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Geometry of weak-bitangent lines to quartic curves and sections on certain rational elliptic surfaces

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Abstract. It is well known that a smooth quartic curve has twenty-eight bitangent lines. For a reduced, possibly singular quartic curve, we introduce the notion of weak-bitangent line. This can be considered as a generalization of bitangent lines. In this article, we consider weak-bitangent lines for certain reduced quartic curves from the viewpoint of rational elliptic surfaces. We utilize Mumford representations of semi-reduced divisors in order to deal with equations of weak-bitangent lines for certain reduced quartic curves. As a result, we can give new proofs for some classical results on singular quartic curves and their bitangent lines.

1. Introduction

Bitangent lines to a smooth quartic curve have been studied by various mathematicians (see [6, Chapter 6] for details). For a reduced, possibly singular quartic curve, we can consider a generalization of bitangent lines as follows:

Definition 1.1. Let $\mathcal{Q}$ be a reduced quartic curve. A line L is said to be a *weak-bitangent line* if for any $p \in \mathcal{Q} \cap L$, the intersection multiplicity of $\mathcal{Q}$ and L at p is even.

In this article, we study weak-bitangent lines for certain reduced quartic curves in $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$ ($\mathbb{C}$ denotes the field of complex numbers). As we will explain later, for a reduced quartic curve $\mathcal{Q}$ which is not the union of four concurrent lines and a smooth point z_o on $\mathcal{Q}$, we can construct a rational elliptic surface in a canonical way. In [18], Shioda studied a smooth quartic curve and its twenty-eight bitangent lines from the viewpoint of the Mordell-Weil lattice of type E_7^*. Also, in [2, 3, 4], Bannai and Tokunaga studied the embedded topology of plane curve arrangements of a certain singular quartic curve, its weak-bitangent lines and conics by using a rational elliptic surface. In this article, we study weak-bitangent lines of a reduced quartic curve $\mathcal{Q}$

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along similar lines to [2, 3, 4, 18] in the case when $\mathcal{Q}$ satisfies the following condition (†):

(†) $\mathcal{Q}$ is irreducible or is the union of smooth conics $\mathcal{C}_1 + \mathcal{C}_2$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ meet transversely.

Before we go on to explain our results in detail, we briefly summarize our construction of a rational surface. (See Section 2.3 for a detailed description of our construction.)

Let $\mathcal{Q}$ be a reduced quartic curve which is not the union of four concurrent lines and let z_o be a smooth point on $\mathcal{Q}$. Let $S_{\mathcal{Q}}$ be the minimal resolution of the double cover of $\mathbb{P}^2$ branched along $\mathcal{Q}$. The pencil of lines passing through z_o induces a pencil of genus 1 curves Λ_{z_o} on $S_{\mathcal{Q}}$, which has a unique base point of multiplicity 2. We resolve the indeterminacy for the rational map induced by Λ_{z_o} and obtain an elliptic fibration $\varphi_{\mathcal{Q},z_o} : S_{\mathcal{Q},z_o} \to \mathbb{P}^1$ with a section O arising from z_o. We denote the canonical map from $S_{\mathcal{Q},z_o}$ to $\mathbb{P}^2$ by $\tilde{f}_{\mathcal{Q},z_o} : S_{\mathcal{Q},z_o} \to \mathbb{P}^2$.

$$\mathbb{P}^2 \leftarrow S_{\mathcal{Q}} \leftarrow S_{\mathcal{Q},z_o}.$$

For a section s $(\neq O)$, $\tilde{f}_{\mathcal{Q},z_o}(s)$ becomes a curve in $\mathbb{P}^2$.

Let $E_{\mathcal{Q},z_o}$ be the generic fiber of $\varphi_{\mathcal{Q},z_o}$. It is well known that the group of sections of $\varphi_{\mathcal{Q},z_o}$ can be canonically identified with the group of $\mathbb{C}(t)$-rational points of $E_{\mathcal{Q},z_o}$. For a rational point P, we denote the corresponding section by s_P. For a section s, we denote the corresponding rational point by P_s.

DEFINITION 1.2. (i) A section s of $S_{\mathcal{Q},z_o}$ is said to be a *line-section* if $\tilde{f}_{\mathcal{Q},z_o}(s)$ is a line in $\mathbb{P}^2$. (ii) A $\mathbb{C}(t)$-rational point P is said to be a *line-point* if s_P is a line-section.

As it is shown in Section 2.4, a weak-bitangent line gives rise to two line-sections of $S_{\mathcal{Q},z_o}$ and vice-versa, if $\mathcal{Q}$ and z_o satisfy (†) and the following condition (‡):

(‡) The tangent line at z_o meets $\mathcal{Q}$ at two distinct points other than z_o.

Then the pull-back of a weak-bitangent line L contains two sections s_L^+ and s_L^- of $S_{\mathcal{Q},z_o}$. In particular, a weak-bitangent line gives rise to two rational points $P_{s_L^+}$ and $P_{s_L^-} = [-1]P_{s_L^+}$.

Under these settings, we obtain the following result:

THEOREM 1.3. *Let $\mathcal{Q}$ be a reduced quartic curve satisfying* (†) *and let z_o be a smooth point on $\mathcal{Q}$ satisfying* (‡). *For three distinct weak-bitangent lines L_1, L_2 and L_3, let P_i $(i = 1, 2, 3)$ be line-points such that $L_i = \tilde{f}_{\mathcal{Q},z_o}(s_{P_i})$. If $P_4 = P_1 \dot{+} P_2 \dot{+} P_3$ is a line-point, then all intersection points of $\mathcal{Q}$ and $L_1 + L_2 + L_3 + L_4$ lie on a conic, where L_4 is the line $\tilde{f}_{\mathcal{Q},z_o}(s_{P_4})$.*

In the proof of Theorem 1.3, we utilize Mumford representations in order to describe divisor classes on elliptic curves. (See Section 3 for the definition and details of Mumford representations.) Mumford representations were first considered in [14] in order to describe the Jacobian of hyperelliptic curves explicitly. They have played important roles in hyperelliptic curve cryptography (see [7]).

Remark 1.4.

(i) *For each* L_i $(i = 1, 2, 3)$ *in Theorem* 1.3, *there are two choices of* P_i *up to* $[\pm 1]$ *since* $L_i = \tilde{f}_{\mathcal{Q}, z_o}(s_{P_i}) = \tilde{f}_{\mathcal{Q}, z_o}(s_{[-1]P_i})$ *holds. Hence, there are eight possibilities for* P_4. *Therefore, since* $\tilde{f}_{\mathcal{Q}, z_o}(s_{P_4}) = \tilde{f}_{\mathcal{Q}, z_o}(s_{[-1]P_4})$, *there are four curves induced by the candidates of* P_4. *When one of the candidates of* P_4 *is a line-point, the assertion of Theorem* 1.3 *holds for its corresponding weak-bitangent line.*

(ii) *Let* L_1, L_2 *and* L_3 *be distinct bitangent lines of a smooth quartic curve* $\mathcal{Q}$. *A triad* (L_1, L_2, L_3) *is said to be a syzygetic triad if the six intersection points of* $\mathcal{Q}$ *and* $L_1 + L_2 + L_3$ *lie on a conic* C. *(It is well-known that the remaining two points in* $\mathcal{Q} \cap C$ *give rise to a bitangent line.) If we can choose rational points* P_1, P_2, *and* P_3 *such that* (i) $L_i = \tilde{f}_{\mathcal{Q}, z_o}(s_{P_i})$ *and* (ii) $P_1 \dot{+} P_2 \dot{+} P_3$ *is a line-point, then* (L_1, L_2, L_3) *becomes a syzygetic triad by Theorem* 1.3. *This means that the existence of such line-points gives a sufficient condition for* (L_1, L_2, L_3) *to be a syzygetic triad.*

Furthermore, we also give a classification (Theorem 5.6) of weak-bitangent lines of singular quartic curves satisfying (†) by using a result of Oguiso-Shioda ([15]) which gives a classification of Mordell-Weil lattices of rational elliptic surfaces. By Theorems 1.3 and 5.6, we have the following classical results:

Corollary 1.5 ([8, §3], [6, Ch. 2], [16, p. 345]). *Let* $C_1, C_2 \subset \mathbb{P}^2$ *be smooth conics meeting transversely and let* $L_1, \dots, L_4$ *be their four common tangent lines. Then the eight points of tangency lie on a conic.*

Corollary 1.6 ([8, §3]). *If* $\mathcal{Q} \subset \mathbb{P}^2$ *is an irreducible quartic with three nodes, then the eight points of contact of* $\mathcal{Q}$ *with its four bitangent lines all lie on a conic.*

Corollary 1.7 ([8, §3]). *An irreducible quartic with an ordinary triple point has four bitangent lines, whose eight points of contact all lie on a conic.*

The organization of this article is as follows: In Section 2, we give a brief summary on concepts and results from the theory of elliptic surfaces necessary for our argument. In Section 3, we explain the Mumford representations of semi-reduced divisors on hyperelliptic curves, which are key tools to prove Theorem 1.3. In Section 4, we prove Theorem 1.3. In Section 5, we classify weak-bitangent lines of certain singular quartic curves under the condition (†). In Section 6, we prove Corollaries 1.5, 1.6 and 1.7.

2. Elliptic surfaces

Throughout this article, all surfaces and curves are defined over $\mathbb{C}$, unless otherwise stated.

2.1. Notation and terminology on elliptic surfaces. We here define some notation and terminology on elliptic surfaces. For general references, we refer to [10, 12, 17].

Let $\varphi : S \to C$ be an elliptic surface over a smooth projective curve C satisfying the following conditions $(*)$:

- φ is relatively minimal.
- φ has a distinguished section $O : C \to S$.
- φ has at least one singular fiber.

Throughout this article, we always assume that an elliptic surface satisfies the conditions $(*)$.

Let E_S be the generic fiber of φ. E_S can be regarded as a curve of genus 1 defined over the field $\mathbb{C}(C)$ of rational functions of C, and we denote the set of $\mathbb{C}(C)$-rational points of E_S by $E_S(\mathbb{C}(C))$. In our setting, S is known as the Kodaira-Néron model of E_S. Let $\mathrm{MW}(S)$ be the set of sections of φ. For any $s \in \mathrm{MW}(S)$, the restriction of s to E_S gives a $\mathbb{C}(C)$-rational point of E_S. Here, we identify a section $s : C \to S$ with its image and we can identify $\mathrm{MW}(S)$ with $E_S(\mathbb{C}(C))$ through this correspondence. For $P \in E_S(\mathbb{C}(C))$, we denote the corresponding section by s_P and for $s \in \mathrm{MW}(S)$ we denote the corresponding rational point by P_s. By abuse of notation, we identify the section O with its restriction to E_S. We can regard E_S as an elliptic curve $(E_S(\mathbb{C}(C)), O)$ having a group structure with O being the identity. We denote the addition with respect to this group structure by $\dot{+}$. Note that, for $P, Q \in E_S$, $P + Q$ denotes the sum as divisors on E_S, while $P \dot{+} Q$ denotes the sum of points in E_S with respect to the group structure. For $P \in E_S$, we denote the inverse of P with respect to $\dot{+}$ by $\dot{-}P$. For $m \in \mathbb{Z}$ and $P \in E_S$, we let

$$[m]P = \overbrace{P \dot{+} \cdots \dot{+} P}^{m \text{ terms if } m>0}, \qquad [m]P = \overbrace{\dot{-}P \dot{-} \cdots \dot{-} P}^{|m| \text{ terms if } m<0} \qquad \text{and} \qquad [0]P = O.$$

Definition 2.1. A section $s \in \mathrm{MW}(S)$ is said to be an integral section if the intersection number $s \cdot O = 0$.

For $v \in C$, we denote the corresponding fiber over v by $F_v = \varphi^{-1}(v)$. We define two finite subsets, $\mathrm{Sing}(\varphi)$ and $\mathrm{Red}(\varphi)$, of C concerning singular fibers as follows:

$$\mathrm{Sing}(\varphi) := \{v \in C \mid F_v \text{ is singular}\},$$

$$\mathrm{Red}(\varphi) := \{v \in \mathrm{Sing}(\varphi) \mid F_v \text{ is reducible}\}.$$

For $v \in \mathrm{Red}(\varphi)$, the irreducible decomposition of F_v is denoted by

$$F_v = \Theta_{v,0} + \sum_{i=1}^{m_v-1} a_{v,i}\Theta_{v,i},$$

where $\Theta_{v,0}$ is the unique component with $\Theta_{v,0} \cdot O = 1$. We call $\Theta_{v,0}$ the *identity component* of F_v. In order to describe the types of singular fibers, we use Kodaira's notation ([10]). Also, irreducible components of singular fibers are labeled as in [21]. For $v \in \mathrm{Red}(\varphi)$, we define

$$\boldsymbol{c}(v, D) := \begin{bmatrix} D \cdot \Theta_{v,1} \\ \vdots \\ D \cdot \Theta_{v,m_v-1} \end{bmatrix} \in \mathbb{Z}^{\oplus(m_v-1)},$$

$$A_v := [\Theta_{v,i} \cdot \Theta_{v,j}]_{1 \le i,j \le m_v-1},$$

$$\mathbb{F}_v := [\Theta_{v,1}, \dots, \Theta_{v,m_v-1}],$$

where D is a divisor on S, and $D \cdot D'$ denotes the intersection number of divisors D and D' on S.

