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Lyubeznik numbers of almost complete intersection and linked ideals

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Abstract. In this work, we examine the Lyubeznik numbers of squarefree monomial ideals that are linked. Also we study these numbers for almost complete intersection ideals.

1. Introduction

In this paper, all rings are commutative and Noetherian. Let A be a local ring containing a field, we consider $\hat{A}$ the completion of A with respect to its maximal ideal. By the Cohen's Structure Theorem, we may write $\hat{A} \cong R/I$ where $(R, \mathfrak{m}, \Bbbk)$ is a complete regular local ring of dimension n and I is an ideal of R. In 1993, Huneke–Sharp [10] and Lyubeznik [11] showed that Bass numbers of local cohomology modules $H_I^i(R)$ for $i \geq 0$ are finite. Followed by that, Lyubeznik introduced some invariants of A called the *Lyubeznik numbers* of A. These numbers were defined as:

$$\lambda_{i,j}(A) = \dim_{\Bbbk} \operatorname{Ext}_R^i(\Bbbk, H_I^{n-j}(R)) = \mu_0(\mathfrak{m}, H_{\mathfrak{m}}^i(H_I^{n-j}(R))) \qquad \forall i, j \in \mathbb{N},$$

depend only on A, i and j. We have the following well known properties for them:

1- For $i > j$ or $j > \dim A$, $\lambda_{i,j}(A) = 0$.

2- $\lambda_{\dim A, \dim A}(A) \neq 0$.

3- Euler characteristic formula [1]

$$\sum_{0 \leq i,j \leq \dim A} (-1)^{i-j} \lambda_{i,j}(A) = 1.$$

4- When A is Cohen-Macaulay, $\lambda_{\dim A, \dim A}(A) = 1$.

By applying these properties we have an upper triangular matrix contains all nonzero Lyubeznik numbers. This matrix is known as *Lyubeznik table* of A

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and it is shown by

$$\Lambda(A) = \begin{bmatrix} \lambda_{0,0} & \cdots & \lambda_{0,\dim A} \\ & \ddots & \vdots \\ & & \lambda_{\dim A,\dim A} \end{bmatrix}.$$

$\Lambda(A)$ is trivial whenever $\lambda_{\dim A,\dim A}(A) = 1$ and the other numbers vanish. These numbers also have some interesting connections with étale cohomology, singular [4, 8] and simplicial complexes [2, 3].

In 1974, the linkage theory of algebraic varieties was introduced by Peskine and Szpiro by using sheaves, duality, and homological tools [17]. This theory has been a very useful tool in the classification of ideals and algebraic varieties. By using linkage, sometimes, we can compute the invariants of an ideal such as height, projective dimension, Hilbert function and other numerical invariants, or find examples of ideals with given invariants. One interesting question is to find the Lyubeznik numbers by using linkage.

In [14], the authors studied the Lyubeznik numbers of linked ideals. Here we will present some new results on these numbers. More precisely, the outline of this paper is as follows:

In Section 2, we will provide the required definitions.

In Section 3, we work on almost complete intersection ideals and their Lyubeznik numbers.

In Section 4, we investigate the Lyubeznik numbers of squarefree monomial ideals which are linked.

2. Preliminaries

In this section, we bring some definitions that we need for the rest of the paper.

Definition 1. A local ring $(A, \mathfrak{n})$ is said to be cohomologically full (C.F for short) if, for every surjection from a local ring $(B, \mathfrak{n}')$ to A such that $A/\sqrt{0} = B/\sqrt{0}$, the map $H^i_{\mathfrak{n}'}(B) \to H^i_{\mathfrak{n}}(A)$ is surjective for all i.

Example 1 ([6, Remark 2.4]). *Cohen-Macaulay local rings, F-pure local rings in positive characteristics and Stanley-Reisner rings are C.F. Also if I is a homogeneous ideal of a standard graded polynomial ring R whose* $\mathrm{in}_{\prec}(I)$ *is radical with respect to some monomial order, then R/I is C.F* [5, Proposition 3.3].

Definition 2. Let I and J be two ideals of a Gorenstein local ring $(R, \mathfrak{m})$. The ideal I is linked to J by a complete intersection ideal $\mathfrak{c}$ of height

g, if I and J are of pure height g, $\mathfrak{c} \subseteq I \cap J$ and $\mathfrak{c} : I = J$ and $\mathfrak{c} : J = I$. We use the notation $I \sim^{\mathfrak{c}} J$ or $I \sim J$ for two linked ideals.

We say that I is evenly linked to J if there exists an odd integer t and ideals $I_1, \ldots, I_t$ of R such that $I \sim I_1, I_1 \sim I_2, \ldots, I_t \sim J$.

