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Borsuk-Ulam type theorems for multivalued maps

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ABSTRACT. Let X_n $(n > 1)$ be a finitistic space with cohomology type $(0, 0)$. Let (X_n, E, π, B) be a fibre bundle and $(\mathbb{R}^k, E', \pi', B)$ be a k-dimensional real vector bundle with fibre preserving $G = \mathbb{Z}_p$, $p > 2$ a prime, action such that G acts freely on E and $E' - \{0\}$, where $\{0\}$ is the zero section of the vector bundle. We determine a lower bound of the cohomological dimension of the set $A_\mu = \{x \in E \mid \mu(x) \cap \mu(gx) \cap \cdots \cap \mu(g^{p-1}x) \neq \phi\}$ for an admissible multivalued fibre preserving map $\mu : E \to E'$.

1. Introduction

For every continuous map $f : \mathbb{S}^n \to \mathbb{R}^n$, the coincidence set $A(f) = \{x \in \mathbb{S}^n \mid f(x) = f(-x)\}$ is nonempty relative to the antipodal action on n-sphere $\mathbb{S}^n$. This result is known as the classical Borsuk-Ulam theorem. Another version of the Borsuk-Ulam theorem states that if $f : \mathbb{S}^n \to \mathbb{R}^k$ is a continuous map with $n \geq k$ then $\mathrm{cd}_2(A(f)) \geq n - k$, where $\mathrm{cd}_2(A(f))$ is the cohomological dimension of $A(f)$ with the coefficient group $\mathbb{Z}_2$. Dold [1] determined the cohomological dimension of the coincidence set $A(f)$ of a fibre preserving $\mathbb{Z}_2$-equivariant map $f : E \to E'$, where E is the total space of a fibre bundle with fibre $\mathbb{S}^n$ and E' is the total space of a k-dimensional real vector bundle with base space a paracompact space B. He proved that $\mathrm{cd}_2(A(f)) \geq \mathrm{cd}_2(B) + n - k$. This result is known as the parameterized version of the Borsuk-Ulam theorem. Dold introduced the concept of Stiefel-Whitney polynomials for vector bundles with the antipodal actions. These polynomials are called the characteristic polynomials. Using these polynomials, Nakaoka [10] proved Dold's result for non-free $\mathbb{Z}_p$ and $\mathbb{S}^1$-actions. Jaworowski [7] established Dold's result for free $\mathbb{Z}_p$-actions, $p > 2$ a prime. The Borsuk-Ulam type theorem of Dold's results were determined for fibre bundles with different fibres, for example: (i) $\mathbb{S}^n \times \mathbb{S}^m$ with free $\mathbb{Z}_p$-actions, $p > 2$ a prime, or $\mathbb{S}^1$-action [9], (ii) spaces of cohomology of type (a, b) with free actions of $\mathbb{Z}_2$ or

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$\mathbb{S}^1$ [8] and $\mathbb{F}P^m \times \mathbb{S}^3$, where $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, with free $\mathbb{Z}_2$-actions [11], etc. Izydorek and Jaworowski [4] extended Dold's result for an admissible multivalued fibre preserving map $\mu : E \to E'$ for $G = \mathbb{Z}_2$-actions and also, for these maps $\mu : E \to E'$, Izydorek and Rybicki [5] proved the parallel result for $G = \mathbb{Z}_p$ actions, $p > 2$ a prime.

Throughout the paper, all spaces are assumed to be paracompact Hausdorff spaces. We use Čech cohomology with coefficient in the group $\mathbb{Z}_p$, where $p > 2$ is a prime.

In this paper, we determine parametrized versions of the Borsuk-Ulam type theorem. We obtain a lower bound of the cohomological dimension of the set $A_\mu = \{x \in E \mid \mu(x) \cap \mu(gx) \cap \cdots \cap \mu(g^{p-1}x) \neq \phi\}$ for an admissible multivalued fibre preserving map $\mu : E \to E'$ for a fibre bundle (X_n, E, π, B) and a k-dimensional (k is odd) real vector bundle $(\mathbb{R}^k, E', \pi', B)$, where X_n is a space of cohomology type $(0, 0)$.

