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Infinite presentations for fundamental groups of surfaces

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Abstract. For any finite type connected surface S, we give an infinite presentation of the fundamental group $\pi_1(S,*)$ of S based at an interior point $* \in S$ whose generators are represented by simple loops. When S is non-orientable, we also give an infinite presentation of the subgroup of $\pi_1(S,*)$ generated by elements which are represented by simple loops whose regular neighborhoods are annuli.

1. Introduction

For any surface S and any point $*$ in the interior of S, let $\pi_1(S,*)$ denote the fundamental group of S based at $*$. When S is non-orientable, we denote by $\pi_1^+(S,*)$ the subgroup of $\pi_1(S,*)$ generated by elements which are represented by simple loops whose regular neighborhoods are annuli, called *two-sided simple loops*. A presentation of $\pi_1(S,*)$ is well known. In particular, $\pi_1(S,*)$, and also $\pi_1^+(S,*)$, are free groups if S has a boundary. For a connected closed orientable surface S, Putman [14] gave an infinite presentation of $\pi_1(S,*)$. In this paper we give infinite presentations of $\pi_1(S,*)$ and $\pi_1^+(S,*)$ whose generators are represented by simple loops, for any finite type connected surface S, as follows.

Theorem 1.1. *For any finite type connected surface S, let π be the group generated by symbols S_α for $\alpha \in \pi_1(S,*)$ which is represented by a non-trivial simple loop, and with the defining relations*
(1) $S_{\alpha^{-1}} = S_\alpha^{-1}$,
(2) $S_\alpha S_\beta = S_\gamma$ *if* $\alpha\beta = \gamma$.
Then π is isomorphic to $\pi_1(S,*)$.

Theorem 1.2. *For any finite type connected non-orientable surface S, let π^+ be the group generated by symbols S_α for $\alpha \in \pi_1^+(S,*)$ which is represented by a non-trivial simple loop, and with the defining relations*

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(1) $S_{\alpha^{-1}} = S_\alpha^{-1}$,
(2) $S_\alpha S_\beta = S_\gamma$ *if* $\alpha\beta = \gamma$,
(3) $S_\alpha S_\beta S_\alpha^{-1} = S_\gamma$ *if* $\alpha\beta\alpha^{-1} = \gamma$.
Then π^+ *is isomorphic to* $\pi_1^+(S, *)$.

These results are useful in studies on the mapping class group of S and its subgroups. For example, in [8], Theorem 1.2 is used to obtain an infinite presentation for the twist subgroup of the mapping class group of a compact non-orientable surface.

In order to prove Theorems 1.1 and 1.2, we use the following lemma.

LEMMA 1.3 (cf. [14]). *Let G and H be groups generated by sets X and Y, respectively, such that H acts on G. Suppose that* $X' \subset X$ *satisfies the following conditions.*

- $H(X') = X$.
- *For any* $x \in X'$ *and* $y \in Y$, $y^{\pm 1}(x)$ *is in the subgroup of G generated by* X'.

Then X' *generates G.*

As groups acting on π and π^+, we consider the *pure mapping class group* of S. Using this lemma, we show that π and π^+ are generated by symbols corresponding to basic generators of $\pi_1(S, *)$ and $\pi_1^+(S, *)$, respectively.

In Section 2, we define mapping class groups and pure mapping class groups of surfaces, and explain their generators. In Sections 3 and 4, we prove Theorems 1.1 and 1.2, respectively.

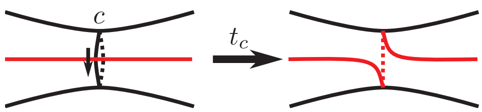
Throughout this paper, we do not distinguish a loop from its homotopy class.

2. On mapping class groups of surfaces

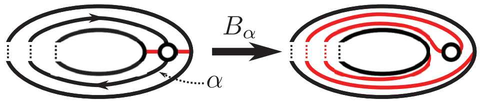
For $g \geq 0$ and $m \geq 0$, let $\Sigma_{g,m}$ be a surface which is obtained by removing m disks from a connected sum of g tori, as shown in Figure 2 (a). We call $\Sigma_{g,m}$ a genus g orientable surface with m boundary components. We define the *mapping class group* $\mathcal{M}(\Sigma_{g,m})$ of $\Sigma_{g,m}$ as the group consisting of isotopy classes of all orientation preserving diffeomorphisms of $\Sigma_{g,m}$. The *pure mapping class group* $\mathcal{PM}(\Sigma_{g,m})$ of $\Sigma_{g,m}$ is the subgroup of $\mathcal{M}(\Sigma_{g,m})$ consisting of elements which do not permute order of the boundary components of $\Sigma_{g,m}$. Regarding some boundary component of $\Sigma_{g,n+1}$ as $*$, we notice that $\mathcal{PM}(\Sigma_{g,n+1})$ acts on $\pi_1(\Sigma_{g,n}, *)$ naturally.

For $g \geq 1$ and $m \geq 0$, let $N_{g,m}$ be a surface which is obtained by removing m disks from a connected sum of g real projective planes. We call $N_{g,m}$ a genus g non-orientable surface with m boundary components. We can regard $N_{g,m}$ as a surface which is obtained by attaching g Möbius bands to g boundary components of $\Sigma_{0,g+m}$, as shown in Figure 2 (b) or (c). We call these attached Möbius bands *crosscaps*. We define the *mapping class group* $\mathcal{M}(N_{g,m})$ of $N_{g,m}$ as the group consisting of isotopy classes of all diffeomorphisms of $N_{g,m}$. The *pure mapping class group* $\mathcal{PM}(N_{g,m})$ of $N_{g,m}$ is the subgroup of $\mathcal{M}(N_{g,m})$ consisting of elements which do not permute order of the boundary components of $N_{g,m}$. Regarding some boundary component of $N_{g,n+1}$ as $*$, we notice that $\mathcal{PM}(N_{g,n+1})$ acts on $\pi_1(N_{g,n},*)$, and also $\pi_1^+(N_{g,n},*)$, naturally.

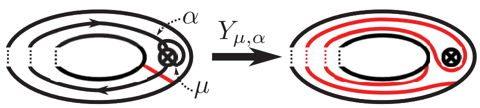
It is well known that $\mathcal{PM}(\Sigma_{g,m})$ can be generated by only *Dehn twists* (for instance see [3, 4, 12]). On the other hand, $\mathcal{PM}(N_{g,m})$ can not be generated by only Dehn twists. We need *boundary pushing maps* and *crosscap pushing maps* as generators of $\mathcal{PM}(N_{g,m})$, in addition to Dehn twists (see [11, 13]). We now define the Dehn twist, the boundary pushing map and the crosscap pushing map. For a two-sided simple closed curve c of a surface S, the Dehn twist t_c about c is the isotopy class of a map as shown in Figure 1 (a). When S is orientable, the direction of t_c is the right side with respect to an orientation of S. When S is non-orientable, the direction of t_c is indicated by an arrow written beside c as shown in Figure 1 (a). Let α be an oriented arc of S with its two endpoints at a boundary component, as shown in Figure 1 (b). The boundary pushing map B_α about α is the isotopy class of a map obtained by pushing the boundary component along α. Let α and μ be an oriented simple closed curve and a simple closed curve whose regular neighborhood is a crosscap, called a *one-sided simple loop*, of a non-orientable surface, respec-



(a) The Dehn twist t_c about c.



(b) The boundary pushing map B_α about α.



(c) The crosscap pushing map $Y_{\mu,\alpha}$ about α and μ.

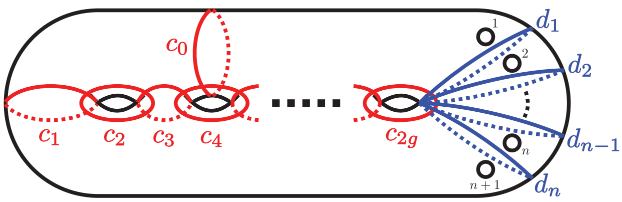
Fig. 1. Elements of mapping class groups of surfaces.

tively, such that α and μ intersect transversally at one point, as shown in Figure 1 (c). The crosscap pushing map $Y_{\mu,\alpha}$ about α and μ is the isotopy class of a map obtained by pushing the crosscap, which is the regular neighborhood of μ, along α.

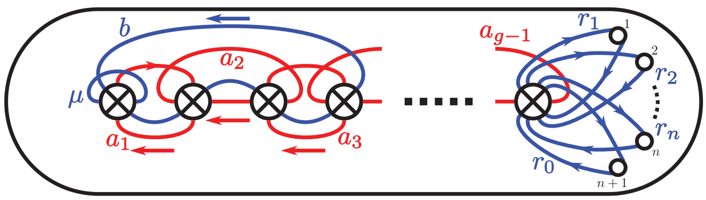
We have the following theorems.

Theorem 2.1 (c.f. [5]). *Let $c_0, c_1, \ldots, c_{2g}$ and $d_1, \ldots, d_n$ be simple closed curves of $\Sigma_{g,n+1}$ as shown in Figure 2 (a). Then $\mathcal{PM}(\Sigma_{g,n+1})$ is generated by $t_{c_0}, t_{c_1}, \ldots, t_{c_{2g}}$ and $t_{d_1}, \ldots, t_{d_n}$.*

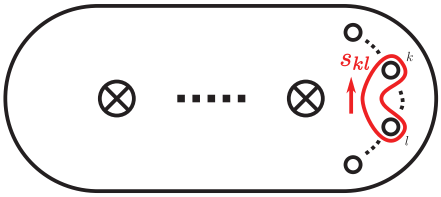
Theorem 2.2. *Let $a_1, \ldots, a_{g-1}$, b, μ, s_{kl} and $r_0, r_1, \ldots, r_n$ be simple closed curves and simple arcs of $N_{g,n+1}$ for $1 \le k < l \le n$, as shown in Figures 2 (b) and (c). Then $\mathcal{PM}(N_{g,n+1})$ is generated by $t_{a_1}, \ldots, t_{a_{g-1}}$, t_b, Y_{μ,a_1}, $t_{s_{kl}}$ and $B_{r_0}, B_{r_1}, \ldots, B_{r_n}$ for $1 \le k < l \le n$.*



(a) Simple closed curves c_i and d_k of $\Sigma_{g,n+1}$ for $0 \le i \le 2g$ and $1 \le k \le n$.



(b) Simple closed curves and simple arcs a_i, b, μ and r_k of $N_{g,n+1}$ for $1 \le i \le g-1$ and $0 \le k \le n$.



(c) A simple closed curve s_{kl} of $N_{g,n+1}$ for $1 \le k < l \le n$.

Fig. 2.

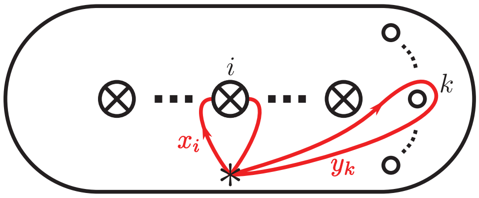


Fig. 3. Oriented simple loops x_i and y_k of $N_{g,n}$ based at $*$ for $1 \leq i \leq g$ and $1 \leq k \leq n$.

Proof. There is an exact sequence

$$\pi_1(N_{g,n}, *) \to \mathscr{PM}(N_{g,n+1}) \to \mathscr{PM}(N_{g,n}) \to 1,$$

introduced by Birman [1] for orientable surfaces. The homomorphism $\pi_1(N_{g,n}, *) \to \mathscr{PM}(N_{g,n+1})$ is defined as $\alpha \mapsto B_{\bar{\alpha}}$, where $\bar{\alpha}$ is an arc which is obtained from α by regarding $*$ as a boundary component. The homomorphism $\mathscr{PM}(N_{g,n+1}) \to \mathscr{PM}(N_{g,n})$ is defined as the map which is induced by capping the boundary component with a disk.

