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On Gosper's Π_q and Lambert series identities

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Abstract. In an interesting article entitled "Experiments and discoveries in q-trigonometry", R. W. Gosper conjectured few beautiful Π_q and Lambert series identities. Many people have attempted confirming some of those identities in the Gosper's list, mainly by using Gosper's q-trigonometric identities. In this paper we either prove or disprove all the Π_q and Lambert series identities in the Gosper's list by mainly using S. Ramanujan's theta function identities and W. N. Bailey's summation formula. In the process, we obtain three new Gosper kind of identities.

1. Introduction

Throughout the paper, let $q = e^{\pi i \tau}$ with $\tau > 0$. As usual for any complex number a, define

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n).$$

In Chapter 16 of his second notebook [15, p. 197], Ramanujan defined his general theta function $f(a,b)$ by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \qquad |ab| < 1.$$

Further, Ramanujan defines

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$$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty,$$

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty,$$

and

$$\chi(q) := (-q;q^2)_\infty.$$

For convenience, we set $f_n := f(-q^n) = (q^n;q^n)_\infty$ for any positive integer n and it is easy to see that

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \qquad \psi(q) = \frac{f_2^2}{f_1}, \qquad \varphi(-q) = \frac{f_1^2}{f_2}, \qquad \psi(-q) = \frac{f_1 f_4}{f_2},$$

$$f(q) = \frac{f_2^3}{f_1 f_4}, \qquad \chi(-q) = \frac{f_1}{f_2}, \qquad \text{and} \qquad \chi(q) = \frac{f_2^2}{f_1 f_4}. \tag{1.1}$$

In an interesting article entitled "Experiments and discoveries in q-trigonometry" by R. W. Gosper [9], introduced a function

$$\Pi_q := q^{1/4} \prod_{n \geq 1} \frac{(1-q^{2n})^2}{(1-q^{2n-1})^2} = q^{1/4}\psi^2(q).$$

Gosper [9] conjectured following 17 Π_q **identities**:

$$\frac{\Pi_q^2}{\Pi_{q^2}\Pi_{q^4}} - \frac{\Pi_{q^2}^2}{\Pi_{q^4}^2} = 4, \tag{1.2}$$

$$\Pi_{q^2}^2 + 2\Pi_{q^2}\Pi_{q^6} = \Pi_q \Pi_{q^3} + 3\Pi_{q^6}^2, \tag{1.3}$$

$$\frac{\Pi_{q^2}\Pi_{q^3}^2}{\Pi_{q^6}\Pi_q^2} = \frac{\Pi_{q^2} - \Pi_{q^6}}{\Pi_{q^2} + 3\Pi_{q^6}}, \tag{1.4}$$

$$\sqrt{\Pi_{q^2}\Pi_{q^6}}(\Pi_q^2 - 3\Pi_{q^3}^2) = \sqrt{\Pi_q\Pi_{q^3}}(\Pi_{q^2}^2 + 3\Pi_{q^6}^2), \tag{1.5}$$

$$\Pi_{q^2}\Pi_{q^3}^4 = \Pi_{q^6}(\Pi_{q^2} - \Pi_{q^6})^3(\Pi_{q^2} + 3\Pi_{q^6}), \tag{1.6}$$

$$\Pi_{q^6}\Pi_q^4 = \Pi_{q^2}(\Pi_{q^2} - \Pi_{q^6})(\Pi_{q^2} + 3\Pi_{q^6})^3, \tag{1.7}$$

$$\Pi_q\Pi_{q^3}(\Pi_q^2 \pm 4\Pi_{q^2}^2)^2 = \Pi_{q^2}^2(\Pi_q \mp \Pi_{q^3})(\Pi_q \pm 3\Pi_{q^3})^3 \tag{1.8}$$

$$\Pi_q\Pi_{q^3}(\Pi_{q^3}^2 \pm 4\Pi_{q^6}^2)^2 = \Pi_{q^6}^2(\Pi_q \mp \Pi_{q^3})^3(\Pi_q \pm 3\Pi_{q^3}), \tag{1.9}$$

$$\Pi_{q^2}^2(\Pi_q^4 + 18\Pi_q^2\Pi_{q^3}^2 - 27\Pi_{q^3}^4) = \Pi_q\Pi_{q^3}(\Pi_q^4 + 16\Pi_{q^2}^4), \tag{1.10}$$

$$\Pi_{q^6}^2(\Pi_q^4 - 6\Pi_q^2\Pi_{q^3}^2 - 3\Pi_{q^3}^4) = \Pi_q\Pi_{q^3}(\Pi_{q^3}^4 + 16\Pi_{q^6}^4), \tag{1.11}$$

$$\Pi_{q^3}^2 + 3\Pi_q\Pi_{q^9} = \sqrt{\Pi_q\Pi_{q^9}}(\Pi_q + 3\Pi_{q^9}), \tag{1.12}$$

$$\sqrt{\frac{\Pi_q}{\Pi_{q^9}}} = 1 + \sqrt[3]{\left(\frac{\Pi_{q^3}}{\Pi_{q^9}}\right)^2 - 1} = \frac{1 + \sqrt[3]{9\left(\frac{\Pi_q}{\Pi_{q^3}}\right)^2 - 1}}{3}, \tag{1.13}$$

$$\Pi_{q^2}\Pi_{q^5}^4[16\Pi_{q^{10}}^4 - \Pi_{q^5}^4] = \Pi_{q^{10}}^3[5\Pi_{q^{10}} - \Pi_{q^2}][\Pi_{q^2} - \Pi_{q^{10}}]^5, \tag{1.14}$$

$$\Pi_{q^{10}}\Pi_q^4[16\Pi_{q^2}^4 - \Pi_q^4] = \Pi_{q^2}^3[5\Pi_{q^{10}} - \Pi_{q^2}]^5[\Pi_{q^2} - \Pi_{q^{10}}], \tag{1.15}$$

$$\Pi_q\Pi_{q^5}[16\Pi_{q^2}^4 - \Pi_q^4]^2 = \Pi_{q^2}^4[5\Pi_{q^5} - \Pi_q]^5[\Pi_{q^5} - \Pi_q], \tag{1.16}$$

$$\Pi_q\Pi_{q^5}[16\Pi_{q^{10}}^4 - \Pi_{q^5}^4]^2 = \Pi_{q^{10}}^4[5\Pi_{q^5} - \Pi_q][\Pi_{q^5} - \Pi_q]^5, \tag{1.17}$$

and

$$\Pi_{q^2}\Pi_{q^{10}}[\Pi_{q^5} - \Pi_q][5\Pi_{q^5} - \Pi_q] = [\Pi_q\Pi_{q^{10}} - \Pi_{q^2}\Pi_{q^5}]^2. \tag{1.18}$$

In [7], M. E. Bachraoui, partially proved the identity (1.4) and showed the equivalence of (1.5) and (1.6) by employing the Z. G. Liu identities on classical theta functions. In [9], Gosper himself confirmed (1.2) through some of his q-trigonometric identities. B. He and H. Zhai [12], have proved (1.12) and the first equality of (1.13) by using existing q-trigonometric identities of Gosper.