2.2. Mordell-Weill lattices. Let $\varphi : S \to C$ be an elliptic surface as before. We denote the Néron-Severi group of S by $\mathrm{NS}(S)$, and the Euler characteristic of its structure sheaf $\mathcal{O}_S$ by $\chi(\mathcal{O}_S)$. We denote a general fiber of φ by F. The following theorems are fundamental.

THEOREM 2.2 ([17, Theorem 1.2]). *Under our setting,* $\mathrm{NS}(S)$ *is finitely generated and torsion-free.*

THEOREM 2.3 ([17, Theorem 1.3]). *Let* T_φ *be the subgroup of* $\mathrm{NS}(S)$ *generated by* O *and the irreducible components of fibers. Then, there is a natural isomorphism*

$$\bar{\psi} : E_S(\mathbb{C}(C)) \to \mathrm{NS}(S)/T_\varphi$$

which maps $P \in E_S(\mathbb{C}(C))$ *to* $s_P \bmod T_\varphi$.

Given a divisor D on S, we denote $\bar{\psi}^{-1}(D \bmod T_\varphi)$ by P_D.

Lemma 2.4 ([17, Lemma 5.1]). *For $D \in \mathrm{Div}(S)$, there exists a unique section $s(D)$ such that*

$$D \approx s(D) + (d-1)O + nF + \sum_{v \in \mathrm{Red}(\varphi)} \mathbb{F}_v A_v^{-1} \boldsymbol{c}(v, D - s(D)),$$

where $\approx$ denotes the algebraic equivalence between divisors, and integers d and n are defined as follows:

$$d = D \cdot F \qquad \text{and} \qquad n = (d-1)\chi(\mathcal{O}_S) + O \cdot (D - s(D)).$$

Remark 2.5. (i) *By Lemma 2.4, for $D \in \mathrm{Div}(S)$, we have $s(D) = s_{P_D}$.* (ii) *Also, we have $A_v^{-1}\boldsymbol{c}(v, D - s(D)) \in \mathbb{Z}^{\oplus(m_v - 1)}$, while entries of A_v^{-1} are not necessarily integers.*

Lemma 2.6 ([1, Lemma 2.1]). *If F_v is a singular fiber of type I_2, $\boldsymbol{c}(v, D) - \boldsymbol{c}(v, s(D))$ is even (Note that $\boldsymbol{c}(v, D)$ becomes an integer in this case).*

By (i) in Remark 2.5 and Lemma 2.6, we also have

Corollary 2.7. *Let F_v be a singular fiber of type I_2. Let $P_1, \dots, P_n$ be elements of $E_S(\mathbb{C}(C))$ and let $c_1, \dots, c_n$ be integers. Put $Q = [c_1]P_1 \dot{+} \cdots \dot{+} [c_n]P_n$ and $D = c_1 s_{P_1} + \cdots + c_n s_{P_n}$. Then, we have*

$$s_Q \cdot \Theta_{v,1} = \begin{cases} 1 & \text{if } D \cdot \Theta_{v,1} \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Let us explain the height pairing on $E_S(\mathbb{C}(t))$ introduced in [17]. Let $\phi : E_S(\mathbb{C}(C)) \to \mathrm{NS}(S) \otimes \mathbb{Q}$ be the homomorphism given in [17, Lemma 8.1] as follows:

$$\phi(P) = s_P - O - (s_P \cdot O + \chi(\mathcal{O}_S))F + \sum_{v \in \mathrm{Red}(\varphi)} \mathbb{F}_v(-A_v^{-1})\boldsymbol{c}(v, s_P).$$

In [17], by using ϕ, the *height pairing* $\langle -, - \rangle$ on $E_S(\mathbb{C}(C))$ is defined as follows:

$$\langle P, Q \rangle = -\phi(P) \cdot \phi(Q).$$

The intersection pairing on $\mathrm{NS}(S)$ induces a pairing on $\mathrm{NS}(S) \otimes \mathbb{Q}$ and $\langle P, Q \rangle$ is explicitly given as follows:

Theorem 2.8 ([17, Theorem 8.6]). *For $P, Q \in E_S(\mathbb{C}(C))$ we have*

$$\langle P, Q \rangle = \chi(\mathcal{O}_S) + s_P \cdot O + s_Q \cdot O - s_P \cdot s_Q - \sum_{v \in \mathrm{Red}(\varphi)} \mathrm{contr}_v(s_P, s_Q),$$

where, for divisors D_1 and D_2 on S, $\mathrm{contr}_v(D_1, D_2)$ *is given by*

$$\mathrm{contr}_v(D_1, D_2) = {}^t\boldsymbol{c}(v, D_1)(-A_v)^{-1}\boldsymbol{c}(v, D_2).$$

Note that, for $s_1, s_2 \in \mathrm{MW}(S)$, we have

$$\langle P_{s_1}, P_{s_2} \rangle = \chi(\mathcal{O}_S) + s_1 \cdot O + s_2 \cdot O - s_1 \cdot s_2 - \sum_{v \in \mathrm{Red}(\varphi)} \mathrm{contr}_v(s_1, s_2).$$

2.3. A rational elliptic surface associated to a reduced quartic curve and a smooth point on the quartic curve. Let us first explain how we obtain a rational elliptic surface from a quartic curve and a smooth point on the quartic curve.

Let $\mathcal{Q}$ be a reduced quartic curve in $\mathbb{P}^2$ which is not the union of four concurrent lines and let z_o be a smooth point on $\mathcal{Q}$. We can associate a rational elliptic surface $S_{\mathcal{Q}, z_o}$ (see [2, 2.2.2], [21, Section 4], [1, Section 1]) from $\mathcal{Q}$ and z_o as follows:

(1) Let $f'_{\mathcal{Q}} : S'_{\mathcal{Q}} \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ with branch locus $\mathcal{Q}$.
(2) Let $\mu : S_{\mathcal{Q}} \to S'_{\mathcal{Q}}$ be the canonical resolution of $S'_{\mathcal{Q}}$ (see [9] for the canonical resolution).
(3) Let Λ_{z_o} be the pencil of genus 1 curves on $S_{\mathcal{Q}}$ induced from the pencil of lines through z_o. The pencil Λ_{z_o} has a unique base point $(f'_{\mathcal{Q}} \circ \mu)^{-1}(z_o)$ with multiplicity 2.
(4) Let $v_{z_o} : S_{\mathcal{Q}, z_o} \to S_{\mathcal{Q}}$ be the resolution of the indeterminacy for the rational map induced by Λ_{z_o}. The induced morphism $\varphi_{\mathcal{Q}, z_o} : S_{\mathcal{Q}, z_o} \to \mathbb{P}^1$ is an elliptic fibration. The map v_{z_o} is a composition of two blowing-ups and the exceptional curve for the second blowing-up is a section of $\varphi_{\mathcal{Q}, z_o}$, which we regard as O. Thus we have a rational elliptic surface $S_{\mathcal{Q}, z_o}$ and the diagram below:

$$\begin{array}{ccccc}
S'_{\mathcal{Q}} & \xleftarrow{\ \mu\ } & S_{\mathcal{Q}} & \xleftarrow{\ v_{z_o}\ } & S_{\mathcal{Q}, z_o} \\
\downarrow{\scriptstyle f'_{\mathcal{Q}}} & & \downarrow{\scriptstyle f_{\mathcal{Q}}} & & \downarrow{\scriptstyle f_{\mathcal{Q}, z_o}} \\
\mathbb{P}^2 & \xleftarrow[\ q\ ]{} & \widehat{\mathbb{P}^2} & \xleftarrow[\ q_{z_o}\ ]{} & (\widehat{\mathbb{P}^2})_{z_o},
\end{array}$$

where q is a composition of a finite number of blowing-ups so that the branch locus becomes smooth and q_{z_o} is the composition of two blowing-ups corresponding to v_{z_o}. The map $f_{\mathcal{Q}, z_o}$ is the double cover induced by the involution $[-1]_{\mathcal{Q}, z_o}$ on $S_{\mathcal{Q}, z_o}$, which is given by the inversion with respect to the group law on the generic fiber.

Remark 2.9. *The above construction is also found in* [11] *and* [18].

Let $\mathrm{Sing}(\mathcal{Q})$ be the set of singularities of $\mathcal{Q}$. For $x \in \mathrm{Sing}(\mathcal{Q})$, a line through x and z_o induces a singular fiber of $S_{\mathcal{Q},z_o}$, which we denote by $F_{v(x)}$.

Put $\tilde{f}_{\mathcal{Q},z_o} = f'_{\mathcal{Q}} \circ \mu \circ \nu_{z_o}$.

Remark 2.10. *For a section s ($\neq O$) and $x \in \mathrm{Sing}(\mathcal{Q})$, the curve $\tilde{f}_{\mathcal{Q},z_o}(s)$ passes through x if and only if $\mathbf{c}(v(x), s) \neq \mathbf{0}$.*

Let l_{z_o} be the tangent line of $\mathcal{Q}$ at z_o. The fiber corresponding to l_{z_o} becomes a singular fiber, which we denote by F_∞. By our construction of $S_{\mathcal{Q},z_o}$, any reducible singular fiber is F_∞ or of the form $F_{v(x)}$. If z_o satisfies (‡), then F_∞ is a singular fiber of type I_2. We denote its irreducible decomposition by $F_\infty = \Theta_{\infty,0} + \Theta_{\infty,1}$, where $\Theta_{\infty,0}$ is the identity component.

In the remaining of this subsection, we assume that (i) $\mathcal{Q}$ is singular and satisfies (†) and (ii) z_o satisfies (‡). Let us introduce $\underline{\mathrm{Sing}}(\mathcal{Q})$ and $R_{\mathcal{Q},z_o}$ as follows:

- $\underline{\mathrm{Sing}}(\mathcal{Q})$: the set of pairs of singularities of $\mathcal{Q}$ and their types. For the types of singularities, we refer to [5, p. 81].
- $R_{\mathcal{Q},z_o}$: the subgroup of $\mathrm{NS}(S_{\mathcal{Q},z_o})$ generated by $\Theta_{v,i}$ ($v \in \mathrm{Red}(\varphi_{\mathcal{Q},z_o})$, $i = 1, \dots, m_v - 1$). We have

$$R_{\mathcal{Q},z_o} = \mathbb{Z}\Theta_{\infty,1} \oplus \bigoplus_{x \in \mathrm{Sing}(\mathcal{Q})} \mathbb{Z}\Theta_{v(x),1} \oplus \cdots \oplus \mathbb{Z}\Theta_{v(x),m_{v(x)}-1}.$$

Here is a table for $\underline{\mathrm{Sing}}(\mathcal{Q})$, $R_{\mathcal{Q},z_o}$ and $E_{\mathcal{Q},z_o}(\mathbb{C}(t))$ after Oguiso-Shioda [15]. We omit cases which do not occur under the assumptions (†) and (‡).

Table 1

Oguiso-Shioda classification	$\underline{\mathrm{Sing}}(\mathcal{Q})$	$R_{\mathcal{Q},z_o}$	$E_{\mathcal{Q},z_o}(\mathbb{C}(t))$
No. 4	(x, A_1)	$A_1^{\oplus 2}$	D_6^*
No. 6	(x, A_2)	$A_2 \oplus A_1$	A_5^*
No. 7	(x, A_1) (y, A_1)	$A_1^{\oplus 3}$	$D_4^* \oplus A_1^*$
No. 10	(x, A_3)	$A_3 \oplus A_1$	$A_3^* \oplus A_1^*$
No. 12	(x, A_2) (y, A_1)	$A_2 \oplus A_1^{\oplus 2}$	$\frac{1}{6}\begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 5 & 3 & 1 \\ 0 & 3 & 6 & 3 \\ -1 & 1 & 3 & 5 \end{bmatrix}$
No. 14	(x, A_1) (y, A_1) (z, A_1)	$A_1^{\oplus 4}$	$(A_1^*)^{\oplus 4}$

Table 1 (cont.)