Let $x_1, \dots, x_g$ and $y_1, \dots, y_{n-1}$ be oriented simple loops of $N_{g,n}$ based at $*$, as shown in Figure 3. It is well known that $\pi_1(N_{g,n}, *)$ is generated by $x_1, \dots, x_g$ and $y_1, \dots, y_{n-1}$. It is easy to check that $t_{a_i} F^{i-1} Y_{\mu,a_1}^{(-1)^{i-1}} F^{1-i}(x_{i+1}) = x_i$ for $i = 1, \dots, g-1$, where $F = t_{a_1} t_{a_2} \cdots t_{a_{g-1}}$. Therefore, since the homomorphism $\pi_1(N_{g,0}, *) \to \mathscr{PM}(N_{g,1})$ sends x_g to B_{r_0}, we see that this homomorphism sends x_i to a conjugate element of B_{r_0} by $t_{a_1}, \dots, t_{a_{g-1}}$ and Y_{μ,a_1} from the relation $B_{f(r_0)} = f B_{r_0} f^{-1}$ (for example see Lemma 2.4 in [10]). In addition, regarding $*$ as the n-th boundary component of $N_{g,n+1}$, it follows that the homomorphism $\pi_1(N_{g,n}, *) \to \mathscr{PM}(N_{g,n+1})$ sends x_i to a conjugate element of B_{r_n} by $t_{a_1}, \dots, t_{a_{g-1}}$ and Y_{μ,a_1} for $n \geq 1$ and $1 \leq i \leq g$ from a similar argument, and y_k to s_{kn} for $n \geq 2$ and $1 \leq k \leq n-1$. It is known that $\mathscr{PM}(N_{g,0})$ is generated by $t_{a_1}, \dots, t_{a_{g-1}}$, t_b and Y_{μ,a_1} (see [2, 16]). Therefore using the exact sequence above, we obtain the generating set inductively.

Note that a finite generating set of $\mathscr{PM}(N_{g,n+1})$ which is different from that of Theorem 2.2 was already given by Korkmaz [10]. However we use the generating set of Theorem 2.2 in this paper.

3. Proof of Theorem 1.1

In Subsection 3.1, we prove Theorem 1.1 of the case where S is orientable. In Subsection 3.2, we prove Theorem 1.1 of the case where S is non-orientable.

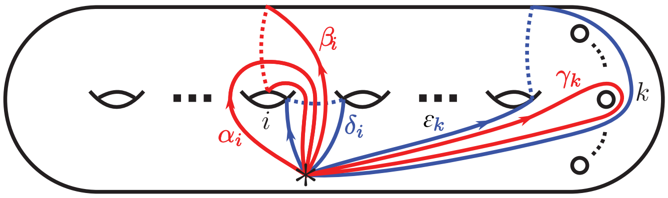


Fig. 4. Oriented simple loops α_i, β_i, γ_k, δ_i and ε_k of $\Sigma_{g,n}$ based at $*$ for $1 \le i \le g$ and $1 \le k \le n$, except for $i = g$ for δ_i.

3.1. The case where S is orientable

Let $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$ and $\gamma_1, \ldots, \gamma_{n-1}$ be oriented simple loops of $\Sigma_{g,n}$ based at $*$, as shown in Figure 4. It is well known that $\pi_1(\Sigma_{g,n}, *)$ is the free group freely generated by these loops for $n \ge 1$ and the group generated by $\alpha_1, \ldots, \alpha_g$ and $\beta_1, \ldots, \beta_g$ which has one relation $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1$ for $n = 0$, where $[x, y] = xyx^{-1}y^{-1}$.

Let X be a set consisting of S_α, where α is a non-separating simple loop or a separating simple loop which bounds the m-th boundary component for $1 \le m \le n-1$, and let X' be the following subset of X:

$$X' = \{S_{\alpha_1}, \ldots, S_{\alpha_g}, S_{\beta_1}, \ldots, S_{\beta_g}, S_{\gamma_1}, \ldots, S_{\gamma_{n-1}}\}.$$

Let Y be the generating set for $\mathscr{PM}(\Sigma_{g,n+1})$ given in Theorem 2.1. In the actions on $\pi_1(\Sigma_{g,n}, *)$ and π by $\mathscr{PM}(\Sigma_{g,n+1})$, we regard the $(n+1)$-st boundary component of $\Sigma_{g,n+1}$ as $*$. We define $f(S_\alpha) = S_{f_\sharp(\alpha)}$ for $S_\alpha \in \pi$ and $f \in \mathscr{PM}(\Sigma_{g,n+1})$, where $f_\sharp$ is the map on $\pi_1(\Sigma_{g,n}, *)$ induced from f. We prove the following proposition.

PROPOSITION 3.1. (1) *X generates π.*
(2) *$\mathscr{PM}(\Sigma_{g,n+1})(X') = X$.*
(3) *For any $x \in X'$ and $y \in Y$, $y^{\pm 1}(x)$ is in the subgroup of π generated by X'.*

In order to prove the proposition, we show the following lemma.

LEMMA 3.2. *For $1 \le i \le g-1$ and $1 \le k \le n$, S_{γ_n}, S_{δ_i} and S_{ε_k} are in the subgroup of π generated by X', where γ_n, δ_i and ε_k are simple loops of $\Sigma_{g,n}$ based at $*$ as shown in Figure 4.*

PROOF. By the relations (1) and (2) of π, we calculate

$$S_{\gamma_n} = S_{([\alpha_1,\beta_1]\cdots[\alpha_g,\beta_g]\gamma_1\cdots\gamma_{n-1})^{-1}} = ([S_{\alpha_1}, S_{\beta_1}] \cdots [S_{\alpha_g}, S_{\beta_g}] S_{\gamma_1} \cdots S_{\gamma_{n-1}})^{-1},$$

$$S_{\delta_i} = S_{\beta_i^{-1}\alpha_{i+1}\beta_{i+1}\alpha_{i+1}^{-1}} = S_{\beta_i}^{-1} S_{\alpha_{i+1}} S_{\beta_{i+1}} S_{\alpha_{i+1}}^{-1},$$

$$S_{\varepsilon_k} = S_{\beta_g^{-1}\gamma_1\cdots\gamma_k} = S_{\beta_g}^{-1} S_{\gamma_1} \cdots S_{\gamma_k}$$

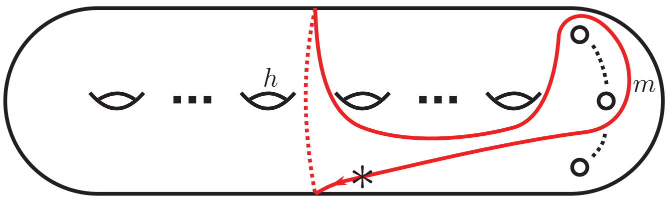


Fig. 5. An oriented simple loop $[\alpha_1, \beta_1] \cdots [\alpha_h, \beta_h]\gamma_1 \cdots \gamma_m$, one of a component of whose complement is homeomorphic to $\Sigma_{h,m+1}$, for $0 \le h \le g$ and $0 \le m \le n$.

for $1 \le i \le g-1$ and $1 \le k \le n$. Since each symbol of the right hand sides is in X', we get the claim.

Proof (Proof of Proposition 3.1). (1) For any generator S_α of π, if α is a non-separating simple loop, S_α is in X clearly. If α is a separating simple loop, one of a component of the complement of α is homeomorphic to $\Sigma_{h,m+1}$ for some $0 \le h \le g$ and $0 \le m \le n$. Therefore, there is $f \in \mathscr{PM}(\Sigma_{g,n+1})$ such that $\alpha = f_\sharp([\alpha_1, \beta_1] \cdots [\alpha_h, \beta_h]\gamma_{k_1} \cdots \gamma_{k_m})$ for some $1 \le k_1 < \cdots < k_m \le n$ (see Figure 5). Then, by the relation (2) of π, we have $S_\alpha = [S_{f_\sharp(\alpha_1)}, S_{f_\sharp(\beta_1)}] \cdots [S_{f_\sharp(\alpha_h)}, S_{f_\sharp(\beta_h)}]S_{f_\sharp(\gamma_{k_1})} \cdots S_{f_\sharp(\gamma_{k_m})}$. Since each symbol of the right hand side is in X, we conclude that X generates π.

(2) For any $S_\alpha \in X$, if α is a non-separating simple loop, there is $f \in \mathscr{PM}(\Sigma_{g,n+1})$ such that $f_\sharp(\alpha_1) = \alpha$, and hence $f(S_{\alpha_1}) = S_\alpha$. If α is a separating simple loop which bounds the m-th boundary component for $1 \le m \le n-1$, there is $f \in \mathscr{PM}(\Sigma_{g,n+1})$ such that $f_\sharp(\gamma_m) = \alpha$, and hence $f(S_{\gamma_m}) = S_\alpha$. Therefore we obtain the claim.

(3) In this proof, we omit details of calculations.

Let $y = t_{c_0}$. We calculate

$$y(S_{\alpha_2}) = S_{\alpha_2\beta_2^{-1}} \overset{(1),(2)}{=} S_{\alpha_2}S_{\beta_2}^{-1}, \qquad y^{-1}(S_{\alpha_2}) = S_{\alpha_2\beta_2} \overset{(2)}{=} S_{\alpha_2}S_{\beta_2}$$

and $y^{\pm 1}(x) = x$ for any other $x \in X'$.

Let $y = t_{c_{2i-1}}$ for $1 \le i \le g$. We calculate

$$y(S_{\alpha_{i-1}}) = S_{\alpha_{i-1}\delta_{i-1}} \overset{(2)}{=} S_{\alpha_{i-1}}S_{\delta_{i-1}},$$

$$y^{-1}(S_{\alpha_{i-1}}) = S_{\alpha_{i-1}\delta_{i-1}^{-1}} \overset{(1),(2)}{=} S_{\alpha_{i-1}}S_{\delta_{i-1}}^{-1},$$

$$y(S_{\alpha_i}) = S_{\delta_{i-1}^{-1}\alpha_i} \overset{(1),(2)}{=} S_{\delta_{i-1}}^{-1}S_{\alpha_i}, \qquad y^{-1}(S_{\alpha_i}) = S_{\delta_{i-1}\alpha_i} \overset{(2)}{=} S_{\delta_{i-1}}S_{\alpha_i},$$

$$y^{\pm 1}(S_{\beta_{i-1}}) = S_{\delta_{i-1}^{\mp 1}\beta_{i-1}\delta_{i-1}^{\pm 1}} \overset{(1),(2)}{=} S_{\delta_{i-1}}^{\mp 1}S_{\beta_{i-1}}S_{\delta_{i-1}}^{\pm 1}$$

and $y^{\pm 1}(x) = x$ for any other $x \in X'$.

Let $y = t_{c_{2i}}$ for $1 \le i \le g$. We calculate

$$y(S_{\beta_i}) = S_{\beta_i \alpha_i} \overset{(2)}{=} S_{\beta_i} S_{\alpha_i}, \qquad y^{-1}(S_{\beta_i}) = S_{\beta_i \alpha_i^{-1}} \overset{(1),(2)}{=} S_{\beta_i} S_{\alpha_i}^{-1}$$

and $y^{\pm 1}(x) = x$ for any other $x \in X'$.

Let $y = t_{d_k}$ for $1 \le k \le n$. We calculate

$$y(S_{\alpha_g}) = S_{\alpha_g \varepsilon_k} \overset{(2)}{=} S_{\alpha_g} S_{\varepsilon_k}, \qquad y^{-1}(S_{\alpha_g}) = S_{\alpha_g \varepsilon_k^{-1}} \overset{(1),(2)}{=} S_{\alpha_g} S_{\varepsilon_k}^{-1},$$

$$y^{\pm 1}(S_{\beta_g}) = S_{\varepsilon_k^{\mp 1} \beta_g \varepsilon_k^{\pm 1}} \overset{(1),(2)}{=} S_{\varepsilon_k}^{\mp 1} S_{\beta_g} S_{\varepsilon_k}^{\pm 1},$$

$$y^{\pm 1}(S_{\gamma_l}) = S_{\varepsilon_k^{\mp 1} \gamma_l \varepsilon_k^{\pm 1}} \overset{(1),(2)}{=} S_{\varepsilon_k}^{\mp 1} S_{\gamma_l} S_{\varepsilon_k}^{\pm 1}$$

for $l \le k$, and $y^{\pm 1}(x) = x$ for any other $x \in X'$.

Hence we have that for any $x \in X'$ and $y \in Y$, $y^{\pm 1}(x)$ is in the subgroup of π generated by X', by Lemma 3.2.

Proof (Proof of Theorem 1.1 of the case where S is orientable). By Lemma 1.3 and Proposition 3.1, it follows that π is generated by X'. There is a natural map $\pi \to \pi_1(\Sigma_{g,n}, *)$. The relations (1) and (2) of π are satisfied in $\pi_1(\Sigma_{g,n}, *)$ clearly. Hence the map is a homomorphism. In addition, the relation $[S_{\alpha_1}, S_{\beta_1}] \cdots [S_{\alpha_g}, S_{\beta_g}] = 1$ is obtained from the relation (2) of π for $n = 0$. Therefore the map is an isomorphism for any $n \ge 0$. Thus we complete the proof.

