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Toroidal handle additions and thrice punctured essential torus

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Abstract. Let M be a simple 3-manifold with F a boundary component of genus at least two. Let α and β be separating slopes on F. It is shown that if both 2-handle attachings $M[\alpha]$ and $M[\beta]$ are toroidal and one of them contains an essential torus whose intersection with M is a thrice punctured essential torus, then $\Delta(\alpha, \beta) \leq 8$.

1. Introduction

This paper studies one of the problems concerning 2-handle additions producing toroidal 3-manifolds, i.e., manifolds that contain essential tori. A compact orientable 3-manifold M is said to be simple if it is irreducible, ∂-irreducible, anannular and atoroidal. Let M be a simple 3-manifold with F a boundary component of genus at least two. Let α be a slope, that is, an isotopy class of a simple closed curve on F. Denote by $M[\alpha]$ the result of attaching a 2-handle to M along a regular neighborhood of a representative of α in F. For two slopes α and β, denoted by $\Delta = \Delta(\alpha, \beta)$ the minimal geometric intersection number between the isotopy classes of α and β. Some work has been done comparing non-simple 2-handle attachments to the boundary of a simple 3-manifold. Earlier, Scharlemann and Wu [4] proved that if $M[\alpha]$ is reducible and $M[\beta]$ is boundary-reducible then $\Delta = 0$. In [5] Qiu and Zhang proved that if α and β are separating slopes such that $M[\alpha]$ and $M[\beta]$ are reducible then $\Delta \leq 2$. Zhang (unpublished) and the author [1], respectively, showed that $\Delta \leq 4$ if $M[\alpha]$ is reducible and $M[\beta]$ is toroidal. H. Lou and Zhang showed that $\Delta \leq 8$ if $M[\alpha]$ and $M[\beta]$ are ∂-reducible in [3].

The case when $M[\alpha]$ and $M[\beta]$ are both toroidal was studied in [1] and we recall the main results.

Theorem 1.1. *Let M be a simple 3-manifold and F a boundary component of genus at least two. Suppose that α and β are two separating slopes on F such that $M[\alpha]$ and $M[\beta]$ are toroidal, anannular and ∂-irreducible then either*

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(1) $\Delta(\alpha, \beta) = 18$, *each of* $M[\alpha]$ *and* $M[\beta]$ *contains an essential torus that intersects the* 2-*handle once and* F *has genus at least* 8, *or*
(2) $\Delta(\alpha, \beta) = 12$, *each of* $M[\alpha]$ *and* $M[\beta]$ *contains an essential torus that intersects the* 2-*handle once and* F *has genus at least* 4, *or*
(3) $\Delta(\alpha, \beta) \leq 10$.

COROLLARY 1.2. *Let* M *be a simple* 3-*manifold and* F *a boundary component with* $g(F) \geq 2$ *where* $g(F)$ *denotes the genus of* F. *Suppose that* α *and* β *are two separating slopes on* F *such that* $M[\alpha]$ *and* $M[\beta]$ *are toroidal, anannular and* ∂-*irreducible.*
(1) *If* $g(F) = 3$ *then* $\Delta(\alpha, \beta) \leq 10$.
(2) *If* $g(F) = 2$ *then* $\Delta(\alpha, \beta) \in \{0, 4, 8\}$.

In this paper we consider the case where a thrice punctured essential torus appears and the main result is the following:

THEOREM 1.3. *Let* M *be a simple* 3-*manifold and* F *a boundary component of genus at least two. Suppose that* α *and* β *are two separating slopes on* F *such that* $M[\alpha]$ *and* $M[\beta]$ *are toroidal, anannular and* ∂-*irreducible. If* $M[\beta]$ *contains an essential torus* $\hat{Q}$ *such that* $\hat{Q} \cap M$ *is a thrice punctured essential torus then* $\Delta(\alpha, \beta) \leq 8$.

It is unknown whether or not the bounds given in Theorem 1.1 and Corollary 1.2 are optimal (we expect that it is not). Thus Theorem 1.3 shows the bound 10 can be improved if we consider certain additional hypotheses.

Theorem 1.3 will be proved by applying the combinatorial techniques developed in [1, 2, 5, 6].

2. Properties of the intersection graphs

In what follows, we shall assume all the conditions listed in Theorem 1.1. We may further assume that $M[\alpha]$ is irreducible, otherwise Theorem 2 in [1] implies $\Delta(\alpha, \beta) \leq 4$.

LEMMA 2.1. *Suppose that* M *is simple and* $M[\alpha]$ *is toroidal. If* $M[\alpha]$ *is* ∂-*irreducible, irreducible and anannular then* M *contains an essential punctured torus* P *with all boundary components of* P *parallel to* α.