Oguiso-Shioda classification	$\underline{\mathrm{Sing}}(\mathcal{Q})$	$R_{\mathcal{Q},z_o}$	$E_{\mathcal{Q},z_o}(\mathbb{C}(t))$
No. 17	(x, A_4)	$A_4 \oplus A_1$	$\frac{1}{10}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 7 & 3 \\ -1 & 3 & 7 \end{bmatrix}$
No. 18	(x, D_4)	$D_4 \oplus A_1$	$(A_1^*)^{\oplus 3}$
No. 20	(x, A_2) (y, A_2)	$A_2^{\oplus 2} \oplus A_1$	$A_2^* \oplus \langle 1/6 \rangle$
No. 22	(x, A_3) (y, A_1)	$A_3 \oplus A_1^{\oplus 2}$	$(A_1^*)^{\oplus 2} \oplus \langle 1/4 \rangle$
No. 23	(x, A_2) (y, A_1) (z, A_1)	$A_2 \oplus A_1^{\oplus 3}$	$A_1^* \oplus \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$
No. 24	(x, A_1) (y, A_1) (z, A_1) (w, A_1)	$A_1^{\oplus 5}$	$(A_1^*)^{\oplus 3} \oplus \mathbb{Z}/2\mathbb{Z}$
No. 29	(x, A_5)	$A_5 \oplus A_1$	$A_1^* \oplus \langle 1/6 \rangle$
No. 30	(x, D_5)	$D_5 \oplus A_1$	$A_1^* \oplus \langle 1/4 \rangle$
No. 33	(x, A_4) (y, A_1)	$A_4 \oplus A_1^{\oplus 2}$	$\frac{1}{10}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$
No. 37	(x, A_3) (y, A_2)	$A_3 \oplus A_2 \oplus A_1$	$A_1^* \oplus \langle 1/12 \rangle$
No. 40	(x, A_2) (y, A_2) (z, A_1)	$A_2^{\oplus 2} \oplus A_1^{\oplus 2}$	$\langle 1/6 \rangle^{\oplus 2}$
No. 47	(x, A_6)	$A_6 \oplus A_1$	$\langle 1/14 \rangle$
No. 49	(x, E_6)	$E_6 \oplus A_1$	$\langle 1/6 \rangle$
No. 56	(x, A_4) (y, A_2)	$A_4 \oplus A_2 \oplus A_1$	$\langle 1/30 \rangle$
No. 61	(x, A_2) (y, A_2) (z, A_2)	$A_2^{\oplus 3} \oplus A_1$	$\langle 1/6 \rangle \oplus \mathbb{Z}/3\mathbb{Z}$

We use the notation in [15] in order to describe the structure of $E_{\mathscr{Q},z_o}(\mathbb{C}(t))$. Also, the Gram matrices of A_n^* and D_m^* $(n \geq 1, m \geq 4)$ are given by the inverses of the following square matrices of sizes n and m, respectively:

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 2 & -1 & -1 \\ \vdots & \ddots & \ddots & 0 & -1 & 2 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 0 & 2 \end{bmatrix}.$$

2.4. Sections arising from lines and conics. Let $\mathscr{Q}$ be a quartic curve satisfying (†) and let z_o be a smooth point on $\mathscr{Q}$ satisfying (‡).

The following lemma gives a characterization of line-sections.

Lemma 2.11 ([3, Lemma 9]). *Let $s \in \mathrm{MW}(S_{\mathscr{Q},z_o})$ be an integral section with $s \cdot \Theta_{\infty,1} = 1$. Then $\tilde{f}_{\mathscr{Q},z_o}(s)$ is a line L_s such that*

(i) *$I_x(\mathscr{Q}, L_s)$ is even for all $x \in \mathscr{Q}$, and*

(ii) *$z_o \notin L_s$.*

Conversely, any line L satisfying the two conditions (i) *and* (ii) *as above gives rise to line-sections s_i $(i = 1, 2)$ such that $s_i \cdot O = 0$, $s_i \cdot \Theta_{\infty,1} = 1$ and $\tilde{f}_{\mathscr{Q},z_o}(s_i) = L$.*

By the choice of z_o, weak-bitangent lines do not pass through z_o. Therefore, by Lemma 2.11, weak-bitangent lines give rise to line-sections of $S_{\mathscr{Q},z_o}$ and vice-versa. Under these settings, for a line L, whether L gives a line-section or not can be determined by how L and $\mathscr{Q}$ intersect. Table 2 shows ten possibilities for how L and $\mathscr{Q}$ intersect.

When we need to describe the type of a weak-bitangent line L and the singularities of $\mathscr{Q}$ on L, we use the following notation:

The type of L	$\mathrm{Sing}(\mathscr{Q}) \cap L$
$Li(x)$ $(i = 3, 5, 6, 7, 8, 9, 10)$	x
$L4(x, y)$	x, y

As for an integral section s with $s \cdot \Theta_{\infty,0} = 1$, we have:

Lemma 2.12. *Let $s \in \mathrm{MW}(S_{\mathscr{Q},z_o})$ be an integral section with $s \cdot \Theta_{\infty,0} = 1$. Then its image $\tilde{f}_{\mathscr{Q},z_o}(s)$ in $\mathbb{P}^2$ is a smooth conic such that either*

Table 2

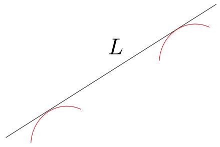
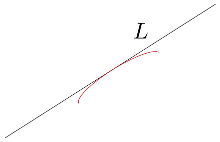
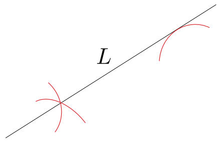
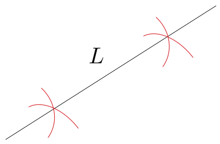
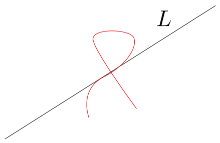
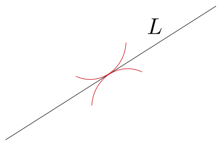
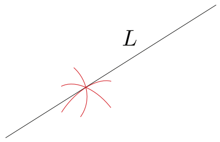
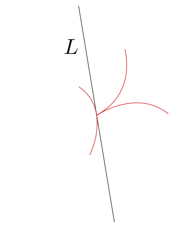
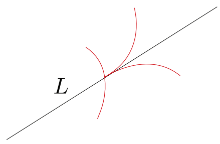
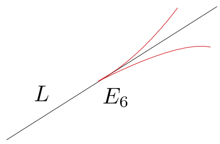
Type	L and $\mathcal{Q}$	How L and $\mathcal{Q}$ intersect
$L1$		L is a bitangent line at distinct smooth points.
$L2$		L is a 4-fold tangent line at a smooth point.
$L3$		L is a line tangent at a smooth point and through a double point.
$L4$		L is a line through distinct double points.
$L5$		L is an inflectional tangent line to one of the branches at an A_1-singularity.
$L6$		L is a unique tangent line to both of the branches (resp. to the branch) at an A_n-singularity if $n \geq 3$ is odd (resp. even).
$L7$		L is a tangent line to one of the branches at a D_4-singularity.

Table 2 (cont.)

Type	L and $\mathcal{Q}$	How L and $\mathcal{Q}$ intersect
$L8$		L is a tangent line to the smooth branch at a D_5-singularity.
$L9$		L is a tangent line to the singular branch at a D_5-singularity.
$L10$		L is a tangent line at an E_6-singularity.

(i) $\tilde{f}_{\mathcal{Q},z_o}(s)$ *is an irreducible component of* $\mathcal{Q}$ *through* z_o, *or*

(ii) $\tilde{f}_{\mathcal{Q},z_o}(s)$ *is tangent to* $\mathcal{Q}$ *at* z_o *and* $I_x(\tilde{f}_{\mathcal{Q},z_o}(s),\mathcal{Q})$ *is even for every* $x \in \tilde{f}_{\mathcal{Q},z_o}(s) \cap \mathcal{Q}$.

Proof. For simplicity, we put $C_s = \tilde{f}_{\mathcal{Q},z_o}(s)$. Since $\tilde{f}_{\mathcal{Q},z_o}(\Theta_{\infty,0} \cup O) = z_o$ and $s \cdot \Theta_{\infty,0} = 1$, $z_o \in C_s$. This means that any line through z_o meets C_s at z_o and another point. As C_s is irreducible, C_s is a smooth conic.

If C_s is an irreducible component of $\mathcal{Q}$, then C_s satisfies the condition (i) in the statement. In the following, we may assume that C_s is not any irreducible component of $\mathcal{Q}$. By our construction of $\tilde{f}_{\mathcal{Q},z_o} : S_{\mathcal{Q}} \to \mathbb{P}^2$, C_s is tangent to $\mathcal{Q}$ at z_o. Choose $x \in C_s \cap \mathcal{Q}$ arbitrary. If $I_x(C_s,\mathcal{Q})$ is odd, the restriction of $\tilde{f}_{\mathcal{Q},z_o}$ to C_s gives rise to a ramified cover of C_s. This means that $\tilde{f}^*_{\mathcal{Q},z_o}(C_s)$ contains a unique irreducible component $\tilde{C}_s$ such that $\tilde{f}_{\mathcal{Q},z_o}|_{\tilde{C}_s} : \tilde{C}_s \to C_s$ is a double cover. On the other hand, $\tilde{f}^*_{\mathcal{Q},z_o}(C_s)$ contains two integral sections s and $[-1]^*_{\mathcal{Q},z_o}s$ as its irreducible components. As $\tilde{f}_{\mathcal{Q},z_o}(s) = \tilde{f}_{\mathcal{Q},z_o}([-1]^*_{\mathcal{Q},z_o}s) = C_s$, this leads us to a contradiction. □

Table 3 lists some cases of conics described in Lemma 2.12 which are necessary for our later argument.

When we need to describe the type of C and the singularities of $\mathcal{Q}$ on C, similarly to lines we use the following notation:

Table 3

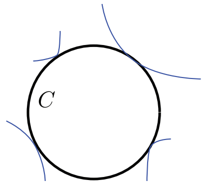
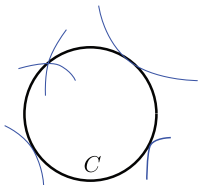
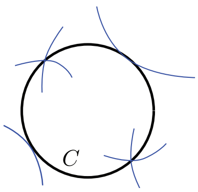
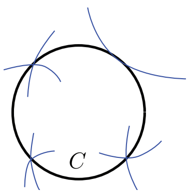
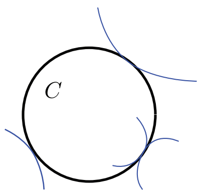
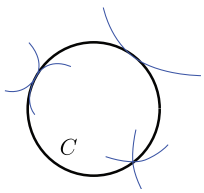
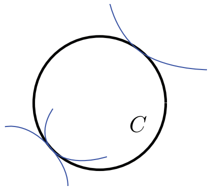
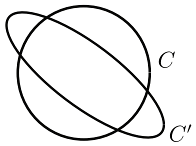
Type	C and $\mathcal{Q}$	How C and $\mathcal{Q}$ intersect
$C1$		C is tangent to $\mathcal{Q}$ at smooth points with even multiplicities.
$C2$		C passes through a double point of $\mathcal{Q}$ and is tangent to $\mathcal{Q}$ at smooth points with even multiplicities.
$C3$		C passes through two distinct double points of $\mathcal{Q}$ and is tangent to $\mathcal{Q}$ at smooth points.
$C4$		C passes through three distinct double points of $\mathcal{Q}$ and is tangent to $\mathcal{Q}$ at a smooth point.
$C5$		C is tangent to $\mathcal{Q}$ at a double point and smooth points with even multiplicities.
$C6$		C is tangent to $\mathcal{Q}$ at a double point with multiplicity 4 and a smooth point, and passes through another double point.

Table 3 (cont.)

Type	C and $\mathscr{Q}$	How C and $\mathscr{Q}$ intersect
$C7$		C is tangent to $\mathscr{Q}$ at a double point with multiplicity 6 and a smooth point.
$C8$		C is a component of two conics.

The type of C	$\mathrm{Sing}(\mathscr{Q}) \cap C$
$Cj(x)$ $(j = 2, 5, 7)$	x
$Cj(x, y)$ $(j = 3, 6)$	x, y
$C4(x, y, z)$	x, y, z

3. The Mumford representations of semi-reduced divisors

In this section, we describe the Mumford representations of semi-reduced divisors on a hyperelliptic curve which are key tools to prove Theorem 1.3.

For terminology and notation for curves and divisors, we refer to [19]. As for details on Mumford representations, we refer to [7, 20]. Let K be a perfect field of $\mathrm{char}(K) \neq 2$ and let $\overline{K}$ be its algebraic closure.

3.1. Mumford representations. Let $\mathscr{C}$ be a hyperelliptic curve of genus g defined over K given by an affine equation

$$y^2 = f(x), \qquad f(x) = x^{2g+1} + c_1 x^{2g} + \cdots + c_{2g+1} \quad (c_i \in K,\ i = 1, \dots, 2g+1).$$

We denote the point of C at infinity by O and the hyperelliptic involution by $\iota : (x, y) \mapsto (x, -y)$. For a divisor $\mathfrak{d} = \sum_{P \in \mathscr{C}} n_P P \in \mathrm{Div}(\mathscr{C})$ on $\mathscr{C}$, we denote the subset $\{P \in \mathscr{C} \mid n_P \neq 0\}$ of $\mathscr{C}$ by $\mathrm{Supp}(\mathfrak{d})$.

Definition 3.1. Let $\mathfrak{d} = \sum_{P \in \mathscr{C}} n_P P \in \mathrm{Div}(\mathscr{C})$ be an effective divisor on $\mathscr{C}$ such that $O \notin \mathrm{Supp}(\mathfrak{d})$. We call $\mathfrak{d}$ a *semi-reduced divisor* if it satisfies the following conditions:

- if $P \in \mathrm{Supp}(\mathfrak{d})$ and $P \neq \iota(P)$, then $\iota(P) \notin \mathrm{Supp}(\mathfrak{d})$, and
- if $P \in \mathrm{Supp}(\mathfrak{d})$ and $P = \iota(P)$, then $n_P = 1$.