3.2. The case where S is non-orientable

Let $x_1, \dots, x_g$ and $y_1, \dots, y_{n-1}$ be oriented simple loops of $N_{g,n}$ based at $*$, as shown in Figure 3. It is well known that $\pi_1(N_{g,n}, *)$ is the free group freely generated by these loops for $n \ge 1$ and the group generated by $x_1, \dots, x_g$ which has one relation $x_1^2 \cdots x_g^2 = 1$ for $n = 0$.

Let X be a set consisting of S_α, where α is a one-sided simple loop whose complement is non-orientable, or a separating simple loop which bounds the m-th boundary component for $1 \le m \le n-1$, and let X' be the following subset of X:

$$X' = \{S_{x_1}, \dots, S_{x_g}, S_{y_1}, \dots, S_{y_{n-1}}\}.$$

Let Y be the generating set for $\mathscr{PM}(N_{g,n+1})$ given in Theorem 2.2. In the actions on $\pi_1(N_{g,n}, *)$ and π by $\mathscr{PM}(N_{g,n+1})$, we regard the $(n+1)$-st boundary component of $N_{g,n+1}$ as $*$. We define $f(S_\alpha) = S_{f_\sharp(\alpha)}$ for $S_\alpha \in \pi$ and $f \in \mathscr{PM}(N_{g,n+1})$, where $f_\sharp$ is the map on $\pi_1(N_{g,n}, *)$ induced from f. We prove the following proposition.

Proposition 3.3. (1) *X generates* π.

(2) $\mathscr{PM}(N_{g,n+1})(X') = X$.

(3) *For any* $x \in X'$ *and* $y \in Y$, $y^{\pm 1}(x)$ *is in the subgroup of* π *generated by* X'.

In order to prove the proposition, we show the following lemma.

Lemma 3.4. S_{y_n} *is in the subgroup of* π *generated by* X', *where* y_n *is a simple loop of* $N_{g,n}$ *as shown in Figure* 3.

Proof. By the relations (1) and (2) of π, we calculate

$$S_{y_n} = S_{(x_1^2 \cdots x_g^2 y_1 \cdots y_{n-1})^{-1}} = (S_{x_1}^2 \cdots S_{x_g}^2 S_{y_1} \cdots S_{y_{n-1}})^{-1}.$$

Since each symbol of the right hand side is in X', we get the claim.

Proof (Proof of Proposition 3.3). (1) For any generator S_α of π, the complement of α is homeomorphic to either

(1) $N_{g-1,n+1}$,

(2) $N_{g-2,n+2}$,

(3) $\Sigma_{h,n+r}$ if $g = 2h + r$ for $r = 1, 2$,

(4) $N_{h,m+1} \sqcup N_{g-h,n-m+1}$ for $1 \le h \le g-1$ and $0 \le m \le n$ or

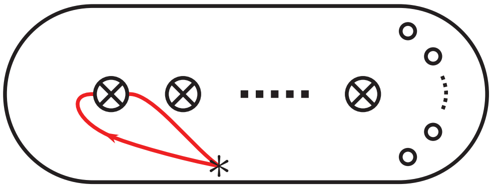
(5) $\Sigma_{h,m+1} \sqcup N_{g-2h,n-m+1}$ for $0 \le h \le \frac{g-1}{2}$ and $0 \le m \le n$

(see [15]). Therefore, there is $f \in \mathscr{PM}(N_{g,n+1})$ such that $\alpha = f_\sharp(\beta)$, where β is either one of the simple loops as in Figure 6. For the case (a), we have $S_\alpha = S_{f_\sharp(x_1)}$. For the case (b), by the relation (2) of π, we have $S_\alpha = S_{f_\sharp(x_1)} S_{f_\sharp(x_2)}$. For the cases (c), by the relation (2) of π, we have $S_\alpha = S_{f_\sharp(x_1)} \cdots S_{f_\sharp(x_g)}$. For the case (d), by the relation (2) of π, we have $S_\alpha = S_{f_\sharp(x_1)}^2 \cdots S_{f_\sharp(x_h)}^2 S_{f_\sharp(y_{k_1})} \cdots S_{f_\sharp(y_{k_m})}$ for some $1 \le k_1 < \cdots < k_m \le n$. For the case (e), by the relations (1) and (2) of π, we have

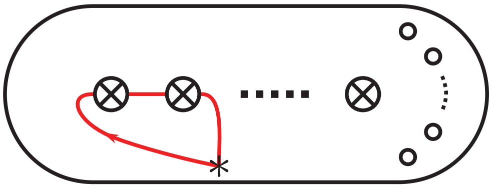
$$S_\alpha = S_{f_\sharp(x_1)} \cdots S_{f_\sharp(x_{2h})} S_{f_\sharp(x_{2h+1})}^{-1} S_{f_\sharp(x_{2h})}^{-2} \cdots S_{f_\sharp(x_2)}^{-2} S_{f_\sharp(x_1)}^{-1}$$
$$\cdot S_{f_\sharp(x_2)} \cdots S_{f_\sharp(x_{2h+1})} S_{f_\sharp(y_{k_1})} \cdots S_{f_\sharp(y_{k_m})}$$

for some $1 \le k_1 < \cdots < k_m \le n$ if $h \ne 0$. If $h = 0$, by the relation (2) of π, we have $S_\alpha = S_{f_\sharp(y_{k_1})} \cdots S_{f_\sharp(y_{k_m})}$ for some $1 \le k_1 < \cdots < k_m \le n$. Since each symbol of the right hand sides is in X, we conclude that X generates π.

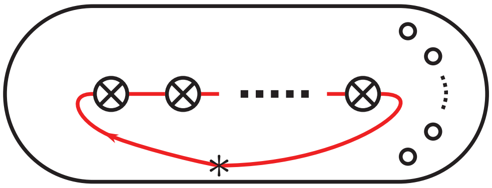
(2) For any $S_\alpha \in X$, if α is a one-sided simple loop whose complement is non-orientable, there is $f \in \mathscr{PM}(N_{g,n+1})$ such that $f_\sharp(x_1) = \alpha$, and hence $f(S_{x_1}) = S_\alpha$. If α is a separating simple loop which bounds the m-th boundary component for $1 \le m \le n-1$, there is $f \in \mathscr{PM}(N_{g,n+1})$ such that $f_\sharp(y_m) = \alpha$, and hence $f(S_{y_m}) = S_\alpha$. Therefore we obtain the claim.



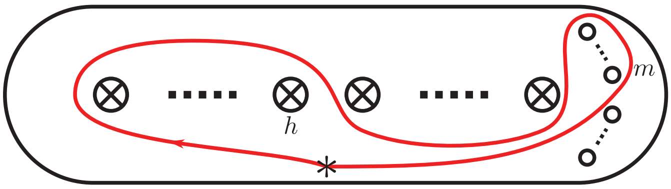
(a) A standard position oriented simple loop whose complement is homeomorphic to $N_{g-1,n+1}$.



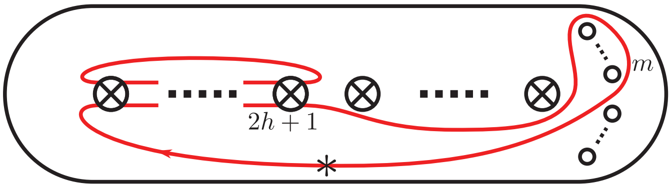
(b) A standard position oriented simple loop whose complement is homeomorphic to $N_{g-2,n+2}$.



(c) A standard position oriented simple loop whose complement is homeomorphic to $\Sigma_{h,n+r}$ if $g = 2h+r$ for $r = 1, 2$.



(d) A standard position oriented simple loop whose complement is homeomorphic to $N_{h,m+1} \sqcup N_{g-h,n-m+1}$ for $1 \leq h \leq g-1$ and $0 \leq m \leq n$.



(e) A standard position oriented simple loop whose complement is homeomorphic to $\Sigma_{h,m+1} \sqcup N_{g-2h,n-m+1}$ for $0 \leq h \leq \dfrac{g-1}{2}$ and $0 \leq m \leq n$.

Fig. 6.

(3) In this proof, we omit details of calculations.

Let $y = t_{a_i}$ for $1 \leq i \leq g$. We calculate

$$y(S_{x_i}) = S_{x_i x_{i+1}^{-1} x_i^{-1}} \overset{(1),(2)}{=} S_{x_i} S_{x_{i+1}}^{-1} S_{x_i}^{-1},$$

$$y^{-1}(S_{x_i}) = S_{x_i^2 x_{i+1}} \overset{(2)}{=} S_{x_i}^2 S_{x_{i+1}},$$

$$y(S_{x_{i+1}}) = S_{x_i x_{i+1}^2} \overset{(2)}{=} S_{x_i} S_{x_{i+1}}^2,$$

$$y^{-1}(S_{x_{i+1}}) = S_{x_{i+1}^{-1} x_i^{-1} x_{i+1}} \overset{(1),(2)}{=} S_{x_{i+1}}^{-1} S_{x_i}^{-1} S_{x_{i+1}}$$

and $y^{\pm 1}(x) = x$ for any other $x \in X'$.

Let $y = t_b$. We calculate

$$y(S_{x_1}) = S_{x_1 x_2 x_3 x_4^{-1} x_3^{-2} x_2^{-2} x_1^{-1}} \overset{(1),(2)}{=} S_{x_1} S_{x_2} S_{x_3} S_{x_4}^{-1} S_{x_3}^{-2} S_{x_2}^{-2} S_{x_1}^{-1},$$

$$y^{-1}(S_{x_1}) = S_{x_1^2 x_2^2 x_3^2 x_4 x_3^{-1} x_2^{-1}} \overset{(1),(2)}{=} S_{x_1}^2 S_{x_2}^2 S_{x_3}^2 S_{x_4} S_{x_3}^{-1} S_{x_2}^{-1},$$

$$y(S_{x_2}) = S_{x_1 x_2^2 x_3^2 x_4 x_3^{-1}} \overset{(1),(2)}{=} S_{x_1} S_{x_2}^2 S_{x_3}^2 S_{x_4} S_{x_3}^{-1},$$

$$y^{-1}(S_{x_2}) = S_{x_2 x_3 x_4^{-1} x_3^{-2} x_2^{-2} x_1^{-1} x_2} \overset{(1),(2)}{=} S_{x_2} S_{x_3} S_{x_4}^{-1} S_{x_3}^{-2} S_{x_2}^{-2} S_{x_1}^{-1} S_{x_2},$$

$$y(S_{x_3}) = S_{x_3 x_4^{-1} x_3^{-2} x_2^{-2} x_1^{-1} x_2 x_3} \overset{(1),(2)}{=} S_{x_3} S_{x_4}^{-1} S_{x_3}^{-2} S_{x_2}^{-2} S_{x_1}^{-1} S_{x_2} S_{x_3},$$

$$y^{-1}(S_{x_3}) = S_{x_2^{-1} x_1 x_2^2 x_3^2 x_4} \overset{(1),(2)}{=} S_{x_2}^{-1} S_{x_1} S_{x_2}^2 S_{x_3}^2 S_{x_4},$$

$$y(S_{x_4}) = S_{x_3^{-1} x_2^{-1} x_1 x_2^2 x_3^2 x_4^2} \overset{(1),(2)}{=} S_{x_3}^{-1} S_{x_2}^{-1} S_{x_1} S_{x_2}^2 S_{x_3}^2 S_{x_4}^2,$$

$$y^{-1}(S_{x_4}) = S_{x_4^{-1} x_3^{-2} x_2^{-2} x_1^{-1} x_2 x_3 x_4} \overset{(1),(2)}{=} S_{x_4}^{-1} S_{x_3}^{-2} S_{x_2}^{-2} S_{x_1}^{-1} S_{x_2} S_{x_3} S_{x_4}$$

and $y^{\pm 1}(x) = x$ for any other $x \in X'$.