2. Preliminaries

A finitistic space X_n ($n > 1$ is a natural number) is said to have cohomology type (a, b) if $H^j(X_n; \mathbb{Z}) \cong \mathbb{Z}$ for $j = 0, n, 2n$ and $3n$ only, and the generators $x \in H^n(X_n; \mathbb{Z})$, $y \in H^{2n}(X_n; \mathbb{Z})$ and $z \in H^{3n}(X_n; \mathbb{Z})$ satisfies $x^2 = ay$ and $xy = bz$, where a and b are integers. For example, $\mathbb{S}^n \vee \mathbb{S}^{2n} \vee \mathbb{S}^{3n}$ and $\mathbb{S}^{2n} \cup_{\mathbb{S}^{n-1}} \mathbb{S}^{3n}$ which is obtained by attaching the spheres $\mathbb{S}^{2n}$ and $\mathbb{S}^{3n}$ along $\mathbb{S}^{n-1}$ are spaces of type $(0, 0)$. The 3-dimensional projective spaces $\mathbb{F}P^3$, where $\mathbb{F} = \mathbb{C}, \mathbb{H}$ are spaces of type $(1, 1)$. Note that if there exists a space of type (a, b) then there are spaces (ma, nb) for all integers m and n. Such spaces were first investigated by James [6] and Toda [12].

We recall some definitions and results which were used to prove our main theorem.

DEFINITION 2.1 ([4]). Let X, Y be spaces and let μ be a multivalued map from X to Y, i.e., a function which assigns to each $x \in X$ a nonempty subset $\mu(x)$ of Y. We say that μ is upper semicontinuous (u.s.c.), if each $\mu(x)$ is compact and if the following condition holds: For every open subset V of Y containing $\mu(x)$ there exists an open subset U of X containing x such that for each $x' \in U$, $\mu(x') \subset V$.

For instance, if X and Y are compact then μ is upper semicontinuous iff its graph is closed in $X \times Y$.

DEFINITION 2.2 ([4]). An u.s.c. map μ from X to Y is said to be $\mathbb{Z}_p$-admissible (briefly admissible), if there exist a space Γ and two single valued continuous maps $\alpha : \Gamma \to X$ and $\beta : \Gamma \to Y$ such that

(i) α is a Vietoris map, i.e., it is surjective, proper and each set $\alpha^{-1}(x)$ is $\mathbb{Z}_p$-acyclic,

(ii) for each $x \in X$, the set $\beta(\alpha^{-1}(x))$ is contained in $\mu(x)$.

We will say that the pair (α, β) is a "selected pair" for μ.

For instance, if each $\mu(x)$ is acyclic (and if μ is u.s.c.) then μ is admissible.

Note that an example of free action of $G = \mathbb{Z}_p$, where p is an odd prime, on spaces of cohomology type $(0,0)$ has been constructed in [2] and the cohomological structure of the orbit space has been discussed in [3].

Proposition 2.1 ([3]). *Let $G = \mathbb{Z}_p$, p an odd prime, act freely on a space X_n (n is an odd) of cohomology type $(0,0)$. Then, as a graded commutative algebra*

$$H^*(X_n/G) = \mathbb{Z}_p[u, v, w]/\langle u^2, w^2, wv^{(n+1)/2}, v^{(3n+1)/2}\rangle,$$

where $\deg u = 1$, $\deg v = 2$, $\deg w = n$ *and* $v = \beta_p(u)$ *(β_p being the mod-p Bockstein).*

3. A lower bound of the cohomological dimension of zero set and coincidence set

Let $G = \mathbb{Z}_p$ (p an odd prime) be a group and X_n (n an odd natural number) be a space of cohomology type $(0,0)$. Let $f : E \to E'$ be a fibre preserving G-equivariant map, where (X_n, E, π, B) is a fibre bundle equipped with fibre preserving free G-action such that the quotient bundle $(X_n/G, \hat{E}, \hat{\pi}, B)$ has the cohomology extension property and $\pi' : E' \to B$ is a k-dimensional real vector bundle equipped with fibrewise free G-action on $E' - \{0\}$, where $\{0\}$ is the zero section of the vector bundle. We denote the zero set $f^{-1}(\{0\})$ by E_f. First, we obtain the characteristic polynomials associated to the fibre bundle and vector bundle, respectively.