We denote the coordinate ring $\overline{K}[x,y]/\langle y^2 - f\rangle$ of $\mathscr{C}$ by $\overline{K}[\mathscr{C}]$ and the image of $g \in \overline{K}[x,y]$ in $\overline{K}[\mathscr{C}]$ by $[g]$. For $P \in \mathscr{C}$, we denote the local ring at P by $\mathcal{O}_P$ and its discrete valuation by ord_P. Let $\mathfrak{d} = \sum_{P\in\mathscr{C}} n_P P$ be a semi-reduced divisor on $\mathscr{C}$. We define ideals $I(\mathfrak{d}) \subset \overline{K}[\mathscr{C}]$ and $\widetilde{I(\mathfrak{d})} \subset \overline{K}[x,y]$ as follows:

$$I(\mathfrak{d}) := \{\xi \in \overline{K}[\mathscr{C}] \mid \mathrm{ord}_P(\xi) \geq n_P,\ \forall P \in \mathrm{Supp}(\mathfrak{d})\},$$

$$\widetilde{I(\mathfrak{d})} := \{g \in \overline{K}[x,y] \mid \mathrm{ord}_P([g]) \geq n_P,\ \forall P \in \mathrm{Supp}(\mathfrak{d})\}.$$

Proposition 3.2 ([20, Proposition 2.1]). *Let $\mathfrak{d}$ be a semi-reduced divisor and let $>_p$ be the pure lexicographical order with $y >_p x$ in $\overline{K}[x,y]$. Then the reduced Gröbner basis of $\widetilde{I(\mathfrak{d})}$ with respect to $>_p$ is of the form $\{a(x), y - b(x)\}$, where $a(x), b(x) \in \overline{K}[x]$ and they satisfy $b(x)^2 - f \in \langle a(x)\rangle$.*

Definition 3.3. Let $\mathfrak{d}$ be a semi-reduced divisor on $\mathscr{C}$ and let $\{a(x), y - b(x)\}$ be as in Proposition 3.2. Then we call the pair (a,b) the *Mumford representation* of $\mathfrak{d}$.

Mumford representations are characterized as follows:

Lemma 3.4. *Let $\mathfrak{d} = \sum_{P\in\mathscr{C}} n_P P$ be a semi-reduced divisor and we put $P = (x_P, y_P)$. Then the pair $(a,b) \in (\overline{K}[x])^2$ is the Mumford representation of $\mathfrak{d}$ if and only if (a,b) satisfies*

(i) $a = \prod_{P\in\mathrm{Supp}(\mathfrak{d})}(x - x_P)^{n_P}$,
(ii) $\deg b < \deg a$, $\mathrm{ord}_P([y-b]) \geq n_P$, *and*
(iii) $a \mid b^2 - f$.

For a proof, see [20, Proposition 2.1].

Remark 3.5. *Let $\mathfrak{d}$ be a semi-reduced divisor. In* [7, 20], *the Mumford representation of $\mathfrak{d}$ is defined by the pair (a,b) satisfying the three conditions in Lemma* 3.4.

A divisor $\mathfrak{d}$ is said to be defined over K if $\mathfrak{d}^\sigma = \mathfrak{d}$ for all $\sigma \in \mathrm{Gal}(\overline{K}/K)$.

Remark 3.6. *Let $\mathfrak{d} = \sum_i n_i P_i$ be a semi-reduced divisor defined over K. Then the Mumford representation (a,b) of $\mathfrak{d}$ belongs to $(K[x])^2$, while the points P_i are not necessarily K-rational points.*

3.2. Semi-reduced divisors of degree 3 on elliptic curves. We refer to [1] for the proof of the lemmas in this section. Let E be an elliptic curve defined over K given by a Weierstrass equation

$$y^2 = f(x), \qquad f(x) = x^3 + c_1x^2 + c_2x + c_3 \quad (c_i \in K,\ i = 1,2,3).$$

Let $\mathfrak{d} = P_1 + P_2 + P_3$ be a semi-reduced divisor of degree 3. We put $P_{\mathfrak{d}} = P_1 \dot{+} P_2 \dot{+} P_3$.

Lemma 3.7 ([1, Lemma 6.2]). *Assume that $P_{\mathfrak{d}} \neq O$ and let (a, b) be the Mumford representation of $\mathfrak{d}$. Then we have*

(i) $P_{\mathfrak{d}} \neq P_i$ $(i = 1, 2, 3)$.

(ii) $\deg b = 2$.

Lemma 3.8 ([1, Lemma 6.3]). *We keep the notation of the previous lemma. Assume that $\mathfrak{d}$ is defined over K. Put $P_{\mathfrak{d}} := (x_{\mathfrak{d}}, y_{\mathfrak{d}})$. Then we have the following:*

(i) *The point $P_{\mathfrak{d}}$ is a K-rational point of E, i.e., $x_{\mathfrak{d}}, y_{\mathfrak{d}} \in K$.*

(ii) *The two polynomials a, b satisfy $a, b \in K[x]$. In particular, b is of the form*

$$b_0(x - x_{\mathfrak{d}})(x - b_1) - y_{\mathfrak{d}} \qquad (b_0, b_1 \in K).$$

4. Proof of Theorem 1.3

Before we prove Theorem 1.3, we prepare two lemmas. Let $[T, X, Z]$ be homogeneous coordinates of $\mathbb{P}^2$ and let $(t, x) = (T/Z, X/Z)$ be affine coordinates for $\mathbb{C}^2 = \mathbb{P}^2 \setminus \{Z = 0\}$.

Lemma 4.1. *Let $\mathcal{Q}$ be a reduced quartic curve that is not the union of four lines and let z_o be a smooth point on $\mathcal{Q}$ satisfying (‡). By choosing suitable homogeneous coordinates $[T, X, Z]$, we may assume that $z_o = [0, 1, 0]$ and $\mathcal{Q}$ is given by an equation of the form*

$$F_{\mathcal{Q}}(T, X, Z) = X^3 Z + A_{\mathcal{Q},2}(T, Z)X^2 + A_{\mathcal{Q},3}(T, Z)X + A_{\mathcal{Q},4}(T, Z),$$

where $A_{\mathcal{Q},i}$ is a binary form of degree i in T and Z such that

$$\deg A_{\mathcal{Q},i}(t, 1) = i \quad (i = 2, 3), \qquad \text{and} \qquad \deg A_{\mathcal{Q},4}(t, 1) \leq 3.$$

Proof. Our statement is immediate if we choose homogeneous coordinates $[T, X, Z]$ such that (i) $z_o = [0, 1, 0]$, (ii) the tangent line l_{z_o} at z_o is given by $Z = 0$ and (iii) $[1, 0, 0] \in \mathcal{Q}$. □

Let E be an elliptic curve given by the Weierstrass equation $y^2 = F_{\mathcal{Q}}(t, x, 1)$. Let $\mathfrak{d} = P_1 + P_2 + P_3 \in \mathrm{Div}(E)$ be a semi-reduced divisor defined over $\mathbb{C}(t)$ whose Mumford representation is given by (a, b). We put $P_{\mathfrak{d}} = P_1 \dot{+} P_2 \dot{+} P_3$ and assume that $P_{\mathfrak{d}} \neq O$. Then we can write $P_{\mathfrak{d}} = (x_{\mathfrak{d}}, y_{\mathfrak{d}})$. By Lemmas 3.4, 3.7 and 3.8, a, b are given as follows:

$$a = x^3 + a_1 x^2 + a_2 x + a_3 \quad (a_i \in \mathbb{C}(t),\ i = 1, 2, 3) \qquad \text{and}$$

$$b = b_0(x - x_{\mathfrak{d}})(x - b_1) - y_{\mathfrak{d}} \quad (b_0 \in \mathbb{C}(t)^{\times},\ b_1 \in \mathbb{C}(t)),$$

where the solutions of $a(x) = 0$ are the x-coordinates of the points P_i. Also, a and b satisfy the following relation

$$b^2 - F_{\mathcal{Q}}(t, x, 1) = {b_0}^2 (x - x_{\mathfrak{d}}) a. \tag{1}$$

Under these circumstances, we have the next lemma.

Lemma 4.2. *If* $x_{\mathfrak{d}} \in \mathbf{C}[t]$ *with* $\deg x_{\mathfrak{d}} \leq 1$, $a \in \mathbf{C}[t, x]$ *and the total degree of* a *is* 3, *then* $b_0 \in \mathbf{C}^{\times}$, $b_1 \in \mathbf{C}[t]$ *and* $\deg b_1 \leq 1$.

Proof. We first prove that b_0 is of the form $1/c$, $c \in \mathbf{C}[t]$. Put $b_0 = c_1/c_2$, where c_1 and c_2 are coprime polynomials. By the relation (1), we have the following two relations:

$$\{(x - x_{\mathfrak{d}})(x - b_1) - y_{\mathfrak{d}}/b_0\}^2 - F_{\mathcal{Q}}/{b_0}^2 = (x - x_{\mathfrak{d}}) a,$$

$$\{c_1 (x - x_{\mathfrak{d}})(x - b_1) - c_2 y_{\mathfrak{d}}\}^2 = {c_1}^2 (x - x_{\mathfrak{d}}) a - {c_2}^2 F_{\mathcal{Q}}.$$

Since the right hand sides of both relations are in $\mathbf{C}[t, x]$, so are the left hand sides. In particular, the coefficient of x^3, $-2(x_{\mathfrak{d}} + b_1) - 1/{b_0}^2$, in the left hand side of the first relation and that of x, $c_1(x_{\mathfrak{d}} + b_1)$, in the left hand side of the second are polynomials.

Since $-2(x_{\mathfrak{d}} + b_1) - 1/{b_0}^2$ and $c_1(x_{\mathfrak{d}} + b_1) \in \mathbf{C}[t]$, we have ${c_2}^2/c_1 \in \mathbf{C}[t]$. Since c_1 and c_2 are coprime to each other, $c_1 \in \mathbf{C}^{\times}$. Hence, $1/b_0 = c_2/c_1 \in \mathbf{C}[t]$ and we have $b_1 \in \mathbf{C}[t]$ as $c_1(x_{\mathfrak{d}} + b_1) \in \mathbf{C}[t]$.

Putting $c = 1/b_0$, we have

$$\{(x - x_{\mathfrak{d}})(x - b_1) - c y_{\mathfrak{d}}\}^2 - c^2 F_{\mathcal{Q}} = (x - x_{\mathfrak{d}}) a.$$

By comparing coefficients of polynomials in $\mathbf{C}[t][x]$, we have the assertion. □

We are now in a position to prove Theorem 1.3.

• Proof of Theorem 1.3. Let us assume that $\mathcal{Q}$ and z_o satisfy (†) and (‡). We may assume that $\mathcal{Q}$ is given by an equation described in Lemma 4.1 and $z_o = [0, 1, 0]$. The generic fiber of $\varphi_{\mathcal{Q}, z_o}$ is an elliptic curve given by $y^2 = F_{\mathcal{Q}}(t, x, 1)$ and L_i $(i = 1, 2, 3, 4)$ are given by $x - x_i(t) = 0$. As L_i $(i = 1, 2, 3)$ are distinct, $P_i \neq [-1]P_j$ $(i \neq j,\ i, j = 1, 2, 3)$. Hence $P_1 + P_2 + P_3$ is a semi-reduced divisor defined over $\mathbf{C}(t)$. We denote its Mumford representation by (a, b). Note that a and b satisfy the relation:

$$b^2 - F_{\mathcal{Q}}(t, x, 1) = {b_0}^2 (x - x_4) a \qquad (b_0 \in \mathbf{C}(t)^{\times}),$$

where $b = b_0(x - x_4)(x - b_1) - y_4$ $(b_1 \in \mathbf{C}(t))$. A polynomial $a = \prod_{i=1}^{3} (x - x_i)$ is of total degree 3. By Lemma 4.2, we have $b_0 \in \mathbf{C}^{\times}$, $b_1 \in \mathbf{C}[t]$ and

$\deg b_1 \leq 1$. Hence, the total degree of b is equal to $\max\{2, \deg y_4\}$. On the other hand, $y_4{}^2 = F_{\mathcal{Q}}(t, x_4, 1)$. By our choice of $F_{\mathcal{Q}}$, we find $\deg y_4{}^2 = \deg F_{\mathcal{Q}}(t, x_4, 1) \leq 4$. Therefore $b(t, x) = 0$ gives rise to the desired conic C. □

5. A classification of weak-bitangent lines

Our goal in this section is to give a list of weak-bitangent lines in terms of Mordell-Weil lattices. Throughout this section, we assume that $\mathcal{Q}$ is a singular quartic curve satisfying (†) and z_o is a smooth point on $\mathcal{Q}$ satisfying (‡), unless otherwise stated.

5.1. Preparations for a classification of weak-bitangent lines. Let us start with the following lemma.

Lemma 5.1. *Choose* $s \in \mathrm{MW}(S_{\mathcal{Q}, z_o})$. *If* $\langle P_s, P_s \rangle < 3/2$ *then* s *is an integral section. Moreover, in the cases of Table* 1 *other than No.* 24 *and* 61, *if* $\langle P_s, P_s \rangle = 3/2$ *then* s *is also an integral section.*

Proof. By Theorem 2.8, we have

$$\langle P_s, P_s \rangle = 2 + 2s \cdot O - \sum_{v \in \mathrm{Red}(\varphi_{\mathcal{Q}, z_o})} \mathrm{contr}_v(s, s).$$

In our setting, the contribution term is of the form

$$\sum_{x \in \mathrm{Sing}(\mathcal{Q})} \mathrm{contr}_{v(x)}(s, s) + \mathrm{contr}_{\infty}(s, s).$$

By straightforward computation with Table 1, we see that the above value is less than or equal to $5/2$. Hence we have

$$\langle P_s, P_s \rangle \geq 2 + 2s \cdot O - 5/2 = 2s \cdot O - 1/2.$$

Hence if $\langle P_s, P_s \rangle < 3/2$, $s \cdot O = 0$.

