prove Theorem 1.1, it suffices to show the existence of a positive solution of (1) defined near ∞. In fact, we can show more precisely the following:

PROPOSITION 2.2. (i) *Equation* (1) *has a positive solution u satisfying* $u' > 0$ *near* ∞.

(ii) *Equation* (1) *has a positive solution u satisfying* $u' < 0$ *near* ∞.

We prepare several lemmas to see the proposition above:

LEMMA 2.3. *Let* $\sigma > 0$ *be a constant. Then*

$$\left|\int_T^t e^{-\sigma(t-s)} b(s)\,ds\right| \le C(p,\sigma)\left(\int_T^\infty |b(s)|^p ds\right)^{1/p}, \qquad t \ge T \tag{7}$$

for some constant $C(p,\sigma) > 0$*; and similarly*

$$\left|\int_t^\infty e^{\sigma(t-s)} b(s)\,ds\right| \le C(p,\sigma)\left(\int_t^\infty |b(s)|^p ds\right)^{1/p}, \qquad t \ge T \tag{8}$$

for some constant $C(p,\sigma) > 0$.

Proof. Since $1/p + (p-1)/p = 1$, Hölder's inequality implies that

$$\left|e^{-\sigma t}\int_T^t e^{\sigma s} b(s)ds\right| \le e^{-\sigma t}\left[\int_T^t (e^{\sigma s})^{p/(p-1)}ds\right]^{(p-1)/p}\left[\int_T^t |b(s)|^p ds\right]^{1/p}$$
$$\le \left(\frac{p-1}{p\sigma}\right)^{(p-1)/p}\left(\int_T^\infty |b(s)|^p ds\right)^{1/p}.$$

So, (7) holds with $C(p,\sigma) = [(p-1)/(p\sigma)]^{(p-1)/p}$.

The estimate (8) can be proved similarly. This completes the proof. □

To see Proposition 2.2 we seek suitable positive solutions u of equation (1) of the form

$$u(t) = \exp\left(\int_T^t |w(s)|^{1/\alpha - 1} w(s)ds\right), \qquad t \ge T, \tag{9}$$

where $T > 0$ is some number and $w(s)$ is a C^1-function. It is easily seen that $u(t)$ given by (9) is a positive solution of (1) if and only if $w(t)$ satisfies the equation

$$w' = \alpha(1 + b(t)) - \alpha|w|^{(\alpha+1)/\alpha}, \qquad t \ge T. \tag{10}$$

We notice that equation (10) is often referred to as the generalized Riccati equation associated with equation (1); see [2].

To solve equation (10) we employ the Taylor's expansion of the function $(1+x)^{(\alpha+1)/\alpha}$ with remainder:

$$(1+x)^{(\alpha+1)/\alpha} = 1 + \frac{\alpha+1}{\alpha}x + \varphi(x), \qquad |x| < 1, \tag{11}$$

where $\varphi(x)$ is a C^1-function satisfying

$$\varphi(x) = O(|x|^2), \qquad \text{and} \qquad \varphi'(x) = O(|x|) \qquad \text{as } x \to 0.$$

More precisely, we can show that there are positive numbers $M = M(\alpha) > 0$ and $M_1 = M_1(\alpha) > 0$ satisfying

$$|\varphi(x)| \le Mx^2, \qquad \text{and} \qquad |\varphi'(x)| \le M_1|x|, \qquad \text{for } |x| \le 1/2. \tag{12}$$

Proof (Proof of Proposition 2.2). (i) We will show that equation (10) has a positive solution $w(t)$ near ∞. In fact, for such a $w(t)$, the function $u(t)$ given by (9) is a positive solution of (1) satisfying $u'(t) = u(t)w(t)^{1/\alpha} > 0$, $t \ge T$.

Further, we put $w(t) = 1 + z(t)$. Then by (11) we find that $z(t)$ is a solution of the equation

$$z' + \beta z = \alpha b(t) - \alpha\varphi(z), \qquad \beta = \alpha + 1, \tag{13}$$

satisfying $2 > 1 + z(t) > 0$. Below we solve this equation.

Let $z_0 \in (0, 1/2]$ be a sufficiently small number satisfying

$$\frac{\alpha M z_0^2}{\beta} < \frac{z_0}{2} \qquad \text{and} \qquad \frac{\alpha M_1 z_0}{\beta} < 1. \tag{14}$$

Here M and M_1 are constants appearing in (12), and $\beta = \alpha + 1$. Further, take a $T = T(z_0) \geq t_0$ so that

$$\left|\alpha \int_T^t e^{-\beta(t-s)} b(s)ds\right| \leq z_0/2, \qquad t \geq T. \tag{15}$$

By Lemma 2.3, there is such a T. Put

$$Z = \left\{ z \in C[T,\infty) \mid \|z\| \equiv \sup_{t \geq T} |z(t)| \leq z_0 \right\}.$$

Then Z is a nonempty closed subset of $BC[T,\infty)$ consisting of all bounded and continuous functions with the supremum norm $\|z\| = \sup_{t \geq T}|z(t)|$ for $z \in BC[T,\infty)$.