The characteristic polynomials associated to the fibre bundle (X_n, E, π, B)**.** Recall that graded algebra of $H^*(X_n/G)$ is generated by the elements

$$v^i, uv^i, wv^j, uv^jw \qquad \text{where } 0 \le j \le \frac{n-1}{2} \text{ and } 0 \le i \le \frac{3n-1}{2}$$

subject to the relations $u^2 = v^{(3n+1)/2} = v^{(n+1)/2}w = w^2 = 0$, where $u \in H^1(X_n/G)$, $v \in H^2(X_n/G)$, $w \in H^n(X_n/G)$. As the quotient bundle $(X_n/G, \hat{E}, \hat{\pi}, B)$ has the cohomology extension property, so by the Leray-Hirsch theorem, there exist elements $a \in H^1(\hat{E})$, $b \in H^2(\hat{E})$ and $c \in H^n(\hat{E})$ such that the natural homomorphism $j^*: H^*(\hat{E}) \to H^*(X_n/G)$ maps $(a,b,c) \mapsto (u,v,w)$. We observe that $H^*(\hat{E})$ is an $H^*(B)$-module, via, the homomorphism $\hat{\pi}^*$ and generated by the basis

$$b^i, ab^i, cb^j, ab^jc \qquad \text{where } 0 \le j \le \frac{n-1}{2} \text{ and } 0 \le i \le \frac{3n-1}{2}.$$

Thus the elements $a^2 \in H^2(\hat{E})$, $b^{(3n+1)/2} \in H^{3n+1}(\hat{E})$, $c^2 \in H^{2n}(\hat{E})$, $b^{(n+1)/2}c \in H^{2n+1}(\hat{E})$ can be expressed as a linear combination of generating elements with coefficients in $H^*(B)$. Thus, there exist unique elements γ_i^j, μ_i^j and $\eta_i^j \in H^i(B)$, where $j = 1, 2$, such that

$$a^2 = 0,$$

$$b^{(3n+1)/2} = \sum_{i=1}^{(3n+1)/2} \mu_{2i}^1 b^{(3n+1)/2-i} + \sum_{i=1}^{(3n+1)/2} \mu_{2i-1}^1 ab^{(3n+1)/2-i} + \sum_{i=1}^{n+1} \mu_{2i-1}^2 cb^{n+1-i} + \sum_{i=1}^{n+1} \mu_{2i-2}^2 ab^{n+1-i}c,$$

$$c^2 = \sum_{i=0}^{n} \eta_{2i}^1 b^{n-i} + \sum_{i=1}^{n} \eta_{2i-1}^1 ab^{n-i} + \sum_{i=0}^{(n-1)/2} \eta_{2i+1}^2 cb^{(n-1)/2-i} + \sum_{i=0}^{(n-1)/2} \eta_{2i}^2 acb^{(n-1)/2-i},$$

$$b^{(n+1)/2}c = \sum_{i=0}^{n} \gamma_{2i+1}^1 b^{n-i} + \sum_{i=0}^{n} \gamma_{2i}^1 ab^{n-i} + \sum_{i=0}^{(n-1)/2} \gamma_{2i+2}^2 cb^{(n-1)/2-i} + \sum_{i=0}^{(n-1)/2} \gamma_{2i+1}^2 acb^{(n-1)/2-i}.$$

The characteristic polynomials associated to the fibre bundle (X_n, E, π, B) in the indeterminates x, y and z of degrees 1, 2 and n, respectively, are x^2, $W_1(x,y,z)$, $W_2(x,y,z)$ and $W_3(x,y,z)$, where

$$W_1(x,y,z) = \sum_{i=1}^{(3n+1)/2} \mu_{2i}^1 y^{(3n+1)/2-i} + \sum_{i=1}^{(3n+1)/2} \mu_{2i-1}^1 xy^{(3n+1)/2-i} + \sum_{i=1}^{n+1} \mu_{2i-1}^2 zy^{n+1-i} + \sum_{i=1}^{n+1} \mu_{2i-2}^2 xy^{n+1-i}z - y^{(3n+1)/2},$$

$$W_2(x,y,z) = \sum_{i=0}^{n} \eta_{2i}^1 y^{n-i} + \sum_{i=1}^{n} \eta_{2i-1}^1 xy^{n-i} + \sum_{i=0}^{(n-1)/2} \eta_{2i+1}^2 zy^{(n-1)/2-i} + \sum_{i=0}^{(n-1)/2} \eta_{2i}^2 xzy^{(n-1)/2-i} - z^2,$$

$$W_3(x,y,z) = \sum_{i=0}^{n} \gamma_{2i+1}^1 y^{n-i} + \sum_{i=0}^{n} \gamma_{2i}^1 xy^{n-i} + \sum_{i=0}^{(n-1)/2} \gamma_{2i+2}^2 zy^{(n-1)/2-i} + \sum_{i=0}^{(n-1)/2} \gamma_{2i+1}^2 xzy^{(n-1)/2-i} - y^{(n+1)/2} z.$$