Let $y = Y_{\mu, a_1}$. We calculate

$$y(S_{x_1}) = S_{x_1^2 x_2 x_1^{-1} x_2^{-1} x_1^{-2}} \overset{(1),(2)}{=} S_{x_1}^2 S_{x_2} S_{x_1}^{-1} S_{x_2}^{-1} S_{x_1}^{-2},$$

$$y^{-1}(S_{x_1}) = S_{x_2^{-1} x_1^{-1} x_2} \overset{(1),(2)}{=} S_{x_2}^{-1} S_{x_1}^{-1} S_{x_2},$$

$$y(S_{x_2}) = S_{x_1^2 x_2} \overset{(2)}{=} S_{x_1}^2 S_{x_2}, \qquad y^{-1}(S_{x_2}) = S_{x_2^{-1} x_1^2 x_2^2} \overset{(1),(2)}{=} S_{x_2}^{-1} S_{x_1}^2 S_{x_2}^2$$

and $y^{\pm 1}(x) = x$ for any other $x \in X'$.

Let $y = B_{r_k}$ for $1 \le k \le n$. We calculate

$$y(S_{x_g}) = S_{x_g^2 y_k x_g^{-1}} \overset{(1),(2)}{=} S_{x_g}^2 S_{y_k} S_{x_g}^{-1}, \qquad y^{-1}(S_{x_g}) = S_{x_g y_k} \overset{(2)}{=} S_{x_g} S_{y_k},$$

$$y(S_{y_k}) = S_{x_g y_k x_g^{-1}} \overset{(1),(2)}{=} S_{x_g} S_{y_k} S_{x_g}^{-1},$$

$$y^{-1}(S_{y_k}) = S_{y_k^{-1} x_g^{-1} y_k x_g y_k} \overset{(1),(2)}{=} S_{y_k}^{-1} S_{x_g}^{-1} S_{y_k} S_{x_g} S_{y_k},$$

$$y(S_{y_l}) = S_{x_g y_k^{-1} x_g^{-1} y_k^{-1} y_l y_k x_g y_k x_g^{-1}} \overset{(1),(2)}{=} S_{x_g} S_{y_k}^{-1} S_{x_g}^{-1} S_{y_k}^{-1} S_{y_l} S_{y_k} S_{x_g} S_{y_k} S_{x_g}^{-1},$$

$$y^{-1}(S_{y_l}) = S_{y_k^{-1} x_g^{-1} y_k^{-1} x_g y_l x_g^{-1} y_k x_g y_k} \overset{(1),(2)}{=} S_{y_k}^{-1} S_{x_g}^{-1} S_{y_k}^{-1} S_{x_g} S_{y_l} S_{x_g}^{-1} S_{y_k} S_{x_g} S_{y_k}$$

for $l < k$, and $y^{\pm 1}(x) = x$ for any other $x \in X'$.

Let $y = B_{r_0}$. We calculate

$$y^{\pm 1}(S_{x_j}) = S_{x_g^{\mp 1} x_j x_g^{\pm 1}} \overset{(1),(2)}{=} S_{x_g}^{\mp 1} S_{x_j} S_{x_g}^{\pm 1},$$

$$y^{\pm 1}(S_{y_l}) = S_{x_g^{\mp 1} y_l x_g^{\pm 1}} \overset{(1),(2)}{=} S_{x_g}^{\mp 1} S_{y_l} S_{x_g}^{\pm 1}$$

for $1 \le j \le g$ and $1 \le l \le n - 1$.

Let $y = t_{s_{kl}}$ for $1 \le k < l \le n$. We calculate

$$y(S_{y_k}) = S_{y_k y_l y_k y_l^{-1} y_k^{-1}} \overset{(1),(2)}{=} S_{y_k} S_{y_l} S_{y_k} S_{y_l}^{-1} S_{y_k}^{-1},$$

$$y^{-1}(S_{y_k}) = S_{y_l^{-1} y_k y_l} \overset{(1),(2)}{=} S_{y_l}^{-1} S_{y_k} S_{y_l},$$

$$y(S_{y_l}) = S_{y_k y_l y_k^{-1}} \overset{(1),(2)}{=} S_{y_k} S_{y_l} S_{y_k}^{-1},$$

$$y^{-1}(S_{y_l}) = S_{y_l^{-1} y_k^{-1} y_l y_k y_l} \overset{(1),(2)}{=} S_{y_l}^{-1} S_{y_k}^{-1} S_{y_l} S_{y_k} S_{y_l},$$

$$y(S_{y_m}) = S_{[y_k, y_l] y_m [y_k, y_l]^{-1}} \overset{(1),(2)}{=} [S_{y_k}, S_{y_l}] S_{y_m} [S_{y_k}, S y_l]^{-1},$$

$$y^{-1}(S_{y_m}) = S_{[y_l^{-1}, y_k^{-1}] y_m [y_l^{-1}, y_k^{-1}]^{-1}} \overset{(1),(2)}{=} [S_{y_l}^{-1}, S_{y_k}^{-1}] S_{y_m} [S_{y_l}^{-1}, S_{y_k}^{-1}]^{-1}$$

for $k < m < l$, and $y^{\pm 1}(x) = x$ for any other $x \in X'$.

Hence we have that for any $x \in X'$ and $y \in Y$, $y^{\pm 1}(x)$ is in the subgroup of π generated by X', by Lemma 3.4.

PROOF (Proof of Theorem 1.1 of the case where S is non-orientable). By Lemma 1.3 and Proposition 3.3, it follows that π is generated by X'. There is a natural map $\pi \to \pi_1(N_{g,n}, *)$. The relations (1) and (2) of π are satisfied in $\pi_1(N_{g,n},*)$ clearly. Hence the map is a homomorphism. In addition, the relation $S_{x_1}^2 \cdots S_{x_g}^2 = 1$ is obtained from the relation (2) of π for $n = 0$. Therefore the map is an isomorphism for any $n \geq 0$. Thus we complete the proof.

4. Proof of Theorem 1.2

Let $x_{ij} = x_i x_j$ and $z_k = x_g y_k x_g^{-1}$ for $1 \leq i, j \leq g$ and $1 \leq k \leq n-1$, where $x_1, \dots, x_g$ and $y_1, \dots, y_{n-1}$ are simple loops of $N_{g,n}$ as shown in Figure 3. We first consider a presentation for $\pi_1^+(N_{g,n},*)$ as follows.

Lemma 4.1. *$\pi_1^+(N_{g,n},*)$ is the free group freely generated by $x_{12}, \dots, x_{g-1\,g}$, $x_{11}, \dots, x_{gg}$, $y_1, \dots, y_{n-1}$ and $z_1, \dots, z_{n-1}$ for $n \geq 1$, and the group generated by $x_{12}, \dots, x_{g-1\,g}$ and $x_{11}, \dots, x_{gg}$ which has two relations $x_{11} \cdots x_{gg} = 1$ and $x_{gg} x_{g-1\,g}^{-1} x_{g-1\,g-1} x_{g-2\,g-1}^{-1} \cdots x_{22} x_{12}^{-1} x_{11} x_{12} \cdots x_{g-1\,g} = 1$ for $n = 0$.*

Proof. It is known that $\pi_1^+(N_{g,n},*)$ is an index two subgroup of $\pi_1(N_{g,n},*)$ (see [8]). Hence we can obtain a presentation of $\pi_1^+(N_{g,n},*)$ by the Reidemeister Schreier method (for details, for instance see [6]). Note that $\pi_1(N_{g,n},*)$ is generated by $x_1, \dots, x_g$ and $y_1, \dots, y_{n-1}$. We chose $\{1, x_g\}$ as a Schreier transversal for $\pi_1^+(N_{g,n},*)$ in $\pi_1(N_{g,n},*)$. Then it follows that $\pi_1^+(N_{g,n},*)$ is generated by $x_1 x_g^{-1}, \dots, x_{g-1} x_g^{-1}$, $x_g x_1, \dots, x_g x_g$, $y_1, \dots, y_{n-1}$ and $z_1, \dots, z_{n-1}$ (see [7]). In addition, we see

$$1(x_1^2 \cdots x_g^2)1^{-1} = x_1 x_g^{-1} \cdot x_g x_1 \cdot x_2 x_g^{-1} \cdot x_g x_2 \cdots x_{g-1} x_g^{-1} \cdot x_g x_{g-1} \cdot x_g x_g,$$

$$x_g(x_1^2 \cdots x_g^2) x_g^{-1} = x_g x_1 \cdot x_1 x_g^{-1} \cdot x_g x_2 \cdot x_2 x_g^{-1} \cdots x_g x_{g-1} \cdot x_{g-1} x_g^{-1} \cdot x_g x_g.$$

Hence when $n = 0$, we have two relations

$$x_1 x_g^{-1} \cdot x_g x_1 \cdot x_2 x_g^{-1} \cdot x_g x_2 \cdots x_{g-1} x_g^{-1} \cdot x_g x_{g-1} \cdot x_g x_g = 1,$$

$$x_g x_1 \cdot x_1 x_g^{-1} \cdot x_g x_2 \cdot x_2 x_g^{-1} \cdots x_g x_{g-1} \cdot x_{g-1} x_g^{-1} \cdot x_g x_g = 1.$$

Let G be the group which has the presentation of the lemma. We next show that G is isomorphic to $\pi_1^+(N_{g,n},*)$. Let $\varphi : G \to \pi_1^+(N_{g,n},*)$ and $\psi : \pi_1^+(N_{g,n},*) \to G$ be homomorphisms defined as

$$\varphi(x_{i\,i+1}) = x_i x_g^{-1} \cdot x_g x_{i+1}, \qquad \varphi(x_{jj}) = x_j x_g^{-1} \cdot x_g x_j, \qquad \varphi(y_k) = y_k, \qquad \varphi(z_k) = z_k,$$

$$\psi(x_i x_g^{-1}) = x_{i\,i+1} x_{i+1\,i+1}^{-1} x_{i+1\,i+2} x_{i+2\,i+2}^{-1} \cdots x_{g-1\,g} x_{gg}^{-1},$$

$$\psi(x_g x_j) = x_{gg} x_{g-1\,g}^{-1} x_{g-1\,g-1} x_{g-2\,g-1}^{-1} \cdots x_{j+1\,j+1} x_{j\,j+1}^{-1} x_{jj},$$

$$\psi(y_k) = y_k, \qquad \psi(z_k) = z_k$$

for $1 \leq i \leq g-1$, $1 \leq j \leq g$ and $1 \leq k \leq n-1$. We calculate

$$\varphi(x_{11}\cdots x_{g-1\,g-1}x_{gg}) = x_1x_g^{-1}\cdot x_gx_1\cdots x_{g-1}x_g^{-1}\cdot x_gx_{g-1}\cdot x_gx_g,$$

$$\begin{aligned}
&\varphi(x_{gg}x_{g-1\,g}^{-1}x_{g-1\,g-1}x_{g-2\,g-1}^{-1}\cdots x_{22}x_{12}^{-1}x_{11}x_{12}\cdots x_{g-1\,g})\\
&\quad = x_gx_g(x_{g-1}x_g^{-1}\cdot x_gx_g)^{-1}(x_{g-1}x_g^{-1}\cdot x_gx_{g-1})(x_{g-2}x_g^{-1}\cdot x_gx_{g-1})^{-1}\\
&\qquad \cdots(x_2x_g^{-1}\cdot x_gx_2)(x_1x_g^{-1}\cdot x_gx_2)^{-1}(x_1x_g^{-1}\cdot x_gx_1)(x_1x_g^{-1}\cdot x_gx_2)\\
&\qquad \cdots(x_{g-1}x_g^{-1}\cdot x_gx_g)\\
&\quad = x_gx_1\cdot x_1x_g^{-1}\cdot x_gx_2\cdot x_2x_g^{-1}\cdots x_gx_{g-1}\cdot x_{g-1}x_g^{-1}\cdot x_gx_g,
\end{aligned}$$

$$\begin{aligned}
&\psi(x_1x_g^{-1}\cdot x_gx_1\cdot x_2x_g^{-1}\cdot x_gx_2\cdots x_{g-1}x_g^{-1}\cdot x_gx_{g-1}\cdot x_gx_g)\\
&\quad = x_{11}x_{22}\cdots x_{g-1\,g-1}x_{gg},\\
&\psi(x_gx_1\cdot x_1x_g^{-1}\cdot x_gx_2\cdot x_2x_g^{-1}\cdots x_gx_{g-1}\cdot x_{g-1}x_g^{-1}\cdot x_gx_g)\\
&\quad = x_{gg}x_{g-1\,g}^{-1}x_{g-1\,g-1}x_{g-2\,g-1}^{-1}\cdots x_{22}x_{12}^{-1}x_{11}x_{12}\cdots x_{g-1\,g}.
\end{aligned}$$

