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A criterion for biholomorphicity of self-mappings of generalized Fock-Bargmann-Hartogs domains

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Abstract. By making use of our previous result on a localization principle for biholomorphic mappings between equidimensional Fock-Bargmann-Hartogs domains in $\mathbb{C}^n \times \mathbb{C}^m$ with $m \geq 2$ and the same technique as in our previous study of the Fock-Bargmann-Hartogs domains in $\mathbb{C}^n \times \mathbb{C}$, in this paper we establish a characterization of biholomorphicity of holomorphic self-mappings of generalized Fock-Bargmann-Hartogs domains. As a special case of this, we obtain the main result of a recent paper by Guo, Feng and Bi.

1. Introduction

This is a continuation of our previous paper [9], and we retain the terminology and notation there.

Let D be a domain in $\mathbb{C}^n$ and $f : D \to D$ a holomorphic self-mapping of D. Then it would be an interesting problem to give a criterion for biholomorphicity of f under some conditions on D or on f. In connection with this problem, in a recent work of Guo-Feng-Bi [4], generalized Fock-Bargmann-Hartogs domains

$$D_{n,m}^p(\mu) = \left\{ (z, w) \in \mathbb{C}^N;\ e^{\mu\|z\|^2} \sum_{j=1}^{m} |w_j|^{2p} < 1 \right\},$$

where $0 < \mu \in \mathbb{R}$, $n, m, p \in \mathbb{N}$, $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m$ $(N = n + m)$, $w = (w_1, \dots, w_m) \in \mathbb{C}^m$, and their holomorphic self-mappings f are studied and the following theorem is proved as their main result:

Theorem G-F-B (Guo-Feng-Bi [4]). *Let f be a holomorphic self-mapping of the generalized Fock-Bargmann-Hartogs domain $D_{n,m}^p(\mu)$. Then f is an*

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automorphism of $D_{n,m}^{p}(\mu)$ *if and only if* f *keeps the function*

$$L(z,w) = e^{\mu\|z\|^2}\sum_{j=1}^{m}|w_j|^{2p}, \qquad (z,w) \in \mathbf{C}^N,$$

invariant, that is, $L(f(z,w)) = L(z,w)$ *on* $D_{n,m}^{p}(\mu)$.

As a special case of $p = 1$, they obtain a criterion for biholomorphicity of holomorphic self-mappings of the Fock-Bargmann-Hartogs domains $D_{n,m}(\mu)$ introduced by Yamamori [14] in 2013.

The main purpose of this paper is to show that the analogue of Theorem G-F-B above is still valid for more general domains under more weak conditions. In order to state our precise result, let us start with introducing our generalized Fock-Bargmann-Hartogs domains: For any $n, m \in \mathbf{N}$, $p = (p_1, \dots, p_m) \in \mathbf{N}^m$ and $0 < \mu \in \mathbf{R}$, we define our generalized Fock-Bargmann-Hartogs domain $\mathscr{D}_{n,m}^{p}(\mu)$ by

$$\mathscr{D}_{n,m}^{p}(\mu) = \left\{(z,w) \in \mathbf{C}^N;\ e^{\mu\|z\|^2}\sum_{j=1}^{m}|w_j|^{2p_j} < 1\right\},$$

where $\mathbf{C}^N = \mathbf{C}^n \times \mathbf{C}^m$ $(N = n+m)$ and $w = (w_1, \dots, w_m) \in \mathbf{C}^m$. This is an unbounded pseudoconvex domain in $\mathbf{C}^N$ with real analytic boundary. Since the complex Euclidean space $\mathbf{C}^n$ is now imbedded in $\mathscr{D}_{n,m}^{p}(\mu)$ in the canonical manner, it is not hyperbolic in the sense of Kobayashi [7]. Of course, in the special case when $p_j = 1$ for all $1 \le j \le m$, this domain reduces to the Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$.

In this notation, we first give the following characterization of biholomorphicity of holomorphic self-mappings of the Fock-Bargmann-Hartogs domains:

Theorem 1. *Let* f *be a holomorphic self-mapping of the Fock-Bargmann-Hartogs domain* $D_{n,m}(\mu)$ *in* $\mathbf{C}^N$. *Assume that* f *is not a constant mapping. Then* f *is an automorphism of* $D_{n,m}(\mu)$ *if and only if there is a real number* r *such that* $0 < r < 1$ *and* f *preserves the real hypersurface* S_r *in* $D_{n,m}(\mu)$ *given by*

$$S_r = \{(z,w) \in \mathbf{C}^N;\ \|w\|^2 e^{\mu\|z\|^2} = r\}.$$

By making use of this, we can establish the following:

Theorem 2. *Let* f *be a holomorphic self-mapping of the generalized Fock-Bargmann-Hartogs domain* $\mathscr{D}_{n,m}^{p}(\mu)$ *in* $\mathbf{C}^N$. *Assume that* f *is not a constant mapping. Then* f *is an automorphism of* $\mathscr{D}_{n,m}^{p}(\mu)$ *if and only if there is a real number* r *such that* $0 < r < 1$ *and* f *preserves the real hypersurface* $\mathscr{S}_r$ *in*

$\mathscr{D}^p_{n,m}(\mu)$ *given by*

$$\mathscr{S}_r = \left\{ (z,w) \in \mathbf{C}^N;\ e^{\mu\|z\|^2} \sum_{j=1}^m |w_j|^{2p_j} = r \right\}.$$

In Theorem G-F-B, define the submanifolds S_t of $D^p_{n,m}(\mu)$ by

$$S_t = \{(z,w) \in \mathbf{C}^N;\ L(z,w) = t\} \qquad \text{for } 0 \le t < 1.$$

Then it is clear that $f(S_t) \subset S_t$ for all $0 \le t < 1$, provided that f keeps L invariant. Of course, f is not constant in this case. Therefore, considering the special case of $p_1 = \cdots = p_m$ in Theorem 2, we obtain Theorem G-F-B.

Here it should be remarked that their techniques used in the proof of Theorem G-F-B are not applicable to the general case when $p_i \ne p_j$ for some $1 \le i, j \le m$ in our Theorem 2. This raises new difficulties to analyse the structure of holomorphic self-mappings of $\mathscr{D}^p_{n,m}(\mu)$ with $p_i \ne p_j$. Finally we would like to point out that the assumption $0 < r < 1$ in Theorem 2 (and in Theorem 1) cannot be replaced by $0 \le r < 1$. In fact, consider the holomorphic self-mapping f of $\mathscr{D}^p_{n,m}(\mu)$ defined by $f(z,w) = (z,0)$ for $(z,w) \in \mathscr{D}^p_{n,m}(\mu)$. Then it is clear that f is non-constant and preserves the complex submanifold

$$\mathscr{S}_0 := \left\{ (z,w) \in \mathbf{C}^N;\ e^{\mu\|z\|^2} \sum_{j=1}^m |w_j|^{2p_j} = 0 \right\} \cong \mathbf{C}^n$$

of $\mathscr{D}^p_{n,m}(\mu)$. But f is not an automorphism of $\mathscr{D}^p_{n,m}(\mu)$. Therefore the condition $r \ne 0$ is essential for Theorem 2.