The map $\sigma : H^*(B)[x,y,z] \to H^*(\hat{E})$ defined by $(x,y,z) \mapsto (a,b,c)$ is a homomorphism of $H^*(B)$-algebras. Clearly, ker σ is generated by the characteristic polynomials x^2, $W_1(x,y,z)$, $W_2(x,y,z)$ and $W_3(x,y,z)$. So, we have

$$H^*(B)[x,y,z]/\langle x^2, W_1(x,y,z), W_2(x,y,z), W_3(x,y,z)\rangle \cong H^*(\hat{E}).$$

The characteristic polynomials associated to $(\mathbb{R}^k, E', \pi', B)$**.** Let $(\mathbb{R}^k, E', \pi', B)$ be a real vector bundle and $G = \mathbb{Z}_p$ acts fibrewise and freely on $E' - \{0\}$. Suppose that SE' is the total space of the sphere bundle associated to $(\mathbb{R}^k, E', \pi', B)$. Note that the quotient bundle of the vector bundle $(\mathbb{R}^k, E', \pi', B)$ is $(k-1)$-dimensional lens space bundle $(L_p^{k-1}, \hat{\pi}', \widehat{SE}', B)$. Let $h : L_p^{k-1} \to B_G$ and $i : \widehat{SE}' \to B_G$ be classifying maps of the principal G-bundles $\mathbb{S}^{k-1} \to L_p^{k-1}$ and $SE' \to \widehat{SE}'$, respectively. We denote $a' = h^*(s)$, $\mathbf{a}' = i^*(s)$, $b' = h^*(t)$ and $\mathbf{b}' = i^*(t)$, where $s \in H^1(B_G)$ and $t = \beta_p(s) \in H^2(B_G)$. Consequently, we have $\beta_p(a') = b'$ and $\beta_p(\mathbf{a}') = \mathbf{b}'$. Then, we get

$$H^*(L_p^{k-1}) = \mathbb{Z}_p[a', b']/\langle a'^2, b'^{(k/2)}\rangle.$$

Define a map $\theta : H^*(L_p^{k-1}) \to H^*(\widehat{SE}')$ by $a' \mapsto \mathbf{a}'$ and $b' \mapsto \mathbf{b}'$. Then θ is a G-module homomorphism and cohomology extension of the fibre bundle $(L_p^{k-1}, \hat{\pi}', \widehat{SE}', B)$. We know that $H^*(\widehat{SE}')$ is an $H^*(B)$-module. By the Leray-Hirsch Theorem, $H^*(\widehat{SE}')$ is generated by the elements

$$\mathbf{b}'^j \text{ and } \mathbf{ab}'^j, \qquad \text{where } 0 \le j \le \frac{k-2}{2}.$$

We can express $\mathbf{b}'^{(k/2)} \in H^k(\widehat{SE}')$ as

$$\mathbf{b}'^{(k/2)} = \tau_k + \tau_{k-1}\mathbf{a}' + \tau_{k-2}\mathbf{b}' + \cdots + \tau_2 \mathbf{b}'^{((k-2)/2)} + \tau_1 \mathbf{a}'\mathbf{b}'^{((k-2)/2)},$$

where τ_i are the unique elements of $H^i(B)$. Clearly, $\mathbf{a}'^2 = 0$. Thus, the characteristic polynomials associated to $(\mathbb{R}^k, E', \pi', B)$ are x^2 and $W'(x,y) = \tau_k +$

$\tau_{k-1}x + \tau_{k-2}y + \cdots + \tau_2 y^{(k-2)/2} + \tau_1 x y^{(k-2)/2} - y^{k/2}$, where the degrees of x and y are 1 and 2, respectively. Clearly, we have

$$H^*(B)[x, y]/\langle x^2, W'(x, y)\rangle \cong H^*(\widehat{SE}').$$

Now, we see that each element $q(x, y, z) \in H^*(B)[x, y, z]$ determines an element of $H^*(\hat{E})$, denote by $q(x, y, z)|_{\hat{E}}$. The image of $q(x, y, z)|_{\hat{E}}$ by the $H^*(B)$-homomorphism $i_1^* : H^*(\hat{E}) \to H^*(\hat{E}_f)$ is denoted by $q(x, y, z)|_{\hat{E}_f}$, where i_1^* is induced by the natural inclusion $i_1 : \hat{E}_f \hookrightarrow \hat{E}$. With these conditions and notations, we have the following lemmas.