PROOF. See Lemma 2.1 in [1]. □

Now let M be a simple 3-manifold and let F be a boundary component of M with genus at least two. Suppose that α and β are separating, toroidal slopes on F. Let $\hat{P}$ (resp. $\hat{Q}$) be an essential torus on $M[\alpha]$ (resp. $M[\beta]$) that minimises the intersection with the 2-handle. By Lemma 2.1, the punctured torus $P = \hat{P} \cap M$ is essential and ∂P has components $\partial_1 P, \dots, \partial_u P, \dots, \partial_p P$, $p \geq 1$ such that $\partial_u P$ and $\partial_{u+1} P$ bound an annulus in F with interior disjoin from P. For $\hat{Q}$ there is a similar punctured torus $Q = \hat{Q} \cap M$ whose boundary components are similarly numbered $\partial_1 Q, \dots, \partial_i Q, \dots, \partial_q Q$, $q \geq 1$.

We isotope P and Q so that ∂P and ∂Q have minimal intersection, and $P \cap Q$ consists of arcs and circles that are essential in both P and Q. The intersection $P \cap Q$ defines two labeled graphs Γ_P on $\hat{P}$ and Γ_Q on $\hat{Q}$. The vertices of the graphs correspond respectively to the boundary components $\partial_u P \subset \partial P$ and $\partial_i Q \subset \partial Q$. Edges of each graph correspond to the arcs of intersection in $P \cap Q$. Circles of intersection are ignored. We need the following results from [1].

Lemma 2.2. (1) *There are no* 1-*sided disk faces in both* Γ_P *and* Γ_Q.
(2) *There are no common parallel edges in both* Γ_P *and* Γ_Q.

Proof. See Lemma 2.2 in [1]. □

If e is an edge of Γ_P with an endpoint x on a vertex $\partial_u P$, then x is labeled i if $x \in \partial_u P \cap \partial_i Q$. In this case i is called the Type A label of x in Γ_P. Thus, when going around $\partial_u P$, the labels of the endpoints of edges appear as $1, 2, \dots, q, q, \dots, 2, 1$ in cyclic order and this sequence being repeated $\Delta(\alpha, \beta)/2$ times. Label the endpoints of edges in Γ_Q similarly.

Now, following [6] we give a sign $g(x) =$ "+" or "−" on x, such that the signed labels $+1, \dots, +q, -q, \dots, -1$ appear in the same direction around all the vertices of Γ_P. The signed label $g(x)i$ is called the Type B label of x in Γ_P. In other words, if $e \in \Gamma_P$ is an edge with its two endpoints x and y labeled (u, i) and (v, j) then (i, j) is called the Type A label pair of e and $(g(x)i, g(y)j)$ is called the Type B label pair of e. Without loss of generality, we take the following assumption.

Assumption 2.3. *The labels* $+1, +2, \dots, +q, -q, \dots, -2, -1$ *appear in the clockwise direction on each vertex of* Γ_P.

Suppose the endpoints of edges in Γ_P are labeled with Type B labels. An edge of Γ_P is called an x-edge if it has label x at its one endpoint. Let Γ_P^x denote the subgraph of Γ_P consisting of all x-edges. A cycle in Γ_P^x is a virtual Scharlemann cycle if it bounds a disk face in Γ_P. Notice by the Assumption 2.3 each edge of a virtual Scharlemann cycle has the same label pair for either Type A or Type B, called the label pair of the virtual Scharlemann cycle. A virtual Scharlemann cycle σ with Type A label pair (i, j) is called a Scharlemann cycle if $i \neq j$. A Scharlemann cycle with only two edges is called an S-cycle. We need some results from [5] and [6].

Lemma 2.4. (1) *Each edge of Γ_P has different Type B labels at its two endpoints.*

(2) *If $S = \{e_1, \dots, e_n\}$ is a set of parallel edges of Γ_P and $n > q$, then there is a virtual S-cycle in S.*

(3) *A virtual S-cycle is either an S-cycle or its Type A label pair is $(1,1)$ or (q,q).*

(4) *Let $S = \{e_1, \dots, e_n\}$ be a set of parallel edges in Γ_P. If one of the edges, say e_k, has opposite Type B (or has the same Type A) labels at its two endpoints, then each edge in S has opposite Type B labels at its two endpoints.*

Proof. (1) is Lemma 2.4 in [5]. See also Lemma 2.2 (3) in [1]. (2) is Lemma 2.9 in [5]. (3) is Lemma 4.1 in [6]. (4) is Lemma 2.5 in [5]. □

Lemma 2.5. *Suppose Γ_P has Type B labels.*

(1) *If $q > 1$ and Γ_P contains $2q$ parallel edges, then Γ_Q contains q length two essential cycles.*

(2) *If Γ_P contains $2q+1$ parallel edges then $q = 1$ or $q = 3$.*

Proof. (1) is Lemma 5.1 in [1]. (2) is Lemma 5.2 in [1]. □

Now we give sufficient conditions for a loop in Γ_P to have identical labels at both of its ends. More precisely, if e is a loop of Γ_P joinning labels x and y, let e^x, e^y denote the endpoints of e at x and y respectively. Let $(e^x e^y)_\alpha$ denotes the arc in ∂P that goes from e^x to e^y with respect to some orientation. Then define

$$\sharp(e^x e^y) = |(e^x e^y)_\alpha \cap \partial Q| - 2$$

to count (the number of labels on the arc $(e^x e^y)_\alpha) - 2$ (depending on orientation).