Our proofs of the theorems are based on our previous result [9] on the localization principle for biholomorphic mappings between the Fock-Bargmann-Hartogs domains $D_{n,m}(\mu)$ in $\mathbf{C}^n \times \mathbf{C}^m$ with $m \ge 2$ and on the same method used in the study of the Fock-Bargmann-Hartogs domains $D_{n,1}(\mu)$ in $\mathbf{C}^n \times \mathbf{C}$ [10]. Hence our proofs here are completely different from that of Theorem G-F-B.

After some preparations in the next Section 2, Theorems 1 and 2 will be proved in Sections 3 and 4, respectively. And, a question related to our results in this paper will be posed in the final Section 5.

2. Preliminaries

Throughout this paper, we usually consider the elements ζ of $\mathbf{C}^N$ as the row vectors. However, we also think of ζ as the column vectors, as the need arises.

For the given generalized Fock-Bargmann-Hartogs domain $\mathscr{D}^p_{n,m}(\mu)$ in $\mathbf{C}^N$ and $0 < r < 1$, we set

$$\mathscr{D} = \mathscr{D}^p_{n,m}(\mu), \qquad \mathscr{D}_r = \left\{(z,w) \in \mathbf{C}^N;\ e^{\mu\|z\|^2} \sum_{j=1}^m |w_j|^{2p_j} < r\right\},$$

$$\mathscr{S}_r = \left\{(z,w) \in \mathbf{C}^N;\ e^{\mu\|z\|^2} \sum_{j=1}^m |w_j|^{2p_j} = r\right\} \qquad \text{as before and}$$

$$\mathscr{S}_r^* = \mathscr{S}_r \cap \{(z,w) \in \mathbf{C}^N;\ w_1 \cdots w_m \neq 0\}.$$

For the Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$, we also define the corresponding objects D, D_r and S_r in the same manner as above.

In this section, we collect some basic facts and results on the generalized Fock-Bargmann-Hartogs domains. For later purpose, we also recall the structure of the holomorphic automorphism group $\mathrm{Aut}(\mathscr{E})$ of an elementary Siegel domain $\mathscr{E}$.

Let us start with recalling the structure of the generalized Fock-Bargmann-Hartogs domain $\mathscr{D}$ in $\mathbf{C}^N = \mathbf{C}^n \times \mathbf{C}^m$. For convenience and with no loss of generality, in the following part we will always assume that there exist positive integers $m_1, \ldots, m_s$ such that

$$\begin{aligned} &m_1 + \cdots + m_s = m, \\ &p_{m_1+\cdots+m_{j-1}+1} = \cdots = p_{m_1+\cdots+m_j} \qquad (1 \le j \le s), \\ &p_{m_1+\cdots+m_j} < p_{m_1+\cdots+m_j+1} \qquad (1 \le j \le s-1), \end{aligned} \tag{2.1}$$

where we put $m_0 = 0$. From now on, we put for $1 \le j \le s$

$$q_j = p_{m_1+\cdots+m_j}, \qquad I_j = \{m_1 + \cdots + m_{j-1} + 1, \ldots, m_1 + \cdots + m_j\}. \tag{2.2}$$

Thus $q_1 < q_2 < \cdots < q_s$ and $\{1, 2, \ldots, m\} = \bigcup_{j=1}^s I_j$ (disjoint union). Moreover, according to (2.1), for the points $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$ and $w = (w_1, \ldots, w_m) \in \mathbf{C}^m$, we set

$$\begin{aligned} &w_{(j)} = (w_{m_1+\cdots+m_{j-1}+1}, \ldots, w_{m_1+\cdots+m_j}) \qquad (1 \le j \le s), \\ &w' = w_{(1)}, \qquad w'' = (w_{(2)}, \ldots, w_{(s)}) \qquad \text{and} \\ &\zeta = (\zeta_1, \ldots, \zeta_N) = (z, w) \in \mathbf{C}^n \times \mathbf{C}^m = \mathbf{C}^N. \end{aligned} \tag{2.3}$$

So we often write $\zeta = (z, w) = (z, w_{(1)}, \ldots, w_{(s)}) = (z, w', w'')$.

Now, let us define a real analytic function u on $\mathbf{C}^N$ by

$$u(\zeta) = e^{\mu\|z\|^2} \sum_{j=1}^{m} |w_j|^{2p_j} \qquad \text{for } \zeta \in \mathbf{C}^N \tag{2.4}$$

and consider the complex Hessian form $H_u(\zeta; \cdot)$ of u at $\zeta \in \mathbf{C}^N$:

$$H_u(\zeta; t) = \sum_{i,j=1}^{N} \frac{\partial^2 u(\zeta)}{\partial \zeta_i \partial \overline{\zeta_j}} t_i \overline{t_j} \qquad \text{for } t = (t_1, \dots, t_N) \in \mathbf{C}^N.$$

For any point $\zeta_o = (a, b) \in \mathbf{C}^n \times \mathbf{C}^m = \mathbf{C}^N$, we then have

$$\begin{aligned} H_u(\zeta_o; t) = e^{\mu\|a\|^2} \Bigg\{ & \mu^2 |\langle a, u\rangle|^2 \sum_{j=1}^{m} |b_j|^{2p_j} + \mu\|u\|^2 \sum_{j=1}^{m} |b_j|^{2p_j} \\ & + 2\mu \operatorname{Re}\left(\langle a, u\rangle \sum_{j=1}^{m} p_j |b_j|^{2(p_j-1)} \overline{b_j} v_j \right) + \sum_{j=1}^{m} p_j^2 |b_j|^{2(p_j-1)} |v_j|^2 \Bigg\} \\ \geq e^{\mu\|a\|^2} & \left\{ \sum_{j=1}^{m} |b_j|^{2(p_j-1)} (\mu |b_j|\, |\langle a, u\rangle| - p_j |v_j|)^2 + \mu \|u\|^2 \sum_{j=1}^{m} |b_j|^{2p_j} \right\} \\ \geq 0 & \qquad \text{for all } t = (u, v) \in \mathbf{C}^n \times \mathbf{C}^m = \mathbf{C}^N. \end{aligned}$$

Thus u is a plurisubharmonic function on $\mathbf{C}^N$; and if $q_1 = 1$ (resp. $q_1 > 1$), then it is a strictly plurisubharmonic function on $\mathbf{C}^{n+m_1} \times (\mathbf{C}^*)^{m-m_1}$ (resp. on $\mathbf{C}^n \times (\mathbf{C}^*)^m$), where $\mathbf{C}^* = \mathbf{C} \backslash \{0\}$ the punctured plane. Notice that the function $\rho(\zeta) := -1 + u(\zeta)$ on $\mathbf{C}^N$ is a global defining function for the generalized Fock-Bargmann-Hartogs domain $\mathscr{D}$. Hence our domain $\mathscr{D}$ is an unbounded pseudoconvex domain in $\mathbf{C}^N$ with real analytic boundary. Moreover, in the special case when all the $p_j = 1$, i.e., $\mathscr{D} = D_{n,m}(\mu)$ the Fock-Bargmann-Hartogs domain in $\mathbf{C}^N$, it is a strictly pseudoconvex domain in $\mathbf{C}^N$ with real analytic boundary. Here it should be emphasized that any point $\zeta^* = (z^*, w^*) \in \partial\mathscr{D}$ with $w_1^* \cdots w_m^* \neq 0$ is a strictly pseudoconvex boundary point of $\mathscr{D}$ in any cases. This fact will be used later.