In [9], Gosper also stated the following 13 **Lambert series identities** without proof:

$$\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2} - 2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2} = \frac{1}{24}\left(\frac{\Pi_q^4}{\Pi_{q^2}^2} - 1\right) + \frac{2}{3}\Pi_{q^2}^2, \tag{1.19}$$

$$\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2} - 3\frac{q^{3n}}{(1-q^{3n})^2} = \frac{(\Pi_q^2 + 3\Pi_{q^3}^2)^2}{12\Pi_q\Pi_{q^3}} - \frac{1}{12}, \tag{1.20}$$

$$\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2} - 4\frac{q^{4n}}{(1-q^{4n})^2} = \frac{1}{8}\left(\frac{\Pi_q^4}{\Pi_{q^2}^2} - 1\right), \tag{1.21}$$

$$\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2} - 9\frac{q^{18n}}{(1-q^{18n})^2} = \frac{\Pi_{q^3}^3}{\Pi_q} + \frac{1}{3}\left(\frac{\Pi_{q^3}^3}{\Pi_{q^9}} - 1\right), \tag{1.22}$$

$$\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2} - 2\sum_{n=1}^{\infty}\frac{q^{4n-2}}{(1-q^{4n-2})^2} = \Pi_{q^2}^2 = \sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{4n-2}}, \tag{1.23}$$

$$\frac{1}{\Pi_{q^3}^2}\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-3\sum_{n=1}^{\infty}\frac{q^{6n-3}}{(1-q^{6n-3})^2}\right)=\frac{\Pi_q}{\Pi_{q^3}}, \tag{1.24}$$

$$\frac{1}{\Pi_{q^5}^2}\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n=1}^{\infty}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\right)=\frac{\frac{\Pi_{q^5}^2}{\Pi_{q^{10}}^2}+16\frac{\Pi_{q^{10}}^2}{\Pi_{q^5}^2}}{\frac{\Pi_q}{\Pi_{q^5}}-4-\frac{\Pi_{q^5}}{\Pi_q}}, \tag{1.25}$$

$$\frac{1}{\Pi_{q^5}^2}\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n=1}^{\infty}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\right)=\sqrt{\frac{\Pi_q^3}{\Pi_{q^5}^3}-2\frac{\Pi_q^2}{\Pi_{q^5}^2}+5\frac{\Pi_q}{\Pi_{q^5}}}, \tag{1.26}$$

$$\frac{1}{\Pi_{q^9}^2}\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-9\sum_{n=1}^{\infty}\frac{q^{18n-9}}{(1-q^{18n-9})^2}\right)$$
$$=\left(\frac{\Pi_q}{\Pi_{q^9}}+3\right)\sqrt{\left(\frac{\Pi_q}{\Pi_{q^9}}\right)^{3/2}-3\left(\frac{\Pi_q}{\Pi_{q^9}}\right)+3\left(\frac{\Pi_q}{\Pi_{q^9}}\right)^{1/2}}, \tag{1.27}$$

$$6\left(\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}-5\frac{q^{5n}}{(1-q^{5n})^2}\right)+1$$
$$=\left(\frac{\Pi_q}{\Pi_{q^5}}+2+5\frac{\Pi_{q^5}}{\Pi_q}\right)\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n=1}^{\infty}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\right), \tag{1.28}$$

$$3\left(\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}-9\frac{q^{9n}}{(1-q^{9n})^2}\right)+1$$
$$=\left(\sqrt{\frac{\Pi_q}{\Pi_{q^9}}}+3\sqrt{\frac{\Pi_{q^9}}{\Pi_q}}\right)\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-9\sum_{n=1}^{\infty}\frac{q^{18n-9}}{(1-q^{18n-9})^2}\right), \tag{1.29}$$

$$3\left(\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}-9\frac{q^{9n}}{(1-q^{9n})^2}\right)+1$$
$$=(\Pi_q+3\Pi_{q^9})3(\Pi_q\Pi_{q^9}+\Pi_{q^3}^2)^2+\frac{(\Pi_q-3\Pi_{q^9})^2}{4\Pi_{q^3}}, \tag{1.30}$$

and

$$\Pi_q^4=6\sum_{n=1}^{\infty}\frac{q^{4n-2}}{(1-q^{2n-1})^4}+\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}=\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^{2n}}. \tag{1.31}$$

Bachraoui proved (1.19)–(1.22), (1.24) and (1.29) in [8] by using certain q-trigonometric identities (of Gosper's kind) except for (1.19) which he obtains using some identities satisfied by the divisor function. In [11], He verifies (1.25) and (1.26) by using the theory of modular equations, for which the identities should be known in advance.

Much of what Gosper has conjectured have appeared in the literature in different forms. In this paper, one of our aims is to bring forth those identities existing in the literature from which many of Gopser's identities follow easily. Apart from this, we also prove some Gosper's Π_q identities and Lambert series identities through classical techniques using Ramanujan's theta function identities. In the process, we obtain one new Gosper kind of Π_q identity and two new Gosper kind of Lambert series identities. Following are they:

$$\frac{(\Pi_{q^2}+3\Pi_{q^{18}})^2}{\sqrt{\Pi_{q^2}\Pi_{q^{18}}}}\Pi_{q^6}=3\frac{\Pi_{q^3}^3}{\Pi_q}+\frac{\Pi_{q^3}^3}{\Pi_{q^9}}, \tag{1.32}$$

$$\begin{aligned}\frac{1}{\Pi_{q^5}^2}&\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n=1}^{\infty}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\right)\\&=\frac{\Pi_{q^{10}}^2}{\Pi_{q^5}^2}\left(\left(\frac{\Pi_{q^2}}{\Pi_{q^{10}}}-1\right)^2+2\left(\frac{\Pi_{q^2}}{\Pi_{q^{10}}}-1\right)\left(\frac{\Pi_q}{\Pi_{q^5}}-1\right)-8\right),\end{aligned} \tag{1.33}$$

and

$$\begin{aligned}\frac{1}{\Pi_{q^5}^2}&\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n=1}^{\infty}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\right)\\&=\frac{\Pi_{q^5}^2}{\Pi_{q^{10}}^2}\left\{-\frac{1}{2}+\frac{\left(\frac{\Pi_q}{\Pi_{q^5}}-1\right)^2}{\left(\frac{\Pi_{q^2}}{\Pi_{q^{10}}}-1\right)^2}+\frac{1}{2}\,\frac{\left(\frac{\Pi_q}{\Pi_{q^5}}-1\right)^2}{\left(\frac{\Pi_{q^2}}{\Pi_{q^{10}}}-1\right)}\right\}.\end{aligned} \tag{1.34}$$

The rest of the paper is structured as follows: In the next section we list out the existing identities and results which are required to prove (1.2)–(1.31), except (1.30), which we show to be wrong, in Section 3. In Section 3, we prove (1.2)–(1.18) and in Section 4, we prove (1.19)–(1.31).

2. Preliminary results

In Chapter 16 of his second notebook [4, p. 40][15, p. 198], Ramanujan recorded following very interesting theta function identities:

$$\varphi^2(q)-\varphi^2(-q)=8q\psi^2(q^4), \tag{2.1}$$

$$\varphi^2(q)+\varphi^2(-q)=2\varphi^2(q^2), \tag{2.2}$$

and

$$\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2). \tag{2.3}$$

Adding (2.1) and (2.2), and then employing the fact that $\varphi(q)\psi(q^2) = \psi^2(q)$, we obtain (1.2), which we rewrite for further convenience as

$$\Pi_{q^2}^2 + 4\Pi_{q^4}^2 = \frac{\Pi_q^2 \Pi_{q^4}}{\Pi_q^2}. \tag{2.4}$$

Changing q to $-q$ in the above, we obtain

$$\Pi_{q^2}^2 - 4\Pi_{q^4}^2 = q\frac{\psi^4(-q)\psi^2(q^4)}{\psi^2(q^2)}. \tag{2.5}$$

Also by using $\varphi(q)\psi(q^2) = \psi^2(q)$ in (2.3) and employing (1.1), we find that

$$\Pi_q^4 - 16\Pi_{q^2}^4 = q\left(\frac{f_1 f_4}{f_2}\right)^8. \tag{2.6}$$

N. D. Baruah and R. Barman [3] deduced,

$$\Pi_{q^2} + \Pi_{q^6} = q^{1/2}\frac{\psi(-q)\varphi^3(-q^3)\psi^2(q^6)}{\varphi(-q)\psi^3(-q^3)}. \tag{2.7}$$

Changing q to $-q$ in the above, we obtain

$$\Pi_{q^2} - \Pi_{q^6} = q^{1/2}\frac{\psi(q)\varphi^3(q^3)\psi^2(q^6)}{\varphi(q)\psi^3(q^3)}. \tag{2.8}$$

K. R. Vasuki, G. Sharath and K. R. Rajanna [18] have deduced the following identity:

$$\Pi_{q^2} - 3\Pi_{q^6} = q^{1/2}\frac{\varphi(-q)\psi(-q)\psi(-q^3)}{\varphi(-q^3)}. \tag{2.9}$$

Changing q to $-q$ in the above, we find that

$$\Pi_{q^2} + 3\Pi_{q^6} = q^{1/2}\frac{\varphi(q)\psi(q)\psi(q^3)}{\varphi(q^3)}. \tag{2.10}$$