Lemma 2.6. *Let e be a loop of Γ_P. Then $x = y$ if and only if $\sharp(e^x e^y) = 2qn - 1$ for some positive integer n.*

Proof. Around the vertex of Γ_P that contain the ends of e we see the signed labels

$$+1, +2, \dots, +q, -q, \dots, -2, -1$$

repeated $\Delta/2$ times. Now, the result follows from the fact that between each pair of identical labels there are exactly $2q - 1 + 2qm = 2q(m+1) - 1$ signed labels for some non-negative integer m. □

Lemma 2.7. *Let Γ be a graph embedded in a torus with V vertices and E edges. If Γ contains no 1-sided disk faces and no 2-sided disk faces, then*

$E \leq 3V$. *Moreover, if each vertex of* Γ *has valency at least* 6 *then* $E = 3V$, *each vertex has valency* 6 *and each face is a disk with* 3 *sides.*

Proof. See Lemma 3.2 in [1]. □

3. Proof of Theorem 1.3: $q = 3$

In this section we prove Theorem 1.3, so we assume that the essential punctured torus $Q = M \cap \hat{Q}$ has exactly three holes. Thus, Γ_Q has exactly three vertices and Γ_P has exactly six labels $+1$, $+2$, $+3$, -3, -2, -1. Recall that $q = 3$ implies that each Type A label pair of a Scharlemann cycle on Γ_P is $(1,2)$ or $(2,3)$. Denote by $A_{t,t+1}$ the part of F between consecutive components $\partial_t Q$ and $\partial_{t+1} Q$ of ∂Q. Then $S_t = Q \cup A_{t,t+1}$ denotes a surface of genus two with a hole for each $t = 1, 2$.

Lemma 3.1. *Let* $q = 3$ *and suppose* Γ_P *contains two Scharlemann cycles* σ_1 *and* σ_2 *with the same Type A labels* $(t, t+1)$. *Then,*

(1) *If* D_1 *and* D_2 *are the faces bounded by* σ_1, σ_2, *respectively, then* ∂D_1 *and* ∂D_2 *are parallel curves on* $S_t = Q \cup A_{t,t+1}$.

(2) σ_1 *and* σ_2 *have the same length.*

Proof. See Lemma 4.1 in [1]. □

Lemma 3.2. *Suppose* $q = 3$ *and let* N *denotes the number of Scharlemann cycles on* Γ_P.

(1) *If* Γ_P *has two Scharleman cycles with distinct Type A label pairs then* F *has genus two.*

(2) *If* Γ_Q *contains no loops and* $\Delta \geq 6$ *then* $\Delta p \leq 2N + 2p$.

Proof. (1) is Lemma 4.2 (1) in [1]. (2) is Lemma 4.4 (3) in [1]. □

Lemma 3.3. *If* $\Delta(\alpha, \beta) = 10$ *then* F *has genus at least* 3.

Proof. See the proof of Corollary 1.1 (2) in [1]. □

Notice that the previous result is the best possible since there exist separating slopes α and β with $\Delta = 10$ on a surface of genus 3 as illustrated in Figure 3.1.

Recall that if $p = 1$ Lemma 5.1 of [2] implies that Γ_P is of the form $H(w_1, w_2, w_3)$, that is, the reduced graph with only one vertex and three edges. Since this graph only have one vertex, Lemma 2.4 (1) implies that the endpoint labels of each edge are different.

Lemma 3.4. *Suppose* $p = 1$ *and* $q = 3$. *Then*

(1) $\Delta = 10$ *implies* $w_i \equiv 1 \pmod 2$ *for each* $i = 1, 2, 3$;

(2) $\Delta = 8$ *implies* $w_i \equiv 0 \pmod 2$ *for each* $i = 1, 2, 3$.