Now let us recall the structure of the holomorphic automorphism group $\mathrm{Aut}(\mathscr{D})$ of the generalized Fock-Bargmann-Hartogs domain $\mathscr{D}$ in $\mathbf{C}^N$. For this purpose, define the holomorphic self-mappings φ_A, φ_B, φ_σ and φ_v of $\mathbf{C}^N$ by

$$\varphi_A : (z, w) \mapsto (Az, w);$$

$$\varphi_B : (z, w) = (z, w', w'') \mapsto (z, B'w', B''w'');$$

$$\varphi_\sigma : (z, w) \mapsto (z, w_{\sigma(1)}, \dots, w_{\sigma(m)});$$

$$\varphi_v : (z, w) = (z, w_{(1)}, \dots, w_{(s)}) \mapsto (\tilde{z}, \tilde{w}) = (\tilde{z}, \tilde{w}_{(1)}, \dots, \tilde{w}_{(s)}),$$
$$\tilde{z} = z + v, \qquad \tilde{w}_{(j)} = (e^{-2\mu\langle z, v\rangle - \mu\|v\|^2})^{1/2q_j} w_{(j)} \qquad (1 \le j \le s),$$

where $A \in U(n)$, B' is a unitary matrix of degree m_1 or a diagonal unitary matrix of degree m_1 according to $q_1 = 1$ or $q_1 > 1$, B'' is a diagonal unitary matrix of degree $m - m_1$, σ is a permutation of $\{1, \dots, m\}$ such that $\sigma(I_j) = I_j$ for all $1 \le j \le s$ and $v \in \mathbf{C}^n$.

With these notations, we have the following fact due to Bi-Tu [3]:

Fact 1. *The automorphism group* $\mathrm{Aut}(\mathscr{D})$ *of the generalized Fock-Bargmann-Hartogs domain* $\mathscr{D}$ *is generated by the mappings* φ_A, φ_B, φ_σ *and* φ_v *as above. More precisely, every automorphism* φ *of* $\mathscr{D}$ *can be written as the composite mapping* $\varphi = \varphi_v \circ \varphi_B \circ \varphi_\sigma \circ \varphi_A$ *of automorphisms* φ_A, φ_B, φ_σ *and* φ_v *of the above type.*

Considering the special case of the Fock-Bargmann-Hartogs domain D in $\mathbf{C}^N$, we have the following fact by Kim-Ninh-Yamamori [5]:

Fact 2. *The automorphism group* $\mathrm{Aut}(D)$ *of the Fock-Bargmann-Hartogs domain* D *is generated by the following mappings:*

$$\varphi_A : (z, w) \mapsto (Az, w), \qquad A \in U(n);$$
$$\varphi_B : (z, w) \mapsto (z, Bw), \qquad B \in U(m);$$
$$\varphi_v : (z, w) \mapsto (z + v, e^{-\mu\langle z, v\rangle - (\mu/2)\|v\|^2} w), \qquad v \in \mathbf{C}^n.$$

More precisely, every automorphism φ *of* D *can be written as the composite mapping* $\varphi = \varphi_v \circ \varphi_B \circ \varphi_A$ *of automorphisms* φ_A, φ_B *and* φ_v *of the above type.*

Next, in the previous paper [9], we proved the following fact on the localization principle for biholomorphic mappings between Fock-Bargmann-Hartogs domains in $\mathbf{C}^N$, which will play a crucial role in our proofs of the theorems:

Fact 3. *Let* $D_1 = D_{n_1, m_1}(\mu_1)$, $D_2 = D_{n_2, m_2}(\mu_2)$ *be two equidimensional Fock-Bargmann-Hartogs domains in* $\mathbf{C}^N$ *with* $\zeta_1 \in \partial D_1$, $\zeta_2 \in \partial D_2$. *Assume that*

(1) $m_1 \ge 2$, $m_2 \ge 2$;

(2) *there are open neighborhoods* U_1 *of* ζ_1, U_2 *of* ζ_2 *in* $\mathbf{C}^N$ *and a biholomorphic mapping* $f : U_1 \to U_2$ *such that* $f(\zeta_1) = \zeta_2$, $f(U_1 \cap D_1) = U_2 \cap D_2$ *and* $f(U_1 \cap \partial D_1) = U_2 \cap \partial D_2$.

Then f *extends to a biholomorphic mapping from* D_1 *onto* D_2. *In particular, we have* $(n_1, m_1) = (n_2, m_2)$.

For later use, let us recall here the structure of the holomorphic automorphism group $\mathrm{Aut}(\mathscr{E})$ of the elementary Siegel domain

$$\mathscr{E} = \{(u, v) \in \mathbf{C} \times \mathbf{C}^n; \operatorname{Im} u - \|v\|^2 > 0\} \qquad \text{in } \mathbf{C}^{n+1}. \tag{2.5}$$

This domain is holomorphically equivalent to the unit ball B^{n+1} in $\mathbf{C}^{n+1}$ via the correspondence $\phi : \mathscr{E} \to B^{n+1}$ given by

$$\phi(u, v) = \left(\frac{u - i}{u + i}, \frac{2v_1}{u + i}, \dots, \frac{2v_n}{u + i}\right) \qquad \text{for } (u, v) = (u, v_1, \dots, v_n) \in \mathscr{E}; \tag{2.6}$$

consequently, $\mathrm{Aut}(\mathscr{E}) = \phi^{-1} \mathrm{Aut}(B^{n+1})\phi$. Thanks to this, every automorphism F of $\mathscr{E}$ can be described explicitly as a linear fractional transformation of $\mathbf{C}^{n+1}$ in terms of the coordinates $(u, v) = (u, v_1, \dots, v_n)$. (For the precise description of $F \in \mathrm{Aut}(\mathscr{E})$, see [8; Section 3].) Let $\mathrm{Aff}(\mathbf{C}^{n+1})$ be the Lie group consisting of all non-singular complex affine transformations of $\mathbf{C}^{n+1}$ and set

$$\mathrm{Aff}(\mathscr{E}) = \{F \in \mathrm{Aff}(\mathbf{C}^{n+1}); F(\mathscr{E}) = \mathscr{E}\}.$$

Then $\mathrm{Aff}(\mathscr{E})$ is a closed subgroup of $\mathrm{Aff}(\mathbf{C}^{n+1})$. We call $\mathrm{Aff}(\mathscr{E})$ the *affine automorphism group of* $\mathscr{E}$ and each element of $\mathrm{Aff}(\mathscr{E})$ is called an *affine automorphism of* $\mathscr{E}$. As for the group $\mathrm{Aff}(\mathscr{E})$, we know the following (cf. [11; Section 2]):

Fact 4. *Every affine automorphism F of the elementary Siegel domain $\mathscr{E}$ in $\mathbf{C} \times \mathbf{C}^n$ can be written in the form*

$$F(u, v) = (ku + a + 2i\langle Bv, b\rangle + i\|b\|^2, Bv + b) \qquad \textit{for } (u, v) \in \mathscr{E},$$

where $a \in \mathbf{R}$, $b \in \mathbf{C}^n$ and $0 < k \in \mathbf{R}$, $B \in GL(n, \mathbf{C})$ with $k\|v\|^2 = \|Bv\|^2$ for all $v \in \mathbf{C}^n$ or $(1/\sqrt{k})B \in U(n)$.