We define the operator $\mathscr{F} : Z \to C[T,\infty)$ by

$$(\mathscr{F}z)(t) = \alpha \int_T^t e^{-\beta(t-s)} b(s)ds - \alpha \int_T^t e^{-\beta(t-s)} \varphi(z(s))ds, \qquad t \geq T$$

for $z \in Z$. Below we will show that $\mathscr{F}$ is a contraction mapping from Z into itself.

To see $\mathscr{F}(Z) \subset Z$, let $z \in Z$. Then by (12), (14) and (15) we find that

$$\begin{aligned} |(\mathscr{F}z)(t)| &\leq \frac{z_0}{2} + \alpha M \int_T^t e^{-\beta(t-s)} z(s)^2 ds \\ &\leq \frac{z_0}{2} + \frac{\alpha M z_0^2}{\beta} < \frac{z_0}{2} + \frac{z_0}{2} = z_0, \qquad t \geq T. \end{aligned}$$

So $\mathscr{F}(Z) \subset Z$.

Next, let $z_1, z_2 \in Z$. The mean value theorem and (12) imply that

$$\begin{aligned} |\varphi(z_1(s)) - \varphi(z_2(s))| &= |\varphi'(\xi(s))(z_1(s) - z_2(s))| \\ &\leq M_1|\xi(s)| \cdot \|z_1 - z_2\| \leq M_1 z_0 \|z_1 - z_2\| \end{aligned}$$

for $s \geq T$, where $\xi(s)$ is some real number between $z_1(s)$ and $z_2(s)$. It follows that

$$|\mathscr{F}z_1(t) - \mathscr{F}z_2(t)| \leq \alpha \int_T^t e^{-\beta(t-s)} |\varphi(z_1(s)) - \varphi(z_2(s))| ds$$
$$\leq \alpha M_1 z_0 \int_T^t e^{-\beta(t-s)} ds \cdot \|z_1 - z_2\|$$
$$\leq \frac{\alpha M_1 z_0}{\beta} \|z_1 - z_2\|$$

for $t \geq T$; and so

$$\|\mathscr{F}z_1 - \mathscr{F}z_2\| \leq \frac{\alpha M_1 z_0}{\beta} \|z_1 - z_2\|.$$

Therefore $\mathscr{F}$ is a contraction by (14).

By the contraction principle, $\mathscr{F}$ has a unique fixed point $z \in Z$, which satisfies the integral equation

$$z(t) = \alpha \int_T^t e^{-\beta(t-s)} b(s) ds - \alpha \int_T^t e^{-\beta(t-s)} \varphi(z(s)) ds, \qquad t \geq T.$$

We can find easily that $z(t)$ is a solution of (13) satisfying

$$\frac{1}{2} \leq 1 - z_0 \leq 1 + z(t) \leq 1 + z_0 \leq \frac{3}{2}, \qquad t \geq T.$$

So, $w(t) \equiv 1 + z(t)$, $t \geq T$, is a positive solution of (10).

(ii) As in the proof of (i) it suffices to show that equation (10) has a negative solution $w(t)$ near ∞. Therefore (10) is rewritten into

$$-w' = \alpha(-w)^{(\alpha+1)/\alpha} - \alpha(1 + b(t)).$$

By putting $w(t) = -1 - z(t)$, and employing (11) it suffices to show the existence of a solution z of the equation

$$z' - \beta z = -\alpha b(t) + \alpha\varphi(z), \qquad \beta = \alpha + 1,$$

satisfying $-1 - z(t) < 0$.

Let $z_0 \in (0, 1/2]$ be a small number satisfying (14), and $T = T(z_0) \geq t_0$ be a large number satisfying

$$\alpha \left| \int_t^\infty e^{-\beta(s-t)} b(s) ds \right| \leq \frac{z_0}{2}, \qquad t \geq T.$$

By Lemma 2.3 there is such a T. Put

$$Z = \{z \in C[T, \infty) \mid \|z\| \le z_0\}.$$

Then as in the proof of (i) we can show that there is a unique solution $z \in Z$ of the integral equation

$$z(t) = \alpha \int_t^\infty e^{-\beta(s-t)} b(s)ds - \alpha \int_t^\infty e^{-\beta(s-t)} \varphi(z(s))ds, \qquad t \ge T,$$

by the contraction principle. It is found that this fixed point $z(t)$ satisfies

$$-\frac{3}{2} \le -1 - z_0 \le -1 - z(t) \le -1 + z_0 \le -\frac{1}{2}, \qquad t \ge T.$$

This complete the proof. □

As mentioned before, we find that Theorem 1.1 is an immediate consequence of Proposition 2.2.