Hence φ and ψ are well defined even if $n=0$. In addition, we have

$$\begin{aligned}
\psi\varphi(x_{i\,i+1}) &= \psi(x_ix_g^{-1}\cdot x_gx_{i+1})\\
&= x_{i\,i+1}x_{i+1\,i+1}^{-1}\cdots x_{g-1\,g}x_{gg}^{-1}\cdot x_{gg}x_{g-1\,g}^{-1}\cdots x_{i+2\,i+2}x_{i+1\,i+2}^{-1}x_{i+1\,i+1}\\
&= x_{i\,i+1},\\
\psi\varphi(x_{jj}) &= \psi(x_jx_g^{-1}\cdot x_gx_j)\\
&= x_{j\,j+1}x_{j+1\,j+1}^{-1}\cdots x_{g-1\,g}x_{gg}^{-1}\cdot x_{gg}x_{g-1\,g}^{-1}\cdots x_{j+1\,j+1}x_{j\,j+1}^{-1}x_{jj}\\
&= x_{jj},\\
\psi\varphi(y_k) &= \psi(y_k) = y_k, \qquad \psi\varphi(z_k) = \psi(z_k) = z_k,\\
\varphi\psi(x_ix_g^{-1}) &= \varphi(x_{i\,i+1}x_{i+1\,i+1}^{-1}\cdots x_{g-1\,g}x_{gg}^{-1})\\
&= (x_ix_g^{-1}\cdot x_gx_{i+1})(x_{i+1}x_g^{-1}\cdot x_gx_{i+1})^{-1}\cdots(x_{g-1}x_g^{-1}\cdot x_gx_g)(x_gx_g)^{-1}\\
&= x_ix_g^{-1},\\
\varphi\psi(x_gx_j) &= \varphi(x_{gg}x_{g-1\,g}^{-1}\cdots x_{j+1\,j+1}x_{j\,j+1}^{-1}x_{jj})\\
&= (x_gx_g)(x_{g-1}x_g^{-1}\cdot x_gx_g)^{-1}\cdots(x_{j+1}x_g^{-1}\cdot x_gx_{j+1})\\
&\quad \cdot(x_jx_g^{-1}\cdot x_gx_{j+1})^{-1}(x_jx_g^{-1}\cdot x_gx_j) = x_gx_j,\\
\varphi\psi(y_k) &= \varphi(y_k) = y_k, \qquad \varphi\psi(z_k) = \varphi(z_k) = z_k.
\end{aligned}$$

Therefore φ and ψ are the isomorphisms. Thus we finish the proof.

Let X be a set consisting of S_α, where α is a non-separating two-sided simple loop whose complement is non-orientable, or a separating simple loop which bounds the m-th boundary component for $1 \le m \le n-1$ or one crosscap whose complement is non-orientable, and let X' be the following subset of X:

$$X' = \{S_{x_{12}}, \dots, S_{x_{g-1\,g}}, S_{x_{11}}, \dots, S_{x_{gg}}, S_{y_1}, \dots, S_{y_{n-1}}, S_{z_1}, \dots, S_{z_{n-1}}\}.$$

Let Y be the generating set for $\mathscr{PM}(N_{g,n+1})$ given in Theorem 2.2. In the actions on $\pi_1^+(N_{g,n}, *)$ and π^+ by $\mathscr{PM}(N_{g,n+1})$, we regard the $(n+1)$-st boundary component of $N_{g,n+1}$ as $*$. Recall that the action $f(S_\alpha)$ of $f \in \mathscr{PM}(N_{g,n+1})$ on $S_\alpha \in \pi^+$ was defined in Subsection 3.2. We prove the following proposition.

Proposition 4.2. (1) *X generates π^+.*
(2) *$\mathscr{PM}(N_{g,n+1})(X') = X$.*
(3) *For any $x \in X'$ and $y \in Y$, $y^{\pm 1}(x)$ is in the subgroup of π^+ generated by X'.*

In order to prove the proposition, we show the following lemma.

Lemma 4.3. *$S_{x_{ij}}$, $S_{x_{ji}}$, S_{y_n} and S_{z_n} are in the subgroup of π^+ generated by X' for $1 \le i < j \le g$, where y_n is a simple loop of $N_{g,n}$ as shown in Figure 3 and $z_n = x_g y_n x_g^{-1}$.*

Proof. For $1 \le i < j \le g$, if $j - i = 1$, then x_{ij} is in X' clearly. If $j - i \ge 2$, we calculate

$$\begin{aligned} S_{x_{ij}} &= S_{x_{i\,j-1}x_{j-1\,j-1}^{-1}x_{j-1\,j}} \overset{(2)}{=} S_{x_{i\,j-1}x_{j-1\,j}} S_{x_{j-1\,j}^{-1}x_{j-1\,j-1}^{-1}x_{j-1\,j}} \\ &\overset{(1),(2),(3)}{=} S_{x_{i\,j-1}} S_{x_{j-1\,j}} S_{x_{j-1\,j}}^{-1} S_{x_{j-1\,j-1}}^{-1} S_{x_{j-1\,j}} = S_{x_{i\,j-1}} S_{x_{j-1\,j-1}}^{-1} S_{x_{j-1\,j}}. \end{aligned}$$

By induction on $j - i$, it follows that $S_{x_{ij}}$ is in the subgroup of π^+ generated by X'. In addition, we calculate

$$\begin{aligned} S_{x_{ji}} &= S_{x_{jj}x_{ij}^{-1}x_{ii}} \overset{(2)}{=} S_{x_{ij}^{-1}} S_{x_{ij}x_{jj}x_{ij}^{-1}x_{ii}} \overset{(1),(2)}{=} S_{x_{ij}}^{-1} S_{x_{ij}x_{jj}x_{ij}^{-1}} S_{x_{ii}} \\ &\overset{(3)}{=} S_{x_{ij}}^{-1} S_{x_{ij}} S_{x_{jj}} S_{x_{ij}}^{-1} S_{x_{ii}} = S_{x_{jj}} S_{x_{ij}}^{-1} S_{x_{ii}}. \end{aligned}$$

Hence $S_{x_{ji}}$ is also in the subgroup of π^+ generated by X'. Moreover, by the relations (1) and (2) of π^+, we calculate

$$\begin{aligned} S_{y_n} &= S_{(x_{11}\cdots x_{gg}y_1\cdots y_{n-1})^{-1}} = (S_{x_{11}} \cdots S_{x_{gg}} S_{y_1} \cdots S_{y_{n-1}})^{-1}, \\ S_{z_n} &= S_{(x_{g1}x_{12}x_{23}\cdots x_{g-1\,g}z_1\cdots z_{n-1})^{-1}} \\ &= (S_{x_{g1}} S_{x_{12}} S_{x_{23}} \cdots S_{x_{g-1\,g}} S_{z_1} \cdots S_{z_{n-1}})^{-1}. \end{aligned}$$

Therefore S_{y_n} and S_{z_n} are also in the subgroup of π^+ generated by X'.

Thus we get the claim.

PROOF (Proof of Proposition 4.2). (1) For any generator S_α of π^+, the complement of α is homeomorphic to either

(b) $N_{g-2,n+2}$,

(c) $\Sigma_{h,n+2}$ only if $g = 2h+2$,

(d) $N_{h,m+1} \sqcup N_{g-h,n-m+1}$ for $1 \le h \le g-1$ and $0 \le m \le n$ or

(e) $\Sigma_{h,m+1} \sqcup N_{g-2h,n-m+1}$ for $0 \le h \le \dfrac{g-1}{2}$ and $0 \le m \le n$.

(see [15]). Therefore, there is $f \in \mathscr{PM}(N_{g,n+1})$ such that $\alpha = f_\sharp(\beta)$, where β is either one of the simple loops as in Figure 6 (b), (c), (d) and (e). For the case (b), we have $S_\alpha = S_{f_\sharp(x_{12})}$. For the case (c), by the relation (2) of π^+, we have $S_\alpha = S_{f_\sharp(x_{12})} \cdots S_{f_\sharp(x_{g-1\,g})}$. For the case (d), by the relation (2) of π^+, we have $S_\alpha = S_{f_\sharp(x_{11})} \cdots S_{f_\sharp(x_{hh})} S_{f_\sharp(y_{k_1})} \cdots S_{f_\sharp(y_{k_m})}$ for some $1 \le k_1 < \cdots < k_m \le n$. For the case (e), by the relation (1) and (2) of π^+, we have

$$S_\alpha = S_{f_\sharp(x_{12})} S_{f_\sharp(x_{34})} \cdots S_{f_\sharp(x_{2h-1\,2h})} \cdot S^{-1}_{f_\sharp(x_{2h\,2h+1})} S^{-1}_{f_\sharp(x_{2h-1\,2h})} \cdots S^{-1}_{f_\sharp(x_{12})}$$
$$\cdot S_{f_\sharp(x_{23})} S_{f_\sharp(x_{45})} \cdots S_{f_\sharp(x_{2h\,2h+1})} S_{f_\sharp(y_{k_1})} \cdots S_{f_\sharp(y_{k_m})}$$

for some $1 \le k_1 < \cdots < k_m \le n$ if $h \ne 0$. If $h = 0$, by the relation (2) of π^+, we have $S_\alpha = S_{f_\sharp(y_{k_1})} \cdots S_{f_\sharp(y_{k_m})}$ for some $1 \le k_1 < \cdots < k_m \le n$. Since each symbol of the right hand sides is in X, we conclude that X generates π.

(2) For any $S_\alpha \in X$, if α is a non-separating two-sided simple loop whose complement is non-orientable, there is $f \in \mathscr{PM}(N_{g,n+1})$ such that $f_\sharp(x_{12}) = \alpha$, and hence $f(S_{x_{12}}) = S_\alpha$. If α is a separating simple loop which bounds the m-th boundary component for $1 \le m \le n-1$, there is $f \in \mathscr{PM}(N_{g,n+1})$ such that $f_\sharp(y_m) = \alpha$, and hence $f(S_{y_m}) = S_\alpha$. If α is a separating simple loop which bounds one crosscap whose complement is non-orientable, there is $f \in \mathscr{PM}(N_{g,n+1})$ such that $f_\sharp(x_{11}) = \alpha$, and hence $f(S_{x_{11}}) = S_\alpha$. Therefore we obtain the claim.

(3) In this proof, we omit details of calculations. In calculations, we use the relation (3) as little as possible (see Remark 4.4).

Let $y = t_{a_i}$ for $1 \le i \le g$. We calculate

$$y(S_{x_{i-1\,i}}) = S_{x_{i-1\,i} x^{-1}_{i\,i+1}} \overset{(1),(2)}{=} S_{x_{i-1\,i}} S^{-1}_{x_{i\,i+1}},$$
$$y^{-1}(S_{x_{i-1\,i}}) = S_{x_{i-1\,i} x_{i\,i+1}} \overset{(2)}{=} S_{x_{i-1\,i}} S_{x_{i\,i+1}},$$
$$y(S_{x_{i+1\,i+2}}) = S_{x_{i\,i+1} x_{i+1\,i+2}} \overset{(2)}{=} S_{x_{i\,i+1}} S_{x_{i+1\,i+2}},$$
$$y^{-1}(S_{x_{i+1\,i+2}}) = S_{x^{-1}_{i\,i+1} x_{i+1\,i+2}} \overset{(1),(2)}{=} S^{-1}_{x_{i\,i+1}} S_{x_{i+1\,i+2}},$$

$$y(S_{x_{ii}}) = S_{x_{i\,i+1}x_{i+1\,i+1}^{-1}x_{i\,i+1}^{-1}} \overset{(1),(3)}{=} S_{x_{i\,i+1}} S_{x_{i+1\,i+1}}^{-1} S_{x_{i\,i+1}}^{-1},$$

$$y^{-1}(S_{x_{ii}}) = S_{x_{ii}x_{i+1\,i+1}x_{i\,i+1}^{-1}x_{ii}x_{i\,i+1}} \overset{(2)}{=} S_{x_{ii}x_{i+1\,i+1}} S_{x_{i\,i+1}^{-1}x_{ii}x_{i\,i+1}}$$

$$\overset{(1),(2),(3)}{=} S_{x_{ii}} S_{x_{i+1\,i+1}} S_{x_{i\,i+1}}^{-1} S_{x_{ii}} S_{x_{i\,i+1}},$$

$$y(S_{x_{i+1\,i+1}}) = S_{x_{i\,i+1}x_{i+1\,i+1}x_{i\,i+1}^{-1}x_{ii}x_{i+1\,i+1}}$$

$$\overset{(2)}{=} S_{x_{i\,i+1}x_{i+1\,i+1}x_{i\,i+1}^{-1}} S_{x_{ii}x_{i+1\,i+1}}$$

$$\overset{(1),(2),(3)}{=} S_{x_{i\,i+1}} S_{x_{i+1\,i+1}} S_{x_{i\,i+1}}^{-1} S_{x_{ii}} S_{x_{i+1\,i+1}},$$

$$y^{-1}(S_{x_{i+1\,i+1}}) = S_{x_{i\,i+1}^{-1}x_{ii}^{-1}x_{i\,i+1}} \overset{(1),(3)}{=} S_{x_{i\,i+1}}^{-1} S_{x_{ii}}^{-1} S_{x_{i\,i+1}},$$

$$t_{a_{g-1}}^{\pm 1}(S_{z_l}) = S_{x_{g-1\,g}^{\pm 1} z_l x_{g-1\,g}^{\mp 1}} \overset{(1),(2)}{=} S_{x_{g-1\,g}}^{\pm 1} S_{z_l} S_{x_{g-1\,g}}^{\mp 1}$$

for $1 \leq l \leq n-1$, and $y^{\pm 1}(x) = x$ for any other $x \in X'$.