Multiplying (2.7) and (2.9) and then changing q to $-q$, and then dividing both sides by $\psi(q^2)\psi(q^6)$, we obtain

$$\frac{\psi^3(q^2)}{\psi(q^6)} - 3q^2\frac{\psi^3(q^6)}{\psi(q^2)} = \phi(q)\phi(q^3) - 2q\psi(q^2)\psi(q^6), \tag{2.11}$$

where we use the relation $\phi(q)\psi(q^2) = \psi^2(q)$. Ramanujan [4, p. 223, 226] has recorded the following series identities,

$$q\psi(q^2)\psi(q^6) = \sum_{n=0}^{\infty} \frac{q^{6n+1}}{1-q^{12n+2}} - \sum_{n=0}^{\infty} \frac{q^{6n+5}}{1-q^{12n+10}}, \tag{2.12}$$

$$\begin{aligned} \varphi(q)\varphi(q^3) &= 1 + 2\left[\sum_{n=0}^{\infty} \frac{q^{3n+1}}{1+(-q)^{3n+1}} - \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1+(-q)^{3n+2}}\right] \\ &= \sum_{n=-\infty}^{\infty} \frac{q^n}{1+(-q)^{3n}}, \end{aligned} \tag{2.13}$$

and

$$\frac{\psi^3(q)}{\psi(q^3)} = 1 + 3\left[\sum_{n=0}^{\infty} \frac{q^{6n+1}}{1-q^{6n+1}} - \sum_{n=0}^{\infty} \frac{q^{6n+5}}{1-q^{6n+5}}\right]. \tag{2.14}$$

Also from [17, p. 30],

$$\frac{\psi^3(q^3)}{\psi(q)} = \sum_{n=0}^{\infty} \frac{q^{3n}}{1-q^{6n+2}} - \sum_{n=0}^{\infty} \frac{q^{3n+1}}{1-q^{6n+4}}. \tag{2.15}$$

The above four identities can also be deduced as particular cases of Ramanujan ${}_1\psi_1$ summation formula,

$$\sum_{n=-\infty}^{\infty} \frac{z^n}{1-aq^n} = \frac{f_1 f\left(-az, \frac{-q}{az}\right)}{f\left(-z, -\frac{q}{z}\right) f\left(-a, -\frac{q}{a}\right)}. \tag{2.16}$$

From (2.12)–(2.15), it easily follows that

$$\frac{\psi^3(q)}{\psi(q^3)} + 3q\frac{\psi^3(q^3)}{\psi(q)} = \varphi(q)\varphi(q^3) + 4q\psi(q^2)\psi(q^6) \tag{2.17}$$

and

$$\varphi(q^4)\varphi(q^{12}) + 4q^4\psi(q^8)\psi(q^{24}) = \varphi(q)\varphi(q^3) - 2q\psi(q^2)\psi(q^6). \tag{2.18}$$

We found from the works of Ramanujan that [4, p. 263]:

$$\Pi_q - \Pi_{q^5} = q^{1/4} \frac{\varphi^2(-q^5)}{\chi(-q)\chi(-q^5)}, \tag{2.19}$$

$$\Pi_q - 5\Pi_{q^5} = q^{1/4} f_1^2 \frac{\chi(-q)}{\chi(-q^5)}, \tag{2.20}$$

and

$$\varphi^2(q) - \varphi^2(q^5) = 4q\frac{f_2^2 f_5 f_{20}}{f_1 f_4}. \tag{2.21}$$

For a simple proof of (2.19) and (2.20), see [5] and [14]. A proof of (2.21) was given by L. C. Shen [16].

The following theta function identities have been recorded by Ramanujan [4, p. 345]:

$$\frac{\psi(q)}{q\psi(q^9)} = \sqrt{\frac{\Pi_q}{\Pi_{q^9}}} = 1 + \frac{\chi^3(-q^9)}{q\chi(-q^3)}, \tag{2.22}$$

$$\frac{\psi^4(q^3)}{q\psi^4(q^9)} = \frac{\Pi_{q^3}^2}{\Pi_{q^9}^2} = 1 + \frac{\chi^9(-q^9)}{q^3\chi^3(-q^3)}, \tag{2.23}$$

$$1 + 3q\frac{\psi(-q^9)}{\psi(-q)} = \left(1 + 9q\frac{\psi^4(-q^3)}{q\psi^4(-q)}\right)^{1/3}, \tag{2.24}$$

and

$$1 + \frac{\psi(-q^{1/3})}{q^{1/3}\psi(-q^3)} = \left(1 + \frac{\psi^4(-q)}{q\psi^4(-q^3)}\right)^{1/3}. \tag{2.25}$$

For proving these Berndt utilizes Entry 31 of Chapter 16 of Ramanujan's second notebook.

We also make use of the following identity due to W. N. Bailey in our proofs:

$$\sum_{n=-\infty}^{\infty}\left[\frac{aq^n}{(1-aq^n)^2} - \frac{bq^n}{(1-bq^n)^2}\right]$$
$$= af^6(-q)\frac{f(-ab,-q/ab)f(-b/a,-aq/b)}{f^2(-a,-q/a)f^2(-b,-q/b)}. \tag{2.26}$$

Bailey proved the above identity by making use of the elliptic function theory. It can also be deduced from his ${}_6\psi_6$ well-poised summation formula [2].

The following identity is due to C. Adiga et al. [1]:

$$f(a,b)f(c,d)f\left(an,\frac{b}{n}\right)f\left(cn,\frac{d}{n}\right)$$
$$- f(-a,-b)f(-c,-d)f\left(-an,-\frac{b}{n}\right)f\left(-cn,-\frac{d}{n}\right)$$
$$= 2af\left(\frac{c}{a},ad\right)f\left(\frac{d}{an},acn\right)f\left(n,\frac{ab}{n}\right)f\left(n,\frac{ab}{n}\right)\psi(ab). \tag{2.27}$$

The Eisenstein series $P_n(q)$ is defined as

$$P_n(q) = 1 - 24\sum_{k=1}^{\infty}\frac{kq^{nk}}{1-q^{nk}}.$$

For convenience, we set P_n for $P_n(q)$. The following relation between Eisenstein series and theta functions holds:

$$-P_1 + 2P_2 = 16q\psi^4(q^2) + \varphi^4(q). \tag{2.28}$$

The above identity can be easily obtained from Bailey formula (2.26), as done in [19]. From ${}_1\psi_1$ summation formula, following can be easily obtained:

$$\varphi^2(q) = 1 + 4\sum_{1}^{\infty}(-1)^k\frac{q^{2k}}{1-q^{2k-1}}, \tag{2.29}$$

$$\psi^2(q^2) = \sum_{0}^{\infty}\frac{q^k}{1+q^{2k+1}}, \tag{2.30}$$

$$\sum_{n=-\infty}^{\infty}\frac{z^n}{1-aq^{kn}} = \frac{f(-az, -q^k(az)^{-1})(q^k;q^k)_\infty^3}{f(-z, -q^kz^{-1})f(-a, -q^ka^{-1})}, \tag{2.31}$$

and

$$2\sum_{n=-\infty}^{\infty}\frac{q^{kn}x^n}{1+q^{2kn}} = \varphi^2(-q^{2k})\frac{f(q^kx, q^kx^{-1})}{f(-q^kx, -q^kx^{-1})}. \tag{2.32}$$

Expanding (2.29) in base 5, subtracting $\phi^2(q^5)$ terms, interchanging the order of summation of each series and then employing (2.32) after obtaining two bilateral series, we arrive at

$$\frac{\varphi^2(q) - \varphi^2(q^5)}{2\varphi^2(-q^{10})} = \frac{f(q, q^9)}{f(-q, -q^9)} - \frac{f(q^3, q^7)}{f(-q^3, -q^7)}. \tag{2.33}$$

In the same way, from (2.30) and (2.31), we arrive at

$$\frac{\psi^2(q^2) - q^2\psi^2(q^{10})}{\psi^2(-q^5)} = \frac{f(q^4, q^6)}{f(-q, -q^9)} - q\frac{f(q^2, q^8)}{f(-q^3, -q^7)}. \tag{2.34}$$

Using the product representation of $f(a,b)$ and using (1.1), (2.19) and (2.21), we easily obtain the following identities:

$$f(q, q^9)f(q^3, q^7) = \frac{\varphi^2(q) - \varphi^2(q^5)}{4q}, \tag{2.35}$$

$$f(q^2, q^8)f(q^4, q^6) = \psi^2(q^2) - q^2\psi^2(q^{10}), \tag{2.36}$$

$$\psi^2(q) - q\psi^2(q^5) = \frac{f_2 f_5^3}{f_1 f_{10}}, \tag{2.37}$$

$$\frac{1}{f(-q,-q^9)f(-q^3,-q^7)} = \frac{\psi^2(q) - q\psi^2(q^5)}{\varphi^2(-q^5)\psi^2(q^5)}, \tag{2.38}$$

and

$$\varphi^2(q) - \varphi^2(q^5) = 4q\psi^2(q^5)\frac{\psi^2(q) - q\psi^2(q^5)}{\psi^2(q^2) - q^2\psi^2(q^{10})}. \tag{2.39}$$

3. Proofs of Π_q identities

In this section we confirm (1.3)–(1.18). It is to be noted in the beginning itself that, (1.10) and (1.11) are not proved in the same spirit as that of the rest. We just verify these two at the end of this section.