3. Asymptotic properties of nontrivial solutions

In the preceding section we have shown that (1) has two types of positive solutions u satisfying

$$0 < \liminf_{t\to\infty} \frac{u'(t)}{u(t)} < \infty \tag{16}$$

and

$$-\infty < \limsup_{t\to\infty} \frac{u'(t)}{u(t)} < 0; \tag{17}$$

respectively, see the proof of Proposition 2.2. In this section, we firstly prove that any nontrivial solution u of (1) satisfies either (16) or (17). Based on this result we will give the proof of Theorem 1.2.

The following simple lemma will be employed very often in what follows:

LEMMA 3.1. *Let $A, B > 0$ be positive constants. Then there is a constant $\varepsilon_0 = \varepsilon_0(A, B) > 0$ such that*

$$Ax > -B + \varepsilon x^{1-(1/p)}, \qquad x \ge 0,$$

if $0 \le \varepsilon \le \varepsilon_0$.

PROPOSITION 3.2. *Any nontrivial solution u of* (1) *satisfies either* (16) *or* (17).

A simple corollary to this properties follows:

Corollary 3.3. *Every nontrivial solution u of* (1) *is a monotone function near* ∞.

Proof (Proof of Proposition 3.2). Let u be a positive solution of (1). Define $w(t)$ by

$$w(t) = \left|\frac{u'(t)}{u(t)}\right|^{\alpha-1}\frac{u'(t)}{u(t)}, \tag{18}$$

and put

$$l = \liminf_{t\to\infty} w(t), \qquad L = \limsup_{t\to\infty} w(t).$$

It suffices to show that

$$0 < l < \infty \qquad \text{or} \qquad -\infty < L < 0. \tag{19}$$

Note that $w(t)$ satisfies the generalized Riccati equation (10) for sufficiently large $T > 0$. We will divide the argument into several cases according to the value of l.

Case 1. *The case where* $0 < l \le \infty$. If $0 < l < \infty$, there is nothing to prove. Let $l = \infty$. Then $L = l = \infty$, that is, $\lim_{t\to\infty} w(t) = \infty$. Integrating (10) on $[T, t]$, we get

$$w(t) = w(T) + \alpha(t - T) + \alpha \int_T^t b(s)ds - \alpha \int_T^t w(s)^{(\alpha+1)/\alpha} ds,$$

where T is a sufficiently large number. So Hölder's inequality implied that

$$\begin{aligned} w(t) \le{}& w(T) + \alpha(t - T) + \alpha\|b\|_{L^p[T,\infty)}(t - T)^{1-(1/p)} \\ & - \alpha t\left(\frac{1}{t}\int_T^t w(s)^{(\alpha+1)/\alpha} ds\right). \end{aligned}$$

Since $\lim_{t\to\infty} t^{-1}\int_T^t w(s)^{(\alpha+1)/\alpha}ds = \infty$, the right-hand side of this inequality tends to $-\infty$ as $t \to \infty$. This is an obvious contradiction. Consequently $l < \infty$.

Case 2. *The case where* $l = 0$. We will show that $L \le 0$. To see this suppose the contrary that $0 < L \le \infty$. Take two number l_1, l_2 satisfying

$$0 < l_1 < l_2 < 1 \qquad \text{and} \qquad 0 < l_1 < l_2 < L.$$

Then we can find two sequences $\{t_n\}_{n=1}^{\infty}$ and $\{\hat{t}_n\}_{n=1}^{\infty}$ such that

$$t_n < \hat{t}_n < t_{n+1} < \hat{t}_{n+1} < \cdots; \tag{20}$$

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \hat{t}_n = \infty; \tag{21}$$

$$w(t_n) = l_2, \qquad w(\hat{t}_n) = l_1; \tag{22}$$

$$l_1 < w(t) < l_2 \qquad \text{for } t \in (t_n, \hat{t}_n). \tag{23}$$

Integrating (10) on $[t_n, \hat{t}_n]$, we obtain

$$w(\hat{t}_n) - w(t_n) = \alpha(\hat{t}_n - t_n) + \alpha \int_{t_n}^{\hat{t}_n} b(s)ds - \alpha \int_{t_n}^{\hat{t}_n} |w(s)|^{(\alpha+1)/\alpha} ds. \tag{24}$$

It follows from (3), (22), (23) and Hölder's inequality that

$$l_1 - l_2 \geq \alpha(\hat{t}_n - t_n) - \alpha\|b\|_{L^p[t_n,\infty)}(\hat{t}_n - t_n)^{1-(1/p)} - \alpha l_2^{(\alpha+1)/\alpha}(\hat{t}_n - t_n);$$

and so

$$l_1 - l_2 \geq \alpha(1 - l_2^{(\alpha+1)/\alpha})(\hat{t}_n - t_n) - \alpha\|b\|_{L^p[t_n,\infty)}(\hat{t}_n - t_n)^{1-(1/p)}. \tag{25}$$

Put

$$A = \alpha(1 - l_2^{(\alpha+1)/\alpha})(> 0), \qquad B = l_2 - l_1(> 0), \qquad x_n = \hat{t}_n - t_n(> 0),$$

and $\qquad \varepsilon_n = \alpha\|b\|_{L^p[t_n,\infty)}.$

Then (25) means that

$$-B \geq Ax_n - \varepsilon_n x_n^{1-(1/p)}.$$

Since $\lim_{n\to\infty} \varepsilon_n = 0$, this contradicts to Lemma 3.1. Therefore, $L \leq 0$. Noting that $l = 0$, we have $l = L = 0$; that is, $\lim_{t\to\infty} w(t) = 0$.