In the cases other than No. 24 and 61, we see that the contribution term is less than $5/2$. In a similar way to the above case, we infer that if $\langle P_s, P_s \rangle \leq 3/2$, $s \cdot O = 0$. □

Choose $P_1, \ldots, P_n$ and $P_\tau \in E_{\mathcal{Q}, z_o}(\mathbb{C}(t))$ such that

(i) $\{P_1, \ldots, P_n\}$ is a basis of the free part of $E_{\mathcal{Q}, z_o}(\mathbb{C}(t))$,

(ii) $P_\tau = O$ if there exists no torsion in $E_{\mathcal{Q}, z_o}(\mathbb{C}(t))$, while $E_{\mathcal{Q}, z_o}(\mathbb{C}(t))_{tor} = \langle P_\tau \rangle$ if $E_{\mathcal{Q}, z_o}(\mathbb{C}(t))_{tor} \neq \{O\}$ and

(iii) the Gram matrix $[\langle P_i, P_j \rangle]_{1 \le i, j \le n}$ coincides with the one given in Table 1.

In the following, we give descriptions for line-points through the above $P_1, \dots, P_n$ and P_τ.

LEMMA 5.2. *Let s_i $(1 \le i \le n)$ be the sections corresponding to P_i $(1 \le i \le n)$ and let s_τ be the section corresponding to P_τ. By relabeling $P_1, \dots, P_n$, for each case in Table 1, $\tilde{f}_{\mathcal{Q}, z_o}(s_i)$ $(1 \le i \le n)$ and $\tilde{f}_{\mathcal{Q}, z_o}(s_\tau)$ are described as in Table 4.*

Table 4

Oguiso-Shioda classification	$\underline{\mathrm{Sing}}(\mathcal{Q})$	*Types of* $(\tilde{f}_{\mathcal{Q}, z_o}(s_1), \dots, \tilde{f}_{\mathcal{Q}, z_o}(s_n), \tilde{f}_{\mathcal{Q}, z_o}(s_\tau))$
No. 4	(x, A_1)	*See the below*[*1]
No. 6	(x, A_2)	$(L3, C2, L1, C2, L3)$
No. 7	(x, A_1) (y, A_1)	$(L3(x), C1, L3(y), C3, L4)$
No. 10	(x, A_3)	$(L3, C5, L3, L6)$
No. 12	(x, A_2) (y, A_1)	$(L4, L3(x), L3(y), L3(x))$ *or* $(L4, C3, L3(y), C3)$
No. 14	(x, A_1) (y, A_1) (z, A_1)	$(L4(x, y), L4(y, z), L4(x, z), C4)$
No. 17	(x, A_4)	$(L6, L3, L3)$
No. 18	(x, D_4)	$(L7, L7, L7)$
No. 20	(x, A_2) (y, A_2)	$(C3, C3, L4)$
No. 22	(x, A_3) (y, A_1)	$(C6(x, y), L6, L4)$[*2]
No. 23	(x, A_2) (y, A_1) (z, A_1)	$(L4(y, z), L4(x, y), L4(x, z))$ *or* $(L4(y, z), L4(x, z), C4)$
No. 24	(x, A_1) (y, A_1) (z, A_1) (w, A_1)	$(L4(x, y), L4(y, z), L4(x, z), C8)$[*3] *or* $(L4(x, w), L4(y, w), L4(z, w), C8)$[*3]

Table 4 (cont.)

Oguiso-Shioda classification	$\underline{\mathrm{Sing}}(\mathcal{Q})$	*Types of* $(\tilde{f}_{\mathcal{Q},z_o}(s_1),\dots,\tilde{f}_{\mathcal{Q},z_o}(s_n),\tilde{f}_{\mathcal{Q},z_o}(s_\tau))$
No. 29	(x,A_5)	$(C7,L6)$
No. 30	(x,D_5)	$(L8,L9)$
No. 33	(x,A_4) (y,A_1)	$(L4,L6)$ *or* $(L4,C6(x,y))$*4
No. 37	(x,A_3) (y,A_2)	$(L6(x),L4)$
No. 40	(x,A_2) (y,A_2) (z,A_1)	$(L4(x,y),C4)$
No. 47	(x,A_6)	$L6$
No. 49	(x,E_6)	$L10$
No. 56	(x,A_4) (y,A_2)	$L4$
No. 61	(x,A_2) (y,A_2) (z,A_2)	$(L4(x,y),C4)$*5

*1 *In the case of No.* 4, *the type of* $(\tilde{f}_{\mathcal{Q},z_o}(s_1),\tilde{f}_{\mathcal{Q},z_o}(s_2),\tilde{f}_{\mathcal{Q},z_o}(s_5),\tilde{f}_{\mathcal{Q},z_o}(s_6))$ *is* $(L3,C1,L1,C2)$. *On the other hand,* s_3 *and* s_4 *satisfy*

$$s_i\cdot O=1\quad(i=3,4)\qquad\text{and}\qquad \boldsymbol{c}(v(x),s_i)=\boldsymbol{c}(\infty,s_i)=\begin{cases}1 & i=3\\ 0 & i=4\end{cases}.$$

*2 *In the case of No.* 22, *if* $\tilde{f}_{\mathcal{Q},z_o}(s_i)$ *is of type* $C6(x,y)$, $I_x(\tilde{f}_{\mathcal{Q},z_o}(s_i),\mathcal{Q})=4$ *and* $I_y(\tilde{f}_{\mathcal{Q},z_o}(s_i),\mathcal{Q})=2$.
*3 *In the case of No.* 24, *we only consider the cases when* (i) *three weak-bitangent lines of type L*4 *are concurrent at w and* (ii) *three weak-bitangent lines of type L*4 *do not pass through w. We omit other cases to avoid redundancy in Table* 4.
*4 *In the case of No.* 33, $I_x(\tilde{f}_{\mathcal{Q},z_o}(s_2),\mathcal{Q})=4$ *and* $I_y(\tilde{f}_{\mathcal{Q},z_o}(s_2),\mathcal{Q})=2$.
*5 *In the case of No.* 61, *we omit weak-bitangent lines of type L*4 *except for* $L4(x,y)$.

Here Li $(1\le i\le 10)$ *are the types of lines in Table* 2 *and* Cj $(1\le j\le 8)$ *are the types of conics in Table* 3. *When* $P_\tau=O$, *we describe types of* $\tilde{f}_{\mathcal{Q},z_o}(s_i)$ $(1\le i\le n)$ *only.*

Proof. We give a proof for the case of No. 4 only as the other cases can be proven similarly. In order to determine types of $\tilde{f}_{\mathcal{Q},z_o}(s_i)$, we need to find $s\cdot O$, $\boldsymbol{c}(v(x),s_i)$ and $\boldsymbol{c}(\infty,s_i)$. First, we have

$$[\langle P_i, P_j\rangle]_{1\le i,j\le 6} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1/2 & 1/2 \\ 1 & 2 & 2 & 2 & 1 & 1 \\ 1 & 2 & 3 & 3 & 3/2 & 3/2 \\ 1 & 2 & 3 & 4 & 2 & 2 \\ 1/2 & 1 & 3/2 & 2 & 3/2 & 1 \\ 1/2 & 1 & 3/2 & 2 & 1 & 3/2 \end{bmatrix}.$$

By Lemma 5.1, the sections s_1, s_5 and s_6 are integral sections. Since the configuration of reducible fibers is either III, I_2 or $2I_2$, we have

$$\langle P_i, P_i\rangle = 2 - \frac{\alpha_i + \beta_i}{2} \qquad (i = 1, 5, 6),$$

$$\langle P_j, P_j\rangle = 2 + 2s_j \cdot O - \frac{\alpha_j + \beta_j}{2} \qquad (j = 2, 3, 4),$$

where $(\alpha_i, \beta_i) = (\boldsymbol{c}(v(x), s_i), \boldsymbol{c}(\infty, s_i))(= (s_i \cdot \Theta_{v(x),1}, s_i \cdot \Theta_{\infty,1}))$. From the matrix $[\langle P_i, P_j\rangle]_{1\le i,j\le 6}$, we infer the following:

s_i	s_1	s_2	s_3	s_4	s_5	s_6
$s_i \cdot O$	0	0	1	1	0	0
$\alpha_i + \beta_i$	2	0	2	0	1	1 .

Hence, s_1, s_2, s_3, s_4, s_5 and s_6 satisfy

$$(\alpha_i, \beta_i) = \begin{cases} (1,1) & \text{for } i = 1, 3 \\ (0,0) & \text{for } i = 2, 4 \\ (0,1) \text{ or } (1,0) & \text{for } i = 5, 6. \end{cases}$$

By Lemmas 2.11 and 2.12, $\tilde{f}_{\mathcal{Q}, z_o}(s_1)$ is a line and $\tilde{f}_{\mathcal{Q}, z_o}(s_2)$ is a smooth conic. In particular, their types are $L3$ and $C1$.

Claim: $(\alpha_5, \beta_5) \neq (\alpha_6, \beta_6)$.

Proof of Claim. Assume that $(\alpha_5, \beta_5) = (\alpha_6, \beta_6)$. By Theorem 2.8, we have $\langle P_5, P_6\rangle = 3/2 - s_5 \cdot s_6$. This is impossible as $\langle P_5, P_6\rangle = 1$.

Therefore, for s_5 and s_6, the following conditions hold:

- $\tilde{f}_{\mathcal{Q}, z_o}(s_i)$ is of type $L1$ if $(\boldsymbol{c}(v(x), s_i), \boldsymbol{c}(\infty, s_i)) = (0, 1)$.
- $\tilde{f}_{\mathcal{Q}, z_o}(s_i)$ is of type $C2$ if $(\boldsymbol{c}(v(x), s_i), \boldsymbol{c}(\infty, s_i)) = (1, 0)$.

Hence, the type of $(\tilde{f}_{\mathcal{Q}, z_o}(s_1), \tilde{f}_{\mathcal{Q}, z_o}(s_2), \tilde{f}_{\mathcal{Q}, z_o}(s_5), \tilde{f}_{\mathcal{Q}, z_o}(s_6))$ is

$$(L3, C1, L1, C2) \text{ or } (L3, C1, C2, L1).$$

By relabeling s_5 and s_6 if necessary, we may assume that they are as in Table 4. As for s_3 and s_4, we have

$$s_i \cdot O = 1 \quad (i = 3, 4) \qquad \text{and} \qquad \boldsymbol{c}(v(x), s_i) = \boldsymbol{c}(\infty, s_i) = \begin{cases} 1 & i = 3 \\ 0 & i = 4 \end{cases}. \qquad \square$$

In the other cases except for No. 40, for weak-bitangent lines of types $L4$ and Li $(6 \le i \le 10)$, we see that all possible cases are classified by Lemma 5.2. In the case of No. 40, weak-bitangent lines of types $L4$ and Li $(6 \le i \le 10)$ are also classified below.

Lemma 5.3. *In the case of No.* 40, *let* P_1, P_2 *be a basis such that types of* $\tilde{f}_{\mathcal{Q},z_o}(s_{P_i})$ *are those indicated in No.* 40 *in Table* 4. *Put* $Q_1 = P_1 \dot{+} P_2$ *and* $Q_2 = P_1 \dot{-} P_2$. *Then* $\tilde{f}_{\mathcal{Q},z_o}(s_{Q_i})$ *are of types* $L4(x,z)$ *and* $L4(y,z)$.

Proof. Before we prove our statement, we start with the following claim.

Claim: If $\langle P_s, P_s\rangle = 1/3$ and $s \cdot \Theta_{\infty,1} = 1$ then $\tilde{f}_{\mathcal{Q},z_o}(s)$ is a line of type $L4$ and passes through a cusp and the node z.

Proof of Claim. If $\mathrm{contr}_v(s,s) \neq 0$, we have

$$\mathrm{contr}_\bullet(s,s) = 2/3 \quad (\bullet = x, y) \qquad \text{and} \qquad \mathrm{contr}_z(s,s) = 1/2.$$

By Lemma 5.1, s is integral. Hence, s is a line-section and we have

$$1/3 = 3/2 - (\mathrm{contr}_{v(x)}(s,s) + \mathrm{contr}_{v(y)}(s,s) + \mathrm{contr}_{v(z)}(s,s)).$$

Hence the possibilities for $\mathrm{contr}_{v(\bullet)}(s,s)$ are as follows:

$$(\mathrm{contr}_{v(x)}(s,s), \mathrm{contr}_{v(y)}(s,s)) = (2/3, 0) \text{ or } (0, 2/3),$$

$$\mathrm{contr}_{v(z)}(s,s) = 1/2.$$

A line $\tilde{f}_{\mathcal{Q},z_o}(s)$ passes through a cusp and the node z in both the cases of $(\mathrm{contr}_{v(x)}(s,s), \mathrm{contr}_{v(y)}(s,s)) = (2/3,0),\ (0,2/3)$.