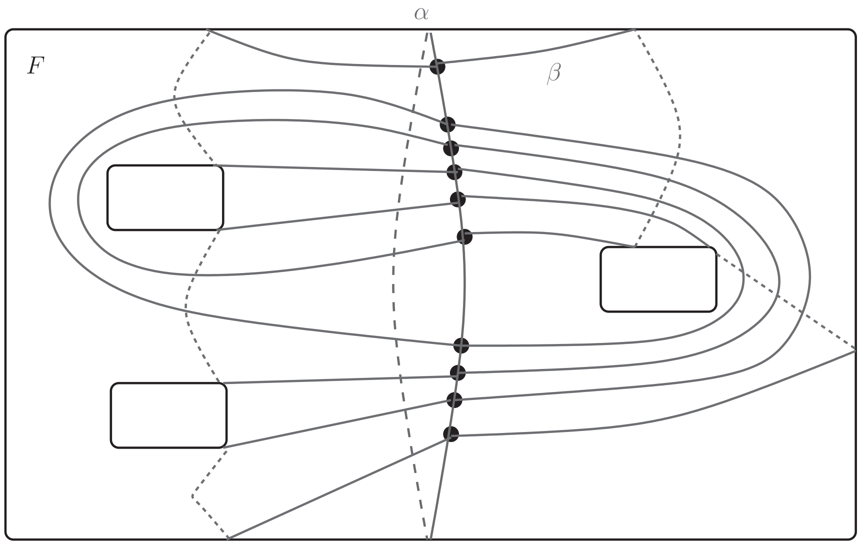

Fig. 3.1. $\Delta(\alpha, \beta) = 10$ on genus 3 surface F.

Proof. (1) Let $\Delta = 10$ and suppose w_i even for some $i = 1, 2, 3$. Since $w_1 + w_2 + w_3 = 3\Delta/2 = 15$, $w_j + w_k$ is odd. By Lemma 2.2 (2) and Lemma 2.7, $0 \le w_l \le 9$ for each $l = 1, 2, 3$. Hence $w_j + w_k \in \{7, 9, 11, 13, 15\}$. We consider three cases separately and we rule out them one by one.

Case 1: $w_j + w_k \in \{7, 9, 11\}$. Then it is easy to see that there exists an edge e of the w_i-collection such that $\sharp(e^x e^y) = 6n - 1$ with $n = 2$.

Case 2: $w_j + w_k = 13$. Then $w_i = 2$ and without loss of generality, suppose w_j odd. Thus $w_j \in \{5, 7, 9\}$. This implies $w_i + w_j \in \{7, 9, 11\}$. Then we can repeat the argument on Case 1 in order to find an edge e of the w_k-collection such that $\sharp(e^x e^y) = 6n - 1$ with $n = 2$.

Case 3: $w_j + w_k = 15$. Without loss of generality, suppose w_j odd. Thus $w_j \in \{7, 9\}$. Then exists an edge e of the w_k-collection such that $\sharp(e^x e^y) = 6n - 1$ with $n = 2$.

In either case, Lemma 2.6 implies that e has identical labels at both of its ends, which contradicts Lemma 2.4 (1).

(2) Let $\Delta = 8$ and suppose w_i odd for some $i = 1, 2, 3$. Since $w_1 + w_2 + w_3 = 3\Delta/2 = 12$, $w_j + w_k$ is odd. By Lemma 2.2 (2) and Lemma 2.7, $0 \le w_l \le 9$ for each $l = 1, 2, 3$. Hence $w_j + w_k \in \{3, 5, 7, 9, 11\}$. This implies that there exists an edge e of the w_i-collection such that $\sharp(e^x e^y) = 6n - 1$ with $n = 2$. Again, Lemma 2.6 contradicts Lemma 2.4 (1). □

We use $\overline{\Gamma}$ to denote the reduced graph of Γ, by which we mean a graph obtained from Γ by amalgamating each set of parallel edges of Γ into a single

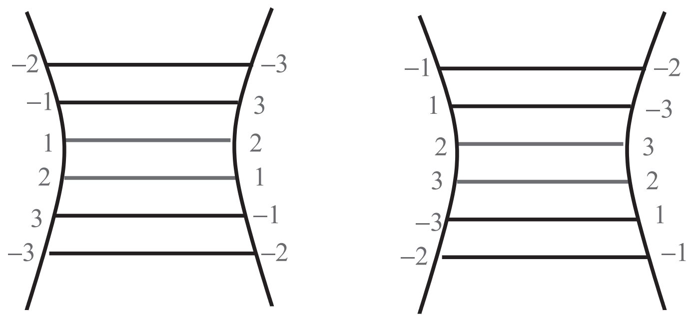

Fig. 3.2. $g(F) \geq 3$ and 6-parallel edges in Γ_P.

edge. The weight of a reduced edge is the number of edges of Γ in the reduced edge.

Lemma 3.5. *Suppose* $q = 3$. *Then,*

(1) *Each set of parallel edges on* Γ_P *contains at most two S-cycles and if it contains two then they have distinct Type A label pairs.*

(2) *If* Γ_P *has at least* 7 *parallel edges then* Γ_P *has two S-cycles with distinct Type A label pairs.*

(3) *Each* 4*-collection of parallel edges on* Γ_P *has an S-cycle.*

(4) *If* $g(F) \geq 3$ *then* Γ_P *cannot contain two Scharlemann cycles with different Type A label pairs and it contains at most* 6 *parallel edges. Moreover, if* $S = \{e_1, \dots, e_6\}$ *is a set of parallel edges in* Γ_P*, then it is one of the* 6*-collections of parallel edges illustrated in Figure* 3.2 *and* $e_3 \cup e_4$ *bounds an S-cycle (with Type A label pair).*

(5) *If* $g(F) \geq 3$ *then each 3-gon of* $\overline{\Gamma}_P$ *cannot have two edges of weight* 6.