Proof of (1.4). Dividing (2.8) by (2.10) and using $\varphi(q) = \frac{\psi^2(q)}{\psi(q^2)}$, we obtain (1.4).

Proof of (1.5). The identity obtained after replacing q by q^2 in (1.5) is equivalent to (2.18) which is evident from (2.17) and (2.11).

Proof of (1.6) and (1.7). Multiplying (2.10) and the identity obtained after cubing (2.8), and then using $\varphi(q) = \frac{\psi^2(q)}{\psi(q^2)}$, we obtain

$$(\Pi_{q^2} - \Pi_{q^6})^3(\Pi_{q^2} + 3\Pi_{q^6}) = \frac{\Pi_{q^3}^4 \Pi_{q^2}}{\Pi_{q^6}},$$

which is nothing but (1.6). Proof of (1.7) is similar to that of (1.6).

Proof of (1.3). Multiplying (1.6) and (1.7), we obtain

$$(\Pi_{q^2} - \Pi_{q^6})^4(\Pi_{q^2} + 3\Pi_{q^6})^4 = \Pi_{q^3}^4 \Pi_q^4.$$

Assuming $0 < q < 1$, we obtain

$$(\Pi_{q^2} - \Pi_{q^6})(\Pi_{q^2} + 3\Pi_{q^6}) = \pm\Pi_{q^3}\Pi_q.$$

Dividing both sides by q and then setting $q = 0$, we observe that the left hand side of the above equation is greater than 0. This forces us to choose $+$ sign, which implies

$$(\Pi_{q^2} - \Pi_{q^6})(\Pi_{q^2} + 3\Pi_{q^6}) = \Pi_{q^3}\Pi_q.$$

By analytic continuation, this holds good for all $|q| < 1$. Hence the proof of (1.3).

Proof of (1.8). From (2.5), (2.7) and (2.9), it follows that

$$\frac{(\Pi_{q^2}+\Pi_{q^6})(\Pi_{q^2}-3\Pi_{q^6})^3}{(\Pi_{q^2}^2-4\Pi_{q^4}^2)^2}=\frac{\Pi_{q^6}\Pi_{q^2}}{\Pi_{q^4}^2}.$$

Changing q to $-q$ in the above equation, we obtain

$$\frac{(\Pi_{q^2}-\Pi_{q^6})(\Pi_{q^2}+3\Pi_{q^6})^3}{(\Pi_{q^2}^2+4\Pi_{q^4}^2)^2}=\frac{\Pi_{q^6}\Pi_{q^2}}{\Pi_{q^4}^2}.$$

Then changing q to $q^{1/2}$ in the above two equations, we obtain (1.8).

Proof of (1.9). From (2.4) and (1.6), it follows that

$$\frac{(\Pi_{q^2}^2+4\Pi_{q^4}^2)^2}{(\Pi_{q^2}-\Pi_{q^6})^3(\Pi_{q^2}+3\Pi_{q^6})}=\frac{\Pi_{q^{12}}^2}{\Pi_{q^2}\Pi_{q^6}}.$$

Changing q to $-q$ in the above equation, we obtain

$$\frac{(\Pi_{q^2}^2-4\Pi_{q^4}^2)^2}{(\Pi_{q^2}+\Pi_{q^6})^3(\Pi_{q^2}-3\Pi_{q^6})}=\frac{\Pi_{q^{12}}^2}{\Pi_{q^2}\Pi_{q^6}}.$$

By changing q to $q^{1/2}$ in the above equations, we obtain (1.9).

Proof of (1.12). Eliminating $\frac{\chi^3(-q^9)}{q\chi(-q^3)}$ between (2.22) and (2.23), it is easy to see that

$$\frac{\Pi_{q^3}^2}{\Pi_{q^9}^2}-3\left(\frac{\Pi_q}{\Pi_{q^9}}\right)=\left(\frac{\Pi_q}{\Pi_{q^9}}\right)^{3/2}+3\left(\frac{\Pi_q}{\Pi_{q^9}}\right)^{1/2}.$$

Multiplying throughout by $\Pi_{q^9}^2$, we complete the proof of (1.12).

Proof of (1.13). The first equality of (1.13) directly follows from (2.22) and (2.23). However, the second equality of (1.13), i.e.

$$\sqrt{\frac{\Pi_q}{\Pi_{q^9}}}=\frac{1+\sqrt[3]{9\left(\frac{\Pi_q}{\Pi_{q^3}}\right)^2-1}}{3},$$

is wrong. This is because

$$\lim_{q\to 1^-}\sqrt{\frac{\Pi_q}{\Pi_{q^9}}}=3,$$

whereas

$$\lim_{q\to 1^-}\frac{1+\sqrt[3]{9\left(\frac{\Pi_q}{\Pi_{q^3}}\right)^2-1}}{3}=\frac{1+\sqrt[3]{80}}{3}.$$

For finding limit as $q \to 1^-$ of the expressions involving theta functions, one can refer [10]. The correct way of putting what Gosper intended might be the following, which is again due to Ramanujan:

$$\sqrt{\frac{\Pi_q}{\Pi_{q^9}}} = \frac{3}{1 - \sqrt[3]{1 - 9\frac{\Pi_{q^3}^2}{\Pi_q^2}}}.$$

Proof of (1.14). From (2.19), (2.20) and (2.6) it follows that

$$\frac{(5\Pi_{q^{10}} - \Pi_{q^2})(\Pi_{q^2} - \Pi_{q^{10}})^5}{(16\Pi_{q^{10}}^4 - \Pi_{q^5}^4)} = \frac{1}{q^7}\left(\frac{f_4^4}{f_2}\right)\left(\frac{f_{10}^{16}}{f_5^8}\right)\left(\frac{f_{10}^6}{f_{20}^{12}}\right).$$

From (1.1), it follows that

$$\frac{(5\Pi_{q^{10}} - \Pi_{q^2})(\Pi_{q^2} - \Pi_{q^{10}})^5}{(16\Pi_{q^{10}}^4 - \Pi_{q^5}^4)} = \frac{\Pi_{q^2}\Pi_{q^5}^4}{\Pi_{q^{10}}^3}.$$

This completes the proof of (1.14).

Proof of (1.15). From (2.19), (2.20) and (2.6), it follows that

$$\frac{(5\Pi_{q^{10}} - \Pi_{q^2})^5(\Pi_{q^2} - \Pi_{q^{10}})}{(16\Pi_{q^2}^4 - \Pi_q^4)} = q^2\frac{f_{12^{16}}}{f_1^8}\frac{f_2^6}{f_4^{12}}\frac{f_{20}^4}{f_{10}^2},$$

where we have used (1.1). Again employing (1.1) to the right-hand side, we obtain

$$\frac{(5\Pi_{q^{10}} - \Pi_{q^2})^5(\Pi_{q^2} - \Pi_{q^{10}})}{(16\Pi_{q^2}^4 - \Pi_q^4)} = \frac{\Pi_{q^2}\Pi_{q^5}^4}{\Pi_{q^{10}}^3},$$

which completes the proof of (1.15).