Let us integrate (10) on $[T, t]$, with T being a sufficiently large number, and divide both sides of the resulting equality to obtain

$$\frac{w(t) - w(T)}{t} = \frac{\alpha(t - T)}{t} + \frac{\alpha}{t}\int_T^t b(s)ds - \frac{\alpha}{t}\int_T^t |w(s)|^{(\alpha+1)/\alpha} ds. \tag{26}$$

Since

$$\left|\frac{1}{t}\int_T^t b(s)ds\right| \leq \frac{1}{t}(t - T)^{1-(1/p)}\left(\int_T^\infty |b(s)|^p ds\right)^{1/p} \to 0 \qquad \text{as } t \to \infty,$$

by letting $t \to \infty$ in (26), we get $0 = \alpha$, which is an obvious contradiction. Therefore this case *Case* 2 never occurs.

Case 3. *The case where* $-\infty \leq l < 0$. We will show $-\infty \leq L < 0$. To this end suppose the contrary that $L \geq 0$. Let l_1 and l_2 be sufficiently small constants satisfying

$$l < l_1 < l_2 < 0, \qquad \text{and} \qquad -1 < l_1 < l_2 < 0.$$

Then, as before, we can find two sequences $\{t_n\}$ and $\{\hat{t}_n\}$ such that (20), (21), (22) and (23) hold. An integration of (10) on $[t_n, \hat{t}_n]$ gives (24). It follows that

$$l_1 - l_2 \geq \alpha(\hat{t}_n - t_n) - \alpha\|b\|_{L^p[t_n,\infty)}(\hat{t}_n - t_n)^{1-(1/p)} - \alpha|l_1|^{(\alpha+1)/\alpha}(\hat{t}_n - t_n),$$

and so

$$l_1 - l_2 \geq \alpha(1 - |l_1|^{(\alpha+1)/\alpha})(\hat{t}_n - t_n) - \alpha\|b\|_{L^p[t_n,\infty)}(\hat{t}_n - t_n)^{1-(1/p)}.$$

Put $A = \alpha(1 - |l_1|^{(\alpha+1)/\alpha})$, $B = l_2 - l_1$, $x_n = \hat{t}_n - t_n$, and $\varepsilon_n = \alpha\|b\|_{L^p[t_n,\infty)}$. Then the inequality above means

$$-B \geq Ax_n - \varepsilon_n x_n^{1-(1/p)}.$$

This contradicts to Lemma 3.1. Therefore $-\infty \leq L < 0$.

To see $-\infty < L < 0$, the latter of (19), suppose the contrary that $L = -\infty$, that is, $\lim_{t\to\infty} w(t) = -\infty$. An integration of (10) on $[T, t]$, with T being sufficiently large, gives

$$w(t) = w(T) + \alpha(t - T) + \alpha\int_T^t b(s)ds - \alpha\int_T^t |w(s)|^{(\alpha+1)/\alpha}ds.$$

Putting $w(t) = -h(t)$, we find that $\lim_{t\to\infty} h(t) = \infty$, and

$$\begin{aligned} h(t) &= c_1 - \alpha t - \alpha\int_T^t b(s)ds + \alpha\int_T^t h(s)^{(\alpha+1)/\alpha}ds \\ &\geq c_1 - \alpha t - \alpha\|b\|_{L^p[T,\infty)}(t - T)^{1-(1/p)} + \alpha\int_T^t h(s)^{(\alpha+1)/\alpha}ds, \end{aligned}$$

where c_1 is a constant. Since $\lim_{t\to\infty} t^{-1}\int_T^t h(s)^{(\alpha+1)/\alpha}ds = \infty$, this inequality implies that

$$h(t) \geq c_2\int_T^t h(s)^{(\alpha+1)/\alpha}ds, \qquad t \geq \tilde{T} > T \tag{27}$$

for some constant $c_2 > 0$ and $\tilde{T}$. Let $H(t) = \int_T^t h(s)^{(\alpha+1)/\alpha}ds$. Then (27) implies that $H(t) > 0$ for $t \geq \tilde{T}$, and

$$H'(t) = h(t)^{(\alpha+1)/\alpha} \geq c_2^{(\alpha+1)/\alpha}H(t)^{(\alpha+1)/\alpha}, \qquad t \geq \tilde{T}.$$

So we obtain

$$H'(t)H(t)^{-(\alpha+1)/\alpha} \geq c_2^{(\alpha+1)/\alpha}, \qquad t \geq \tilde{T}.$$

An integration of this inequality on $[\tilde{T}, t]$ gives

$$\alpha H(\tilde{T})^{-1/\alpha} - \alpha H(t)^{-1/\alpha} \geq c_2^{(\alpha+1)/\alpha}(t - \tilde{T}), \qquad t \geq \tilde{T}.$$

This is a contradiction. Consequently the case $L = -\infty$ (and $l = -\infty$) never occurs.