(6) *If* $\Delta \geq 8$ *then* Γ_P *contains an S-cycle.*

(7) *Let N denotes the number of Scharlemann cycles of* Γ_P. *If* $\Delta \geq 8$ *and* $g(F) \geq 3$ *then* $1 \leq N \leq 2p$ *and each Scharlemann cycle is an S-cycle with the same Type A label pair.*

Proof. (1)–(2) See Corollary 4.5 (1)–(2) in [1].

(3) Let $C = \{e_1, \dots, e_4\}$ be a 4-collection of parallel edges of Γ_P. By Lemma 2.4 (2) each 4-collection of parallel edges has a virtual S-cycle σ. If σ has Type A label pair $(1,1)$ or $(3,3)$, then Lemma 2.4 (4) implies that the edges of the 4-collection of parallel edges represent loops in Γ_Q. Then Γ_Q has two loops C_1 and C_2 at vertex 1 or vertex 3 by Assumption 2.3. In each case Γ_Q has at least one loop C_3 at vertex 2. Cutting the torus $\hat{Q}$ along C_3 gives an annulus containing vertices 1 and 3 in its interior. This implies that $C_1 \cup C_2$ bounds a bigon D. By Lemma 2.2 (2), D contains a vertex. It is now easy to check that in each case the fourth loop on Γ_Q is an inessential loop or it is

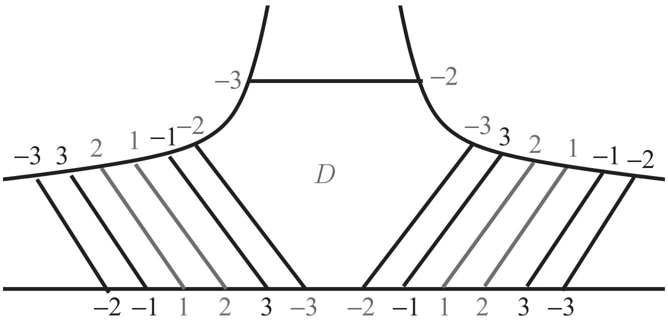

Fig. 3.3. A 3-gon D of Γ_P with two edges adjacent to a set of 6-parallel edges.

parallel to C_3 in Γ_Q, contradicting Lemma 2.2. Therefore σ is an S-cycle by Lemma 2.4 (3).

(4) First note that by Lemma 3.2 (1), Γ_P cannot contain two Scharlemann cycles with different Type A label pairs. Then, Γ_P contains at most 6-parallel edges by (2). Now, let $S = \{e_1, \dots, e_6\}$ be a set of 6 parallel edges in Γ_P. By (3), each set of 4 parallel edges $\{e_1, \dots, e_4\}$ and $\{e_3, \dots, e_6\}$ contains an S-cycle. Then, by (1) and Lemma 3.2 (1), $e_3 \cup e_4$ bounds an S-cycle σ. Since $q = 3$, σ has Type A label pair $(1, 2)$ or $(2, 3)$. Thus, there are two possibilities (up to sign) for the labels of the set of 6-parallel edges as shown in Figure 3.2.

(5) Suppose that $\overline{\Gamma}_P$ has a 3-gon D with two edges of weight 6. Then (4) implies that each 6-collection of parallel edges can be obtained from one of the collections illustrated in Figure 3.2 (up to signs). Notice that $q = 3$ implies that the two 6-collections of parallel edges must have the same labels. Without loss of generality, assume that each one contains an S-cycle with Type B label pair $(+1, +2)$. This implies that each corner of D has label pair $(-2, -3)$ as illustrated in Figure 3.3. Then D bounds a Scharleman cycle and Lemma 3.2 (1) implies $g(F) = 2$ which is a contradiction.

(6) On the contrary assume that Γ_P does not contain an S-cycle. By (3), each collection of parallel edges on Γ_P contains at most 3 edges. Now, because each vertex of Γ_P has valency $3\Delta \geq 24$ then each vertex of the reduced graph $\overline{\Gamma}_P$ has valency at least 8. Now, if V and E denote the vertices and edges of $\overline{\Gamma}_P$ then by taking the sum over all the vertices we obtain $2E \geq 8V$ which contradicts Lemma 2.7.