We finish this section by the following lemma, which will be important in our proofs of the theorems. Before proceeding, we introduce the following functions ρ and $\hat{\rho}$ defined by

$$\rho(\zeta) = -r + u(\zeta) \quad \text{for } \zeta \in \mathbf{C}^N \qquad \text{and} \qquad \hat{\rho}(\zeta) = \rho \circ f(\zeta) \quad \text{for } \zeta \in \mathscr{D},$$

where u is the plurisubharmonic function introduced in (2.4) and f is a holomorphic self-mapping of $\mathscr{D}$. Thus ρ is a defining function for $\mathscr{D}_r$ and so $\mathscr{D}_r$ is a pseudoconvex subdomain of $\mathscr{D}$ with real analytic boundary $\partial\mathscr{D}_r = \mathscr{S}_r$. Moreover every point of $\mathscr{S}_r^*$ is a strictly pseudoconvex boundary point of $\mathscr{D}_r$.

Under the same situation as in Theorem 2, we prove the following:

Lemma. *Let ζ^* be an arbitrary point of $\mathscr{S}_r^*$. Then there exists an open neighborhood U of ζ^* in $\mathscr{D}$ such that f gives a biholomorphic mapping from U into $\mathscr{D}$ with $f(U \cap \mathscr{D}_r) = f(U) \cap \mathscr{D}_r$ and $f(U \cap \mathscr{S}_r) = f(U) \cap \mathscr{S}_r$.*

Proof. First of all, we choose a small open neighborhood U of ζ^* in $\mathscr{D}$ in such a way that $U \subset \mathbf{C}^n \times (\mathbf{C}^*)^m$ and $U \cap \mathscr{D}_r$ is connected. Thus every point of $U \cap \mathscr{S}_r$ is a strictly pseudoconvex boundary point of $\mathscr{D}_r$. After shrinking U sufficiently small, if necessary, we would like to prove that this U satisfies all the requirements in the lemma. The proof will be carried out by steps as follows:

(1) *We assert that $f(U \cap \mathscr{D}_r) \subset \overline{\mathscr{D}_r}$, the closure of $\mathscr{D}_r$ in $\mathbf{C}^N$.* For the verification of this, we shall employ an idea used in the proof of Pinchuk [12; Lemma 1.3]. Since every point of $U \cap \mathscr{S}_r$ is a strictly pseudoconvex boundary point of $\mathscr{D}_r$, we may assume that $U \cap \mathscr{S}_r$ is a strictly convex real analytic hypersurface in $\mathbf{C}^N$ (see, e.g. [13; p. 61]). Thus, for any point $\zeta_0 \in U \cap \mathscr{D}_r$, there exists a complex line L in $\mathbf{C}^N$ passing through ζ_0 such that $L \cap \mathscr{D}_r$ is a convex domain in L containing ζ_0 and $\partial(L \cap \mathscr{D}_r) \subset \mathscr{S}_r$ (after shrinking U sufficiently small, if necessary). Since $f(\partial(L \cap \mathscr{D}_r)) \subset f(\mathscr{S}_r) \subset \mathscr{S}_r$ by our assumption, it then follows that $\hat{\rho}(\zeta) \equiv 0$ on $\partial(L \cap \mathscr{D}_r)$; and hence, $\rho(f(\zeta_0)) = \hat{\rho}(\zeta_0) \leq 0$ by the maximum principle for the subharmonic function $\hat{\rho}|_{L \cap \mathscr{D}}$, the restriction of $\hat{\rho}$ to $L \cap \mathscr{D}$. Therefore we have shown that $f(U \cap \mathscr{D}_r) \subset \overline{\mathscr{D}_r}$, as asserted.

(2) *We assert that $f(U \cap \mathscr{D}_r) \subset \mathscr{D}_r$.* To prove this, assume to the contrary that $f(\zeta_1) \in \mathscr{S}_r$ for some point $\zeta_1 \in U \cap \mathscr{D}_r$. Since $\hat{\rho}(\zeta) \leq 0$ on $U \cap \mathscr{D}_r$ by the step (1) and $\hat{\rho}(\zeta_1) = 0$, we have $\hat{\rho}(\zeta) \equiv 0$ on $U \cap \mathscr{D}_r$ by the maximum principle for the plurisubharmonic function $\hat{\rho}$. Consequently we have $f(U \cap \mathscr{D}_r) \subset \mathscr{S}_r$. However, this is impossible. To verify this, represent f by coordinates $f = (g, h) = (g, h_1, \dots, h_m)$ and assume that there exists a point $\zeta_2 \in U \cap \mathscr{D}_r$ with $h_1(\zeta_2) \cdots h_m(\zeta_2) \neq 0$. In such a case, since $\eta_2 := f(\zeta_2)$ is a strictly pseudoconvex boundary point of $\mathscr{D}_r$, there exists a local holomorphic peaking function for $\mathscr{D}_r$ at η_2, that is, there is an open neighborhood W_2 of η_2 and a holomorphic function $\Psi_{\eta_2} : W_2 \cap \mathscr{D}_r \to \mathbf{C}$ with a continuous extension to $\overline{W_2 \cap \mathscr{D}_r}$ such that

$$\Psi_{\eta_2}(\eta_2) = 1 \qquad \text{and} \qquad |\Psi_{\eta_2}(\eta)| < 1 \qquad \text{for all } \eta \in \overline{W_2 \cap \mathscr{D}_r} \backslash \{\eta_2\}$$

(cf. [13; p. 222]). Fix a connected open neighborhood V_2 of ζ_2 such that $V_2 \subset U \cap \mathscr{D}_r$ and $f(V_2) \subset W_2$. It then follows at once by the maximum principle for the holomorphic function $\Psi_{\eta_2} \circ f$ defined on V_2 that $f(\zeta) = \eta_2$ for all $\zeta \in V_2$; so that f is constant on $\mathscr{D}$ by analytic continuation. But this contradicts our assumption. Therefore we conclude that $h_1(\zeta) \cdots h_m(\zeta) \equiv 0$ on $U \cap \mathscr{D}_r$ and so on $\mathscr{D}$ by analytic continuation. Together with the facts that

$$\mathscr{S}_r \cap \{(z, w) \in \mathbf{C}^N;\ w = 0\} = \varnothing \qquad \text{and} \qquad f(U \cap \mathscr{D}_r) \subset \mathscr{S}_r,$$

this yields that there are at least two component functions h_i, h_j of f with $h_i \neq 0$, $h_j = 0$ on $\mathscr{D}$, respectively. Accordingly we may rename the indices so that, for some $1 \le \ell < m$, one has

$$h_i \neq 0 \quad \text{for } 1 \le i \le \ell, \text{ while } h_j = 0 \quad \text{for } \ell + 1 \le j \le m.$$

