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Local theory of singularities of three functions and the product maps

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ABSTRACT. Suppose that a smooth map $(f, g, h) : \mathbb{R}^n \to \mathbb{R}^3$, where $n \geq 3$, has a stable singularity at the origin. We characterize the stability of the function $f : \mathbb{R}^n \to \mathbb{R}$ and the map $(f, g) : \mathbb{R}^n \to \mathbb{R}^2$ at the origin in terms of the discriminant set of (f, g, h).

1. Introduction

We consider the relationships among singularities of multiple functions and the product maps, with the following notation. Let n be an integer such that $n \geq 3$, and M be a smooth n-dimensional manifold, and $f, g, h : M \to \mathbb{R}$ be smooth functions. By the product maps, we mean the maps $(f, g) : M \to \mathbb{R}^2$, $x \mapsto (f(x), g(x))$ and $(f, g, h) : M \to \mathbb{R}^3$, $x \mapsto (f(x), g(x), h(x))$. Let $\pi : \mathbb{R}^3 \to \mathbb{R}$ and $\Pi : \mathbb{R}^3 \to \mathbb{R}^2$ denote the projections such that $f = \pi \circ (f, g, h)$ and $(f, g) = \Pi \circ (f, g, h)$. Let p be an interior point in M, and U be a sufficiently small neighborhood of p in M. Let d and D denote the discriminant sets of $(f, g)|_U$ and $(f, g, h)|_U$, respectively.

In the case of two functions, we already know characterizations of stable singularities as follows. Johnson [3, Section 6] gave those in the case where $n = 3$, and the author [8] did in the general case.

FACT 1. *If p is a regular point of (f, g), then p is a regular point of f.*

PROPOSITION 2. *If p is a fold point of (f, g), then we have the following.*
- *p is a regular point of f if and only if $(f, g)(p)$ is a regular point of $\pi|_d$.*
- *p is a fold point of f if and only if $(f, g)(p)$ is a fold point of $\pi|_d$.*

They also had the corresponding characterization in the case where p is a cusp point of (f, g), though we omit it here. It is in terms of singularity of $\pi|_d$ as

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well, but not in the usual sense, as d is not a smooth manifold in that case. See [8] for the details.

In the case of three functions, we give characterizations of stable singularities as follows.

Fact 3. *If p is a regular point of (f, g, h), then p is a regular point of f and (f, g).*

Proposition 4. *If p is a fold point of (f, g, h), then we have the following.*

- *p is a regular point of f if and only if $(f, g, h)(p)$ is a regular point of $\pi|_D$.*
- *p is a fold point of f if and only if $(f, g, h)(p)$ is a fold point of $\pi|_D$.*
- *p is a regular point of (f, g) if and only if $(f, g, h)(p)$ is a regular point of $\Pi|_D$.*
- *p is a fold point of (f, g) if and only if $(f, g, h)(p)$ is a fold point of $\Pi|_D$.*
- *p is a cusp point of (f, g) if and only if $(f, g, h)(p)$ is a cusp point of $\Pi|_D$.*

We also have the corresponding characterizations in the cases where p is a cusp point or a swallow-tail point of (f, g, h), though we omit them here. They are in terms of singularity of $\pi|_D$ or $\Pi|_D$ as well, and the normal curvature of D, but not in the usual sense, as D is not a smooth manifold in those cases. See Subsection 2.2 for the details.

Those characterizations may be compared to various works in differential geometry of singular surfaces. In the cusp and swallow-tail cases, D is a certain kind of singular surface (see Subsection 2.1). We can think of singularity of $\pi|_D$ or $\Pi|_D$ as contact between D and a family of planes or lines, respectively, in $\mathbb{R}^3$. Such contacts were studied by Oset Sinha and Tari [7], and Francisco [1].