The proof of Proposition 3.2 is complete. □

Now we are in a position to prove Theorem 1.2.

Proof (Proof of Theorem 1.2). As in the proof of Proposition 3.2, define $w(t)$ by (18), and put $l = \liminf_{t\to\infty} w(t)$ and $L = \limsup_{t\to\infty} w(t)$. We will show that

$$l = L = 1 \qquad \text{or} \qquad l = L = -1.$$

Note that, by Proposition 3.2, we have already established

$$0 < l < \infty \qquad \text{or} \qquad -\infty < L < 0.$$

(I) *The case where* $0 < l < \infty$. In this case $u' > 0$. So $w(t) = (u'(t)/u(t))^{\alpha} > 0$ and w satisfies the equation

$$w' = \alpha + \alpha b(t) - \alpha w^{(\alpha+1)/\alpha}. \tag{28}$$

The proof is further divided into several cases according to the value of l.

Case 1. *The case where* $l > 1$. We obviously have

$$w(t) \geq c_1 > 1, \qquad t \geq T,$$

for some constant $c_1 > 0$ and some sufficiently large number $T > 0$. Then an integration of (28) on $[T, t]$ gives for some constant c_2

$$\begin{aligned} w(t) &= \alpha t + c_2 + \alpha \int_T^t b(s)ds - \alpha \int_T^t w(s)^{(\alpha+1)/\alpha} ds \\ &\leq \alpha t + c_2 + \alpha \|b\|_{L^p[T,\infty)}(t-T)^{1-(1/p)} - \alpha c_1^{(\alpha+1)/\alpha}(t-T) \\ &\to -\infty \qquad \text{as } t \to \infty, \end{aligned}$$

which is an obvious contradiction. Consequently this case does not occur.

Case 2. *The case where* $0 < l < 1$. To see $L \geq 1$, suppose the contrary that $L < 1$. Then

$$w(t) \leq c_3 < 1, \qquad t \geq T$$

for some constants c_3 and T. An integration of (28) on $[T, t]$, as before, gives for some constant c_4

$$w(t) \geq \alpha t + c_4 - \alpha \|b\|_{L^p[T,\infty)}(t-T)^{1-(1/p)} - \alpha c_3^{(\alpha+1)/\alpha}(t-T)$$

$$\to \infty \qquad \text{as } t \to \infty,$$

which is an obvious contradiction. Therefore, $L \geq 1$, and so $0 < l < 1 \leq L$.

Let $l_1, l_2 > 0$ be two constants such that $l < l_1 < l_2 < 1 \leq L$. Then there are two sequences $\{t_n\}$ and $\{\hat{t}_n\}$ satisfying (20), (21), (22) and (23). Arguing as in *Case* 2 of Proof of Proposition 3.2, we can get a contradiction.

Consequently *Case* 2 does not occur.

Case 3. *The case where* $l = 1$. In this case $1 = l \leq L$. If we can show that $L = 1$, then $\lim_{t\to\infty} u'(t)/u(t) = 1$, and so (i) holds.

Suppose to the contrary that $L > 1 (= l)$. Let l_1 and l_2 be constants satisfying $l = 1 < l_1 < l_2 < L$. Then there are two sequences $\{t_n\}$ and $\{\hat{t}_n\}$ such that

$$t_n < \hat{t}_n < t_{n+1} < \hat{t}_{n+1} < \cdots; \tag{29}$$

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \hat{t}_n = \infty; \tag{30}$$

$$w(t_n) = l_1, \qquad w(\hat{t}_n) = l_2; \tag{31}$$

$$l_1 < w(t) < l_2 \qquad \text{for } t \in (t_n, \hat{t}_n). \tag{32}$$

An integration of (10) on $[t_n, \hat{t}_n]$ yields

$$l_2 - l_1 = \alpha(\hat{t}_n - t_n) + \alpha \int_{t_n}^{\hat{t}_n} b(s)ds - \alpha \int_{t_n}^{\hat{t}_n} w(s)^{(\alpha+1)/\alpha} ds$$

$$\leq \alpha(\hat{t}_n - t_n) + \alpha \|b\|_{L^p[t_n,\infty)}(\hat{t}_n - t_n)^{1-(1/p)} - \alpha l_1^{(\alpha+1)/\alpha}(\hat{t}_n - t_n),$$

which implies

$$l_2 - l_1 \leq -\alpha(l_1^{(\alpha+1)/\alpha} - 1)(\hat{t}_n - t_n) + \alpha \|b\|_{L^p[t_n,\infty)}(\hat{t}_n - t_n)^{1-(1/p)}.$$

By Lemma 3.1 this is a contradiction. So we can conclude $L = 1$.