Hence f has the form $f = (g, h_1, \dots, h_\ell, 0, \dots, 0)$. Set $w^{[\ell]} = (w_1, \dots, w_\ell)$ and

$$u^{[\ell]}(z, w^{[\ell]}) = e^{\mu\|z\|^2} \sum_{j=1}^{\ell} |w_j|^{2p_j} \qquad \text{on } \mathbf{C}^n \times \mathbf{C}^\ell,$$

$$\mathscr{D}^{[\ell]} = \{(z, w^{[\ell]}) \in \mathbf{C}^n \times \mathbf{C}^\ell;\ u^{[\ell]}(z, w^{[\ell]}) < 1\},$$

$$\mathscr{D}_r^{[\ell]} = \{(z, w^{[\ell]}) \in \mathbf{C}^n \times \mathbf{C}^\ell;\ u^{[\ell]}(z, w^{[\ell]}) < r\},$$

$$\mathscr{S}_r^{[\ell]} = \{(z, w^{[\ell]}) \in \mathbf{C}^n \times \mathbf{C}^\ell;\ u^{[\ell]}(z, w^{[\ell]}) = r\}.$$

Then $\partial \mathscr{D}_r^{[\ell]} = \mathscr{S}_r^{[\ell]}$ and our f may be naturally regarded as a holomorphic mapping from $\mathscr{D}$ into $\mathscr{D}^{[\ell]}$ with $f(U \cap \mathscr{D}_r) \subset \mathscr{S}_r^{[\ell]}$. Moreover, since $h_1 \cdots h_\ell \neq 0$ on $\mathscr{D}$, there exists a point $\zeta_3 \in U \cap \mathscr{D}_r$ such that $h_1(\zeta_3) \cdots h_\ell(\zeta_3) \neq 0$. Thus $\eta_3 := f(\zeta_3)$ is a strictly pseudoconvex boundary point of $\mathscr{D}_r^{[\ell]}$; consequently, by the same reasoning as above, we conclude that $f(\zeta) = \eta_3$ on $\mathscr{D}$, a contradiction. Eventually we arrive at a contradiction in any case; completing the proof of our assertion $f(U \cap \mathscr{D}_r) \subset \mathscr{D}_r$.

(3) *The neighborhood U of ζ^* satisfies all the requirements in Lemma.* We know already that $\hat{\rho}(\zeta) = 0$ on $U \cap \mathscr{S}_r$ and $\hat{\rho}(\zeta) < 0$ on $U \cap \mathscr{D}_r$ by the step (2). It then follows from the Hopf lemma that $d\hat{\rho}(\zeta) \neq 0$ for all $\zeta \in U \cap \mathscr{S}_r$; accordingly, $\hat{\rho}$ gives a local defining function for $\mathscr{D}_r$ around the strictly pseudoconvex boundary point ζ^*. Thus, by using the same method as in the last paragraph of the proof of Bell [2; Theorem 2], it can be checked that the complex Jacobian determinant $J_f(\zeta^*)$ of f at ζ^* cannot vanish. Hence the inverse mapping theorem guarantees that f is injective on some open neighborhood of ζ^*, so that U satisfies all the requirements in Lemma (after shrinking U again, if necessary).

Therefore our proof is now completed. □

Remark. A glance at the proof above tells us that this Lemma is valid for any strictly pseudoconvex boundary point ζ^* of $\mathscr{D}_r$. In particular, in the special case when $\mathscr{D} = D_{n,m}(\mu)$, the Fock-Bargmann-Hartogs domain in $\mathbf{C}^N$, this Lemma is valid for every boundary point ζ^* of D_r.

3. Proof of Theorem 1

Let f be a holomorphic automorphism of $D = D_{n,m}(\mu)$. Then, by using the explicit description of the generators of $\mathrm{Aut}(D)$ given in Fact 2, it is easily seen that $f(S_r) \subset S_r$ for all $0 < r < 1$ (even for $r = 0$).

Conversely, taking a non-constant holomorphic self-mapping f of D and assuming that $f(S_r) \subset S_r$ for some $0 < r < 1$ as in Theorem 1, we would like to show that f is an automorphism of D. The proof is now divided into two cases when $m = 1$ and $m \geq 2$.

Case 1. $m = 1$: In this case, we put

$$D_r^* = \{(z, w) \in \mathbf{C}^n \times \mathbf{C};\ 0 < |w|^2 e^{\mu \|z\|^2} < r\}.$$

Thus our real hypersurface $S_r = \partial D_r$ is the subset of ∂D_r^* consisting of all strictly pseudoconvex boundary points of D_r^*.

Let $\mathscr{E}$ be the elementary Siegel domain in $\mathbf{C}^{n+1} = \mathbf{C} \times \mathbf{C}^n$ appearing in (2.5) and consider a holomorphic mapping ϖ from $\mathscr{E}$ into $\mathbf{C}^n \times \mathbf{C}^*$ defined by

$$\varpi(u, v) = ((1/\sqrt{\mu})v, \sqrt{r}e^{iu/2}) \qquad \text{for } (u, v) \in \mathscr{E}.$$

Then it is easily seen that $\varpi(\mathscr{E}) = D_r^*$ and $\mathscr{E}$ is the universal covering of D_r^* with the covering projection ϖ. Clearly, ϖ is, in fact, defined on $\mathbf{C} \times \mathbf{C}^n$ and $\varpi(\partial\mathscr{E}) = S_r$.

Now, pick a point $\zeta_1^* \in S_r$ arbitrarily and put $\zeta_2^* = f(\zeta_1^*) \in S_r$. Then, by Lemma in Section 2, we can choose connected open neighborhoods V_1, V_2 of ζ_1^*, ζ_2^* in D, respectively, such that f gives rise to a biholomorphic mapping, say again f, from V_1 onto V_2 with $f(V_1 \cap D_r) = V_2 \cap D_r$ and $f(V_1 \cap S_r) = V_2 \cap S_r$. Let $\xi_1^*, \xi_2^* \in \partial\mathscr{E}$ be any two points such that $\varpi(\xi_j^*) = \zeta_j^*$ for $j = 1, 2$. Since ϖ is a covering projection from $\mathbf{C} \times \mathbf{C}^n$ onto $\mathbf{C}^n \times \mathbf{C}^*$ with $\varpi(\partial\mathscr{E}) = S_r$, we can find connected open neighborhoods W_1, W_2 of ξ_1^*, ξ_2^*, respectively, such that $\varpi(W_j) = V_j$ and both the restrictions

$$\Pi_j := \varpi|_{W_j} : W_j \to V_j \qquad \text{for } j = 1, 2$$

are biholomorphic mappings, after shrinking V_1 sufficiently small, if necessary. Thus we obtain a biholomorphic mapping $F := \Pi_2^{-1} \circ f \circ \Pi_1 : W_1 \to W_2$ with

$$F(W_1 \cap \mathscr{E}) = W_2 \cap \mathscr{E} \qquad \text{and} \qquad F(W_1 \cap \partial\mathscr{E}) = W_2 \cap \partial\mathscr{E}.$$

Recall that $\mathscr{E}$ is biholomorphically equivalent to the unit ball B^{n+1} in $\mathbf{C}^{n+1}$ via the correspondence ϕ defined in (2.6). Then, as an immediate consequence of the main result of Alexander [1], F now extends to a holomorphic automorphism, denoted by the same letter F, of $\mathscr{E}$. Thus

$$\varpi(F(\xi)) = f(\varpi(\xi)) \qquad \text{for all } \xi \in \mathscr{E} \tag{3.1}$$

by analytic continuation. We here assert that F is an affine automorphism of $\mathscr{E}$. The following proof of this will be presented only in outline, since the details of the steps can be filled in by consulting the corresponding passages in