Proof of (1.16). From (1.15), we have

$$\frac{(16\Pi_{q^4}^4 - \Pi_{q^2}^4)^2}{(5\Pi_{q^{10}} - \Pi_{q^2})^5(\Pi_{q^2} - \Pi_{q^{10}})} = \frac{(16\Pi_{q^4}^4 - \Pi_{q^2}^4)^2}{(16\Pi_{q^2}^4 - \Pi_q^4)}\frac{\Pi_{q^2}^3}{\Pi_{q^{10}}\Pi_q^4}.$$

From (2.6), we have

$$\frac{(16\Pi_{q^4}^4 - \Pi_{q^2}^4)^2}{(16\Pi_{q^2}^4 - \Pi_q^4)} = q\left(\frac{f_2 f_8}{f_4}\right)^{16}\left(\frac{f_2}{f_1 f_4}\right)^8.$$

On using (1.1), we observe that

$$q\left(\frac{f_2 f_8}{f_4}\right)^{16}\left(\frac{f_2}{f_1 f_4}\right)^8 = \frac{\Pi_q^4\Pi_{q^4}^4}{\Pi_{q^2}^4}.$$

Using this in the above, we obtain

$$\frac{(16\Pi_{q^4}^4 - \Pi_{q^2}^4)^2}{(5\Pi_{q^{10}} - \Pi_{q^2})^5(\Pi_{q^2} - \Pi_{q^{10}})} = \frac{\Pi_{q^4}^4}{\Pi_{q^2}\Pi_{q^{10}}}.$$

Changing q to $q^{1/2}$, we obtain (1.16).

Proof of (1.17). From (1.14), we observe that

$$\frac{(16\Pi_{q^{20}}^4 - \Pi_{q^{10}}^4)^2}{(5\Pi_{q^{10}} - \Pi_{q^2})(\Pi_{q^{10}} - \Pi_{q^2})^5} = \frac{\Pi_{q^{10}}^3(16\Pi_{q^{20}}^4 - \Pi_{q^{10}}^4)^2}{(16\Pi_{q^{10}}^4 - \Pi_{q^5}^4)\Pi_{q^{12}}\Pi_{q^5}^4}.$$

From the proof of (1.16) above, we have

$$\frac{(16\Pi_{q^{20}}^4 - \Pi_{q^{10}}^4)^2}{(16\Pi_{q^{10}}^4 - \Pi_{q^5}^4)} = \frac{\Pi_{q^5}^4\Pi_{q^{20}}^4}{\Pi_{q^{10}}^4}.$$

From the above two identities, we find that

$$\frac{(16\Pi_{q^{20}}^4 - \Pi_{q^{10}}^4)^2}{(5\Pi_{q^{10}} - \Pi_{q^2})(\Pi_{q^{10}} - \Pi_{q^2})^5} = \frac{\Pi_{q^{20}}^4}{\Pi_{q^2}\Pi_{q^{10}}}.$$

Changing q to $q^{1/2}$, we obtain (1.17).

We now introduce some terminologies related to modular equations and list few related results which will be used in the verification of (1.10) and (1.11).

If $0 < \alpha, \beta < 1$, and the equality

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)}$$

holds, then any relationship between α and β induced by the above equation is called a modular equation of degree n. In such equations, we say that β is of degree n over α. We define the multiplier m connecting α and β by

$$m = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)}.$$

Following theorem due to Ramanujan serves as a bridge between the theory of modular equations and the theory of theta functions:

THEOREM 3.1. *Suppose* $0 < \alpha < 1$, $y = \pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}$, *and* $q = e^{-y}$ *then* ${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right) = z$, *where* $z = \varphi^2(q)$.

For a proof of the above, see [4, p. 101]. We require following theorems, which are due to Ramanujan:

Theorem 3.2 ([4, p. 123]). *If α, q and z are as defined in Theorem 3.1, then*

(1) $\psi(q) = \sqrt{\frac{1}{2} z \left(\frac{\alpha}{q}\right)^{1/8}}$,

(2) $\psi(q^2) = \frac{1}{2}\sqrt{z}\left(\frac{\alpha}{q}\right)^{1/4}$.

Theorem 3.3 ([4, p. 232]). *If β is of degree 3 over α, and m is the multiplier connecting α and β, then*

(1) $\left(\frac{\beta^3}{\alpha}\right)^{1/8} = \frac{m-1}{2}$.

(2) $\left(\frac{\alpha^3}{\beta}\right)^{1/8} = \frac{3+m}{2m}$.

Now, we will move to the verification of (1.10) and (1.11).

Verification of (1.10). From Theorem 3.2 and Theorem 3.3, it is easy to see that

$$\frac{\psi^4(q)}{q\psi^4(q^3)} = \frac{m(3+m)}{m-1}.$$

Hence, we have

$$\frac{\psi^4(q)}{q\psi^4(q^3)} + 18 - 27\frac{q\psi^4(q^3)}{\psi^4(q)} = \frac{24m^3 + m^4 + 18m^2 - 27}{m(m-1)(3+m)}. \tag{3.1}$$

From Theorem 3.2, we have

$$\frac{\psi^8(q)}{q\psi^8(q^2)} = \frac{16}{\alpha}.$$

Also from Theorem 3.1 and 3.2, we have

$$\alpha = \frac{(3+m)^3(m-1)}{16m^3}, \qquad \text{and} \qquad \frac{\psi^4(q^2)}{\psi^2(q)\psi^2(q^3)} = \frac{(3+m)^2}{16m}.$$

Thus

$$\frac{\psi^4(q^2)}{\psi^2(q)\psi^2(q^3)}\left(\frac{\psi^8(q)}{q\psi^8(q^2)} + 16\right) = \frac{(3+m)^2}{m}\left[\frac{16m^3}{(3+m)^3(m-1)} + 1\right],$$

which on simplification yields

$$\frac{\psi^4(q^2)}{\psi^2(q)\psi^2(q^3)}\left(\frac{\psi^8(q)}{q\psi^8(q^2)} + 16\right) = \frac{24m^3 + m^4 + 18m^2 - 27}{m(m-1)(3+m)}. \tag{3.2}$$

From (3.1) and (3.2), the verification of (1.10) is completed.

Verification of (1.11). From Theorem 3.2 and Theorem 3.3, it is easy to see the following identities

$$\frac{\psi^4(q)}{q\psi^4(q^3)} = \frac{m(3+m)}{m-1}.$$

We have

$$\frac{\psi^4(q)}{q\psi^4(q^3)} + 18 - 27\frac{q\psi^4(q^3)}{\psi^4(q)} = \frac{m(3+m)}{m-1} - 6 - \frac{m-1}{m(3+m)}.$$

Which on simplification yields

$$\frac{\psi^4(q)}{q\psi^4(q^3)} - 6 - 3\frac{q\psi^4(q^3)}{\psi^4(q)} = \frac{m^4 - 6m^2 + 24m - 3}{m(m-1)(3+m)}. \tag{3.3}$$

From Theorem 3.2, we obtain

$$\frac{\psi^8(q^3)}{q^3\psi^8(q^6)} = \frac{16}{\beta}.$$

Also from Theorem 3.1 and 3.2, we have

$$\beta = \frac{(m-1)^3(3+m)}{16m} \qquad \text{and} \qquad \frac{q^2\psi^4(q^6)}{\psi^2(q)\psi^2(q^3)} = \frac{(m-1)^2}{16m}.$$

Hence

$$\frac{q^2\psi^4(q^6)}{\psi^2(q)\psi^2(q^3)}\left(\frac{\psi^8(q^3)}{q^3\psi^8(q^6)} + 16\right) = \frac{(m-1)^2}{m}\left(\frac{16m}{(m-1)^3(3+m)} + 1\right),$$

which on simplification yields

$$\frac{q^2\psi^4(q^6)}{\psi^2(q)\psi^2(q^3)}\left(\frac{\psi^8(q^3)}{q^3\psi^8(q^6)} + 16\right) = \frac{m^4 - 6m^2 + 24m - 3}{m(m-1)(3+m)}. \tag{3.4}$$

From (3.3) and (3.4), verification of (1.11) is completed.

4. Proofs of Lambert series identities

Proof of (1.19). It is easy to observe from the definition of P_k's that

$$\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2} - 2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2} = \frac{1}{24}(1-P_1) - \frac{2}{24}(1-P_2)$$
$$= \frac{1}{24}(-1 + 2P_2 - P_1).$$

(1.19) now follows easily from (2.28) and the fact that $\varphi(q) = \frac{\psi(q)^2}{\psi(q^2)}$.

Proof of (1.20). Ramanujan has recorded in [4, p. 460],

$$1+12\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-36\sum_{k=1}^{\infty}\frac{kq^{3k}}{1-q^{3k}}=\left\{\frac{\psi^4(q)+3q\psi^4(q^3)}{\psi(q)\psi(q^3)}\right\}^2.$$

This is equivalent to (1.20).