(II) *The case where* $-\infty < L < 0$. In this case $u' < 0$. So $w(t) = |u'/u|^{\alpha-1} \cdot (u'/u) < 0$ and w satisfies the equation

$$w' = \alpha + \alpha b(t) - \alpha |w|^{(\alpha+1)/\alpha}.$$

The proof is further divided into several cases according to the value of L.

Case 1. *The case where* $L < -1$. There are some constants $c_1 > 1$ and T such that

$$w(t) \leq -c_1 \qquad \text{for } t \geq T.$$

An integration of (10) on $[T, t]$ gives for some constant c_2

$$w(t) = c_2 + \alpha t + \alpha \int_T^t b(s)ds - \alpha \int_T^t |w(s)|^{(\alpha+1)/\alpha} ds, \tag{33}$$

that is

$$\begin{aligned} w(t) &\leq c_2 + \alpha t + \alpha \|b\|_{L^p[T,\infty)} (t-T)^{1-(1/p)} - \alpha c_1^{(\alpha+1)/\alpha}(t-T) \\ &\to -\infty \qquad \text{as } t \to \infty. \end{aligned}$$

This is a contradiction. Consequently this case never occurs.

Case 2. *The case where* $-1 < L < 0$. Suppose further that $-1 < l$ $(\leq L < 0)$. Then there are constants $c_1 \in (0,1)$ and $T > 0$ such that

$$w(t) \geq -c_1, \qquad t \geq T.$$

Therefore from (10), as before, we obtain for some constant c_2

$$\begin{aligned} w(t) &\geq c_2 + \alpha t - \alpha \|b\|_{L^p[T,\infty)} (t-T)^{1-(1/p)} - \alpha c_1^{(\alpha+1)/\alpha}(t-T) \\ &\to \infty \qquad \text{as } t \to \infty. \end{aligned}$$

This is a contradiction. So $l \leq -1$ $(< L < 0)$.

Let $l_1, l_2 < 0$ be numbers satisfying

$$l \leq -1 < l_1 < l_2 < L < 0.$$

Then, there are two sequences $\{t_n\}$, $\{\hat{t}_n\}$ satisfying (20), (21), (22) and (23). An integration of (10) on $[t_n, \hat{t}_n]$ gives

$$\begin{aligned} (0 >)\ l_1 - l_2 &= \alpha(\hat{t}_n - t_n) + \alpha \int_{t_n}^{\hat{t}_n} b(s)ds - \alpha \int_{t_n}^{\hat{t}_n} |w(s)|^{(\alpha+1)/\alpha} ds \\ &\geq \alpha(\hat{t}_n - t_n) - \alpha \|b\|_{L^p[t_n,\infty)} (\hat{t}_n - t_n)^{1-(1/p)} - \alpha |l_1|^{(\alpha+1)/\alpha} (\hat{t}_n - t_n) \\ &= \alpha(1 - |l_1|^{(\alpha+1)/\alpha})(\hat{t}_n - t_n) - \alpha \|b\|_{L^p[t_n,\infty)} (\hat{t}_n - t_n)^{1-(1/p)}. \end{aligned}$$

By Lemma 3.1 this is a contradiction.

Consequently, this case *Case* 2 never occurs.

Case 3. *The case where* $L = -1$. (By the above consideration this case must occur.) To see $l = -1$ $(= L)$, suppose to the contrary that $l < -1 = L$.

Let l_1, l_2 be a numbers such that $l < l_1 < l_2 < -1 = L$. Then there are two sequences $\{t_n\}$ and $\{\hat{t}_n\}$ satisfying (29), (30), (31) and (32). An integration of (10) on $[t_n, \hat{t}_n]$ gives, as before,

$$l_2 - l_1 \le \alpha(\hat{t}_n - t_n) + \alpha\|b\|_{L^p[t_n,\infty)}(\hat{t}_n - t_n)^{1-(1/p)} - \alpha|l_2|^{(\alpha+1)/\alpha}(\hat{t}_n - t_n);$$

so

$$l_2 - l_1 \le -\alpha(|l_2|^{(\alpha+1)/\alpha} - 1)(\hat{t}_n - t_n) + \alpha\|b\|_{L^p[t_n,\infty)}(\hat{t}_n - t_n)^{1-(1/p)}.$$

By Lemma 3.1 this is a contradiction. Therefore $l = L = -1$.

This completes the proof of Theorem 1.2. □

Remark 3.4. The arguments in Proofs of Proposition 3.2 and Theorem 1.2 are motivated by [6].

4. Asymptotic forms of nontrivial solutions

In this section we give the proof of Theorem 1.3, which is the main result of this paper.