Case 1 in the proof of the assertion (II) of [10; Theorem 1]. First of all, notice that

$$\varpi^{-1}(\varpi(\xi)) = \{(u + 4\pi\nu, v);\ \nu \in \mathbb{Z}\} \qquad \text{for any } \xi = (u, v) \in \mathscr{E}.$$

Thus, representing $F = (F_0, F_1, \dots, F_n)$ by coordinates $(u, v) = (u, v_1, \dots, v_n)$ in $\mathbb{C} \times \mathbb{C}^n = \mathbb{C}^{n+1}$, we obtain from (3.1) that

$$F_0(u + 4\pi\nu, v) = F_0(u, v) + 4\pi n(\xi, \nu);$$

$$F_i(u + 4\pi\nu, v) = F_i(u, v), \qquad 1 \le i \le n,$$

for any point $\xi = (u, v) \in \mathscr{E}$ and any integer ν, where $n(\xi, \nu)$ is an integer uniquely determined by (ξ, ν). For each fixed $\nu \in \mathbb{Z}$, being a continuous integer-valued function in ξ defined on the domain $\mathscr{E}$, $n(\xi, \nu)$ is independent on ξ; and so, we may write $n(\xi, \nu) = n(\nu)$. Moreover, since $\mathscr{E}$ is a complete hyperbolic manifold in the sense of Kobayashi [7], we see that

$$|n(\nu)| \to +\infty \qquad \text{if and only if} \qquad |\nu| \to +\infty.$$

Therefore, by repeating exactly the same argument as in the proof of [10; Theorem 1], it can be checked that F is, in fact, an affine automorphism of $\mathscr{E}$, as asserted.

Now let us express the affine automorphism F of $\mathscr{E}$ as in Fact 4 in Section 2 and write $M = (1/\sqrt{k})B \in U(n)$, $b^* = (1/\sqrt{\mu})b$. It then follows from (3.1) that

$$f(z, w) = (M\sqrt{k}z + b^*, (1/\sqrt{r})^{k-1} e^{-\mu\langle M\sqrt{k}z, b^*\rangle - (\mu/2)\|b^*\|^2} e^{(a/2)i} w^k)$$

on D_r^* and so on D by analytic continuation. Here, since f is a single-valued holomorphic mapping defined on D, the positive real number k has to be an integer. Moreover, f may be regarded as a holomorphic self-mapping of $\mathbb{C}^{n+1}$. Thus $f(\partial D) \subset \bar{D}$ and so

$$1 \ge u(f(z, w)) = 1/r^{k-1} \qquad \text{whenever } u(z, w) = 1.$$

Since $0 < r < 1$ and $k \ge 1$, this can only happen when $k = 1$. This combined with Fact 2 assures us that f is, in fact, an automorphism of D; thereby the proof of Theorem 1 is completed in the case when $m = 1$.

Case 2. $m \ge 2$: We now proceed to define a non-singular linear transformation L of $\mathbb{C}^n \times \mathbb{C}^m$ by

$$L(z, w) = (z, (1/\sqrt{r})w) \qquad \text{for } (z, w) \in \mathbb{C}^n \times \mathbb{C}^m.$$

Notice that $L(D_r) = D$ and $L(S_r) = \partial D$.

Pick a point $\zeta_1^* \in S_r$ and put $\zeta_2^* = f(\zeta_1^*) \in S_r$. Then, thanks to Lemma in Section 2, one can choose connected open neighborhoods U_1, U_2 of ζ_1^*, ζ_2^* in D, respectively, such that f gives a biholomorphic mapping, say again f, from U_1 onto U_2 such that $f(U_1 \cap D_r) = U_2 \cap D_r$ and $f(U_1 \cap S_r) = U_2 \cap S_r$. Here, setting $V_i = L(U_i)$ for $i = 1, 2$ and considering the biholomorphic mapping $F := L \circ f \circ L^{-1} : V_1 \to V_2$, we have

$$F(V_1 \cap D) = V_2 \cap D \qquad \text{and} \qquad F(V_1 \cap \partial D) = V_2 \cap \partial D.$$

Recall that D is now a Fock-Bargmann-Hartogs domain in $\mathbf{C}^n \times \mathbf{C}^m$ with $m \geq 2$. Then, as a direct consequence of Fact 3, F extends to a holomorphic automorphism, denoted by the same letter F, of D. Hence

$$L(f(z,w)) = F(L(z,w)) \qquad \text{for all } (z,w) \in D_r \tag{3.2}$$

by analytic continuation. In particular, we see that $f(D_r) \subset D_r$. Recall that our automorphism F is now expressed as the composite mapping $F = \varphi_v \circ \varphi_B \circ \varphi_A$ of automorphisms φ_A, φ_B and φ_v defined in Fact 2. Then the relation (3.2) tells us that f has the form

$$f(z,w) = (Az + v, e^{-\mu\langle Az, v\rangle - (\mu/2)\|v\|^2} Bw) \qquad \text{on } D_r$$

and so on D by analytic continuation. Thus f is an automorphism of D by Fact 2; thereby the proof of Theorem 1 is completed in the case when $m \geq 2$.

Therefore the proof of Theorem 1 is now completed. □

4. Proof of Theorem 2

Let $\mathscr{D}$ be the generalized Fock-Bargmann-Hartogs domain in $\mathbf{C}^N$ and let f be a holomorphic self-mapping of $\mathscr{D}$ as in Theorem 2.

Since every holomorphic automorphism of $\mathscr{D}$ preserves the real hypersurface $\mathscr{S}_r$ by Fact 1, we have only to prove the converse assertion. So, assuming that $f(\mathscr{S}_r) \subset \mathscr{S}_r$ for some $0 < r < 1$ as in Theorem 2, we wish to show that f is an automorphism of $\mathscr{D}$. We have now two cases to consider.

Case 1. $m = 1$: Putting $p = p_1$, $\mu^* = \mu/p$ and $r^* = \sqrt[p]{r}$, we have

$$\mathscr{D} = \{(z,w) \in \mathbf{C}^N;\ |w|^2 e^{\mu^*\|z\|^2} < 1\} = D_{n,1}(\mu^*),$$

$$\mathscr{S}_r = \{(z,w) \in \mathbf{C}^N;\ |w|^2 e^{\mu^*\|z\|^2} = r^*\} = S_{r^*}$$

as sets. Thus f is a holomorphic self-mapping of the Fock-Bargmann-Hartogs domain $D_{n,1}(\mu^*)$ preserving the real hypersurface S_{r^*} in $D_{n,1}(\mu^*)$. Hence, as an immediate consequence of Theorem 1, f is an automorphism of $\mathscr{D}$, as desired.

Case 2. $m \geq 2$: If both the integers s and q_1 appearing in (2.1) and (2.2), respectively, are equal to one, then $\mathscr{D}$ is just the Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$; consequently, f is an automorphism of $\mathscr{D}$ by Theorem 1.