Let $y = t_b$. We calculate

$$y(S_{x_{45}}) = S_{x_{23}^{-1}x_{12}x_{23}x_{34}x_{45}} \overset{(1),(2)}{=} S_{x_{23}}^{-1} S_{x_{12}} S_{x_{23}} S_{x_{34}} S_{x_{45}},$$

$$y^{-1}(S_{x_{45}}) = S_{x_{34}^{-1}x_{23}^{-1}x_{12}^{-1}x_{23}x_{45}} \overset{(1),(2)}{=} S_{x_{34}}^{-1} S_{x_{23}}^{-1} S_{x_{12}}^{-1} S_{x_{23}} S_{x_{45}},$$

$$y(S_{x_{11}}) = S_{x_{12}x_{34}x_{44}^{-1}x_{33}^{-1}x_{12}^{-1}x_{13}x_{34}^{-1}x_{23}^{-1}x_{12}^{-1}} \overset{(2)}{=} S_{x_{12}x_{34}x_{44}^{-1}x_{33}^{-1}x_{12}^{-1}} S_{x_{13}x_{34}^{-1}x_{23}^{-1}x_{12}^{-1}}$$

$$\overset{(1),(2)}{=} S_{x_{12}} S_{x_{34}} S_{x_{44}}^{-1} S_{x_{33}}^{-1} S_{x_{12}}^{-1} S_{x_{13}} S_{x_{34}}^{-1} S_{x_{23}}^{-1} S_{x_{12}}^{-1},$$

$$y^{-1}(S_{x_{11}}) = S_{x_{11}x_{22}x_{33}x_{44}x_{34}^{-1}x_{12}^{-1}x_{11}x_{12}x_{23}x_{34}x_{23}^{-1}}$$

$$\overset{(2)}{=} S_{x_{11}x_{22}x_{33}x_{44}} S_{x_{34}^{-1}x_{12}^{-1}x_{11}x_{12}x_{23}x_{34}x_{23}^{-1}}$$

$$\overset{(1),(2)}{=} S_{x_{11}} S_{x_{22}} S_{x_{33}} S_{x_{44}} S_{x_{34}}^{-1} S_{x_{12}^{-1}x_{11}x_{12}x_{23}x_{34}x_{23}^{-1}}$$

$$\overset{(2)}{=} S_{x_{11}} S_{x_{22}} S_{x_{33}} S_{x_{44}} S_{x_{34}}^{-1} S_{x_{12}^{-1}x_{11}x_{12}} S_{x_{23}x_{34}x_{23}^{-1}}$$

$$\overset{(1),(2),(3)}{=} S_{x_{11}} S_{x_{22}} S_{x_{33}} S_{x_{44}} S_{x_{34}}^{-1} S_{x_{12}}^{-1} S_{x_{11}} S_{x_{12}} S_{x_{23}} S_{x_{34}} S_{x_{23}}^{-1},$$

$$y(S_{x_{22}}) = S_{x_{12}x_{23}x_{34}x_{13}^{-1}x_{11}x_{22}x_{33}x_{44}x_{34}^{-1}} \overset{(2)}{=} S_{x_{12}x_{23}} S_{x_{34}x_{13}^{-1}x_{11}x_{22}x_{33}x_{44}x_{34}^{-1}}$$

$$\overset{(1),(2)}{=} S_{x_{12}} S_{x_{23}} S_{x_{34}} S_{x_{13}}^{-1} S_{x_{11}x_{22}x_{33}x_{44}x_{34}^{-1}}$$

$$\overset{(2)}{=} S_{x_{12}} S_{x_{23}} S_{x_{34}} S_{x_{13}}^{-1} S_{x_{11}} S_{x_{22}} S_{x_{33}x_{44}x_{34}^{-1}}$$

$$
\begin{aligned}
&\overset{(1),(2)}{=} S_{x_{12}} S_{x_{23}} S_{x_{34}} S_{x_{13}}^{-1} S_{x_{11}} S_{x_{22}} S_{x_{33}x_{44}} S_{x_{34}}^{-1} \\
&\overset{(2)}{=} S_{x_{12}} S_{x_{23}} S_{x_{34}} S_{x_{13}}^{-1} S_{x_{11}} S_{x_{22}} S_{x_{33}} S_{x_{44}} S_{x_{34}}^{-1},
\end{aligned}
$$

$$
\begin{aligned}
y^{-1}(S_{x_{22}}) &= S_{x_{23}x_{34}^{-1}x_{23}^{-1}x_{12}^{-1}x_{22}x_{34}x_{44}^{-1}x_{33}^{-1}x_{22}^{-1}x_{11}^{-1}x_{12}} \\
&\overset{(2)}{=} S_{x_{23}x_{42}} S_{x_{42}^{-1}x_{34}^{-1}x_{23}^{-1}x_{12}^{-1}x_{22}x_{34}x_{44}^{-1}x_{33}^{-1}x_{22}^{-1}x_{11}^{-1}x_{12}} \\
&\overset{(2)}{=} S_{x_{23}} S_{x_{42}} S_{x_{42}^{-1}x_{34}^{-1}x_{23}^{-1}x_{12}^{-1}x_{22}} S_{x_{34}x_{44}^{-1}x_{33}^{-1}x_{22}^{-1}x_{11}^{-1}x_{12}} \\
&\overset{(1),(2)}{=} S_{x_{23}} S_{x_{42}} S_{x_{42}}^{-1} S_{x_{34}}^{-1} S_{x_{23}^{-1}x_{12}^{-1}x_{22}} S_{x_{34}} S_{x_{44}^{-1}x_{33}^{-1}x_{22}^{-1}x_{11}^{-1}x_{12}} \\
&\overset{(1),(2)}{=} S_{x_{23}} S_{x_{34}}^{-1} S_{x_{23}^{-1}x_{12}^{-1}} S_{x_{22}} S_{x_{34}} S_{x_{44}}^{-1} S_{x_{33}}^{-1} S_{x_{22}^{-1}x_{11}^{-1}x_{12}} \\
&\overset{(1),(2)}{=} S_{x_{23}} S_{x_{34}}^{-1} S_{x_{23}}^{-1} S_{x_{12}}^{-1} S_{x_{22}} S_{x_{34}} S_{x_{44}}^{-1} S_{x_{33}}^{-1} S_{x_{22}^{-1}x_{11}^{-1}} S_{x_{12}} \\
&\overset{(1),(2)}{=} S_{x_{23}} S_{x_{34}}^{-1} S_{x_{23}}^{-1} S_{x_{12}}^{-1} S_{x_{22}} S_{x_{34}} S_{x_{44}}^{-1} S_{x_{33}}^{-1} S_{x_{22}}^{-1} S_{x_{11}}^{-1} S_{x_{12}},
\end{aligned}
$$

$$
\begin{aligned}
y(S_{x_{33}}) &= S_{x_{34}x_{44}^{-1}x_{33}^{-1}x_{22}^{-1}x_{11}^{-1}x_{13}x_{23}^{-1}x_{22}x_{33}x_{34}^{-1}x_{23}^{-1}x_{12}^{-1}x_{23}} \\
&\overset{(2)}{=} S_{x_{34}x_{44}^{-1}x_{33}^{-1}x_{22}^{-1}x_{11}^{-1}x_{13}} S_{x_{23}^{-1}x_{22}x_{33}x_{34}^{-1}x_{23}^{-1}x_{12}^{-1}x_{23}} \\
&\overset{(2)}{=} S_{x_{34}} S_{x_{44}^{-1}x_{33}^{-1}x_{22}^{-1}x_{11}^{-1}x_{13}} S_{x_{23}^{-1}x_{22}x_{33}} S_{x_{34}^{-1}x_{23}^{-1}x_{12}^{-1}x_{23}} \\
&\overset{(1),(2)}{=} S_{x_{34}} S_{x_{44}}^{-1} S_{x_{33}^{-1}x_{22}^{-1}x_{11}^{-1}x_{13}} S_{x_{23}}^{-1} S_{x_{22}x_{33}} S_{x_{34}}^{-1} S_{x_{23}}^{-1} S_{x_{12}}^{-1} S_{x_{23}} \\
&\overset{(2)}{=} S_{x_{34}} S_{x_{44}}^{-1} S_{x_{33}^{-1}x_{22}^{-1}x_{11}^{-1}} S_{x_{13}} S_{x_{23}}^{-1} S_{x_{22}} S_{x_{33}} S_{x_{34}}^{-1} S_{x_{23}}^{-1} S_{x_{12}}^{-1} S_{x_{23}} \\
&\overset{(1),(2)}{=} S_{x_{34}} S_{x_{44}}^{-1} S_{x_{33}}^{-1} S_{x_{22}}^{-1} S_{x_{11}}^{-1} S_{x_{13}} S_{x_{23}}^{-1} S_{x_{22}} S_{x_{33}} S_{x_{34}}^{-1} S_{x_{23}}^{-1} S_{x_{12}}^{-1} S_{x_{23}},
\end{aligned}
$$

$$
\begin{aligned}
y^{-1}(S_{x_{33}}) &= S_{x_{12}^{-1}x_{11}x_{22}x_{33}x_{44}x_{24}^{-1}x_{12}x_{23}x_{34}} \\
&\overset{(1),(2)}{=} S_{x_{12}}^{-1} S_{x_{11}x_{22}x_{33}x_{44}x_{24}^{-1}x_{12}x_{23}x_{34}} \\
&\overset{(2)}{=} S_{x_{12}}^{-1} S_{x_{11}x_{22}x_{33}x_{44}x_{24}^{-1}} S_{x_{12}x_{23}x_{34}} \\
&\overset{(2)}{=} S_{x_{12}}^{-1} S_{x_{11}} S_{x_{22}x_{33}x_{44}x_{24}^{-1}} S_{x_{12}} S_{x_{23}} S_{x_{34}} \\
&\overset{(1),(2)}{=} S_{x_{12}}^{-1} S_{x_{11}} S_{x_{22}x_{33}x_{44}} S_{x_{24}}^{-1} S_{x_{12}} S_{x_{23}} S_{x_{34}} \\
&\overset{(2)}{=} S_{x_{12}}^{-1} S_{x_{11}} S_{x_{22}} S_{x_{33}} S_{x_{44}} S_{x_{24}}^{-1} S_{x_{12}} S_{x_{23}} S_{x_{34}},
\end{aligned}
$$