We hope that the above local theory can be applied to some global theory. Suppose that M is a closed manifold, and f, g are Morse functions. Johnson [3] and the author [9] gave upper bounds for the minimal number of birth-death singularities over all generic homotopies connecting f and g. To do so, they read off the behavior of certain homotopies connecting f and g, from the discriminant set of (f, g), by using Proposition 2 and its variation. This suggests that such a local theory is useful for a global theory. Still, we could much improve the upper bounds, if we could simplify the singularities of (f, g) by isotopies of both or one of f and g, say, an isotopy $\{g_t : M \to \mathbb{R}\}_{t \in [0,1]}$ such that $g_0 = g$ and $g_1 = h$. To do so, we might read off the behavior of the homotopy $\{(f, g_t) : M \to \mathbb{R}^2\}_{t \in [0,1]}$, from the discriminant set of (f, g, h), by using

Proposition 4 and its variations. In that way, or any other way, we hope that the present results also have applications.

2. Preliminaries and results

In this section, we give preliminary definitions and facts, state the main results of this paper, and review general methods which we use in our proofs.

2.1. Definitions and facts. We use the following notation. Let m and n be positive integers, let X and Y be C^∞ manifolds of dimensions m and n, respectively, and let $f : X \to Y$ be a C^∞ map. Let p be an interior point in X, and suppose that $f(p)$ is an interior point in Y. Let U be a sufficiently small neighborhood of p in X.

Some basic notions concerning singularity are defined as follows. The point p is said to be a *regular point* of f if there are local coordinate systems of X and Y with respect to which $p = \mathbf{0}$ and

$$f(x_1, x_2, \dots, x_m) = \begin{cases} (x_1, x_2, \dots, x_n) & (m \geq n) \\ (x_1, x_2, \dots, x_m, 0, \dots, 0) & (m \leq n), \end{cases}$$

and a *singular point* otherwise. The set of singular points of f is called the *singular set* of f, and its image by f is called the *discriminant set* of f.

Some notions concerning fold singularity are defined as follows. Suppose that $m \geq n$. The point p is said to be a *fold point* of f if there are local coordinate systems of X and Y with respect to which $p = \mathbf{0}$ and

$$f(x_1, x_2, \dots, x_m) = (x_1, x_2, \dots, x_{n-1}, -x_n^2 - \cdots - x_{n+\lambda-1}^2 + x_{n+\lambda}^2 + \cdots + x_m^2),$$

where λ is an integer such that $0 \leq \lambda \leq m - n + 1$. A fold point in the case where $n = 1$ is a so-called Morse critical point. The minimum of $\{\lambda, m - n - \lambda + 1\}$ does not depend on the choice of coordinate systems, and is called the *absolute index* of the fold point p. After coordinate transformations if necessary, we can arrange the above local form so that λ attains the absolute index. Suppose that $n = 3$ and p is a fold point of f. Then, we can see that the singular set of $f|_U$ consists only of fold points, and its discriminant set is a regular surface.

Some notions concerning cusp singularity are defined as follows. Suppose that $m \geq n \geq 2$. The point p is said to be a *cusp point* of f if there are local coordinate systems of X and Y with respect to which $p = \mathbf{0}$ and

$$f(x_1, x_2, \dots, x_m) = (x_1, x_2, \dots, x_{n-1}, x_n^3 + x_1 x_n - x_{n+1}^2 - \cdots - x_{n+\lambda}^2 + x_{n+\lambda+1}^2 + \cdots + x_m^2),$$

where λ is an integer such that $0 \le \lambda \le m - n$. The minimum of $\{\lambda, m - n - \lambda\}$ does not depend on the choice of coordinate systems, and is called the *absolute index* of the cusp point p. After coordinate transformations if necessary, we can arrange the above local form so that λ attains the absolute index. Suppose that $n = 3$ and p is a cusp point of f. Then, we can see that the singular set of $f|_U$ consists only of fold points and cusp points, and its discriminant set, denoted by D, is a singular surface as in the left of Figure 1. (See the second paragraph in Section 3 for more details.) The images of the cusp points form the cuspidal edge, which is a regular arc, denoted by E. There exists a C^∞ disk in Y with the same limiting tangent plane in $T_{f(p)}Y$ as each component of $D \backslash E$, which we call a *tangent disk* of D at $f(p)$.