We will employ the following Taylor's expansion:

$$(1 + x)^{1/\alpha} = 1 + \frac{1}{\alpha}x + \rho(x), \qquad |x| < 1, \tag{34}$$

where ρ is a continuous function satisfying

$$\rho(x) = O(|x|^2) \qquad \text{as } x \to 0.$$

Proof (Proof of Theorem 1.3). Recall that, by Theorem 1.2, every positive solution u of (1) satisfies either (i) or (ii) of Theorem 1.2. We will show that u has the asymptotic form (4) if $u' > 0$, and that u has the asymptotic form (5) if $u' < 0$.

(I) Let $u' > 0$. Put

$$w(t) = \left|\frac{u'(t)}{u(t)}\right|^{\alpha-1}\frac{u'(t)}{u(t)} = \left(\frac{u'(t)}{u(t)}\right)^{\alpha},$$

and

$$w(t) = 1 + z(t).$$

Then, by Theorem 1.2

$$\lim_{t\to\infty} z(t) = 0$$

and we have shown

$$z' = -\beta z + \alpha b(t) - \alpha\varphi(z), \qquad t \ge T, \tag{35}$$

where $\beta = \alpha + 1$, and φ is the function introduced in (11). Since $u'/u = (1+z)^{1/\alpha}$, $t \geq T$, for some large $T > t_0$, we have

$$\log \frac{u(t)}{u(T)} = \int_T^t (1 + z(s))^{1/\alpha} ds, \qquad t \geq T.$$

By (34) we obtain

$$\log \frac{u(t)}{u(T)} = t - T + \frac{1}{\alpha} \int_T^t z(s)ds + \int_T^t \rho(z(s))ds. \tag{36}$$

On the other hand an integration of (35) gives

$$z(t) = z(T) - \beta \int_T^t z(s)ds + \alpha \int_T^t b(s)ds - \alpha \int_T^t \varphi(z(s))ds;$$

namely

$$\frac{1}{\alpha} \int_T^t z(s)ds = \frac{z(T)}{\alpha\beta} - \frac{z(t)}{\alpha\beta} + \frac{1}{\beta} \int_T^t b(s)ds - \frac{1}{\beta} \int_T^t \varphi(z(s))ds. \tag{37}$$

From (36) and (37) we obtain

$$u(t) = u(T) \exp\biggl(t - T + \frac{1}{\beta} \int_T^t b(s)ds + \frac{z(T)}{\alpha\beta} - \frac{z(t)}{\alpha\beta} - \frac{1}{\beta} \int_T^t \varphi(z(s))ds + \int_T^t \rho(z(s))ds \biggr).$$

Since $\varphi(x) = O(|x|^2)$, $\rho(x) = O(|x|^2)$ as $x \to 0$ and $\lim_{t\to\infty} z(t) = 0$, to see (4) we will show

$$\int^\infty z(t)^2 dt < \infty.$$

Recall the assumption $1 < p \leq 2$. Since

$$\int^\infty |z(t)|^2 dt = \int^\infty |z(t)|^{2-p} \cdot |z(t)|^p dt \leq c_1 \int^\infty |z(t)|^p dt$$

for some constant $c_1 > 0$, it suffices to show

$$\int^\infty |z(t)|^p dt < \infty. \tag{38}$$

(It is here that the restriction $1 < p \leq 2$ is employed.)

Multiplying both sides of (35) by $|z(t)|^{p-2} z(t)$ and integrating the resulting expression on $[T, t]$, we obtain

$$\frac{1}{p}(|z(t)|^p - |z(T)|^p) = -\beta \int_T^t |z(s)|^p ds + \alpha \int_T^t b(s)|z(s)|^{p-2} z(s) ds$$
$$- \alpha \int_T^t \varphi(z(s))|z(s)|^{p-2} z(s) ds. \qquad (39)$$

Let $\varepsilon > 0$ be a small number satisfying $\beta - \alpha\varepsilon > 0$. Since $\lim_{t\to\infty} z(t) = 0$, there is a sufficiently large $T = T_\varepsilon \geq t_0$ such that

$$|\varphi(z(s))| \cdot |z(s)|^{p-1} \leq \varepsilon |z(s)|^p, \qquad s \geq T.$$

We may assume that T in (39) is such a number T. Then for $t \geq T$ we have

$$\beta \int_T^t |z(s)|^p ds \leq c_1 + \alpha \int_T^t |b(s)| \cdot |z(s)|^{p-1} ds + \alpha \int_T^t |\varphi(z(s))| \cdot |z(s)|^{p-1} ds$$
$$\leq c_1 + \alpha \left(\int_T^t |b(s)|^p ds \right)^{1/p} \left(\int_T^t |z(s)|^p ds \right)^{1-(1/p)} + \alpha\varepsilon \int_T^t |z(s)|^p ds,$$

for some constant $c_1 > 0$. It follows that

$$(\beta - \alpha\varepsilon) \int_T^t |z(s)|^p ds \leq c_1 + \alpha \|b\|_{L^p[T,\infty)} \left(\int_T^t |z(s)|^p ds \right)^{1-(1/p)}, \qquad t \geq T.$$