$$
\begin{aligned}
y(S_{x_{44}}) &= S_{x_{23}^{-1}x_{12}x_{23}x_{34}x_{44}x_{34}^{-1}x_{12}^{-1}x_{11}x_{22}x_{33}x_{44}} \\
&\overset{(1),(2)}{=} S_{x_{23}}^{-1}S_{x_{12}x_{23}x_{34}x_{44}x_{34}^{-1}x_{12}^{-1}x_{11}x_{22}x_{33}x_{44}} \\
&\overset{(2)}{=} S_{x_{23}}^{-1}S_{x_{12}}S_{x_{23}}S_{x_{34}x_{44}x_{34}^{-1}x_{12}^{-1}x_{11}x_{22}x_{33}x_{44}} \\
&\overset{(2)}{=} S_{x_{23}}^{-1}S_{x_{12}}S_{x_{23}}S_{x_{34}x_{44}x_{34}^{-1}}S_{x_{12}^{-1}x_{11}x_{22}x_{33}x_{44}} \\
&\overset{(1),(2),(3)}{=} S_{x_{23}}^{-1}S_{x_{12}}S_{x_{23}}S_{x_{34}}S_{x_{44}}S_{x_{34}}^{-1}S_{x_{12}}^{-1}S_{x_{11}x_{22}x_{33}x_{44}} \\
&\overset{(2)}{=} S_{x_{23}}^{-1}S_{x_{12}}S_{x_{23}}S_{x_{34}}S_{x_{44}}S_{x_{34}}^{-1}S_{x_{12}}^{-1}S_{x_{11}}S_{x_{22}}S_{x_{33}}S_{x_{44}}, \\
y^{-1}(S_{x_{44}}) &= S_{x_{34}^{-1}x_{23}^{-1}x_{12}^{-1}x_{24}x_{34}^{-1}x_{22}^{-1}x_{11}^{-1}x_{12}x_{34}} \\
&\overset{(2)}{=} S_{x_{34}^{-1}x_{23}^{-1}x_{12}^{-1}x_{24}}S_{x_{34}^{-1}x_{22}^{-1}x_{11}^{-1}x_{12}x_{34}} \\
&\overset{(2)}{=} S_{x_{34}^{-1}x_{23}^{-1}}S_{x_{12}^{-1}x_{24}}S_{x_{34}^{-1}x_{22}^{-1}x_{11}^{-1}}S_{x_{12}x_{34}} \\
&\overset{(1),(2)}{=} S_{x_{34}}^{-1}S_{x_{23}}^{-1}S_{x_{12}}^{-1}S_{x_{24}}S_{x_{34}}^{-1}S_{x_{22}}^{-1}S_{x_{11}}^{-1}S_{x_{12}}S_{x_{34}},
\end{aligned}
$$

$$
\begin{aligned}
y(S_{z_k}) &= S_{x_{23}^{-1}x_{12}x_{23}x_{34}z_kx_{34}^{-1}x_{23}^{-1}x_{12}^{-1}x_{23}} \overset{(2)}{=} S_{x_{23}^{-1}x_{12}x_{23}x_{34}z_k}S_{x_{34}^{-1}x_{23}^{-1}x_{12}^{-1}x_{23}} \\
&\overset{(1),(2)}{=} S_{x_{23}^{-1}x_{12}x_{23}x_{34}}S_{z_k}S_{x_{34}}^{-1}S_{x_{23}}^{-1}S_{x_{12}}^{-1}S_{x_{23}} \\
&\overset{(1),(2)}{=} S_{x_{23}}^{-1}S_{x_{12}}S_{x_{23}}S_{x_{34}}S_{z_k}S_{x_{34}}^{-1}S_{x_{23}}^{-1}S_{x_{12}}^{-1}S_{x_{23}}, \\
y^{-1}(S_{z_k}) &= S_{x_{34}^{-1}x_{23}^{-1}x_{12}^{-1}x_{23}z_kx_{23}^{-1}x_{12}x_{23}x_{34}} \overset{(2)}{=} S_{x_{34}^{-1}x_{23}^{-1}x_{12}^{-1}}S_{x_{23}z_kx_{23}^{-1}x_{12}x_{23}x_{34}} \\
&\overset{(1),(2)}{=} S_{x_{34}}^{-1}S_{x_{23}}^{-1}S_{x_{12}}^{-1}S_{x_{23}}S_{z_k}S_{x_{23}}^{-1}S_{x_{12}}S_{x_{23}}S_{x_{34}}
\end{aligned}
$$

for $1 \le k \le n-1$ only if $g = 4$, and $y^{\pm 1}(x) = x$ for any other $x \in X'$.

Let $y = Y_{\mu, a_1}$. We calculate

$$
\begin{aligned}
y(S_{x_{12}}) = S_{x_{11}x_{22}x_{12}^{-1}} &\overset{(1),(2)}{=} S_{x_{11}}S_{x_{22}}S_{x_{12}}^{-1}, \\
y^{-1}(S_{x_{12}}) = S_{x_{12}^{-1}x_{11}x_{22}} &\overset{(1),(2)}{=} S_{x_{12}}^{-1}S_{x_{11}}S_{x_{22}}, \\
y(S_{x_{23}}) = S_{x_{11}x_{23}} &\overset{(2)}{=} S_{x_{11}}S_{x_{23}}, \\
y^{-1}(S_{x_{23}}) = S_{x_{12}^{-1}x_{11}x_{12}x_{23}} \overset{(2)}{=} S_{x_{12}^{-1}x_{11}x_{12}}S_{x_{23}} &\overset{(1),(3)}{=} S_{x_{12}}^{-1}S_{x_{11}}S_{x_{12}}S_{x_{23}},
\end{aligned}
$$

$$
\begin{aligned}
y(S_{x_{11}}) &= S_{x_{11}x_{22}x_{12}^{-1}x_{11}^{-1}x_{12}x_{22}^{-1}x_{11}^{-1}} \overset{(2)}{=} S_{x_{11}x_{22}} S_{x_{12}^{-1}x_{11}^{-1}x_{12}x_{22}^{-1}x_{11}^{-1}} \\
&\overset{(2)}{=} S_{x_{11}} S_{x_{22}} S_{x_{12}^{-1}x_{11}^{-1}x_{12}} S_{x_{22}^{-1}x_{11}^{-1}} \\
&\overset{(1),(2),(3)}{=} S_{x_{11}} S_{x_{22}} S_{x_{12}}^{-1} S_{x_{11}}^{-1} S_{x_{12}} S_{x_{22}}^{-1} S_{x_{11}}^{-1},
\end{aligned}
$$

$$
y^{-1}(S_{x_{11}}) = S_{x_{12}^{-1}x_{11}^{-1}x_{12}} \overset{(1),(3)}{=} S_{x_{12}}^{-1} S_{x_{11}}^{-1} S_{x_{12}},
$$

$$
\begin{aligned}
y(S_{x_{22}}) &= S_{x_{11}x_{22}x_{12}^{-1}x_{11}x_{12}} \overset{(2)}{=} S_{x_{11}x_{22}} S_{x_{12}^{-1}x_{11}x_{12}} \\
&\overset{(1),(2),(3)}{=} S_{x_{11}} S_{x_{22}} S_{x_{12}}^{-1} S_{x_{11}} S_{x_{12}},
\end{aligned}
$$

$$
\begin{aligned}
y^{-1}(S_{x_{22}}) &= S_{x_{12}^{-1}x_{11}x_{12}x_{11}x_{22}} \overset{(2)}{=} S_{x_{12}^{-1}x_{11}x_{12}} S_{x_{11}x_{22}} \\
&\overset{(1),(2),(3)}{=} S_{x_{12}}^{-1} S_{x_{11}} S_{x_{12}} S_{x_{11}} S_{x_{22}},
\end{aligned}
$$

$$
y(S_{z_k}) = S_{x_{11}z_k x_{11}^{-1}} \overset{(1),(2)}{=} S_{x_{11}} S_{z_k} S_{x_{11}}^{-1}
$$

$$
\begin{aligned}
y^{-1}(S_{z_k}) &= S_{x_{12}^{-1}x_{11}x_{12}z_k x_{12}^{-1}x_{11}^{-1}x_{12}} \overset{(2)}{=} S_{x_{12}^{-1}x_{11}x_{12}z_k} S_{x_{12}^{-1}x_{11}^{-1}x_{12}} \\
&\overset{(1),(2),(3)}{=} S_{x_{12}^{-1}x_{11}x_{12}} S_{z_k} S_{x_{12}}^{-1} S_{x_{11}}^{-1} S_{x_{12}} \\
&\overset{(1),(3)}{=} S_{x_{12}}^{-1} S_{x_{11}} S_{x_{12}} S_{z_k} S_{x_{12}}^{-1} S_{x_{11}}^{-1} S_{x_{12}},
\end{aligned}
$$

for $1 \le k \le n-1$ only if $g = 2$, and $y^{\pm 1}(x) = x$ for any other $x \in X'$.

Let $y = B_{r_k}$ for $1 \le k \le n$. We calculate

$$
y(S_{x_{g-1\,g}}) = S_{x_{g-1\,g}z_k} \overset{(2)}{=} S_{x_{g-1\,g}} S_{z_k},
$$

$$
y^{-1}(S_{x_{g-1\,g}}) = S_{x_{g-1\,g}y_k} \overset{(2)}{=} S_{x_{g-1\,g}} S_{y_k},
$$

$$
y(S_{x_{gg}}) = S_{x_{gg}y_k z_k} \overset{(2)}{=} S_{x_{gg}y_k} S_{z_k} \overset{(2)}{=} S_{x_{gg}} S_{y_k} S_{z_k},
$$

$$
y^{-1}(S_{x_{gg}}) = S_{z_k x_{gg} y_k} \overset{(2)}{=} S_{z_k} S_{x_{gg}} S_{y_k},
$$

$$
y(S_{y_k}) = S_{z_k^{-1}} \overset{(1)}{=} S_{z_k}^{-1},
$$

$$
\begin{aligned}
y^{-1}(S_{y_k}) &= S_{y_k^{-1}x_{gg}^{-1}z_k^{-1}x_{gg}y_k} \overset{(2)}{=} S_{y_k^{-1}x_{gg}^{-1}z_k^{-1}} S_{x_{gg}y_k} \\
&\overset{(1),(2)}{=} S_{y_k^{-1}x_{gg}^{-1}} S_{z_k}^{-1} S_{x_{gg}} S_{y_k} \overset{(1),(2)}{=} S_{y_k}^{-1} S_{x_{gg}}^{-1} S_{z_k}^{-1} S_{x_{gg}} S_{y_k},
\end{aligned}
$$

$$
y(S_{y_m}) = S_{z_k^{-1}y_k^{-1}y_m y_k z_k} \overset{(2)}{=} S_{z_k^{-1}y_k^{-1}y_m y_k} S_{z_k} \overset{(1),(2)}{=} S_{z_k}^{-1} S_{y_k}^{-1} S_{y_m} S_{y_k} S_{z_k},
$$

$$\begin{aligned} y^{-1}(S_{y_m}) &= S_{y_k^{-1}x_{gg}^{-1}z_k^{-1}x_{gg}y_m x_{gg}^{-1}z_k x_{gg}y_k} \overset{(2)}{=} S_{y_k^{-1}x_{gg}^{-1}z_k^{-1}}S_{x_{gg}y_m x_{gg}^{-1}z_k x_{gg}y_k} \\ &\overset{(1),(2)}{=} S_{y_k}^{-1}S_{x_{gg}}^{-1}S_{z_k}^{-1}S_{x_{gg}y_m x_{gg}^{-1}}S_{z_k x_{gg}y_k} \\ &\overset{(1),(2)}{=} S_{y_k}^{-1}S_{x_{gg}}^{-1}S_{z_k}^{-1}S_{x_{gg}y_m}S_{x_{gg}}^{-1}S_{z_k}S_{x_{gg}}S_{y_k} \\ &\overset{(2)}{=} S_{y_k}^{-1}S_{x_{gg}}^{-1}S_{z_k}^{-1}S_{x_{gg}}S_{y_m}S_{x_{gg}}^{-1}S_{z_k}S_{x_{gg}}S_{y_k}, \end{aligned}$$

$$y(S_{z_k}) = S_{x_{gg}y_k^{-1}x_{gg}^{-1}} \overset{(1),(2)}{=} S_{x_{gg}}S_{y_k}^{-1}S_{x_{gg}}^{-1},$$

$$y^{-1}(S_{z_k}) = S_{y_k^{-1}} \overset{(1)}{=} S_{y_k}^{-1},$$

$$y(S_{z_m}) = S_{z_k^{-1}z_m z_k} \overset{(1),(2)}{=} S_{z_k}^{-1}S_{z_m}S_{z_k},$$

$$y^{-1}(S_{z_m}) = S_{y_k^{-1}z_m y_k} \overset{(2)}{=} S_{y_k^{-1}z_m}S_{y_k} \overset{(1),(2)}{=} S_{y_k}^{-1}S_{z_m}S_{y_k},$$

$$\begin{aligned} y(S_{z_{m'}}) &= S_{x_{gg}y_k x_{gg}^{-1}z_{m'}x_{gg}y_k^{-1}x_{gg}^{-1}} \overset{(2)}{=} S_{x_{gg}y_k x_{gg}^{-1}}S_{z_{m'}x_{gg}y_k^{-1}x_{gg}^{-1}} \\ &\overset{(1),(2)}{=} S_{x_{gg}y_k}S_{x_{gg}}^{-1}S_{z_{m'}}S_{x_{gg}}S_{y_k}^{-1}S_{x_{gg}}^{-1} \overset{(2)}{=} S_{x_{gg}}S_{y_k}S_{x_{gg}}^{-1}S_{z_{m'}}S_{x_{gg}}S_{y_k}^{-1}S_{x_{gg}}^{-1}, \end{aligned}$$

$$y^{-1}(S_{z_{m'}}) = S_{z_k z_{m'}z_k^{-1}} \overset{(1),(2)}{=} S_{z_k z_{m'}}S_{z_k}^{-1} \overset{(1),(2)}{=} S_{z_k}S_{z_{m'}}S_{z_k}^{-1}$$

for $1 \le m < k$ and $k < m' \le n-1$, and $y^{\pm 1}(x) = x$ for any other $x \in X'$.