Proof of (1.21). In [4, p. 114], Ramanujan has recorded the following identity:

$$\varphi^4(q)=1+8\sum_{k=1}^{\infty}\frac{kq^k}{1+(-q)^k},$$

which is same as

$$\varphi^4(q)=\frac{1}{3}(-P_1+4P_4).$$

By the definition of P_k, we have

$$\begin{aligned}\sum\frac{q^n}{(1-q^n)^2}-4\sum\frac{q^{4n}}{(1-q^{4n})^2}&=\frac{1}{24}(1-P_1-4+4P_4)\\&=\frac{1}{24}(-3+3\varphi^4(q))\\&=\frac{1}{8}\left(\frac{\psi^8(q)}{\psi^4(q^2)}-1\right),\end{aligned}$$

where we have used $\varphi(q)=\frac{\psi(q)^2}{\psi(q^2)}$. Which completes the proof of (1.21).

Proof of (1.23). Changing q to q^4, followed by setting $a=q$ and $b=q^2$ in (2.26), and then expanding the bilateral series, we obtain

$$\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-2\sum_{n=1}^{\infty}\frac{q^{4n-2}}{(1-q^{4n-2})^2}=\Pi_{q^2}^2,$$

where we have used, $\frac{f_4^6}{\varphi^2(-q^2)}=\psi^4(q^2)$. Now using the fact that

$$\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}=\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n},$$

we observe that

$$\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-2\sum_{n=1}^{\infty}\frac{q^{4n-2}}{(1-q^{4n-2})^2}=\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}$$
$$-2\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}+2\sum_{n=1}^{\infty}\frac{nq^{4n}}{1-q^{4n}}.$$

Now using the facts that

$$\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-2\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}=\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}$$

and

$$\frac{q}{1-q^2}=\frac{q}{1-q}-\frac{q^2}{1-q^2},$$

in the above equation, we obtain

$$\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-2\sum_{n=1}^{\infty}\frac{q^{4n-2}}{(1-q^{4n-2})^2}=\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{4n-2}},$$

which completes the proof of (1.23).

Before going to the proof of (1.25) and (1.26) we shall first establish the following equality

$$\frac{\frac{\Pi_{q^5}^2}{\Pi_{q^{10}}^2}+16\frac{\Pi_{q^{10}}^2}{\Pi_{q^5}^2}}{\frac{\Pi_q}{\Pi_{q^5}}-4-\frac{\Pi_{q^5}}{\Pi_q}}=\sqrt{\frac{\Pi_q^3}{\Pi_{q^5}^3}-2\frac{\Pi_q^2}{\Pi_{q^5}^2}+5\frac{\Pi_q}{\Pi_{q^5}}}. \tag{4.1}$$

From (1.17) and the formula $(a+b)^2=(a-b)^2+4ab$ it follows that,

$$[16\Pi_{q^{10}}^4-\Pi_{q^5}^4]^2=\frac{\Pi_{q^{10}}^4}{\Pi_q\Pi_{q^5}}\{[5\Pi_{q^5}-\Pi_q][\Pi_{q^5}-\Pi_q]^5\}+64\Pi_{q^{10}}^4\Pi_{q^5}^4.$$

Expanding the right hand side and then factoring it yields,

$$(16\Pi_{q^{10}}^4-\Pi_{q^5}^4)^2=\left(\frac{\Pi_q^2}{\Pi_{q^5}^2}-4-\frac{\Pi_{q^5}^2}{\Pi_q^2}\right)^2\left(\frac{\Pi_q^6}{\Pi_{q^5}^6}-2\frac{\Pi_q^4}{\Pi_{q^5}^4}+5\frac{\Pi_q^2}{\Pi_{q^5}^2}\right).$$

Proof of (1.26). Changing q to q^{10} in (2.26) and setting $b=q^5$, we observe that

$$\sum_{n=-\infty}^{\infty}\left[\frac{aq^{10n}}{(1-aq^{10n})^2}-\frac{q^{10n+5}}{(1-q^{10n+5})^2}\right]=a\psi^4(q^5)\frac{f^2\left(-aq^5,-\frac{q^5}{a}\right)}{f^2\left(-a,-\frac{q^{10}}{a}\right)}.$$

Setting $a = q$ first and then $a = q^3$ in the above equation, and adding the resultant identities, we obtain

$$\sum_{n=1}^{\infty} \frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5\sum_{n=1}^{\infty} \frac{q^{10n-5}}{(1-q^{10n-5})^2} = \frac{\Pi_{q^5}^2}{q^{3/2}}\left(\frac{f^2(-q^4,-q^6)}{f^2(-q,-q^9)} + q^2\frac{f^2(-q^2,-q^8)}{f^2(-q^3,-q^7)}\right). \tag{4.2}$$

Setting $a = b = c = d = q^5$ and $n = -q$ in (2.27), we see that

$$\varphi^2(q^5)f^2(-q^4,-q^6) - f^2(q^4,q^6)\varphi^2(-q^5) = -4q^4\psi^2(q^{10})f^2(-q,-q^9),$$

where we have used $f(1,q) = 2\psi(q)$. This implies

$$\frac{f^2(-q^4,-q^6)}{f^2(-q,-q^9)} = \frac{\varphi^2(-q^5)}{\varphi^2(q^5)}\frac{f^2(q^4,q^6)}{f^2(-q,-q^9)} - 4q^4\frac{\psi^2(q^{10})}{\varphi^2(q^5)}.$$

Similarly by setting $a = b = c = d = q^5$ and $n = -q^3$ in (2.27), we obtain

$$q^2\frac{f^2(-q^2,-q^8)}{f^2(-q^3,-q^7)} = q^2\frac{\varphi^2(-q^5)}{\varphi^2(q^5)}\frac{f^2(q^2,q^8)}{f^2(-q^3,-q^7)} - 4q^4\frac{\psi^2(q^{10})}{\varphi^2(q^5)}.$$

Adding the above two, and using (4.2), we obtain

$$\frac{1}{\Pi_{q^5}^2}\left(\sum_{n=1}^{\infty} \frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5\sum_{n=1}^{\infty} \frac{q^{10n-5}}{(1-q^{10n-5})^2}\right) = \frac{1}{q^{3/2}}\left(\frac{\varphi^2(-q^5)}{\varphi^2(q^5)}\left\{\frac{f^2(q^4,q^6)}{f^2(-q,-q^9)} + q^2\frac{f^2(q^2,q^8)}{f^2(-q^3,-q^7)}\right\} - 8q^4\frac{\psi^2(q^{10})}{\varphi^2(q^5)}\right).$$

Now using (2.34), (2.36) and the identity $a^2 + b^2 = (a-b)^2 + 2ab$, we observe that

$$\frac{1}{\Pi_{q^5}^2}\left(\sum_{n=1}^{\infty} \frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5\sum_{n=1}^{\infty} \frac{q^{10n-5}}{(1-q^{10n-5})^2}\right) = \frac{1}{q^{3/2}}\frac{\varphi^2(-q^5)}{\varphi^2(q^5)}\left\{\frac{(\psi^2(q^2) - q^2\psi^2(q^{10}))^2}{\psi^4(-q^5)} + 2q\frac{(\psi^2(q^2) - q^2\psi^2(q^{10}))(\psi^2(q) - q\psi^2(q^5))}{\varphi^2(-q^5)\psi^2(q)}\right\} - 8q^{5/2}\frac{\psi^2(q^{10})}{\varphi^2(q^5)}.$$

Now simplifying the above using $\psi(q^2)\varphi^2(q) = \psi^2(q)$, it reduces to

$$\frac{1}{\Pi_{q^5}^2}\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n=1}^{\infty}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\right)$$
$$=\frac{q^{5/2}\psi^4(q^{10})}{\psi^4(q^5)}\left\{\left(\frac{\psi^2(q^2)}{q^2\psi^2(q^{10})}-1\right)^2\right.$$
$$\left.+2\left(\frac{\psi^2(q^2)}{q^2\psi^2(q^{10})}-1\right)\left(\frac{\psi^2(q)}{q^2\psi^2(q^5)}-1\right)-8\right\},$$

which is nothing but

$$\frac{1}{\Pi_{q^5}^2}\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n=1}^{\infty}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\right)$$
$$=\frac{\Pi_{q^{10}}^2}{\Pi_{q^5}^2}\left(\left(\frac{\Pi_{q^2}}{\Pi_{q^{10}}}-1\right)^2+2\left(\frac{\Pi_{q^2}}{\Pi_{q^{10}}}-1\right)\left(\frac{\Pi_q}{\Pi_{q^5}}-1\right)-8\right). \tag{4.3}$$

Now, by setting $a=b=c=d=q^5$ and $n=-q^2$ in (2.27), we obtain

$$q^3\frac{f^2(-q^2,-q^8)}{f^2(-q^3,-q^7)}=-\frac{\varphi^2(q^5)}{4\psi^2(q^{10})}+\frac{\varphi^2(-q^5)}{4\psi^2(q^{10})}\frac{f^2(q^3,q^7)}{f^2(-q^3,-q^7)}.$$

The above is nothing but (1.33).