Lemma 3.1. *Let X_n (n an odd natural number) be a space of cohomology type $(0,0)$ and let $q(x, y, z) \in H^*(B)[x, y, z]$ be a polynomial such that $q(x, y, z)|_{\hat{E}_f} = 0$. Then there exist polynomials $r_i(x, y, z) \in H^*(B)[x, y, z]$ $(i = 1, 2, 3, 4)$ such that*

$$q(x, y, z)W'(x, y) = r_1(x, y, z)W_1(x, y, z) + r_2(x, y, z)W_2(x, y, z)$$
$$+ r_3(x, y, z)W_3(x, y, z) + r_4(x, y, z)x^2,$$

where W_i's and W' are characteristic polynomials defined above.

Proof. We have a polynomial $q(x, y, z)$ in $H^*(B)[x, y, z]$ such that $q(x, y, z)|_{\hat{E}_f} = 0$. Thus, the continuity property of Čech cohomology, implies that there exist an open subset $V \subset \hat{E}$ such that $\hat{E}_f \subset V$ and $q(x, y, z)|_V = 0$. From the cohomology exact sequence

$$\cdots \to H^*(\hat{E}, V) \xrightarrow{j_1^*} H^*(\hat{E}) \to H^*(V) \to H^{*+1}(\hat{E}, V) \to \cdots$$

of the pair $(\hat{E}, V)$, there exists $\zeta \in H^*(\hat{E}, V)$ such that $j_1^*(\zeta) = q(x, y, z)|_{\hat{E}}$, where $j_1 : \hat{E} \hookrightarrow (\hat{E}, V)$ is the natural inclusion. The G-equivariant map $f : E \to E'$ gives the map $\hat{f} : \hat{E} - \hat{E}_f \to \hat{E}' - \{0\}$ which induces $H^*(B)$-homomorphism. We know that $\widehat{SE}'$ is homotopically equivalent to $E' - \{0\}$, so we get $\hat{f}^*(\mathbf{a}') = i_2^*(u)$ and $\hat{f}^*(\mathbf{b}') = i_2^*(v)$, where $i_2 : \hat{E} - \hat{E}_f \hookrightarrow \hat{E}$ is the natural inclusion map and $W'(\mathbf{a}', \mathbf{b}') = 0$. Therefore,

$$W'(x, y)|_{\hat{E}-\hat{E}_f} = W'(i_2^*(u), i_2^*(v)) = W'(\hat{f}^*(\mathbf{a}'), \hat{f}^*(\mathbf{b}')) = \hat{f}^*(W'(\mathbf{a}', \mathbf{b}')) = 0.$$

Next, we consider the long exact cohomology sequence

$$\cdots \to H^*(\hat{E}, \hat{E} - \hat{E}_f) \xrightarrow{j_2^*} H^*(\hat{E}) \to H^*(\hat{E} - \hat{E}_f) \to \cdots$$

for the pair $(\hat{E}, \hat{E} - \hat{E}_f)$. By the property of exactness, there exists $\xi \in H^*(\hat{E}, \hat{E} - \hat{E}_f)$ such that $j_2^*(\xi) = W'(x, y)|_{\hat{E}}$, where $j_2 : \hat{E} \hookrightarrow (\hat{E}, \hat{E} - \hat{E}_f)$ is the inclusion map. By the naturality of the cup product, we get

$$q(x, y, z)W'|_{\hat{E}} = j_1^*(\varsigma) \cup j_2^*(\xi) = j^*(\varsigma \cup \xi),$$

where $j : \hat{E} \hookrightarrow (\hat{E}, V \cup \hat{E} - \hat{E}_f)$ is the inclusion map. Note that

$$\varsigma \cup \xi \in H^*(\hat{E}, V \cup (\hat{E} - \hat{E}_f)) = H^*(\hat{E}, \hat{E}),$$

which gives $\varsigma \cup \xi = 0$. Therefore, $q(x, y, z)W'|_{\hat{E}} = 0$. $q(x, y, z)W'$ belongs to the kernel of σ. Therefore, there exist polynomials $r_i(x, y, z) \in H^*(B)[x, y, z]$ $(i = 1, 2, 3, 4)$ such that

$$\begin{aligned} q(x, y, z)W'(x, y) = {} & r_1(x, y, z)W_1(x, y, z) + r_2(x, y, z)W_2(x, y, z) \\ & + r_3(x, y, z)W_3(x, y, z) + r_4(x, y, z)x^2. \end{aligned}$$