Put $X(t) = \int_T^t |z(s)|^p ds$, $t \geq T$. Then this inequality asserts

$$AX(t) \leq c_1 + BX(t)^{1-(1/p)}, \qquad t \geq T,$$

with some positive constants A and B. Since $\lim_{y\to\infty}[Ay - (c_1 + By^{1-(1/p)})] = \infty$, the set

$$\{ y \geq 0 \mid Ay \leq c_1 + By^{1-(1/p)} \}$$

is bounded. Therefore $X(t) = O(1)$ as $t \to \infty$; and so (4) holds.

(II) Let $u' < 0$. The proof is parallel to that of the case $u' > 0$ above. Put

$$w(t) = \left| \frac{u'(t)}{u(t)} \right|^{\alpha-1} \frac{u'(t)}{u(t)} = -\left(-\frac{u'(t)}{u(t)} \right)^{\alpha} < 0,$$

and

$$w(t) = -1 - z(t).$$

Then, by Theorem 1.2, $\lim_{t\to\infty} z(t) = 0$, and

$$z' = \beta z + \alpha\varphi(z) - \alpha b(t), \qquad t \geq T, \qquad (40)$$

where $\beta = \alpha + 1$, and φ is the function introduced in (11). Since $-u'/u = (1+z)^{1/\alpha}$, $t \geq T$, for large T, we have

$$\log \frac{u(T)}{u(t)} = \int_T^t (1 + z(s))^{1/\alpha} ds.$$

By (34) we obtain

$$\log \frac{u(T)}{u(t)} = t - T + \frac{1}{\alpha} \int_T^t z(s) ds + \int_T^t \rho(z(s)) ds.$$

On the other hand an integration of (40) gives

$$z(t) = z(T) + \beta \int_T^t z(s) ds + \alpha \int_T^t \varphi(z(s)) ds - \alpha \int_T^t b(s) ds.$$

So arguing as in (I), we obtain

$$u(t) = u(T) \exp\bigg(-t + T - \frac{1}{\beta} \int_T^t b(s) ds - \frac{z(t)}{\alpha\beta} + \frac{z(T)}{\alpha\beta} + \frac{1}{\beta} \int_T^t \varphi(z(s)) ds - \int_T^t \rho(z(s)) ds \bigg).$$

To prove (5), it suffices to show that (38) holds as before. Multiplying both sides of (40) by $|z(t)|^{p-2} z(t)$ and integrating the resulting expression on $[T, t]$, we obtain

$$\frac{1}{p}(|z(t)|^p - |z(T)|^p) = \beta \int_T^t |z(s)|^p ds - \alpha \int_T^t b(s)|z(s)|^{p-2} z(s) ds + \alpha \int_T^t \varphi(z(s))|z(s)|^{p-2} z(s) ds.$$

So

$$\beta \int_T^t |z(s)|^p ds \leq c_2 + \alpha \int_T^t |b(s)| \cdot |z(s)|^{p-1} ds + \alpha \int_T^t |\varphi(z(s))| \cdot |z(s)|^{p-1} ds$$

for some constant $c_2 > 0$. Arguing as in (I), we can show that (38) holds.

This completes the proof of Theorem 1.3. □

Acknowledgement

This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. The authors also would like to thank the referee for giving them appropriate comments.

References

[1] W. A. Coppel, Stability and Asymptotic Behavior of Differential Equations, D.C. Heath and Co., 1965.
[2] O. Došlý and P. Řehák, Half-linear Differential Equations, Elsevier, 2005.
[3] P. Hartman, Ordinary Differential Equations, Birkhäuser, 1982.
[4] S. Luey and H. Usami, Asymptotic forms of solutions of perturbed half-linear ordinary differential equations, Arch. Math. (Brno), 57 (2021), 27–39.
[5] S. Luey and H. Usami, Application of generalized Riccati equations to analysis of asymptotic forms of solutions of perturbed half-linear ordinary differential equations, Int. J. Dynamical Systems and Differential Equations, 11 (2021), Nos. 3/4, 378–390.
[6] M. Naito and H. Usami, On the existence and asymptotic behavior of solutions of half-linear ordinary differential equations, J. Differential Equations, 318 (2022), 359–383.
[7] P. Řehák, Nonlinear Poincaré-Perron theorem, Appl. Math. Lett., 121 (2021), 107425.

Sokea Luey
Department of Mathematics
National Institute of Education
80, *Preah Nordom Blvd, Phnom Penh, Cambodia*
E-mail: luey.sokea@nie.edu.kh

Hiroyuki Usami
Faculty of Engineering
Gifu University
501-1193, *Gifu Japan*
E-mail: husami@gifu-u.ac.jp