Some notions concerning swallow-tail singularity are defined as follows. Suppose that $m \ge n = 3$. The point p is said to be a *swallow-tail point* of f if there are local coordinate systems of X and Y with respect to which $p = \mathbf{0}$ and

$$f(x_1, x_2, \dots, x_m) = (x_1, x_2, x_3^4 + x_1 x_3^2 + x_2 x_3 - x_4^2 - \cdots - x_{\lambda+3}^2 + x_{\lambda+4}^2 + \cdots + x_m^2),$$

where λ is an integer such that $0 \le \lambda \le m - 3$. The minimum of $\{\lambda, m - \lambda - 3\}$ does not depend on the choice of coordinate systems, and is called the *absolute index* of the swallow-tail point p. After coordinate transformations if necessary, we can arrange the above local form so that λ attains the absolute index. Suppose that p is a swallow-tail point of f. Then, we can see that the singular set of $f|_U$ consists only of fold points, cusp points and the swallow-tail point p, and its discriminant set, denoted by D, is a singular surface as in the right of Figure 1. (See the second paragraph in Section 4 for more details.) The images of the cusp points form the twin cuspidal edges, which together with

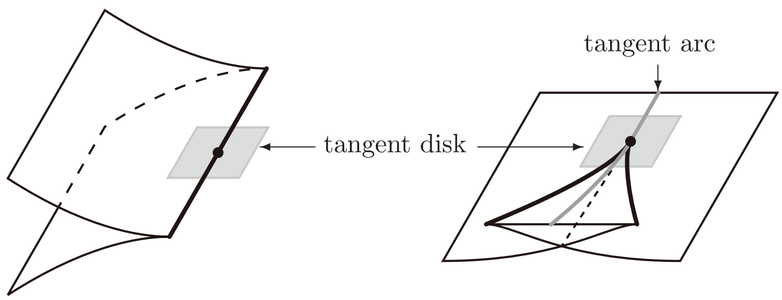

Fig. 1. The local structures of the discriminant set at the images of a cusp point and a swallow-tail point.

$f(p)$ form the cusped arc, denoted by E. There exists a C^∞ disk in Y with the same limiting tangent plane in $T_{f(p)}Y$ as each component of $D \backslash E$, which we call a *tangent disk* of D at $f(p)$. There also exists a C^∞ arc in D with the same limiting tangent line in $T_{f(p)}Y$ as each component of $E \backslash \{f(p)\}$, which we call a *tangent arc* of E at $f(p)$ in D.

We remark that fold, cusp and swallow-tail singularities are stable. For descriptions of stability, see [2] for example. Stable singular points of f are classified into fold points if $n = 1$, into fold points and cusp points if $m \geq n = 2$, and into fold points, cusp points and swallow-tail points if $m \geq n = 3$.

The notion of limiting normal curvature for certain kinds of singular surfaces is defined as follows. This is due to Martins–Saji–Umehara–Yamada [4]. Typical examples of the kinds of singular surfaces are the discriminant sets in the above paragraphs, provided that the target manifold Y is Riemannian. Suppose that $m = 2$, $n = 3$, and let g be a Riemannian metric of Y, and ∇ denote the Levi-Civita connection. Suppose that f has a unit normal vector field, that is to say, a C^∞ map $v : U \to TY$ such that $v(x)$ is a unit vector in $T_{f(x)}Y$ perpendicular to $(df)_x(T_x U)$ for $x \in U$. Suppose also that there is a coordinate system (x_1, x_2) of U with respect to which $p = \mathbf{0}$ and

$$(df)_p\left(\frac{\partial}{\partial x_1}\right) \neq \mathbf{0} \qquad \text{and} \qquad (df)_p\left(\frac{\partial}{\partial x_2}\right) = \mathbf{0}.$$