To prove our main result, we show that Lemma 3.1 holds true for following more general conditions:

Suppose that Z is any space with a free $G = \mathbb{Z}_p$ (p an odd prime) action and $v : Z \to E$ is an equivariant Vietoris map. Let $\delta : Z \to E'$ be a single valued, equivariant map which makes the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\ \delta\ } & E' \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \pi'} \\ E & \xrightarrow{\ \pi\ } & B \end{array}$$

commutative. Note that the zero set $Z_\delta = \delta^{-1}(\{0\})$ is invariant under the action of G and $H^*(\hat{Z}_\delta)$ is an $H^*(B)$-module, via, the homomorphism $\hat{v}^*\hat{\pi}^* : H^*(B) \to H^*(\hat{Z}_\delta)$. As v is a Vietoris map, it is easy to see that $\hat{v} : \hat{Z} \to \hat{E}$ is also a Vietoris map. Then the homomorphism $\hat{v}^*$ induced by the Vietoris map $\hat{v}$ is an isomorphism. Let $q(x, y, z)|_{\hat{Z}_\delta}$ denote the image of $q(x, y, z)$ by the $H^*(B)$-homomorphism $i_2^* : H^*(\hat{Z}) \to H^*(\hat{Z}_\delta)$, where i_2^* is induced by the natural inclusion $i_2 : \hat{Z}_\delta \hookrightarrow \hat{Z}$. We have the following lemma.

Lemma 3.2. *Let X_n (n an odd natural number) be a space of cohomology type $(0, 0)$ and let $q(x, y, z) \in H^*(B)[x, y, z]$ be polynomial such that $q(x, y, z)|_{\hat{Z}_\delta} = 0$. Then there exist polynomials $r_i(x, y, z) \in H^*(B)[x, y, z]$ $(i = 1, 2, 3, 4)$ such that*

$$\begin{aligned} q(x, y, z)W'(x, y) = {} & r_1(x, y, z)W_1(x, y, z) + r_2(x, y, z)W_2(x, y, z) \\ & + r_3(x, y, z)W_3(x, y, z) + r_4(x, y, z)x^2, \end{aligned}$$

where W_i's and W' are characteristic polynomials defined above.

Note that the proof of the above lemma follows from the fact that the homomorphism induced by the arrows $E' \xleftarrow{\delta} Z \xrightarrow{v} E$ works in the same way as it works for a single arrow $E \to E'$.

With above notations and lemma, we are interested in determining a lower bound of the cohomological dimension of the zero set Z_δ.

Theorem 3.1. *Let (X_n, E, π, B) be a fibre bundle equipped with a fiber-wise free $G = \mathbb{Z}_p$ (p an odd prime) action such that the quotient bundle $(X_n/G, \hat{E}, \hat{\pi}, B)$ has the cohomology extension property. Let $(\mathbb{R}^k, E', \pi', B)$ be a k-dimensional real vector bundle and $\delta : Z \to E'$ be a G-equivariant map such that $\pi'\delta = \pi v$, where $v : Z \to E$ is a Vietoris map. Then,* $\mathrm{cd}_p(Z_\delta) \geq \mathrm{cd}_p(B) + 3n + 1 - k$.

Proof. Suppose $\deg q(x, y, z) < 3n + 1 - k$, where $q(x, y, z) \in H^*(B)[x, y, z]$ is a nonzero polynomial. We observe that $q(x, y, z)|_{\hat{Z}_\delta} \neq 0$. If $q(x, y, z)|_{\hat{Z}_\delta} = 0$ then by Lemma 3.2, we have

$$q(x, y, z)W'(x, y) = r_1(x, y, z)W_1(x, y, z) + r_2(x, y, z)W_2(x, y, z) + r_3(x, y, z)W_3(x, y, z) + r_4(x, y, z)x^2,$$

where $\deg W_1(x, y, z) = 3n + 1$, $\deg W_2(x, y, z) = 2n$ and $\deg W_3(x, y, z) = 2n + 1$. Thus,

$$\deg q(x, y, z) + k = \max_{1 \leq i \leq 3}\{\deg r_i(x, y, z) + \deg W_i, \deg r_4(x, y, z) + 2\}.$$