Now we go back to prove our statement. As $(s_{P_1} + s_{P_2}) \cdot \Theta_{\infty,1} = 1$, we have $s \cdot \Theta_{\infty,1} = 1$ by Corollary 2.7. Since $s_{Q_i} \cdot \Theta_{\infty,1} = 1$ and $\langle Q_i, Q_i\rangle = 1/3$, any $\tilde{f}_{\mathcal{Q},z_o}(s_{Q_i})$ is of type $L4$ through a cusp and z. Also, as $Q_1 \neq \dot{\pm} Q_2$, the $\tilde{f}_{\mathcal{Q},z_o}(s_{Q_i})$ are distinct lines. Hence, we obtain lines of types $L4(x,z)$ and $L4(y,z)$. □

In the next section, for our classification of weak-bitangent lines, we consider weak-bitangent lines of types $L1$, $L2$, $L3$ and $L5$.

5.2. A classification of weak-bitangent lines via Mordell-Weil lattices. We next consider characterizations of weak-bitangent lines via Mordell-Weil lattices. Let us start with the following proposition.

Proposition 5.4. *Let* $\mathcal{Q}$ *be an irreducible quartic curve with double points only. For* $s \in \mathrm{MW}(S_{\mathcal{Q},z_o})$, *the following conditions* (i) *and* (ii) *are equivalent:*

(i) $\tilde{f}_{\mathcal{Q},z_o}(s)$ *is a weak-bitangent line of type* $L3$ *or* $L5$.

(ii) $s \cdot \Theta_{\infty,1} = 1$ *and there exists a positive integer* n_s *such that* $\langle P_s, P_s\rangle = 3/2 - n_s/(n_s+1)$.

Proof. In the case when $\mathcal{Q}$ has three cusps, there exists no weak-bitangent line of type $L3$. In fact, if such a line exists, it gives rise to a section s with $\langle P_s, P_s \rangle = 5/6$. On the other hand, as $E_{\mathcal{Q},z_o}(\mathbf{C}(t)) \simeq \langle 1/6 \rangle \oplus \mathbf{Z}/3\mathbf{Z}$, there exists no $\mathbf{C}(t)$-rational point such that its height pairing equals $5/6$. This leads us to a contradiction. Therefore, we omit the case of No. 61.

By our choice of z_o, $\varphi_{\mathcal{Q},z_o}$ has a singular fiber F_∞ of type I_2. By [13, Table 6.2], the other reduced fibers of $\varphi_{\mathcal{Q},z_o}$ are of types III, IV and I_b $(b \geq 2)$. For each case, if $\mathrm{contr}_{v(x)}(s,s) \neq 0$, it is as follows:

Type of $F_{v(x)}$	$\mathrm{contr}_{v(x)}(s,s)$
III	$1/2$
IV	$2/3$
I_b	$k(b-k)/b$ (if $s \cdot \Theta_{v(x),k} = 1$)

Assume that $\tilde{f}_{\mathcal{Q},z_o}(s)$ is a weak-bitangent line of type $L3$ or $L5$. Then $s \cdot \Theta_{\infty,1} = 1$ and there exists a unique $x_0 \in \mathrm{Sing}(\mathcal{Q}) \cap \tilde{f}_{\mathcal{Q},z_o}(s)$. Then by our construction of $S_{\mathcal{Q},z_o}$, we have

$$\mathrm{contr}_{v(x_0)}(s,s) = \begin{cases} 1/2 & \text{if } F_{v(x_0)} \text{ is of type III,} \\ 2/3 & \text{if } F_{v(x_0)} \text{ is of type IV,} \\ k(b-k)/b & \text{if } F_{v(x_0)} \text{ is of type } \mathrm{I}_b \ (b \geq 2) \text{ and} \\ & \quad s \cdot \Theta_{v(x_0),k} = 1. \end{cases}$$

For weak-bitangent lines of types $L3$ and $L5$, the following conditions hold:

- $s \cdot \Theta_{v(x_0),1} = 1$ if $F_{v(x_0)}$ is of type III,
- $s \cdot \Theta_{v(x_0),1} = 1$ or $s \cdot \Theta_{v(x_0),2} = 1$ if $F_{v(x_0)}$ is of type IV, and
- $s \cdot \Theta_{v(x_0),1} = 1$ or $s \cdot \Theta_{v(x_0),b-1} = 1$ if $F_{v(x_0)}$ is of type I_b.

Hence $n_s = 1, 2$ or $b-1$ if $F_{v(x_0)}$ is of type III, IV or I_b, respectively.

Conversely, assume that the condition (ii) in the statement holds. Then as $s \cdot \Theta_{\infty,1} = 1$, s is an integral section by Lemma 5.1. Hence we have

$$\langle P_s, P_s \rangle = 3/2 - n_s/(n_s+1) = 3/2 - \sum_{x \in \mathrm{Sing}(\mathcal{Q})} n_x/(n_x+1).$$

Hence, $\sum_{x \in \mathrm{Sing}(\mathcal{Q})} n_x/(n_x+1) = n_s/(n_s+1) < 1$. From the above possible values of $\mathrm{contr}_{v(x)}(s,s)$, there exists a unique $x_0 \in \mathrm{Sing}(\mathcal{Q}) \cap \tilde{f}_{\mathcal{Q},z_o}(s)$. Also $s \cdot \Theta_{v(x_0),1} = 1$ or $s \cdot \Theta_{v(x_0),b-1} = 1$ if $F_{v(x)}$ is of type I_b. By our construction of $S_{\mathcal{Q},z_o}$, $\tilde{f}_{\mathcal{Q},z_o}(s)$ is of type $L5$, if x_0 is a node and $\tilde{f}_{\mathcal{Q},z_o}(s)$ is an inflectional tangent to one of the branches, while $\tilde{f}_{\mathcal{Q},z_o}(s)$ is of type $L3$ for the remaining cases. □

Similarly, we obtain the following proposition.

Proposition 5.5. *Let $\mathcal{Q}$ be a singular quartic curve satisfying* (†). *For* $s \in \mathrm{MW}(S_{\mathcal{Q},z_o})$, *the following conditions* (i) *and* (ii) *are equivalent:*

(i) $\tilde{f}_{\mathcal{Q},z_o}(s)$ *is a weak-bitangent line of type* $L1$ *or* $L2$.

(ii) $s \cdot \Theta_{\infty,1} = 1$, $\langle P_s, P_s \rangle = 3/2$ *and* $s \cdot O = 0$.

Moreover, in the cases other than No. 24 *and* 61, (i) *is equivalent to the following condition* (ii)′:

(ii)′ $s \cdot \Theta_{\infty,1} = 1$ *and* $\langle P_s, P_s \rangle = 3/2$.

We next classify weak-bitangent lines of types $L1$, $L2$, $L3$ and $L5$.

Let $P_1, \dots, P_n$ and P_τ be generators of $E_{\mathcal{Q},z_o}(\mathbb{C}(t))$ described just after Lemma 5.1. For $Q \in E_{\mathcal{Q},z_o}(\mathbb{C}(t))$, we put

$$Q = [c_1]P_1 \dot{+} \cdots \dot{+} [c_n]P_n \dot{+} [c_\tau]P_\tau,$$

where c_i $(1 \le i \le n)$, $c_\tau \in \mathbb{Z}$. Note that $c_\tau = 0$ if $P_\tau = O$. We classify weak-bitangent lines of types $L1$ and $L2$ by vectors ${}^t[c_1, \dots, c_n]$ if $P_\tau = O$ and ${}^t[c_1, \dots, c_n, c_\tau]$ if $P_\tau \neq O$. Similarly, ${}^t[c_1, \dots, c_n]_x$ and ${}^t[c_1, \dots, c_n, c_\tau]_x$ denote weak-bitangent lines of types $L3(x)$ and $L5(x)$.

Theorem 5.6. *If $\tilde{f}_{\mathcal{Q},z_o}(s_Q)$ is of type $L1$, $L2$, $L3$ or $L5$, then* ${}^t[c_1, \dots, c_n]$, ${}^t[c_1, \dots, c_n, c_\tau]$, ${}^t[c_1, \dots, c_n]_x$ *and* ${}^t[c_1, \dots, c_n, c_\tau]_x$ *are given as in Table* 5.

Table 5

No.	$\underline{\mathrm{Sing}}(\mathcal{Q})$	$L1$ *or* $L2$	$L3$ *or* $L5$
No. 4	(x, A_1)	${}^t[1,-1,1,-1,1,0]$, ${}^t[1,0,-1,0,1,0]$, ${}^t[1,-1,1,0,-1,0]$, ${}^t[1,0,-1,1,-1,0]$, ${}^t[1,-1,0,1,0,-1]$, ${}^t[1,0,0,-1,0,1]$, ${}^t[1,0,0,0,0,-1]$, ${}^t[1,-1,0,0,0,1]$, ${}^t[0,1,0,-1,1,0]$, ${}^t[0,-1,0,0,1,0]$, ${}^t[0,0,0,0,1,0]$, ${}^t[0,0,0,-1,1,0]$, ${}^t[0,0,1,-1,0,1]$, ${}^t[0,-1,1,-1,0,1]$, ${}^t[0,0,1,0,0,-1]$, ${}^t[0,-1,1,0,0,-1]$	${}^t[1,0,0,0,0,0]_x$, ${}^t[1,-1,0,0,0,0]_x$, ${}^t[0,0,0,-1,1,1]_x$, ${}^t[0,0,0,0,1,-1]_x$, ${}^t[0,-1,1,0,0,0]_x$, ${}^t[0,0,1,-1,0,0]_x$

Table 5 (cont.)

No.	$\underline{\mathrm{Sing}}(\mathcal{Q})$	$L1$ *or* $L2$	$L3$ *or* $L5$
No. 6	(x, A_2)	$\begin{bmatrix}1\\-1\\1\\-1\\1\end{bmatrix} \begin{bmatrix}1\\-1\\1\\0\\-1\end{bmatrix} \begin{bmatrix}1\\0\\-1\\0\\1\end{bmatrix} \begin{bmatrix}1\\0\\-1\\1\\-1\end{bmatrix} \begin{bmatrix}1\\-1\\0\\1\\0\end{bmatrix}$ $\begin{bmatrix}1\\0\\0\\-1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix} \begin{bmatrix}0\\-1\\1\\-1\\0\end{bmatrix} \begin{bmatrix}0\\-1\\0\\0\\1\end{bmatrix} \begin{bmatrix}0\\1\\0\\-1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}_x \begin{bmatrix}1\\-1\\0\\0\\0\end{bmatrix}_x$ $\begin{bmatrix}0\\-1\\1\\0\\0\end{bmatrix}_x \begin{bmatrix}0\\0\\1\\-1\\0\end{bmatrix}_x$ $\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}_x \begin{bmatrix}0\\0\\0\\-1\\1\end{bmatrix}_x$
No. 7	(x, A_1) (y, A_1)	$\begin{bmatrix}0\\0\\0\\\pm1\\1\end{bmatrix} \begin{bmatrix}0\\-1\\0\\1\\\pm1\end{bmatrix} \begin{bmatrix}-1\\0\\1\\0\\\pm1\end{bmatrix} \begin{bmatrix}1\\-1\\1\\0\\\pm1\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}_x \begin{bmatrix}0\\-1\\1\\1\\0\end{bmatrix}_x$ $\begin{bmatrix}0\\0\\-1\\1\\0\end{bmatrix}_x \begin{bmatrix}-1\\1\\0\\0\\0\end{bmatrix}_x$ $\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}_y \begin{bmatrix}-1\\0\\0\\1\\0\end{bmatrix}_y$ $\begin{bmatrix}1\\-1\\0\\1\\0\end{bmatrix}_y \begin{bmatrix}0\\-1\\1\\0\\0\end{bmatrix}_y$

Table 5 (cont.)

No.	$\underline{\mathrm{Sing}}(\mathscr{Q})$	*L*1 *or* *L*2	*L*3 *or* *L*5
No. 10	(x, A_3)	$\begin{bmatrix}0\\1\\0\\\pm1\end{bmatrix}\begin{bmatrix}1\\-1\\1\\\pm1\end{bmatrix}\begin{bmatrix}-1\\0\\1\\\pm1\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}_x\begin{bmatrix}0\\0\\1\\0\end{bmatrix}_x$ $\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}_x\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}_x$
No. 12	(x, A_2) (y, A_1)	$\begin{bmatrix}2\\0\\0\\1\end{bmatrix}\begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\\0\\0\end{bmatrix}_x\begin{bmatrix}0\\0\\0\\1\end{bmatrix}_x$ $\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}_x\begin{bmatrix}-1\\1\\-1\\0\end{bmatrix}_x$ $\begin{bmatrix}0\\0\\1\\0\end{bmatrix}_y\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}_y$ $\begin{bmatrix}1\\-1\\0\\1\end{bmatrix}_y$
No. 14	(x, A_1) (y, A_1) (z, A_1)	$\begin{bmatrix}\pm1\\1\\1\\0\end{bmatrix}\begin{bmatrix}\pm1\\-1\\1\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\1\end{bmatrix}_z\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}_z$ $\begin{bmatrix}0\\1\\0\\1\end{bmatrix}_x\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}_x$ $\begin{bmatrix}0\\0\\1\\1\end{bmatrix}_y\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}_y$

Table 5 (cont.)