Definition 3. Let $(R, \mathfrak{m})$ be a local ring and N be a Noetherian R-module. N is generalized Cohen-Macaulay if for $i < \dim N$, the i-th local cohomology module $H^i_{\mathfrak{m}}(N)$ has finite length.

Definition 4 ([18]). Let $(R, \mathfrak{m})$ be an n-dimensional Gorenstein local ring and N be an R-module.

1- N is canonically Cohen-Macaulay if and only if $K_N = \mathrm{Ext}_R^{n-\dim N}(N, R)$ is Cohen-Macaulay.

2- N is sequentially Cohen-Macaulay if and only if for all $0 \le i \le \dim N$, $\mathrm{Ext}_R^{n-i}(N, R)$ is zero or an i-dimensional Cohen-Macaulay module.

Definition 5. Let R be a ring and I an ideal of R. The unmixed part of I, denoted by I^u, is the intersection of those primary components Q of I with $\mathrm{ht}(Q) = \mathrm{ht}(I)$.

Definition 6. Let R be a ring and I an ideal of R. I is an almost complete intersection (A.C.I for short) ideal if $\mu(I) = \mathrm{ht}(I) + 1$, where $\mu(I)$ is the cardinality of a minimal set of generators of I.

Definition 7 ([20]). Let $R = \Bbbk[x_1, \ldots, x_n]$ be a positively graded polynomial ring over a field $\Bbbk$. Suppose that N is a finitely graded R-module. The Castelnuovo-Mumford regularity of N is defined as

$$\mathrm{reg}(N) = \max\{a^i(N) + i : i \ge 0\}, \tag{1}$$

where $a^i(N) = \max\{t : H^i_{\mathfrak{m}}(N)_t \ne 0\}$ and $\mathfrak{m} = (x_1, \ldots, x_n)$. Also, the regularity index of N is

$$r(N) = \min\{t \in \mathbb{Z} : h_N(i) = p_N(i),\ i \ge t\},$$

$h_N(-)$ and $p_N(-)$ are the Hilbert function and the Hilbert polynomial of N respectively.

Notation 1. *By $(-)^v$ we mean $\Bbbk$-dual of N as*

$$N^v := \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\Bbbk}(N_{-j}, \Bbbk).$$

3. Lyubeznik numbers

In this section, we will study A.C.I ideals and their Lyubeznik numbers.

LEMMA 1. *Let $(R, \mathfrak{m})$ be a regular local ring and I be an A.C.I ideal of R. Assume that R/I is a C.F ring which is not Cohen-Macaulay. Then*

$$\operatorname{ara}(I) = \operatorname{pd} R/I.$$

PROOF. Since R/I is C.F, by using [6, Proposition 2.6], $\operatorname{cd}(R, I) \geq \operatorname{pd} R/I$. In addition, $\operatorname{ht}(I) + 1 \leq \operatorname{pd} R/I$ as R/I is not Cohen-Macaulay. So,

$$\operatorname{pd}(R/I) \leq \operatorname{cd}(R, I) \leq \operatorname{ara}(I) \leq \mu(I) = \operatorname{ht}(I) + 1 \leq \operatorname{pd}(R/I),$$

the second inequality comes from [9, Theorem 5.4]. Therefore $\operatorname{ht}(I) + 1 = \operatorname{pd} R/I = \operatorname{ara}(I)$.

Lemma 1 is not true without this assumption that S/I is C.F:

EXAMPLE 2. *Let $R = \Bbbk[x_1, \ldots, x_6]_{(x_1, \ldots, x_6)}$ and I be a homogeneous ideal generated by*

$$x_4^3 + x_5^3 + x_6^3,\ x_4^2x_1 + x_5^2x_2 + x_6^2x_3,\ x_1^2x_4 + x_2^2x_5 + x_3^2x_6,\ x_1^3 + x_2^3 + x_3^3.$$

R/I is a non-Cohen-Macaulay ring of dimension 3 and $4 = \mu(I) = \operatorname{ht}(I) + 1$. We see that $\operatorname{pd}(R/I) = 6$ but $\operatorname{ara}(I) \leq \mu(I) = 4$. R/I is not C.F because $\operatorname{cd}(R, I) < \operatorname{pd} R/I$ ([6, Proposition 2.6]).