(7) By (6), $1 \leq N$. Also (4), (6) and Lemma 3.1 (2) imply each Scharlemann cycle in Γ_P is an S-cycle with the same Type A label pair. Now, Lemma 2.7 implies $N \leq 3p$. Suppose $2p + 1 \leq N \leq 3p$. Then Lemma 3.1 (1) implies that Γ_Q has two collections of at least $2p + 1$ parallel edges. Thus, Lemma 2.5 (2) implies $p = 1$ or $p = 3$. If $p = 3$ then Γ_Q has a 7-collection of parallel

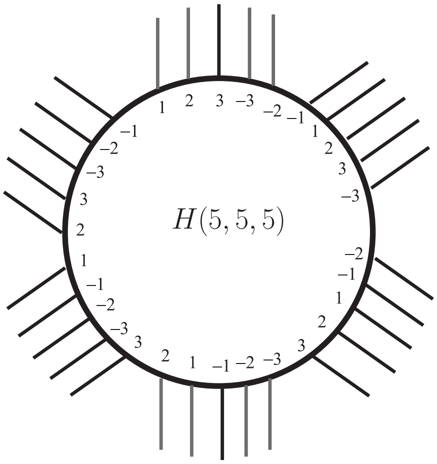

Fig. 3.4. $\Delta = 10$, $q = 3$ and $\Gamma_P = H(5,5,5)$.

edges and (2) contradicts (4). Therefore $p = 1$, $N = 3$ and $\Gamma_P = H(w_1, w_2, w_3)$. By Theorem 1.1 (3), we distinguish the two cases $\Delta = 10$ and $\Delta = 8$ (See also Lemma 7.8 in [1]).

Case 1: $\Delta = 10$. Then $w_1 + w_2 + w_3 = \Delta q/2 = 15$. By (4) and Lemma 3.4 (1), $w_i \leq 5$. Thus $\Gamma_P = H(5,5,5)$. Let $C = \{e_1, \ldots, e_5\}$ be a set of edges of Γ_P which form one of the 5-parallel edges. Let e_i^x, e_i^y denote the endpoints of e_i. Since for each $i = 1, 5$ we have $\sharp(e_i^x e_i^y) = 18$, Assumption 2.3 implies $e_1 \cup e_2$ and $e_4 \cup e_5$ bound virtual S-cycles σ_1 and σ_2, respectively (see Figure 3.4). By the proof of (3), each σ_i is an S-cycle. Then $q = 3$ implies they have different Type A label pair. Finally, Lemma 3.2 (1) implies $g(F) = 2$, a contradiction.

Case 2: $\Delta = 8$. Then $w_1 + w_2 + w_3 = 12$. By, (4) and Lemma 3.4 (2) this case is subdivided into three subcases: $\Gamma_P = H(6,6,0), H(6,4,2), H(4,4,4)$.

If $\Gamma_P = H(6,6,0)$ then (4) implies $N = 2$, a contradiction.

If $\Gamma_P = H(6,4,2)$ or $H(4,4,4)$, let $C = \{e_1, \ldots, e_4\}$ be a set of edges of Γ_P which form one of the 4-parallel edges. Since for each $i = 2, 3$ we have $\sharp(e_i^x e_i^y) = 12$, Assumption 2.3 implies $e_2 \cup e_3$ bounds a virtual S-cycle σ (see Figure 3.5). By (3), σ is an S-cycle. Then if $\Gamma_P = H(6,4,2)$, σ has different Type A label pair than the S-cycle given by (4), a contradiction. Finally, if $\Gamma_P = H(4,4,4)$ then it is easy to see that one of the 4-collection of parallel edges does not contain an S-cycle. This contradicts (3). □

Now, let $\{i, j, k\}$ denote the three vertices of Γ_Q. In the following two lemmas we use the convention that Γ_Q^{ij} denotes the subgraph induced by

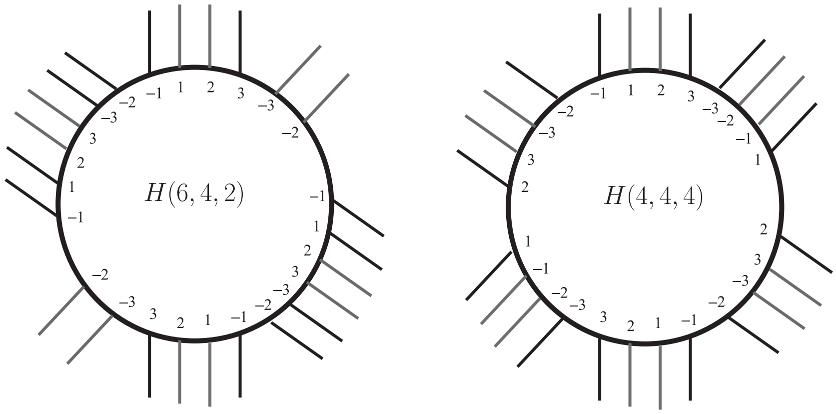

Fig. 3.5. $\Delta = 8$, $q = 3$ and $\Gamma_P = H(6,4,2)$ or $H(4,4,4)$.

the vertices i and j in the graph Γ_Q and $\overline{\Gamma}_Q^{ij}$ denotes the subgraph induced by the vertices i and j in the reduced graph $\overline{\Gamma}_Q$.