No.	$\underline{\mathrm{Sing}}(\mathcal{Q})$	$L1$ or $L2$	$L3$ or $L5$
No. 17	(x, A_4)	$\begin{bmatrix}-1\\-1\\1\end{bmatrix}\begin{bmatrix}2\\0\\1\end{bmatrix}\begin{bmatrix}-2\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\\0\end{bmatrix}_x\begin{bmatrix}0\\0\\1\end{bmatrix}_x\begin{bmatrix}1\\-1\\1\end{bmatrix}_x$
No. 18	(x, D_4)	$\begin{bmatrix}\pm1\\1\\1\\0\end{bmatrix}\begin{bmatrix}\pm1\\-1\\1\\0\end{bmatrix}$	N/A
No. 20	(x, A_2) (y, A_2)	$\begin{bmatrix}0\\0\\3\end{bmatrix}$	$\begin{bmatrix}1\\0\\1\end{bmatrix}_x\begin{bmatrix}0\\1\\-1\end{bmatrix}_x\begin{bmatrix}1\\-1\\-1\end{bmatrix}_x\begin{bmatrix}1\\0\\-1\end{bmatrix}_y\begin{bmatrix}0\\1\\1\end{bmatrix}_y\begin{bmatrix}1\\-1\\1\end{bmatrix}_y$
No. 22	(x, A_3) (y, A_1)	$\begin{bmatrix}0\\\pm1\\2\end{bmatrix}$	$\begin{bmatrix}1\\0\\1\end{bmatrix}_x\begin{bmatrix}-1\\0\\1\end{bmatrix}_x\begin{bmatrix}1\\1\\0\end{bmatrix}_y\begin{bmatrix}-1\\1\\0\end{bmatrix}_y$
No. 23	(x, A_2) (y, A_1) (z, A_1)	$\begin{bmatrix}\pm1\\1\\1\end{bmatrix}$	$\begin{bmatrix}1\\-1\\1\end{bmatrix}_x\begin{bmatrix}-1\\-1\\1\end{bmatrix}_x\begin{bmatrix}0\\-1\\2\end{bmatrix}_y\begin{bmatrix}0\\-2\\1\end{bmatrix}_z$
No. 24	(x, A_1) (y, A_1) (z, A_1) (w, A_1)	$\begin{bmatrix}\pm1\\1\\1\\0\end{bmatrix}\begin{bmatrix}\pm1\\-1\\1\\0\end{bmatrix}$	N/A
No. 29	(x, A_5)	$\begin{bmatrix}0\\3\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}_x\begin{bmatrix}-1\\1\end{bmatrix}_x$

Table 5 (cont.)

No.	$\underline{\mathrm{Sing}(\mathcal{Q})}$	*L*1 *or* *L*2	*L*3 *or* *L*5
No. 30	(x, D_5)	$\begin{bmatrix} 1 \\ \pm 2 \end{bmatrix}$	N/A
No. 33	(x, A_4) (y, A_1)	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}_x \begin{bmatrix} -1 \\ 2 \end{bmatrix}_y$
No. 37	(x, A_3) (y, A_2)	N/A	$\begin{bmatrix} 0 \\ 3 \end{bmatrix}_x \begin{bmatrix} 1 \\ 2 \end{bmatrix}_y$ $\begin{bmatrix} -1 \\ 2 \end{bmatrix}_y$
No. 40	(x, A_2) (y, A_2) (z, A_1)	$\begin{bmatrix} 3 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 2 \end{bmatrix}_x \begin{bmatrix} 1 \\ 2 \end{bmatrix}_y$
No. 47	(x, A_6)	N/A	$[3]_x$
No. 49	(x, E_6)	$[3]$	N/A
No. 56	(x, A_4) (y, A_2)	N/A	$[5]_y$
No. 61	(x, A_2) (y, A_2) (z, A_2)	$\begin{bmatrix} 3 \\ 0 \end{bmatrix}$	N/A

(We give either ${}^t[c_1, \ldots, c_n]$ *or* ${}^t[-c_1, \ldots, -c_n]$ *since they give the same line* $\tilde{f}_{\mathcal{Q}, z_o}(s_Q)$*.) Here,* $P_1, \ldots, P_n$ *and* P_τ *are chosen in the following manner:*

- *For No.* 12, 23, 24 *and* 33*, types of* $\tilde{f}_{\mathcal{Q}, z_o}(s_i)$ *are the first types indicated in the corresponding no. in Table* 4*.*
- *For the other remaining cases, the types of* $\tilde{f}_{\mathcal{Q}, z_o}(s_i)$ *are those indicated in the corresponding no. in Table* 4*.*

Proof. <u>The case No. 4.</u> Let G be the Gram matrix $[\langle P_i, P_j \rangle]_{1 \le i, j \le 6}$ and let $\boldsymbol{c} = {}^t[c_1, \ldots, c_6]$. As for $\boldsymbol{c}(\infty, s_i)$, the following holds:

s_i	s_1	s_2	s_3	s_4	s_5	s_6
$\boldsymbol{c}(\infty, s_i)$	1	0	1	0	1	0.

We remark that $s_Q \cdot \Theta_{\infty, 1} = 1$ if and only if $\sum_{i=1}^{6} c_i \boldsymbol{c}(\infty, s_i) = c_1 + c_3 + c_5$ is odd by Corollary 2.7.

a) The case when $\tilde{f}_{\mathcal{Q}, z_o}(s_Q)$ is of type $L1$ or $L2$:

In this case, by Proposition 5.5, $c_1 + c_3 + c_5$ is odd and $c_i \in \mathbb{Z}$ $(i = 1, \ldots, 6)$ satisfy the following equality:

$$\begin{aligned}3/2 &= {}^t\boldsymbol{c}G\boldsymbol{c}\\ &= \left(c_1+c_2+c_3+c_4+\frac{c_5+c_6}{2}\right)^2+\left(c_2+c_3+c_4+\frac{c_5+c_6}{2}\right)^2\\ &\quad +2\left(\frac{c_3}{2}+c_4+\frac{c_5+c_6}{2}\right)^2+\frac{c_3{}^2}{2}+\frac{c_5{}^2}{2}+\frac{c_6{}^2}{2}. \end{aligned}\tag{2}$$

From the above equality, we see that $|c_i| \le 1$ $(i = 3, 5, 6)$.

Claim: $|c_5| \neq |c_6|$.

Proof of Claim. If $|c_5| = |c_6|$, we see that both $(c_5 + c_6)/2$ and $(c_5{}^2 + c_6{}^2)/2$ are integers. Hence, the right hand side of (2) becomes an integer but this is impossible. Therefore, $|c_5| \neq |c_6|$.

- The case $(c_3, c_5, c_6) = (1, 1, 0)$. In this case, the equality (2) becomes

$$\frac{1}{2} = \left(c_1+c_2+c_4+\frac{3}{2}\right)^2+\left(c_2+c_4+\frac{3}{2}\right)^2+2(c_4+1)^2.$$

Hence, we have $c_4 = -1$ and

$$\frac{1}{2} = \left(c_1+c_2+\frac{1}{2}\right)^2+\left(c_2+\frac{1}{2}\right)^2.$$

This implies that the possibilities for (c_1, c_2) are

$$(0,0), (-1,0), (0,-1), (1,-1).$$

Since $c_1 + c_3 + c_5$ is odd, $\boldsymbol{c} = {}^t[-1, 0, 1, -1, 1, 0], {}^t[1, -1, 1, -1, 1, 0]$ in this case.

- The case $(c_3, c_5, c_6) = (0, 1, 0)$. In this case, we have

$$1 = \left(c_1+c_2+c_4+\frac{1}{2}\right)^2+\left(c_2+c_4+\frac{1}{2}\right)^2+2\left(c_4+\frac{1}{2}\right)^2.$$

Hence c_4 must be 0 or -1 and we have

c_4	(c_1, c_2)
0	$(0,0)$, $(-1,0)$, $(0,-1)$, $(1,-1)$
-1	$(1,0)$, $(0,0)$, $(0,1)$, $(-1,1)$.

Since $c_1 + c_3 + c_5$ is odd, $\boldsymbol{c} = {}^t[0, 0, 0, 0, 1, 0]$, ${}^t[0, -1, 0, 0, 1, 0]$, ${}^t[0, 0, 0, -1, 1, 0]$, ${}^t[0, 1, 0, -1, 1, 0]$ in this case.

- The case $(c_3, c_5, c_6) = (-1, 1, 0)$. In this case, we have

$$\frac{1}{2} = \left(c_1+c_2+c_4-\frac{1}{2}\right)^2+\left(c_2+c_4-\frac{1}{2}\right)^2+2c_4^2.$$

Hence $c_4 = 0$ and $(c_1, c_2) = (0,0), (1,0), (0,1), (-1,1)$. As $c_1 + c_3 + c_5$ is odd, $\boldsymbol{c} = {}^t[1,0,-1,0,1,0], {}^t[-1,1,-1,0,1,0]$. For the cases $c_5 = 0$ and -1, we can compute $\boldsymbol{c}$ similarly and we have the list for No. 4, $L1$ and $L2$.

b) The case when $\tilde{f}_{\mathcal{Q},z_o}(s_Q)$ is of type $L3$ or $L5$:

In this case, such a line passes through the A_1-singularity x. By Proposition 5.4 and its proof, $\tilde{f}_{\mathcal{Q},z_o}(s_Q)$ is of type $L3$ or $L5$ if and only if c_i $(i = 1, \dots, 6)$ satisfy the following equality and $c_1 + c_3 + c_5$ is odd:

$$1 = \left(c_1 + c_2 + c_3 + c_4 + \frac{c_5 + c_6}{2}\right)^2 + \left(c_2 + c_3 + c_4 + \frac{c_5 + c_6}{2}\right)^2$$
$$+ 2\left(\frac{c_3}{2} + c_4 + \frac{c_5 + c_6}{2}\right)^2 + \frac{{c_3}^2}{2} + \frac{{c_5}^2}{2} + \frac{{c_6}^2}{2}.$$

By a similar argument to the above case for $L1$ and $L2$, we have the list for No. 4, $L3$ and $L5$. Note that the assertion in other cases except for No. 24 and 61 can be proven similarly. See Remark 5.7 below.

The case No. 24. There exists no weak-bitangent line of type $L3$ or $L5$. Therefore, in this case, we only need to consider the case when $\tilde{f}_{\mathcal{Q},z_o}(s_Q)$ is of type $L1$ or $L2$. Let $\boldsymbol{c} = {}^t[c_1, c_2, c_3, c_\tau]$ and put $a_v = s_Q \cdot \Theta_{v,1}$. By Proposition 5.5, s_Q is a line-section for a line of type $L1$ or $L2$ if and only if Q satisfies the following conditions:

(i) $\langle Q, Q\rangle = 3/2$, (ii) $s_Q \cdot O = 0$ and (iii) $a_\infty = 1$.

Claim 1: $\langle Q, Q\rangle = 3/2$ if and only if $|c_i| = 1$ $(i = 1,2,3)$.

PROOF OF CLAIM. Since $\langle Q, Q\rangle = ({c_1}^2 + {c_2}^2 + {c_3}^2)/2$, our claim follows.

Claim 2: If $\langle Q, Q\rangle = 3/2$, then $a_{v(\bullet)} = 0$ $(\bullet = x, y, z, w)$ if and only if Q satisfies (ii) and (iii).

PROOF OF CLAIM. Recall

$$\langle Q, Q\rangle = 2 + 2s_Q \cdot O - \frac{1}{2}(a_{v(x)} + a_{v(y)} + a_{v(z)} + a_{v(w)} + a_\infty).$$

Since $\langle Q, Q\rangle = 3/2$, the above equality becomes

$$-\frac{1}{2} = 2s_Q \cdot O - \frac{1}{2}(a_{v(x)} + a_{v(y)} + a_{v(z)} + a_{v(w)} + a_\infty).$$

As $a_v = 0$ or 1, possibilities for $(s_Q \cdot O, a_{v(x)}, a_{v(y)}, a_{v(z)}, a_{v(w)}, a_\infty)$ are

$$(0,1,0,0,0,0),\ (0,0,1,0,0,0),\ (0,0,0,1,0,0),$$
$$(0,0,0,0,1,0),\ (0,0,0,0,0,1),\ (1,1,1,1,1,1).$$

Hence, $s_Q \cdot O = 0$ and $a_\infty = 1$ if and only if $(a_{v(x)}, a_{v(y)}, a_{v(z)}, a_{v(w)}) = (0,0,0,0)$.

By Claims 1 and 2, $\tilde{f}_{\mathcal{Q},z_o}(s_Q)$ is of type $L1$ or $L2$ if and only if $|c_i| = 1$ $(i = 1,2,3)$ and $a_{v(\bullet)} = 0$ $(\bullet = x, y, z, w)$. In the following, consider a condition for c_i to satisfy $a_{v(\bullet)} = 0$ $(\bullet = x, y, z, w)$ under $|c_i| = 1$ $(i = 1,2,3)$. As for $\boldsymbol{c}(v, s_i)$, we have the following table:

	$\boldsymbol{c}(v(x), s_i)$	$\boldsymbol{c}(v(y), s_i)$	$\boldsymbol{c}(v(z), s_i)$	$\boldsymbol{c}(v(w), s_i)$	$\boldsymbol{c}(\infty, s_i)$
s_1	1	1	0	0	1
s_2	0	1	1	0	1
s_3	1	0	1	0	1
s_τ	1	1	1	1	0

By our construction of $S_{\mathcal{Q},z_o}$, singular fibers of $\varphi_{\mathcal{Q},z_o}$ are of type I_2 or III. Hence, by Corollary 2.7, we have

- $a_{v(x)} = 0$ if and only if $c_1 + c_3 + c_\tau$ is even,
- $a_{v(y)} = 0$ if and only if $c_1 + c_2 + c_\tau$ is even,
- $a_{v(z)} = 0$ if and only if $c_2 + c_3 + c_\tau$ is even, and
- $a_{v(w)} = 0$ if and only if c_τ is even.