Consequently, $\deg q(x, y, z) + k \geq \deg r_1(x, y, z) + 3n + 1$. If $r_1(x, y, z) = 0$ then $\deg q(x, y, z) \geq 3n + 1 - k$, a contradiction. Thus, $q(x, y, z)|_{\hat{Z}_\delta} \neq 0$. Then the $H^*(B)$-module homomorphism

$$\bigoplus_{i=0}^{(3n+1-k)/2} H^*(B)y^i \to H^*(\hat{Z}_\delta)$$

defined by $y^i \mapsto y^i|_{\hat{Z}_\delta}$ is a monomorphism. Thus, for $3n + 1 \geq k$, we obtain

$$\mathrm{cd}_p(\hat{Z}_\delta) \geq \mathrm{cd}_p(B) + 3n + 1 - k.$$

As defined above, we are interested in determining a lower bound of the cohomological dimension of the coincidence set $A(\delta)$ of δ, where δ is a map such that $\pi'\delta = \pi v$.

Corollary 3.1. *Let (X_n, E, π, B) be a fibre bundle equipped with a fiber-wise free $G = \mathbb{Z}_p$ (p an odd prime) action such that the quotient bundle $(X_n/G, \hat{E}, \hat{\pi}, B)$ has the cohomology extension property. Let $(\mathbb{R}^k, E', \pi', B)$ be a k-dimensional real vector bundle and $\delta : Z \to E'$ be a map such that $\pi'\delta = \pi v$, where $v : Z \to E$ is a Vietoris map. Let $A(\delta) = \{x \in Z \mid \delta(x) = \delta(gx) = \cdots = \delta(g^{p-1}x)$, g is a generator of $G\}$. Then,* $\mathrm{cd}_p(A(\delta)) \geq \mathrm{cd}_p(B) + 3n + 1 - k$*, where $3n + 1 \geq k$.*

Proof. Let $M = E' \oplus \cdots \oplus E'$ be the total space of Whitney sum of p-copies of the k-dimensional real vector bundle $\pi' : E' \to B$. A map $\psi : M \to M$ defined by $(e_1', e_2', \dots, e_p') \mapsto (e_p', e_1', \dots, e_{p-1}')$ generates a fibre preserving G-action on M with fixed point set diagonal $\triangle$. It is easy to see that the orthogonal $\triangle^\perp$ is invariant under the action induced by ψ on M. This action is free outside the zero section $\{0\}$. A map $q : Z \to M$ defined by $q(x) = (\delta(x), \delta(gx), \dots, \delta(g^{p-1}x))$ is a fiber preserving G-equivariant map. Thus, the diagonal $\triangle$ and its orthogonal $\triangle^\perp$ are the total spaces of k-dimensional and $k(p-1)$-dimensional sub-bundles of Whitey sum $\pi' \oplus \cdots \oplus \pi'$ (p copies). The linear projection $r : (M, M - \triangle) \to (\triangle^\perp, \triangle^\perp - \{0\})$ along the diagonal $\triangle$ is also a fiber preserving map. This implies that r is a fibre preserving equivariant map. Therefore, the map $h : Z \to \Delta^\perp$ is fibre preserving G-equivariant map, where $h = rq$. Consequently, the zero set $Z_h = A(\delta)$. If $3n + 1 \geq k$ then by applying Theorem 3.1, we get $\mathrm{cd}_p(A(\delta)) \geq \mathrm{cd}_p(B) + 3n + 1 - k$. Hence, our claim holds.

Now, we prove our main result:

Theorem 3.2. *Let (X_n, E, π, B) be a fibre bundle equipped with a fiber-wise free $G = \mathbb{Z}_p$ (p an odd prime) action such that the quotient bundle $(X_n/G, \hat{E}, \hat{\pi}, B)$ has the cohomology extension property. Let $(\mathbb{R}^k, E', \pi', B)$ be a k-dimensional real vector bundle and $\mu : E \to E'$ be an admissible multivalued fibre preserving map. Let $A_\mu = \{x \in E \mid \mu(x) \cap \mu(gx) \cdots \cap \mu(g^{p-1}x) \neq \phi,\ g$ is a generator of $G\}$. Then, $\mathrm{cd}_p(A_\mu) \geq \mathrm{cd}_p(B) + 3n + 1 - k$, where $3n + 1 \geq k$.*