LEMMA 3.6. (1) *If Γ_Q contains a loop at i, then* $E(\overline{\Gamma}_Q^{jk}) \leq 2$.
(2) *If* $E(\overline{\Gamma}_Q^{ij}) \geq 4$, *then* $E(\overline{\Gamma}_Q^{ik}) \leq 2$.

PROOF. See Lemma 7.4 in [2]. □

LEMMA 3.7. *Suppose* $q = 3$ *and* $\Delta = 10$. *Then,*
(1) *Γ_Q has at least one loop and it has at most $2p$ parallel edges.*
(2) *Γ_P contains 6 parallel edges as illustrated in Figure 3.2.*
(3) *Γ_Q is the graph $R(w_1, \dots, w_9)$ illustrated in Figure 3.6 with $w_i \neq 0$ for each $i = 1, \dots, 9$ and $p \leq w_i \leq 2p$ for each $i = 1, 2, 3$.*

PROOF. (1) If Γ_Q has no loops then Lemma 3.2 (2) implies

$$10p = \Delta p \leq 2N + 2p \quad \Rightarrow \quad 4p \leq N.$$

But, Lemma 3.3 and Lemma 3.5 (7) imply $N \leq 2p$, which is a contradiction.

Suppose, for a contradiction, that Γ_Q has $2p+1$ parallel edges. By Lemma 2.5 (2), $p = 1$ or $p = 3$. Also, Lemma 3.3 implies $g(F) \geq 3$. If $p = 1$ then $\Gamma_P = H(w_1, w_2, w_3)$ and $w_1 + w_2 + w_3 = \Delta q/2 = 15$. Then Lemma 3.4 (1) and Lemma 3.5 (4) imply $w_i \in \{1, 3, 5\}$. Hence $\Gamma_P = H(5,5,5)$. By the proof of Lemma 3.5 (3), this implies Γ_Q has no loops which is a contradiction. If $p = 3$ then Γ_Q has $2p + 1 = 7$ parallel edges, which contradicts Lemma 3.5 (4).

(2) Suppose that $\overline{\Gamma}_P$ has no edges with weight 6. Then Lemma 3.3 and Lemma 3.5 (4) imply each edge has weight at most 5. $\Delta = 10$ implies each vertex of $\overline{\Gamma}_P$ has valency at least 6. Hence Lemma 2.7 implies each vertex of

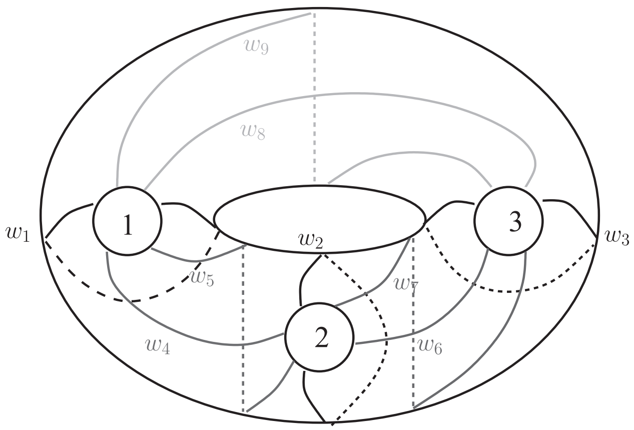

Fig. 3.6. $\Delta = 10$ and $\Gamma_Q = R(w_1, \dots, w_9)$.

$\overline{\Gamma}_P$ has valency 6. Thus, each edge of $\overline{\Gamma}_P$ has weight 5. This implies that Γ_Q has no loops by the proof of Lemma 3.5 (3). But this contradicts (1). Therefore Γ_P contains 6 parallel edges.

(3) First we prove the following two claims.

Claim 1. Each vertex of $\overline{\Gamma}_Q$ has valency at least 5.

Proof. This follows from (1) and $\Delta = 10$. □

Claim 2. $E(\overline{\Gamma}_Q^{ij}) \geq 2$.

Proof. This follows immediately from (2) and Lemma 2.5 (1). □

Now, by analysing the number of loops in $\overline{\Gamma}_Q$ we consider three cases.