From now on, we always assume that $(s, q_1) \neq (1, 1)$. In this case, the following holomorphic self-mapping Π of $\mathbf{C}^N = \mathbf{C}^n \times \mathbf{C}^m$ will play an important role in our proof: For the given $p = (p_1, \dots, p_m) \in \mathbf{N}^m$ and $0 < r < 1$, we set

$$\Pi(z, w) = (z, (w_1)^{p_1}/\sqrt{r}, \dots, (w_m)^{p_m}/\sqrt{r}) \qquad \text{for } (z, w) \in \mathbf{C}^N.$$

Notice that Π induces a proper holomorphic mapping from $\mathscr{D}_r$ onto the Fock-Bargmann-Hartogs domain $D = D_{n,m}(\mu)$ with $\Pi(\mathscr{S}_r) = \partial D$. Moreover, for any point $\zeta^* \in \mathscr{S}_r^*$, Π is injective on some open neighborhood of ζ^*.

Let ζ_1^* be an arbitrary point of $\mathscr{S}_r^*$ and put $\zeta_2^* = f(\zeta_1^*)$. Then, by Lemma in Section 2, we can choose connected open neighborhoods V_1, V_2 of ζ_1^*, ζ_2^* in $\mathscr{D}$ such that f gives a biholomorphic mapping from V_1 onto V_2 with $f(V_1 \cap \mathscr{D}_r) = V_2 \cap \mathscr{D}_r$ and $f(V_1 \cap \mathscr{S}_r) = V_2 \cap \mathscr{S}_r$. In particular, replacing ζ_1^* by a nearby point, if necessary, we may assume that ζ_2^* is also contained in $\mathscr{S}_r^*$. Now we set $W_i = \Pi(V_i)$ for $i = 1, 2$. We may assume that both the restrictions

$$\Pi_i := \Pi|_{V_i} : V_i \to W_i \qquad \text{for } i = 1, 2$$

are biholomorphic mappings, after shrinking V_1 sufficiently small, if necessary. Thus we obtain a biholomorphic mapping $F := \Pi_2 \circ f \circ \Pi_1^{-1} : W_1 \to W_2$ with

$$F(W_1 \cap D) = W_2 \cap D \qquad \text{and} \qquad F(W_1 \cap \partial D) = W_2 \cap \partial D.$$

Recall that $D = D_{n,m}(\mu)$ is a Fock-Bargmann-Hartogs domain with $m \geq 2$. It then follows from Fact 3 that F extends to a holomorphic automorphism, say again F, of D. Hence

$$\Pi(f(\zeta)) = F(\Pi(\zeta)) \qquad \text{for all } \zeta \in \mathscr{D}_r \tag{4.1}$$

by analytic continuation. On the other hand, by virtue of Fact 2, F can be written in the form

$$F(z, w) = (Az + v, e^{-\mu\langle Az, v\rangle - (\mu/2)\|v\|^2} Bw) \qquad \text{on } D,$$

where $A \in U(n)$, $B \in U(m)$ and $v \in \mathbf{C}^n$. Making use of this, we wish to show that f is an automorphism of $\mathscr{D}$. For this purpose, we here introduce the following notation: Let q_j be the positive integer defined in (2.2) and represent f as $f = (g, h) = (g, h_1, \dots, h_m)$ by coordinates $\zeta = (z, w) = (z, w_1, \dots, w_m)$ in $\mathbf{C}^n \times \mathbf{C}^m = \mathbf{C}^N$. Then, according to the notations in (2.1) and (2.3), we set

$$w_{(j)}^{q_j} = ((w_{m_1+\cdots+m_{j-1}+1})^{q_j}, \ldots, (w_{m_1+\cdots+m_j})^{q_j}),$$

$$h_{(j)} = (h_{m_1+\cdots+m_{j-1}+1}, \ldots, h_{m_1+\cdots+m_j}),$$

$$h_{(j)}^{q_j} = ((h_{m_1+\cdots+m_{j-1}+1})^{q_j}, \ldots, (h_{m_1+\cdots+m_j})^{q_j})$$

for $1 \leq j \leq s$. In this notation, the relation (4.1) can be rewritten in the form

$$g(\zeta) = Az + v, \qquad \begin{pmatrix} h_{(1)}^{q_1}(\zeta) \\ \vdots \\ h_{(s)}^{q_s}(\zeta) \end{pmatrix} = \lambda(z) \begin{pmatrix} B_{11} & \ldots & B_{1s} \\ \vdots & \ddots & \vdots \\ B_{s1} & \ldots & B_{ss} \end{pmatrix} \begin{pmatrix} w_{(1)}^{q_1} \\ \vdots \\ w_{(s)}^{q_s} \end{pmatrix} \tag{4.2}$$

for $\zeta = (z, w) \in \mathscr{D}_r$, where $\lambda(z) = e^{-\mu\langle Az, v\rangle - (\mu/2)\|v\|^2}$ and B_{ij} is an $m_i \times m_j$ matrix for $1 \leq i, j \leq s$ with $B = (B_{ij})_{1 \leq i, j \leq s}$.

First of all, we consider the case $s = 1$, so that $m_1 = m$ and $q_1 > 1$ by our assumption. In this case, we assert that $B = B_{11}$ has the following property:

($\star$) *Every row and every column of B contain exactly one entry of modulus* 1.

Recall that B is a unitary matrix. Then the verification of our assertion ($\star$) is now reduced to showing that every row of B contains exactly one nonzero entry. To this end, let us define the holomorphic functions $\hat{h}_j$ $(1 \leq j \leq m)$ by

$$\hat{h}_j(w) = h_j(0, w) \qquad \text{on } E = \{w \in \mathbf{C}^m;\, |w_1|^{2q_1} + \cdots |w_m|^{2q_1} < r\},$$

where 0 denotes the origin of $\mathbf{C}^n$. Then, being a holomorphic function on the complete Reinhardt domain E, each $\hat{h}_j$ can be expanded uniquely as

$$\hat{h}_j(w) = \sum_{k=0}^{\infty} P_k(w) \qquad \text{on } E, \tag{4.3}$$

which converges absolutely and uniformly on compact subsets of E, where $P_k(w)$ is a homogeneous polynomial of degree k in $w = (w_1, \ldots, w_m)$. Notice that $\hat{h}_j(0) = 0$ for all $1 \leq j \leq m$ in our case. Hence, together with (4.3), the second equality in (4.2) tells us that each $\hat{h}_j(w)$ is a homogeneous polynomial of degree 1 in w. Represent

$$B = (b_{ij})_{1 \leq i, j \leq m} \qquad \text{and} \qquad \hat{h}_j(w) = \sum_{\nu=1}^{m} c_{j\nu} w_\nu \qquad \text{for } 1 \leq j \leq m,$$

where $c_{j\nu}$ are complex constants. Now, since B is non-singular, it is obvious that every row of B contains at least one nonzero entry. So, fix an arbitrary index j and assume that there are two entries b_{jk}, $b_{j\ell}$ $(k < \ell)$ with $b_{jk} b_{j\ell} \neq 0$.

It then follows from (4.2) that

$$(c_{jk}w_k + c_{j\ell}w_\ell)^{q_1} = \hat{h}_j^{q_1}(w^*) = \lambda(0)(b_{jk}(w_k)^{q_1} + b_{j\ell}(w_\ell)^{q_1})$$

for all $w^* = (0, \dots, 0, w_k, 0, \dots, 0, w_\ell, 0, \dots, 0) \in E$. However, this is impossible because $\lambda(0) \neq 0$ and $q_1 > 1$. As a result, we have shown that every row of B contains exactly one nonzero entry; proving the assertion ($\star$).