Proof. As $\mu : E \to E'$ is an admissible multivalued map, then there exists a space Γ and single valued maps $\alpha : \Gamma \to E$ and $\beta : \Gamma \to E'$ such that (α, β) is a selected pair for μ. Let $Z = \{(\gamma_1, \gamma_2, \dots, \gamma_p) \in \Gamma \times \Gamma \times \cdots \times \Gamma$ (p copies) $\mid \alpha(\gamma_1) = g\alpha(\gamma_2) = \cdots = g^{p-1}\alpha(\gamma_p)\}$. Now, we have the following commutative diagram:

$$
\begin{array}{ccccc}
Z & \xrightarrow{q} & \Gamma & \xrightarrow{\beta} & E' \\
 & \searrow^{v} & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\pi'} \\
 & & E & \xrightarrow{\pi} & B
\end{array}
$$

where q is the first protection $(\gamma_1, \gamma_2, \dots, \gamma_p) \mapsto \gamma_1$ and $v = \alpha q$. Note that $v : Z \to E$ is a Vietoris map since for each $e \in E$, we have

$$v^{-1}(e) = \alpha^{-1}(e) \times \alpha^{-1}(ge) \times \cdots \times \alpha^{-1}(g^{p-1}e)$$

is acyclic being the cartesian product of acyclic sets. A map defined on Z by $(\gamma_1, \gamma_2, \dots, \gamma_p) \mapsto (\gamma_p, \gamma_1, \dots, \gamma_{p-1})$ induces a free G-action on Z and v is a G-equivariant map with respect to this action. Let $\delta = \beta q : Z \to E'$. Note that if $\delta(\gamma_1, \gamma_2, \dots, \gamma_p) = \delta(\gamma_p, \gamma_1, \dots, \gamma_{p-1}) = \cdots = \delta(\gamma_2, \gamma_3, \dots, \gamma_p, \gamma_1)$ for some $(\gamma_1, \gamma_2, \dots, \gamma_p) \in Z$ then $\mu(\alpha(\gamma_1)) \cap \mu(\alpha(\gamma_2)) \cap \cdots \cap \mu(\alpha(\gamma_p)) \neq \phi$. Thus, $v(A(\delta)) \subset A_\mu$. Let $u' = \hat{i}_2^* h'^*(s)$ and $v' = \hat{i}_2^* h'^*(t)$, where $s \in H^1(B_G)$ and $t = \beta_p(s) \in H^2(B_G)$, $h' : \hat{Z} \to B_G$ is a characteristic map of the principle G-bundle $Z \to \hat{Z}$ and $i_2 : A(\delta) \hookrightarrow Z$ is the inclusion map. Let $u'' = \hat{i}_4^*(a)$ and $v'' = \hat{i}_4^*(b)$, where a, b are characteristic classes of the principle G-bundle $E \to \hat{E}$ and $i_4 : A_\mu \hookrightarrow E$ is the inclusion map. As $v : A(\delta) \to A_\mu$ is a G-equivariant map, we have $\hat{v}^*(u'') = u'$ and $\hat{v}^*(v'') = v'$. Thus

$$q(x, y)|_{\hat{A}(\delta)} = q(u', v') = q(\hat{v}^*(u''), \hat{v}^*(v'')) = \hat{v}^*(q(u'', v'')) = \hat{v}^*(q(x, y)|_{\hat{A}_\mu}).$$

Thus, if $\hat{v}^*(q(x, y)|_{\hat{A}_\mu}) = 0$ then $q(x, y)|_{\hat{A}(\delta)} = 0$. Therefore, by Lemma 3.2 and Corollary 3.1, we have $\mathrm{cd}_p(A_\mu) \geq \mathrm{cd}_p(B) + 3n + 1 - k$.

Taking B as a singleton set in the previous theorem, we have

COROLLARY 3.2. *Let* $G = \mathbb{Z}_p$ *(p an odd prime) act freely on a space* X_n *of cohomology type* $(0, 0)$. *Let* $\mu : X_n \to \mathbb{R}^k$ *be an admissible multivalued map. Then,* $\mathrm{cd}_p(A_\mu) \geq 3n + 1 - k$, *where* $3n + 1 \geq k$.

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