Case 1. $\overline{\Gamma}_Q$ has one loop. Suppose $\overline{\Gamma}_Q$ contains a loop at vertex i. Then Lemma 3.6 (1) and Claim 2 imply $E(\overline{\Gamma}_Q^{jk}) = 2$. This implies $E(\overline{\Gamma}_Q^{ij}), E(\overline{\Gamma}_Q^{ik}) \geq 3$, by Claim 1. If $E(\overline{\Gamma}_Q^{ik}) \geq 4$, Lemma 3.6 (2) implies $E(\overline{\Gamma}_Q^{ij}) \leq 2$ which is a contradiction. Similarly, $E(\overline{\Gamma}_Q^{ij}) \geq 4$ is impossible. Therefore $E(\overline{\Gamma}_Q^{ij}) = 3 = E(\overline{\Gamma}_Q^{ik})$. Hence the vertices j and k have valency 5 and each edge in $\overline{\Gamma}_Q$ has weight $2p$ by (1). By counting edges at i we have

$$\Delta p = 10p > 2p[E(\overline{\Gamma}_Q^{ij}) + E(\overline{\Gamma}_Q^{ik})] = 2p(6) = 12p,$$

a contradiction.

Case 2. $\overline{\Gamma}_Q$ has two loops. Subcase 1: two loops at i. Cutting the torus $\hat{Q}$ along these two loops gives a disk. Now, by (2) and Lemma 2.5 (1), it is easy to see that an innermost length 2 cycle in Γ_Q contradicts Lemma 2.2 (2). Subcase 2: one loop at i and one loop at j. By Lemma 3.6 (1), $E(\overline{\Gamma}_Q^{jk}) \leq 2$ and $E(\overline{\Gamma}_Q^{ik}) \leq 2$. This implies k has valency 4 which contradicts Claim 1.

Case 3. $\overline{\Gamma}_Q$ has three loops. Suppose there are two loops at vertex k and one loop at vertex j. This implies that these loops are parallel on the torus $\hat{Q}$. Then vertex i is on the bigon bounded by the loops incident at vertex k. Hence, $E(\overline{\Gamma}_Q^{ij}) = 0$ which contradicts Claim 2. Therefore, $\overline{\Gamma}_Q$ has one loop at each vertex and by Claim 2 we obtain that Γ_Q is the graph illustrated in Figure 3.6.

Finally, we only prove $p \le w_2 \le 2p$ because the other two inequalities are similar. Now, (1) implies $w_i \le 2p$ for each $i = 1, \dots, 9$ and by counting edges at vertex 2 we have

$$10p = \Delta p = 2w_2 + \sum_{k=4}^{7} w_k \le 2w_2 + 8p.$$

Therefore $p \le w_2 \le 2p$. □

PROOF (Proof of Theorem 1.3). We assume that $\Delta \ge 10$ so as to obtain a contradiction. By Theorem 1.1 (3), $\Delta = 10$. Hence, Lemma 3.7 (3) implies $\Gamma_Q = R(w_1, \dots, w_9)$ with $p \le w_i \le 2p$ for each $i = 1, 2, 3$.

Claim 1: $\overline{\Gamma}_P$ has at least p edges with weight at most 3.

PROOF. By Lemma 2.4 (4), a parallel family of edges of Γ_P corresponds to either loops in Γ_Q, or edges in Γ_Q. But, if it corresponds to loops then the number of parallel edges is at most 3 by Lemma 3.5 (3). Now the result follows from the fact that the number of loops in Γ_Q is at least $3p \le \sum_1^3 w_i$ by Lemma 3.7 (3). □

Claim 2: $\overline{\Gamma}_P$ has p edges of weight 3 and $2p$ edges of weight 6.

PROOF. Lemma 3.5 (4) implies that the edges of $\overline{\Gamma}_P$ have weight at most 6. Also, we have $15p = 3\Delta p/2 = E(\Gamma_P)$. Let a be the number of edges with weight at most 3 of $\overline{\Gamma}_P$. Then $p \le a$, by Claim 1. Lemma 2.7 implies that the number of edges of $\overline{\Gamma}_P$ with weight at least 4 is at most $3p - a$. Then we obtain the following

$$15p = E(\Gamma_P) \le 3a + 6(3p - a) = 18p - 3a = 15p + 3(p - a).$$

This implies $a = p$ and the above inequality is an equality. In particular, the claim must hold. □

Claim 3: $\overline{\Gamma}_P$ contains a 3-gon with two edges of weight 6.

PROOF. Suppose for a contradiction that $\overline{\Gamma}_P$ contains 3-gons with at most one edge of weight 6. By Claim 2, the sum of the weights of the edges on each 3-gon is at most 12. Let V, E, F denote the number of the vertices, edges and disk faces of the reduced graph $\overline{\Gamma}_P$, respectively. Then $V = p$. Also, Claim 2 implies $E = 3p$ and Lemma 2.7 implies that each face of $\overline{\varGamma}_P$ is a 3-gon. By calculating the Euler number of the torus $\hat{P}$ we have $F = 2p$. Hence

$$2E(\varGamma_P) \le 12F(\overline{\varGamma}_P) = 12(2p) = 24p.$$

This implies $15p = E(\varGamma_P) \le 12p$, a contradiction. □

Finally, Lemma 3.3 implies $g(F) \ge 3$. Then Claim 3 contradicts Lemma 3.5 (5). Therefore $\varDelta \le 8$. □

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