Now it is easily seen by using ($\star$) that there exist a permutation σ of $\{1, \dots, m\}$ and a diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_m)$ with diagonal entries d_j of modulus 1 such that

$$h(\zeta) = \lambda(z)^{1/q_1} D\psi_\sigma(w) \qquad \text{for } \zeta = (z, w) \in \mathscr{D}_r,$$

where ψ_σ is the automorphism of $\mathbb{C}^m$ induced from σ in the canonical fashion:

$$\psi_\sigma(w) = (w_{\sigma(1)}, \dots, w_{\sigma(m)}) \qquad \text{for } w \in \mathbb{C}^m. \tag{4.4}$$

Therefore we have shown that f has the form

$$f(\zeta) = (Az + v, \lambda(z)^{1/q_1} D\psi_\sigma(w)) \qquad \text{on } \mathscr{D}$$

by analytic continuation. Thus f is an automorphism of $\mathscr{D}$ by Fact 1; thereby the proof of Theorem 2 is completed in the case when $s = 1$ and $q_1 > 1$.

Next we consider the case when $s > 1$, so that $m > m_1$ and $q_j \geq 2$ for all $2 \leq j \leq s$. In this case, we first claim that $B_{ij} = 0$ for all $i \neq j$. To this end, we have only to show that $B_{ij} = 0$ for $i > j$, since B is unitary. To derive a contradiction, assume that there exists a submatrix $B_{ij} = (b_{\mu\nu})_{1 \leq \mu \leq m_i, 1 \leq \nu \leq m_j}$, $i > j$, with a nonzero entry $b_{\mu\nu}$. Put for a while

$$\alpha = m_1 + \cdots + m_{i-1} + \mu \qquad \text{and} \qquad \beta = m_1 + \cdots + m_{j-1} + \nu.$$

It then follows from (4.2) that

$$(h_\alpha(\zeta^*))^{q_i} = \lambda(0) b_{\mu\nu}(w_\beta)^{q_j} \qquad \text{for all } \zeta^* = (0, \dots, 0, w_\beta, 0, \dots, 0) \in \mathscr{D}_r.$$

However, this is impossible. Indeed, since $\lambda(0) b_{\mu\nu} \neq 0$, the left-hand side of the above equation is a polynomial in w_β of degree at least q_i, while the right-hand side is a monomial of degree q_j. Hence we arrive at a contradiction because $q_i > q_j$ for $i > j$. Thus $B_{ij} = 0$ for all $i \neq j$, as claimed. In particular, we see that $B_{jj} \in U(m_j)$ for all $1 \leq j \leq s$. Moreover, recall that $q_j > 1$ for all $2 \leq j \leq s$. Then, by repeating exactly the same argument as in the case when $s = 1$ and $q_1 > 1$, it can be seen that every submatrix B_{jj} $(2 \leq j \leq s)$ has the same property as stated in ($\star$) above. Thus, for every $2 \leq j \leq s$, there exist a permutation σ_j of I_j and a diagonal matrix $D_j = \operatorname{diag}(d_{j1}, \dots, d_{jm_j})$ with

$|d_{j\ell}| = 1$ for all $1 \le \ell \le m_j$ such that

$$h_{(j)}(\zeta) = \lambda(z)^{1/q_j} D_j \psi_{\sigma_j}(w_{(j)}) \qquad \text{for } \zeta = (z, w) \in \mathscr{D}_r,$$

where ψ_{σ_j} is the automorphism of $\mathbf{C}^{m_j}$ naturally induced from σ_j. Of course, the same is true for B_{11}, provided that $q_1 > 1$.

Summarizing the above, we obtain that f can be written in the form

$$f(\zeta) = (Az + v, h_{(1)}(\zeta), \lambda(z)^{1/q_2} D_2 \psi_{\sigma_2}(w_{(2)}), \dots, \lambda(z)^{1/q_s} D_s \psi_{\sigma_s}(w_{(s)})),$$

$$h_{(1)}(\zeta) = \lambda(z) B_1 w_{(1)} \qquad \text{or} \qquad h_{(1)}(\zeta) = \lambda(z)^{1/q_1} D_1 \psi_{\sigma_1}(w_{(1)})$$

according to $q_1 = 1$ or $q_1 > 1$ for $\zeta = (z, w_{(1)}, \dots, w_{(s)}) \in \mathbf{C}^n \times \mathbf{C}^{m_1} \times \cdots \times \mathbf{C}^{m_s} = \mathbf{C}^N$, where $\lambda(z) = e^{-\mu\langle Az, v\rangle - (\mu/2)\|v\|^2}$, $A \in U(n)$, $B_1 \in U(m_1)$, D_j is a diagonal unitary matrix of degree m_j, σ_j is a permutation of the index set I_j and ψ_{σ_j} is the automorphism of $\mathbf{C}^{m_j}$ induced from σ_j as in (4.4) for $1 \le j \le s$. Anyway we conclude by Fact 1 that f is, in fact, a holomorphic automorphism of $\mathscr{D}$ in the case when $s > 1$.

Therefore the proof of Theorem 2 is now completed. □

5. A concluding remark

Let us define the twisted Fock-Bargmann-Hartogs domains $\mathbb{D}_{n,m,k}^{\mu,p}$ according to Kim-Yamamori [6] as follows:

$$\mathbb{D}_{n,m,k}^{\mu,p} = \left\{ (z, w_1, \dots, w_m) \in \mathbf{C}^n \times \mathbf{C}^{k_1} \times \cdots \times \mathbf{C}^{k_m};\ \sum_{j=1}^{m} \frac{\|w_j\|^{2p_j}}{e^{-\mu_j \|z\|^2}} < 1 \right\},$$

where

$$k = (k_1, \dots, k_m) \in \mathbf{N}^m \qquad \text{and} \qquad \mu = (\mu_1, \dots, \mu_m),\ p = (p_1, \dots, p_m) \in (\mathbf{R}_+)^m.$$

Clearly our generalized Fock-Bargmann-Hartogs domain $\mathscr{D}_{n,m}^{p}(\mu)$ is a special case of the twisted Fock-Bargmann-Hartogs domains $\mathbb{D}_{n,m,k}^{\mu,p}$. Hence we wish to generalize our results to the domains $\mathbb{D}_{n,m,k}^{\mu,p}$. However, it does not seem easy to achieve this although the structure of the holomorphic automorphism groups $\mathrm{Aut}(\mathbb{D}_{n,m,k}^{\mu,p})$ of the twisted Fock-Bargmann-Hartogs domains $\mathbb{D}_{n,m,k}^{\mu,p}$ is completely determined by Kim-Yamamori [6]. In fact, the author does not know at this writing how to generalize our results to the twisted Fock-Bargmann-Hartogs domains $\mathbb{D}_{n,m,k}^{\mu,p}$ even in the case when $m \ge 2$, $\mu_1 = \cdots = \mu_m$, $p \in \mathbf{N}^m$ with $p_j \ge 2$ and $k_j \ge 2$ for some $1 \le j \le m$, for instance. Thus we would like to finish this article by posing the following:

Question. *Can one generalize the results in this paper to the twisted Fock-Bargmann-Hartogs domains* $\mathbb{D}_{n,m,k}^{\mu,p}$*?*

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