Similarly by setting $a=b=c=d=q^5$ and $n=-q^2$ in (2.27), we obtain

$$q\frac{f^2(-q^4,-q^6)}{f^2(-q,-q^9)}=-\frac{\varphi^2(q^5)}{4\psi^2(q^{10})}+\frac{\varphi^2(-q^5)}{4\psi^2(q^{10})}\frac{f^2(q,q^9)}{f^2(-q,-q^9)}.$$

Adding the above two equation and substituting the resultant one in (4.2), we see that

$$\frac{1}{\Pi_{q^5}^2}\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n=1}^{\infty}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\right)$$
$$=\frac{1}{q^{5/2}}\left\{-\frac{\varphi^2(q^5)}{4\psi^2(q^{10})}+\frac{\varphi^2(-q^5)}{4\psi^2(q^{10})}\left(\frac{f^2(q^3,q^7)}{f^2(-q^3,-q^7)}+\frac{f^2(q,q^9)}{f^2(-q,-q^9)}\right)\right\}.$$

Now using (2.33), (2.35), (2.39) and the formula $a^2+b^2=(a-b)^2+2ab$ in the above equation, we obtain

$$\frac{1}{\Pi_{q^5}^2}\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n=1}^{\infty}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\right)$$
$$=-\frac{1}{q}\frac{\varphi^2(q^5)}{2\psi^2(q^{10})}+\frac{1}{q}\frac{\varphi^2(-q^5)}{4\psi^2(q^{10})}$$
$$\times\left(\frac{4q^2\psi^4(q^5)}{\varphi^4(-q^{10})}\frac{(\psi^2(q)-q\psi^2(q^5))^2}{(\psi^2(q^2)-q^2\psi^2(q^{10}))^2}+2\frac{(\psi^2(q)-q\psi^2(q^5))^2}{\varphi^2(-q^5)(\psi^2(q^2)-q^2\psi^2(q^{10}))}\right).$$

Simplifying the above equation using $\varphi(q)\psi(q^2)=\psi^2(q)$, we obtain

$$\frac{1}{\Pi_{q^5}^2}\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n=1}^{\infty}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\right)$$
$$=\frac{1}{q}\left\{-\frac{\psi^4(q^5)}{2\psi^4(q^{10})}+\frac{\psi^4(q^5)}{4\psi^4(q^{10})}\left\{4\frac{\left(\frac{\psi^2(q)}{q\psi^2(q^5)}-1\right)^2}{\left(\frac{\psi^2(q^2)}{q^2\psi^2(q^{10})}-1\right)^2}+2\frac{\left(\frac{\psi^2(q)}{q\psi^2(q^5)}-1\right)^2}{\left(\frac{\psi^2(q^2)}{q^2\psi^2(q^{10})}-1\right)}\right\}\right\},$$

which upon conversion to Π_q's becomes,

$$\frac{1}{\Pi_{q^5}^2}\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n=1}^{\infty}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\right)$$
$$=\frac{\Pi_{q^5}^2}{\Pi_{q^{10}}^2}\left\{-\frac{1}{2}+\frac{\left(\frac{\Pi_q}{\Pi_{q^5}}-1\right)^2}{\left(\frac{\Pi_{q^2}}{\Pi_{q^{10}}}-1\right)^2}+\frac{1}{2}\frac{\left(\frac{\Pi_q}{\Pi_{q^5}}-1\right)^2}{\left(\frac{\Pi_{q^2}}{\Pi_{q^{10}}}-1\right)}\right\}. \tag{4.4}$$

The above is nothing but (1.34). From [13, p. 33], we have

$$P^2Q-4PQ+5Q-P^2-Q^2=0,$$

where $P=\frac{\Pi_q}{\Pi_{q^5}}$ and $Q=\frac{\Pi_{q^2}}{\Pi_{q^{10}}}$. This implies

$$Q=\frac{(P^2-4P+5)\pm\sqrt{(P^2-4P+5)^2-4P^2}}{2}.$$

Since $q^2Q(q)$ at $q=0$ takes the value 1, we must choose positive sign in the above equation. Also, it is easy to see that

$$Q-1=\frac{(P^2-4P+3)+\sqrt{(P^2-4P+5)^2-4P^2}}{2} \tag{4.5}$$

and

$$\frac{P-1}{Q-1} = \frac{(P^2-4P+3) - \sqrt{(P^2-4P+5)^2 - 4P^2}}{8}. \tag{4.6}$$

Hence, we have

$$\frac{1}{\Pi_{q^5}^2}\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5\sum_{n=1}^{\infty}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\right) = \frac{\Pi_{q^{10}}^2}{\Pi_{q^5}^2}\{(Q-1)^2 + 2(Q-1)(P-1) - 8\}, \tag{4.7}$$

and

$$\frac{1}{\Pi_{q^5}^2}\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2} - 5\sum_{n=1}^{\infty}\frac{q^{10n-5}}{(1-q^{10n-5})^2}\right) = \frac{\Pi_{q^5}^2}{\Pi_{q^{10}}^2}\left\{-\frac{1}{2} + \frac{(P-1)^2}{(Q-1)^2} + \frac{1}{2}\frac{(P-1)^2}{(Q-1)}\right\}. \tag{4.8}$$

Using (4.5) and (4.6) in (4.7) and (4.8) and then multiplying the resulting equations, we obtain (1.26).

Proofs of (1.24) and (1.27). The following identity can be found in [6, p. 197]:

$$-P_1 + P_2 + 3P_3 - 3P_6 = 24q\psi^2(q)\psi^2(q^3). \tag{4.9}$$

This can be proved from (2.26) as done in [20, p. 88]. By the definition of P_n, the above can be written as

$$\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2} - 3\sum_{n=1}^{\infty}\frac{q^{6n-3}}{(1-q^{6n-3})^2} = \Pi_q\Pi_{q^3},$$

which is nothing but (1.24). Changing q to q^3 in the above equation and then adding 3 times of the resulting equation to it, we obtain

$$-P_1 + P_2 + 9P_9 - 9P_{18} = 24(q\psi^2(q)\psi^2(q^3) + 3q^3\psi^2(q^3)\psi^2(q^9)).$$

By the definition of P_n, the above equation is same as

$$\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2} - 9\sum_{n=1}^{\infty}\frac{q^{18n-9}}{(1-q^{18n-9})^2} = q\psi^2(q)\psi^2(q^3) + 3q^3\psi^2(q^3)\psi^2(q^9).$$

Hence

$$\frac{1}{\Pi_{q^9}^2}\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-9\sum_{n=1}^{\infty}\frac{q^{18n-9}}{(1-q^{18n-9})^2}\right)=\frac{\Pi_{q^3}}{\Pi_{q^9}}\left(\frac{\Pi_q}{\Pi_{q^9}}+3\right). \quad (4.10)$$

Eliminating $\frac{\chi^3(-q^9)}{q\chi(-q^3)}$ between (2.22) and (2.23), it is easy to see that

$$\frac{\Pi_{q^3}}{\Pi_{q^9}}=\sqrt{\left(\frac{\Pi_q}{\Pi_{q^9}}\right)^{3/2}-3\left(\frac{\Pi_q}{\Pi_{q^9}}\right)+3\left(\frac{\Pi_q}{\Pi_{q^9}}\right)^{1/2}}. \quad (4.11)$$

Using this in the above, we obtain (1.27).