By Claim 1, $(a_{v(x)}, a_{v(y)}, a_{v(z)}, a_{v(w)}) = (0,0,0,0)$ if and only if c_τ is even. Therefore, $|c_i| = 1$ $(i = 1,2,3)$ and c_τ is even if and only if $\tilde{f}_{\mathcal{Q},z_o}(s_Q)$ is of type $L1$ or $L2$. Since $P_\tau = O$ is a 2-torsion, we may assume $c_\tau = 0$. Hence, $\tilde{f}_{\mathcal{Q},z_o}(s_Q)$ depends on c_1, c_2 and c_3 only. Therefore, line-points for weak-bitangent lines of type $L1$ or $L2$ are given by $\pm{}^t[1,1,1,0]$, $\pm{}^t[1,-1,1,0]$ and $\pm{}^t[1,1,-1,0]$.

We omit our proof for the case of No. 61 as we can prove it similarly. □

Remark 5.7. *Except for the cases No.* 24 *and* 61, *our proof is based on the following form of* ${}^t\boldsymbol{c}G\boldsymbol{c}$ *(we omit those cases of rank* ≤ 2, *and some obvious cases):*

No. 4 $\left(c_1 + c_2 + c_3 + c_4 + \frac{c_5}{2} + \frac{c_6}{2}\right)^2 + \left(c_2 + c_3 + c_4 + \frac{c_5}{2} + \frac{c_6}{2}\right)^2 + 2\left(\frac{c_3}{2} + c_4 + \frac{c_5}{2} + \frac{c_6}{2}\right)^2 + \frac{c_3^2}{2} + \frac{c_5^2}{2} + \frac{c_6^2}{2}$

No. 6 $\frac{4}{3}\left(\frac{c_1}{2} + c_2 + \frac{3}{4}c_3 + \frac{c_4}{2} + \frac{c_5}{4}\right)^2 + \frac{c_1^2}{2} + \left(\frac{c_3}{2} + c_4 + \frac{c_5}{2}\right)^2 + \frac{c_3^2}{2} + \frac{c_5^2}{2}$

No. 7 $\left(c_1 + c_2 + \frac{c_3}{2} + \frac{c_4}{2}\right)^2 + \left(c_2 + \frac{c_3}{2} + \frac{c_4}{2}\right)^2 + \frac{c_3^2}{2} + \frac{c_4^2}{2} + \frac{c_5^2}{2}$

No. 10 $\left(\frac{c_1}{2} + c_2 + \frac{c_3}{2}\right)^2 + \frac{c_1^2}{2} + \frac{c_3^2}{2} + \frac{c_4^2}{2}$

No. 12 $\frac{1}{3}\left(c_1 + \frac{c_2}{2} - \frac{c_4}{2}\right)^2 + \left(\frac{c_2}{2} + c_3 + \frac{c_4}{2}\right)^2 + \frac{c_2^2}{2} + \frac{c_4^2}{2}$

No. 17 $\frac{3}{10}\left(c_1 + \frac{c_2}{3} - \frac{c_3}{3}\right)^2 + \frac{2}{3}\left(c_2 + \frac{c_3}{2}\right)^2 + \frac{c_3^2}{2}$

No. 20 $\frac{2}{3}\left(c_1 + \frac{c_2}{2}\right)^2 + \frac{c_2^2}{2} + \frac{c_3^2}{6}$

No. 23 $\frac{c_1^2}{2} + \frac{1}{3}\left(c_2 + \frac{c_3}{2}\right)^2 + \frac{c_3^2}{4}$.

Remark 5.8. *From Table* 5, *we see that there are many examples that satisfy the assumption of Theorem* 1.3.

6. Applications of Theorems 1.3 and 5.6

6.1. Proof of Corollary 1.5. We may assume that $C_1 + C_2$ is given by an equation described in Lemma 4.1, and let $z_o = [0, 1, 0]$. Then the structure of $E_{C_1+C_2, z_o}(\mathbf{C}(t))$ corresponds to that of No. 24 in Table 1. Choose a basis, $\{P_1, P_2, P_3\}$, of the free part of $E_{C_1+C_2, z_o}(\mathbf{C}(t))$ such that $\tilde{f}_{C_1+C_2, z_o}(s_{P_i})$ are the first types indicated in No. 24 in Table 4. Define

$$Q_1 := [-1]P_1 \dot{+} P_2 \dot{+} P_3, \qquad Q_2 := P_1 \dot{+} [-1]P_2 \dot{+} P_3,$$

$$Q_3 := P_1 \dot{+} P_2 \dot{+} [-1]P_3, \qquad Q_4 := P_1 \dot{+} P_2 \dot{+} P_3.$$

Then, from Theorem 5.6, $\tilde{f}_{C_1+C_2, z_o}(s_{Q_i})$ $(i = 1, 2, 3, 4)$ are distinct bitangent lines of $C_1 + C_2$. On the other hand, $Q_4 = Q_1 \dot{+} Q_2 \dot{+} Q_3$ holds. By Theorem 1.3, the eight points of $(C_1 + C_2) \cap (\bigcup_{i=1}^{4} \tilde{f}_{C_1+C_2, z_o}(s_{Q_i}))$ lie on a conic C. Hence our statement follows. □

Remark 6.1. *Corollary* 1.5 *is well-known as Salmon's theorem. Its history and references to this well-known result can be found in* [6, Chapter 2].

6.2. Proofs of Corollaries 1.6 and 1.7. We may assume that the quartic curves are given by equations described in Lemma 4.1. Let $z_o = [0, 1, 0]$. We choose bases of $E_{\mathcal{Q}, z_o}(\mathbf{C}(t))$ as follows:

Corollary 1.6: The structure of $E_{\mathcal{Q}, z_o}(\mathbf{C}(t))$ corresponds to that of No. 14. By Lemma 5.2, we can choose a basis $\{P_1, P_2, P_3, P_4\}$ as follows:

	$E_{\mathcal{Q}, z_o}(\mathbf{C}(t))$	$(\tilde{f}_{\mathcal{Q}, z_o}(s_{P_1}), \tilde{f}_{\mathcal{Q}, z_o}(s_{P_2}), \tilde{f}_{\mathcal{Q}, z_o}(s_{P_3}), \tilde{f}_{\mathcal{Q}, z_o}(s_{P_4}))$
No. 14	$(A_1^*)^{\oplus 4}$	$(L4(x, y), L4(y, z), L4(x, z), C4)$

.

Corollary 1.7: The structure of $E_{\mathcal{Q}, z_o}(\mathbf{C}(t))$ corresponds to that of No. 18. Choose its basis as in Table 4. By abuse of notation, we denote it by $\{P_1, P_2, P_3\}$.

For each case, we define

$$Q_1 := [-1]P_1 \dot{+} P_2 \dot{+} P_3, \qquad Q_2 := P_1 \dot{+} [-1]P_2 \dot{+} P_3,$$

$$Q_3 := P_1 \dot{+} P_2 \dot{+} [-1]P_3, \qquad Q_4 := P_1 \dot{+} P_2 \dot{+} P_3.$$

By a similar argument to the previous section, our statements follow. □

6.3. Another application. We give another application.

Corollary 6.2. *Let $\mathcal{Q}$ be an irreducible quartic curve with exactly two singularities x and y such that x is a simple cusp and y is a node. Then*

(i) *there exist four weak-bitangent lines L_1, L_2, L_3 and L_4 of type $L3(x)$, and there exist three weak-bitangent lines M_1, M_2 and M_3 of type $L3(y)$ or $L5$.*

(ii) *If M_i $(i = 1, 2, 3)$ are of type $L3$, then for each pair (L_i, L_j) $(1 \leq i < j \leq 4)$, there exists a unique pair $(M_{a_{ij}}, M_{b_{ij}})$ $(1 \leq a_{ij} < b_{ij} \leq 3)$ such that*

($\star$) *the six points in $\mathcal{Q} \cap (L_i + L_j + M_{a_{ij}} + M_{b_{ij}})$ all lie on a conic.*

Proof. (i) We may assume that $\mathcal{Q}$ is given by an equation described in Lemma 4.1. Let $z_o = [0, 1, 0]$. Then the structure of $E_{\mathcal{Q}, z_o}(\mathbb{C}(t))$ corresponds to that of No. 12 in Table 1. By Lemma 5.2, we choose a basis, $\{P_1, P_2, P_3, P_4\}$, of $E_{\mathcal{Q}, z_o}(\mathbb{C}(t))$ such that the type of $(\tilde{f}_{\mathcal{Q}, z_o}(s_{P_1}), \tilde{f}_{\mathcal{Q}, z_o}(s_{P_2}), \tilde{f}_{\mathcal{Q}, z_o}(s_{P_3}), \tilde{f}_{\mathcal{Q}, z_o}(s_{P_4}))$ is the first type indicated in No. 12 in Table 4. Define $Q_1 := P_2$, $Q_2 := P_4$, $Q_3 := P_1 \dot{+} [-1]P_3 \dot{+} P_4$, $Q_4 := [-1]P_1 \dot{+} P_2 \dot{+} [-1]P_3$, $R_1 := P_3$, $R_2 := P_2 \dot{+} [-1]P_3 \dot{+} P_4$ and $R_3 := P_1 \dot{+} [-1]P_2 \dot{+} P_4$. From Theorem 5.6, we have

- $\tilde{f}_{\mathcal{Q}, z_o}(s_{Q_l})$ $(l = 1, 2, 3, 4)$ are of type $L3(x)$,
- $\tilde{f}_{\mathcal{Q}, z_o}(s_{R_m})$ $(m = 1, 2, 3)$ are of type $L3(y)$ or $L5$, and
- the seven lines are distinct.

Put $L_l = \tilde{f}_{\mathcal{Q}, z_o}(s_{Q_l})$ and $M_m = \tilde{f}_{\mathcal{Q}, z_o}(s_{R_m})$ $(l = 1, 2, 3, 4,\ m = 1, 2, 3)$.

(ii) Suppose that M_i $(i = 1, 2, 3)$ are of type $L3$.

Claim 1: For (L_i, L_j) $(1 \leq i < j \leq 4)$, there exists $(M_{a_{ij}}, M_{b_{ij}})$ $(1 \leq a_{ij} < b_{ij} \leq 3)$ satisfying ($\star$).

Proof of Claim. Let us only consider the case when $i = 1$ and $j = 2$, since the other cases follow similarly. We have $R_2 = Q_1 \dot{+} Q_2 \dot{+} [-1]R_1$. By Theorem 1.3, the six points of $\mathcal{Q} \cap (L_1 + L_2 + M_1 + M_2)$ lie on a conic. Hence, (M_1, M_2) satisfies ($\star$) for (L_1, L_2).

Claim 2: For (L_i, L_j), there exists a unique pair satisfying ($\star$).

Proof of Claim. Suppose that there exist two pairs as in Claim 1. Since there exist three lines of type $L3(y)$, two pairs of weak-bitangent lines of type $L3(y)$ have at least one common line. Hence we may assume that two pairs satisfying ($\star$) for (L_i, L_j) are either $(M_{a_{ij}}, M_{b_{ij}})$ or $(M_{a_{ij}}, M_{c_{ij}})$. Let C_{ij} and C'_{ij} be two conics such that

$$(L_i + L_j + M_{a_{ij}} + M_{b_{ij}}) \cap \mathcal{Q} \subset C_{ij} \qquad \text{and} \qquad (L_i + L_j + M_{a_{ij}} + M_{c_{ij}}) \cap \mathcal{Q} \subset C'_{ij}.$$

Putting $\{x, p_i, p_j\} = \mathcal{Q} \cap (L_i + L_j)$, $\{y, q_{a_{ij}}\} = \mathcal{Q} \cap M_{a_{ij}}$, $\{y, q_{b_{ij}}\} = \mathcal{Q} \cap M_{b_{ij}}$ and $\{y, q_{c_{ij}}\} = \mathcal{Q} \cap M_{c_{ij}}$, we have

$$C_{ij}|_{\mathscr{Q}} = 2x + 2y + p_i + p_j + q_{a_{ij}} + q_{b_{ij}} \qquad \text{and}$$

$$C'_{ij}|_{\mathscr{Q}} = 2x + 2y + p_i + p_j + q_{a_{ij}} + q_{c_{ij}},$$

where $C|_{\mathscr{Q}}$ denotes the divisor on a curve C cut out by $\mathscr{Q}$. Then C_{ij} and C'_{ij} pass through the five points x, y, p_i, p_j and $q_{a_{ij}}$. Since there are no four colinear points among the above five points, we have $C_{ij} = C'_{ij}$. Therefore $q_{b_{ij}} = q_{c_{ij}}$ and $M_{b_{ij}} = M_{c_{ij}}$. □

Remark 6.3. *The referee informed the author that Corollary* 6.2 *is more obvious than Corollaries* 1.5, 1.6 *and* 1.7. *In fact, we find this theorem from the application of a standard qudratic transformation, centered at the two singularities and a smooth point, and the group law on the resulting smooth cubic.*

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