PROPOSITION 1. *Let $(R, \mathfrak{m})$ be a regular local ring, and I be an ideal of R such that R/I is C.F. If $\mu(I) = \operatorname{pd} R/I$, then*

$$\operatorname{ara}(I) = \operatorname{pd} R/I.$$

PROOF. By [6, Proposition 2.6], $\operatorname{pd} R/I \leq \operatorname{cd}(R, I)$. So $\operatorname{pd} R/I \leq \operatorname{ara}(I) \leq \mu(I) = \operatorname{pd} R/I$. Thus $\operatorname{pd} R/I = \operatorname{ara}(I)$.

Proposition 1 fails if R/I is not C.F:

EXAMPLE 3. *Let $R = \Bbbk[x_1, \ldots, x_6]_{(x_1, \ldots, x_6)}$ and I be a homogeneous ideal generated by*

$$x_4^3 + x_5^3 + x_6^3,\ x_4^2x_1 + x_5^2x_2 + x_6^2x_3,\ x_1^2x_4 + x_2^2x_5 + x_3^2x_6,\ x_1^3 + x_2^3 + x_3^3,$$
$$x_5x_3 - x_6x_2,\ x_6x_1 - x_4x_3.$$

R/I is a non-Cohen-Macaulay ring of dimension 3. We have $\mu(I) = \operatorname{pd}(R/I) = 6$ and $\operatorname{ara}(I) \leq 5$, because

$$\sqrt{(x_4^3 + x_5^3 + x_6^3, x_4^2x_1 + x_5^2x_2 + x_6^2x_3, x_1^2x_4 + x_2^2x_5 + x_3^2x_6, x_1^3 + x_2^3 + x_3^3, x_5x_3 - x_6x_2)}$$
$$= \sqrt{I}.$$

R/I is not C.F because $\operatorname{cd}(R, I) < \operatorname{pd} R/I$.

Theorem 1. *Let $(R, \mathfrak{m})$ be a regular local ring and I be an A.C.I ideal of R such that R/I is C.F. Then, R/I is sequentially Cohen-Macaulay.*

Proof. If R/I is Cohen-Macaulay, then R/I is sequentially Cohen-Macaulay. Assume that R/I is not Cohen-Macaulay. Let J be an ideal linked to I^u, by [7, Lemma 2 and Theorem 3], $\mathrm{pd}(R/I) = \mathrm{pd}(R/J) + 1$. We observe also that $\mathrm{pd}(R/I) = \mathrm{ara}(I)$ by Lemma 1. Hence

$$\mathrm{ht}(I) < \mathrm{pd}(R/I) = \mathrm{ara}(I) \leq \mu(I) = \mathrm{ht}(I) + 1.$$

Then, $\mathrm{pd}(R/I) = \mathrm{ht}(I) + 1$. It concludes that R/J is Cohen-Macaulay. As I^u is linked to J and Cohen-Macaulay property is preserved under linkage, I^u is also Cohen-Macaulay. So the assertion follows from [23, Theorem 1.6].

Corollary 1. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field $\Bbbk$ of positive characteristic p. Let I be an A.C.I ideal and R/I is C.F. Then $\Lambda(R/I)$ is trivial.*

Proof. By Theorem 1, R/I is sequentially Cohen-Macaulay. So we have the result by [1, Corollary 3.4].

Corollary 2. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field and I be an A.C.I ideal of R. If R/I is C.F, then*

$$\lambda_{d,d}(R/I) = 1.$$

$d = \dim R/I$.

Proof. It is easy to see that $H_I^{n-d}(R) \cong H_{I^u}^{n-d}(R)$. So, by the definition of Lyubeznik numbers, $\lambda_{d,d}(R/I) = \lambda_{d,d}(R/I^u)$. Since by Theorem 1, R/I^u is Cohen-Macaulay, $\lambda_{d,d}(R/I^u) = 1$.

4. Lyubeznik numbers under linkage

Let $R = \Bbbk[x_1, \ldots, x_n]$ be the standard graded polynomial ring over a field $\Bbbk$ with the unique homogeneous maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$ and $\omega_R = R(-n)$ its canonical module. In [14], the authors studied the Lyubeznik numbers of evenly linked ideals I and J that are canonically Cohen-Macaulay or generalized Cohen-Macaulay and showed that under some conditions we have, $\Lambda(R/I) = \Lambda(R/J)$. In this section, we will bring some other results of these numbers by using the fact that for any radical monomial ideal L of R,

$$\lambda_{i,j}(R/L) = \dim_{\Bbbk} \mathrm{Ext}_R^{n-i}(\mathrm{Ext}_R^{n-j}(R/L, \omega_R), \omega_R)_0$$

[21, Corollary 3.10].