Then, the *limiting normal curvature* of $f(U)$ at $f(p)$ is

$$\frac{g\left(\left(\nabla_{(df)(\partial/(\partial x_1))}(df)\left(\frac{\partial}{\partial x_1}\right)\right)_p, v(p)\right)}{g\left((df)_p\left(\frac{\partial}{\partial x_1}\right), (df)_p\left(\frac{\partial}{\partial x_1}\right)\right)}.$$

This is an invariant of the singular surface $f(U)$ and the metric g, up to sign corresponding to the two possibilities for v.

2.2. Main results. We work in the following setting, slightly different from that in Introduction for generality and convenience. Let m be an integer such that $m \geq 3$, and X be an m-dimensional C^∞ manifold. Let Y_1 and $Y_{2,3}$ be Riemannian manifolds of dimensions 1 and 2, respectively, and Y denote the product Riemannian manifold $Y_1 \times Y_{2,3}$. Let $f_1 : X \to Y_1$ and $f_{2,3} : X \to Y_{2,3}$ be C^∞ maps, and f denote the product map $(f_1, f_{2,3}) : X \to Y$, $x \mapsto (f_1(x), f_{2,3}(x))$. Let $\pi : Y \to Y_1$ and $\Pi : Y \to Y_{2,3}$ denote the projections. Let p be an interior point in X, and suppose that $f_1(p)$ and $f_{2,3}(p)$ are interior points in Y_1 and $Y_{2,3}$, respectively, and let $q = f(p)$. Let U be a sufficiently small neighborhood of p in X, and let D denote the discriminant set of $f|_U$. Let E

denote the subset of D consisting of the images of singular points of $f|_U$ other than fold points. We note the following immediate fact.

Fact 5. *If p is a regular point of f, then p is a regular point of f_1 and $f_{2,3}$.*

The following three propositions are the main results of this paper.

Proposition 6. *Suppose that p is a fold point of f of absolute index λ. Then we have the following.*

- *p is a regular point of f_1 if and only if q is a regular point of $\pi|_D$.*
- *p is a fold point of f_1 if and only if q is a fold point of $\pi|_D$. Moreover, the absolute index of the fold point p of f_1 is equal to either $\min\{\lambda+\mu, m-\lambda-\mu\}$ or $\min\{\lambda-\mu+2, m-\lambda+\mu-2\}$, where μ is the absolute index of the fold point q of $\pi|_D$.*
- *p is a regular point of $f_{2,3}$ if and only if q is a regular point of $\Pi|_D$.*
- *p is a fold point of $f_{2,3}$ if and only if q is a fold point of $\Pi|_D$. Moreover, the absolute index of the fold point p of $f_{2,3}$ is equal to either λ or $\min\{\lambda+1, m-\lambda-2\}$.*
- *p is a cusp point of $f_{2,3}$ if and only if q is a cusp point of $\Pi|_D$. Moreover, the absolute index of the cusp point p of $f_{2,3}$ is equal to λ.*

Proposition 7. *Suppose that p is a cusp point of f of absolute index λ. Let $\tilde{D}$ be a tangent disk of D at q. Then we have the following.*

- *p is a regular point of f_1 if and only if q is a regular point of $\pi|_{\tilde{D}}$.*
- *p is a fold point of f_1 if and only if q is a singular point of $\pi|_{\tilde{D}}$ and a fold point of $\pi|_E$. Moreover, the absolute index of the fold point p of f_1 is equal to either $\min\{\lambda+1, m-\lambda-1\}$ or $\min\{\lambda+2, m-\lambda-2\}$.*
- *p is a regular point of $f_{2,3}$ if and only if q is a regular point of $\Pi|_{\tilde{D}}$.*
- *p is a fold point of $f_{2,3}$ if and only if q is a singular point of $\Pi|_{\tilde{D}}$ and a regular point of $\Pi|_E$. Moreover, the absolute index of the fold point p of $f_{2,3}$ is equal to $\min\{\lambda+1, m-\lambda-2\}$.*
- *p is a cusp point of $f_{2,3}$ if and only if q is a singular point of $\Pi|_E$, and the limiting normal curvature of D at q is non-zero. Moreover, the absolute index of the cusp point p of $f_{2,3}$ is equal to either λ or $\min\{\lambda+1, m-\lambda-3\}$.*