Proof of (1.22). The following can be found in [4, p. 475]:

$$1+3\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-27\sum_{k=1}^{\infty}\frac{kq^{9k}}{1-q^{9k}}$$
$$=\left(\frac{\psi^4(q^3)+3q\psi^2(q)\psi^2(q^9)}{\psi(q)\psi(q^9)}\right)^2\frac{\psi^2(q^3)}{\psi(q)\psi(q^9)}. \quad (4.12)$$

Using the fact that $\Pi_q=q^{1/4}\psi^2(q)$, above equation can be rewritten as

$$\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-9\sum_{k=1}^{\infty}\frac{kq^{9k}}{1-q^{9k}}=\frac{1}{3}\left((\Pi_{q^3}^2+3\Pi_q\Pi_{q^9})^2\frac{\Pi_{q^3}}{(\Pi_q\Pi_{q^9})^{3/2}}-1\right).$$

From the above and (4.10), we have

$$\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}-9\sum_{k=1}^{\infty}\frac{kq^{18k}}{1-q^{18k}}=\frac{1}{3}\left((\Pi_{q^3}^2+3\Pi_q\Pi_{q^9})^2\frac{\Pi_{q^3}}{(\Pi_q\Pi_{q^9})^{3/2}}-1\right)$$
$$-\Pi_{q^3}(\Pi_q+3\Pi_{q^9}).$$

Now employing (1.12), the above can be written as

$$\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}-9\sum_{k=1}^{\infty}\frac{kq^{18k}}{1-q^{18k}}=\frac{1}{3}\left((\Pi_{q^3}^2+3\Pi_q\Pi_{q^9})(\Pi_q+3\Pi_{q^9})\frac{\Pi_{q^3}}{(\Pi_q\Pi_{q^9})}-1\right)$$
$$-\Pi_{q^3}(\Pi_q+3\Pi_{q^9}).$$

Simplifying the above, we obtain (1.22). Now, from (4.12), (1.21) and (1.11), the identity (1.32) follows.

Proof of (1.28). Ramanujan has recorded the following [4, p. 463]:

$$1+6\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-30\sum_{k=1}^{\infty}\frac{kq^{5k}}{1-q^{5k}}=\frac{\psi^4(q)+2q\psi^2(q)\psi^2(q^5)+5q^2\psi^4(q^5)}{\psi(q)\psi(q^5)}\times\sqrt{\psi^4(q)-2q\psi^2(q)\psi^2(q^5)+5q^2\psi^4(q^5)}.$$

Employing (1.26) along with the fact that $\Pi_q=q^{1/4}\psi^2(q)$, we obtain (1.28).

Proof of (1.29). Using the fact that $\Pi_q=q^{1/4}\psi^2(q)$, (4.12) can be rewritten as

$$1+3\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-27\sum_{k=1}^{\infty}\frac{kq^{9k}}{1-q^{9k}}=\left(\frac{\Pi_{q^3}^2}{\Pi_{q^9}^2}\frac{\Pi_{q^9}^{3/2}}{\Pi_q^{1/2}}+3\Pi_q^{1/2}\Pi_{q^9}^{1/2}\right)^2\frac{\Pi_{q^3}}{\Pi_{q^9}}\sqrt{\frac{\Pi_{q^9}}{\Pi_q}}.$$

Now using (4.11) for the term $\frac{\Pi_{q^3}^2}{\Pi_{q^9}^2}$ inside the square and using (4.10) for the term $\frac{\Pi_{q^3}}{\Pi_{q^9}}$ outside the square in the above equation and then simplifying the resulting equation, we obtain (1.29).

Disproving (1.30). Setting $q=0$ in (4.12) yields 1, while at $q=0$, right hand side expression of (1.30) is not defined, which shows that (1.30) must be wrong. We are unable to guess what Gosper might have intended with respect to (1.30).

Proof of (1.31). Ramanujan has recorded the following [4, p. 139]:

$$q\psi^8(q)=\Pi_q^4=\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}. \tag{4.13}$$

Using the facts that

$$\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}=\sum_{n=1}^{\infty}\frac{q^n+4q^{2n}+q^{3n}}{(1-q^n)^4}$$

and

$$\frac{q^n}{1-q^{2n}}=\frac{q^n}{1-q^n}-\frac{q^{2n}}{1-q^{2n}},$$

we find that

$$\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^{2n}}=\sum_{n=1}^{\infty}\frac{q^{2n-1}+4q^{4n-2}+q^{6n-3}}{(1-q^{2n-1})^4}.$$

But,

$$q^{2n-1} + 4q^{4n-2} + q^{6n-3} = q^{2n-1}(1 - q^{2n-1})^2 + 6q^{4n-2}.$$

This implies,

$$\sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^{2n}} = 6 \sum_{n=1}^{\infty} \frac{q^{4n-2}}{(1 - q^{4n-2})^4} + \sum_{n=1}^{\infty} \frac{q^{2n-1}}{(1 - q^{2n-1})^2}. \tag{4.14}$$

(4.13) and (4.14) together imply (1.31).

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References

[1] C. Adiga, B. C. Berndt, S. Bhargava, and G. N. Watson. Chapter 16 of Ramanujan's Second Notebook: Theta-Functions and q-Series. Amer Mathematical Society, 1985.

[2] W. N. Bailey. A further note on two of Ramanujan's formulae. Quart. J. Math. Oxford Ser. (2), 3:158–160, 1952.

[3] N. D. Baruah and R. Barman. Certain theta-function identities and Ramanujan's modular equations of degree 3. Indian J. Math., 48(1):113–133, 2006.

[4] B. C. Berndt. Ramanujan's Notebooks: Part III. Springer New York, 1991.

[5] S. Bhargava, K. R. Vasuki, and K. R. Rajanna. On some Ramanujan identities for the ratios of eta-functions. Ukrainian Math. J., 66(8):1131–1151, 2015. Reprint of Ukraïn. Mat. Zh. 66 (2014), no. 8, 1011–1028.

[6] S. Cooper. Ramanujan's theta functions. Springer, Cham, 2017.

[7] M. El Bachraoui. On the Gosper's q-constant Π_q. Acta Math. Sin. (Engl. Ser.), 34(11):1755–1764, 2018.

[8] M. El Bachraoui. On series identities of Gosper and integrals of Ramanujan theta function $\psi(q)$. Proc. Amer. Math. Soc., 147(10):4451–4464, 2019.

[9] R. W. Gosper. Experiments and discoveries in q-trigonometry. In Symbolic computation, number theory, special functions, physics and combinatorics (Gainesville, FL, 1999), volume 4 of Dev. Math., pages 79–105. Kluwer Acad. Publ., Dordrecht, 2001.

[10] K. N. Harshitha, K. R. Vasuki, and M. V. Yathirajsharma. Trigonometric sums through Ramanujan's theory of theta functions. The Ramanujan Journal (To appear).

[11] B. He. Proofs for certain Π_q-identities of W. Gosper. J. Math. Anal. Appl., 492(2):124486, 14, 2020.

[12] B. He, F. He, and H. Zhai. Proof for a q-trigonometric identity of Gosper. Mediterr. J. Math., 17(5):Art. 161, 8, 2020.
[13] A. A. A. Kahtan. Some studies in q-series and modular equations influenced by works of Ramanujan. A thesis submitted to the University of Mysore, 2010.
[14] S.-Y. Kang. Some theorems on the Rogers-Ramanujan continued fraction and associated theta function identities in Ramanujan's lost notebook. Ramanujan J., 3(1):91–111, 1999.
[15] S. Ramanujan. Notebooks (2 volumes). Tata Institute of Fundamental Research Bombay, 1957.
[16] L.-C. Shen. On some modular equations of degree 5. Proc. Amer. Math. Soc., 123(5):1521–1526, 1995.
[17] K. R. Vasuki. Some studies in the theory of special functions and number theory. A thesis submitted to the University of Mysore, 2000.
[18] K. R. Vasuki, G. Sharath, and Rajanna. Two modular equations for square of the cubic-functions with applications. Note Di Mathematica, 30(2):61–71, 2010.
[19] K. R. Vasuki and R. G. Veeresha. Ramanujan's Eisenstein series of level 7 and 14. J. Number Theory, 159:59–75, 2016.
[20] R. G. Veeresha. An elementary approach to Ramanujan's modular equations of degree 7 and its applications. A thesis submitted to the University of Mysore, 2015.

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