Theorem 2. *Let I and J be two squarefree monomial ideals of R with* $\dim R/I = d$. *Assume that* $\mathfrak{c} = (c_1, \ldots, c_{n-d})$ *and* $\mathfrak{c}' = (c'_1, \ldots, c'_{n-d})$ *are two complete intersection ideals of R such that* $I \sim^{\mathfrak{c}} L$ *and* $J \sim^{\mathfrak{c}'} L$. *If* $\sum_{i=1}^{n-d} \deg(c_i) = \sum_{i=1}^{n-d} \deg(c'_i)$, *then*

$$\Lambda(R/I) = \Lambda(R/J).$$

Proof. As $\mathfrak{c}$ and $\mathfrak{c}'$ are complete intersection ideals, by [16, Lemma 2.3],

$$r(R/\mathfrak{c}) = r(R/\mathfrak{c}') = \sum_{i=1}^{n-d} \deg(c_i) - n + 1.$$

Hence, [12, Theorem 5.3.1] and [13, Example 5.7] imply that

$$H^j_{\mathfrak{m}}(R/J) \cong H^j_{\mathfrak{m}}(R/I), \qquad \forall 1 \le j < d.$$

So, by duality theorem [19, Theorem 4.14, p. 58],

$$\mathrm{Ext}^{n-i}_R(\mathrm{Ext}^{n-j}_R(R/I, \omega_R), \omega_R) \cong \mathrm{Ext}^{n-i}_R(\mathrm{Ext}^{n-j}_R(R/J, \omega_R), \omega_R), \qquad \forall 0 \le i, j < d.$$

On the other hand, by [13, Lemma 5.10] , for two ideals $L_1 \sim^{\mathfrak{b}} L_2$ there is an isomorphism

$$H^t_{\mathfrak{m}}(K_{R/L_1}) \cong H^{t-1}_{\mathfrak{m}}(R/L_2)(r(R/\mathfrak{b}) - 1) \qquad \forall t \le d - 1, \tag{2}$$

which concludes that

$$H^i_{\mathfrak{m}}(K_{R/I}) \cong H^i_{\mathfrak{m}}(K_{R/J}), \qquad \forall i \le d - 1.$$

Finally by Euler characteristic formula $\Lambda(R/I) = \Lambda(R/J)$.

Example 4. *Let* $R = \Bbbk[x_1, \ldots, x_{10}]$ *and* $I = (x_1, x_2, x_3) \cap (x_3, x_4, x_5) \cap (x_5, x_6, x_7) \cap (x_7, x_8, x_9)$. *Consider the two complete intersection ideals* $\mathfrak{c} = (x_3x_7, x_2x_5x_9, x_1x_4x_6x_8)$ *and* $\mathfrak{c}' = (x_1x_5x_9, x_2x_4x_7, x_3x_6x_8)$. *Suppose that* $J = \mathfrak{c} : I$ *and* $L = \mathfrak{c}' : I$. *By using Macaulay 2 we have*

$$\Lambda(R/J) = \Lambda(R/L) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 1 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 1 \\ & & & & & & 0 & 0 \\ & & & & & & & 1 \end{bmatrix}.$$

Remark 1. *In Theorem 2, if* $\sum_{i=1}^{n-d} \deg(c_i) \neq \sum_{i=1}^{n-d} \deg(c_i')$ *we can not have the result in general. For instance, in Example 4, let* $\mathfrak{c}'' = (x_3x_7x_{10}, x_2x_5x_9, x_1x_4x_6x_8)$ *and* $H = \mathfrak{c}'' : I$. *Now* $H \sim^{\mathfrak{c}''} I$ *and* $J \sim^{\mathfrak{c}} I$ *but again using Macaulay 2,* $\Lambda(H)$ *is trivial.*

Lemma 2. *Let* I *be a squarefree monomial ideal of* R. *Then*

$$\lambda_{0,i}(R/I) = 0, \qquad \forall i > \operatorname{reg}(R/I).$$

Proof. Suppose that $a^i(R/I) = \max\{t : H_{\mathfrak{m}}^i(R/I)_t \neq 0\}$. So, we have

$$a^i(R/I) = -\min\{t : \operatorname{Ext}_R^{n-i}(R/I, \omega_R)_t \neq 0\} = r_i.$$

By Definition 7, $-r_i \geq i - k > 0$. Hence, $\operatorname{Ext}_R^{n-i}(R/I, \omega_R)_0 = 0$. Because $\operatorname{Ext}_R^{n-i}(R/I, \omega_R)$ is a squarefree module by [22, Theorem 2.6], [15, Lemma 4.1] implies that $\operatorname{depth} \operatorname{Ext}_R^{n-i}(R/I, \omega_R) > 0$. So,

$$\lambda_{0,i}(R/I) = \dim_{\mathbb{k}} \operatorname{Ext}_R^n(\operatorname{Ext}_R^{n-i}(R/I, \omega_R), \omega_R)_0 = 0.$$