Let $y = B_{r_0}$. We calculate

$$y(S_{x_{i\,i+1}}) = S_{x_{ig}^{-1}x_{ii}x_{i+1\,g}} \overset{(1),(2)}{=} S_{x_{ig}}^{-1}S_{x_{ii}}S_{x_{i+1\,g}},$$

$$\begin{aligned} y^{-1}(S_{x_{i\,i+1}}) &= S_{x_{gi}x_{i+1\,i+1}x_{g\,i+1}^{-1}} \overset{(1),(2)}{=} S_{x_{gi}x_{i+1\,i+1}}S_{x_{g\,i+1}}^{-1} \\ &\overset{(2)}{=} S_{x_{gi}}S_{x_{i+1\,i+1}}S_{x_{g\,i+1}}^{-1}, \end{aligned}$$

$$y(S_{x_{g-1\,g}}) = S_{x_{g-1\,g}^{-1}x_{g-1\,g-1}x_{gg}} \overset{(1),(2)}{=} S_{x_{g-1\,g}}^{-1}S_{x_{g-1\,g-1}}S_{x_{gg}},$$

$$y^{-1}(S_{x_{g-1\,g}}) = S_{x_{g\,g-1}},$$

$$y(S_{x_{jj}}) = S_{x_{jg}^{-1}x_{jj}x_{jg}} \overset{(1),(3)}{=} S_{x_{jg}}^{-1}S_{x_{jj}}S_{x_{jg}},$$

$$y^{-1}(S_{x_{jj}}) = S_{x_{gj}x_{jj}x_{gj}^{-1}} \overset{(1),(3)}{=} S_{x_{gj}}S_{x_{jj}}S_{x_{gj}}^{-1},$$

$$y^{\pm 1}(S_{x_{gg}}) = S_{x_{gg}},$$

$$y(S_{y_l}) = S_{x_{gg}^{-1}z_l x_{gg}} \overset{(1),(2)}{=} S_{x_{gg}}^{-1}S_{z_l}S_{x_{gg}},$$

$$y^{-1}(S_{y_l}) = S_{z_l},$$

$$y(S_{z_l}) = S_{y_l},$$
$$y^{-1}(S_{z_l}) = S_{x_{gg}y_lx_{gg}^{-1}} \overset{(1),(2)}{=} S_{x_{gg}y_l}S_{x_{gg}}^{-1} \overset{(2)}{=} S_{x_{gg}}S_{y_l}S_{x_{gg}}^{-1}$$

for $1 \leq i \leq g-2$, $1 \leq j \leq g-1$ and $1 \leq l \leq n-1$.

Let $y = t_{s_{kl}}$ for $1 \leq k < l \leq n$. We calculate

$$y(S_{y_k}) = S_{y_ky_ly_ky_l^{-1}y_k^{-1}} \overset{(2)}{=} S_{y_ky_l}S_{y_ky_l^{-1}y_k^{-1}} \overset{(1),(2)}{=} S_{y_k}S_{y_l}S_{y_k}S_{y_l}^{-1}S_{y_k}^{-1},$$
$$y^{-1}(S_{y_k}) = S_{y_l^{-1}y_ky_l} \overset{(1),(2)}{=} S_{y_l}^{-1}S_{y_k}S_{y_l},$$
$$y(S_{y_l}) = S_{y_ky_ly_k^{-1}} \overset{(1),(2)}{=} S_{y_ky_l}S_{y_k}^{-1} \overset{(2)}{=} S_{y_k}S_{y_l}S_{y_k}^{-1},$$
$$\begin{aligned} y^{-1}(S_{y_l}) &= S_{y_l^{-1}y_k^{-1}y_ly_ky_l} \overset{(2)}{=} S_{y_l^{-1}y_k^{-1}y_l}S_{y_ky_l} \\ &\overset{(2)}{=} S_{y_l^{-1}y_k^{-1}}S_{y_l}S_{y_k}S_{y_l} \overset{(1),(2)}{=} S_{y_l}^{-1}S_{y_k}^{-1}S_{y_l}S_{y_k}S_{y_l}, \end{aligned}$$
$$\begin{aligned} y(S_{y_m}) &= S_{y_ky_ly_k^{-1}y_l^{-1}y_my_ly_ky_l^{-1}y_k^{-1}} \overset{(2)}{=} S_{y_ky_ly_k^{-1}y_l^{-1}y_my_l}S_{y_ky_l^{-1}y_k^{-1}} \\ &\overset{(1),(2)}{=} S_{y_ky_ly_k^{-1}}S_{y_l^{-1}y_my_l}S_{y_k}S_{y_l}^{-1}S_{y_k}^{-1} \\ &\overset{(1),(2)}{=} S_{y_ky_l}S_{y_k}^{-1}S_{y_l}^{-1}S_{y_m}S_{y_l}S_{y_k}S_{y_l}^{-1}S_{y_k}^{-1} \\ &\overset{(2)}{=} S_{y_k}S_{y_l}S_{y_k}^{-1}S_{y_l}^{-1}S_{y_m}S_{y_l}S_{y_k}S_{y_l}^{-1}S_{y_k}^{-1}, \end{aligned}$$
$$\begin{aligned} y^{-1}(S_{y_m}) &= S_{y_l^{-1}y_k^{-1}y_ly_ky_my_k^{-1}y_l^{-1}y_ky_l} \overset{(2)}{=} S_{y_l^{-1}y_k^{-1}y_l}S_{y_ky_my_k^{-1}y_l^{-1}y_ky_l} \\ &\overset{(1),(2)}{=} S_{y_l}^{-1}S_{y_k}^{-1}S_{y_l}S_{y_ky_my_k^{-1}}S_{y_l^{-1}y_ky_l} \\ &\overset{(1),(2)}{=} S_{y_l}^{-1}S_{y_k}^{-1}S_{y_l}S_{y_ky_m}S_{y_k}^{-1}S_{y_l}^{-1}S_{y_k}S_{y_l} \\ &\overset{(2)}{=} S_{y_l}^{-1}S_{y_k}^{-1}S_{y_l}S_{y_k}S_{y_m}S_{y_k}^{-1}S_{y_l}^{-1}S_{y_k}S_{y_l}, \end{aligned}$$
$$y(S_{z_k}) = S_{z_kz_lz_kz_l^{-1}z_k^{-1}} \overset{(2)}{=} S_{z_kz_l}S_{z_kz_l^{-1}z_k^{-1}} \overset{(1),(2)}{=} S_{z_k}S_{z_l}S_{z_k}S_{z_l}^{-1}S_{z_k}^{-1},$$
$$y^{-1}(S_{z_k}) = S_{z_l^{-1}z_kz_l} \overset{(1),(2)}{=} S_{z_l}^{-1}S_{z_k}S_{z_l},$$
$$y(S_{z_l}) = S_{z_kz_lz_k^{-1}} \overset{(1),(2)}{=} S_{z_kz_l}S_{z_k}^{-1} \overset{(2)}{=} S_{z_k}S_{z_l}S_{z_k}^{-1},$$
$$\begin{aligned} y^{-1}(S_{z_l}) &= S_{z_l^{-1}z_k^{-1}z_lz_kz_l} \overset{(2)}{=} S_{z_l^{-1}z_k^{-1}z_l}S_{z_kz_l} \\ &\overset{(2)}{=} S_{z_l^{-1}z_k^{-1}}S_{z_l}S_{z_k}S_{z_l} \overset{(1),(2)}{=} S_{z_l}^{-1}S_{z_k}^{-1}S_{z_l}S_{z_k}S_{z_l}, \end{aligned}$$

$$\begin{aligned}
y(S_{z_m}) &= S_{z_k z_l z_k^{-1} z_l^{-1} z_m z_l z_k z_l^{-1} z_k^{-1}} \overset{(2)}{=} S_{z_k z_l z_k^{-1} z_l^{-1} z_m z_l} S_{z_k z_l^{-1} z_k^{-1}} \\
&\overset{(1),(2)}{=} S_{z_k z_l z_k^{-1}} S_{z_l^{-1} z_m z_l} S_{z_k} S_{z_l}^{-1} S_{z_k}^{-1} \\
&\overset{(1),(2)}{=} S_{z_k z_l} S_{z_k}^{-1} S_{z_l}^{-1} S_{z_m} S_{z_l} S_{z_k} S_{z_l}^{-1} S_{z_k}^{-1} \\
&\overset{(2)}{=} S_{z_k} S_{z_l} S_{z_k}^{-1} S_{z_l}^{-1} S_{z_m} S_{z_l} S_{z_k} S_{z_l}^{-1} S_{z_k}^{-1}, \\
y^{-1}(S_{z_m}) &= S_{z_l^{-1} z_k^{-1} z_l z_k z_m z_k^{-1} z_l^{-1} z_k z_l} \overset{(2)}{=} S_{z_l^{-1} z_k^{-1} z_l} S_{z_k z_m z_k^{-1} z_l^{-1} z_k z_l} \\
&\overset{(1),(2)}{=} S_{z_l}^{-1} S_{z_k}^{-1} S_{z_l} S_{z_k z_m z_k^{-1}} S_{z_l^{-1} z_k z_l} \\
&\overset{(1),(2)}{=} S_{z_l}^{-1} S_{z_k}^{-1} S_{z_l} S_{z_k z_m} S_{z_k}^{-1} S_{z_l}^{-1} S_{z_k} S_{z_l} \\
&\overset{(2)}{=} S_{z_l}^{-1} S_{z_k}^{-1} S_{z_l} S_{z_k} S_{z_m} S_{z_k}^{-1} S_{z_l}^{-1} S_{z_k} S_{z_l}
\end{aligned}$$

for $k < m < l$, and $y^{\pm 1}(x) = x$ for any other $x \in X'$.

Hence we have that for any $x \in X'$ and $y \in Y^{\pm 1}$, $y(x)$ is in the subgroup of π^+ generated by X', by Lemma 4.3.

Proof (Proof of Theorem 1.2). By Lemma 1.3 and Proposition 4.2, it follows that π^+ is generated by X'. There is a natural map $\pi^+ \to \pi_1^+(N_{g,n}, *)$. The relations (1), (2) and (3) of π^+ are satisfied in $\pi_1^+(N_{g,n}, *)$ clearly. Hence the map is a homomorphism. In addition, the relations $S_{x_{11}} \cdots S_{x_{gg}} = 1$ and $S_{x_{gg}} S_{x_{g-1\,g}}^{-1} S_{x_{g-1\,g-1}} S_{x_{g-2\,g-1}}^{-1} \cdots S_{x_{22}} S_{x_{12}}^{-1} S_{x_{11}} S_{x_{12}} \cdots S_{x_{g-1\,g}} = 1$ are obtained from the relations (1), (2) and (3) of π^+ for $n = 0$. Therefore the map is an isomorphism for any $n \geq 0$. Thus we complete the proof.

Remark 4.4. Simple loops α, β and γ of the relation (3) in Theorem 1.2 can be reduced to the form as shown in Figure 7. In fact, we used only this reduced relation as the relation (3), in the proof of Theorem 1.2. We do not know whether the relation (3) in Theorem 1.2 can be obtained from the relations (1) and (2) there or not.

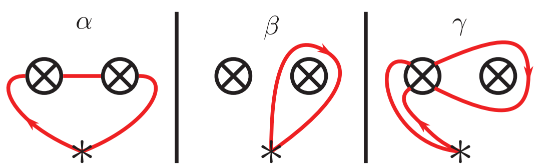


Fig. 7. The reduced relation $S_\alpha S_\beta S_\alpha^{-1} = S_\gamma$.

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