Corollary 3. *Let* I *be a homogeneous ideal of* R *such that* $\operatorname{in}_{\prec}(I)$ *is squarefree with respect to some monomial order* $\prec$. *Assume that* $\operatorname{char}(\mathbb{k}) > 0$. *Then for all* $i > \operatorname{reg}(R/I)$,

$$\lambda_{0,i}(R/I) = 0.$$

Proof. By virtue of [5, Corollary 2.7], $\operatorname{reg}(R/I) = \operatorname{reg}(R/\operatorname{in}_{\prec}(I))$. Since by Lemma 2, for each $i > \operatorname{reg}(R/I)$, $\lambda_{0,i}(R/\operatorname{in}_{\prec}(I)) = 0$, [15, Corollary 2.5] follows that for all $i > \operatorname{reg}(R/I)$, $\lambda_{0,i}(R/I) = 0$.

Theorem 3. *Let* I *and* J *be radical monomial ideals of* R *such that* $I \sim^{\mathfrak{c}} J$ *and* $\dim R/I = d$. *Assume that* $\mathfrak{c} = (c_1, \ldots, c_{n-d})$ *and* $\sum_{i=1}^{n-d} \deg(c_i) = n$. *For all* i *with* $\operatorname{reg}(R/J) < i \leq d-2$,

$$\lambda_{i+1,d}(R/I) = 0.$$

Proof. Using Equation (2), for all $i \leq d-2$, $H_{\mathfrak{m}}^{i+1}(K_{R/I}) \cong H_{\mathfrak{m}}^i(R/J)(r(R/\mathfrak{c}) - 1)$. In the proof of Lemma 2, we showed that

$$\operatorname{Ext}_R^{n-i}(R/J, \omega_R)_0 = 0, \qquad \forall i > \operatorname{reg}(R/J).$$

Therefore,

$$\operatorname{Ext}_R^{n-i-1}(K_{R/I}, \omega_R)_0 = 0, \qquad \operatorname{reg}(R/J) < \forall i \leq d-2.$$

It means that $\lambda_{i+1,d}(R/I) = 0$ for all i with $\operatorname{reg}(R/J) < i \leq d-2$.

Remark 2. *Theorem* 3 *does not hold if* $\sum_{i=1}^{n-d} \deg(c_i) \neq n$. *Because in Example* 4, $I \sim^{\mathrm{c}} J$ *and* $\operatorname{reg}(R/I) = 3$ *but* $\lambda_{5,7}(R/J) \neq 0$.

Theorem 4. *Let* I *and* J *be two* d*-dimensional squarefree monomial ideals of* R *and* $I \sim^{\mathrm{c}} J$. *If* R/I *is generalized Cohen-Macaulay, then*

$$\lambda_{0,d-i}(R/I) = \lambda_{0,i}(R/J), \qquad \forall 1 \leq i \leq d-1.$$

Proof. Let $1 \leq i \leq d-1$. Because R/I is generalized Cohen-Macaulay, $\operatorname{Ext}_R^{n-d+i}(R/I, \omega_R)$ is of finite length. Thus,

$$H_{\mathfrak{m}}^0(\operatorname{Ext}_R^{n-d+i}(R/I, \omega_R)) = \operatorname{Ext}_R^{n-d+i}(R/I, \omega_R) = \operatorname{Ext}_R^{n-d+i}(R/I, \omega_R)_0.$$

Due to the fact that generalized Cohen-Macaulay is preserved under linkage,

$$H_{\mathfrak{m}}^0(\operatorname{Ext}_R^{n-i}(R/J, \omega_R)) = \operatorname{Ext}_R^{n-i}(R/J, \omega_R) = \operatorname{Ext}_R^{n-i}(R/J, \omega_R)_0.$$

So $\operatorname{Ext}_R^{n-d+i}(R/I, \omega_R)$ and $\operatorname{Ext}_R^{n-i}(R/J, \omega_R)$ are both finitely generated $\Bbbk$-vector spaces. Moreover, by [13, Corollary 5.11], for all $1 \leq t \leq d-1$,

$$H_{\mathfrak{m}}^{d-t}(R/I) \cong (H_{\mathfrak{m}}^t(R/J))^v(1 - r(R/\mathfrak{c})).$$

Therefore,

$$\dim_{\Bbbk} \operatorname{Ext}_R^{n-i}(R/J, \omega_R) = \dim_{\Bbbk} \operatorname{Ext}_R^{n-d+i}(R/I, \omega_R),$$

and this completes the proof.

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