Proposition 8. *Suppose that p is a swallow-tail point of f of absolute index λ. Let $\tilde{D}$ be a tangent disk of D at q, and $\tilde{E}$ be a tangent arc of E at q in D. Then we have the following.*

- *p is a regular point of f_1 if and only if q is a regular point of $\pi|_{\tilde{D}}$.*
- *p is a fold point of f_1 if and only if q is a singular point of $\pi|_{\tilde{D}}$, and the limiting normal curvature of D at q is non-zero. Moreover, the absolute index of the fold point p of f_1 is equal to either* $\min\{\lambda+1, m-\lambda-1\}$ *or* $\min\{\lambda+2, m-\lambda-2\}$.
- *p is a regular point of $f_{2,3}$ if and only if q is a regular point of $\Pi|_{\tilde{D}}$.*
- *p is a fold point of $f_{2,3}$ if and only if q is a singular point of $\Pi|_{\tilde{D}}$ and a regular point of $\Pi|_{\tilde{E}}$. Moreover, the absolute index of the fold point p of $f_{2,3}$ is equal to* $\min\{\lambda+1, m-\lambda-2\}$.
- *p is not a cusp point of $f_{2,3}$.*

Proposition 6 can be interpreted as Proposition 4. We remark that some of the assertions about the indices can be refined by considering stable singularities together with normal vectors on the discriminant sets, which is left to the reader.

2.3. Methods. We use the following notation. Let m and n be positive integers, let X and Y be C^∞ manifolds of dimensions m and n, respectively, and let $f : X \to Y$ be a C^∞ map. Let p be an interior point in X, and suppose that $f(p)$ is an interior point in Y.

It is standard to use Jacobian matrix for distinguishing between regular and singular points of maps. The point p is a regular point of f if and only if $(df)_p : T_pX \to T_{f(p)}Y$ has maximal rank. Rather this is usually regarded as the definition. If f has a local form:

$$f(x_1, x_2, \dots, x_m) = (f_1(x_1, x_2, \dots, x_m), f_2(x_1, x_2, \dots, x_m), \dots, f_n(x_1, x_2, \dots, x_m)),$$

then $(df)_p$ is represented by, and hence identified with, the Jacobian matrix

$$\begin{pmatrix} \left(\dfrac{\partial f_1}{\partial x_1}\right)_p & \left(\dfrac{\partial f_1}{\partial x_2}\right)_p & \cdots & \left(\dfrac{\partial f_1}{\partial x_m}\right)_p \\ \left(\dfrac{\partial f_2}{\partial x_1}\right)_p & \left(\dfrac{\partial f_2}{\partial x_2}\right)_p & \cdots & \left(\dfrac{\partial f_2}{\partial x_m}\right)_p \\ \vdots & \vdots & & \vdots \\ \left(\dfrac{\partial f_n}{\partial x_1}\right)_p & \left(\dfrac{\partial f_n}{\partial x_2}\right)_p & \cdots & \left(\dfrac{\partial f_n}{\partial x_m}\right)_p \end{pmatrix}.$$

It is also standard to use Hessian matrix for recognizing stable singularities of functions. Suppose that $n = 1$ and p is a singular point of f. For a local coordinate system $(x_1, x_2, \dots, x_m)$ of X at p, let $(\mathbf{H}_{x_1, x_2, \dots, x_m} f)_p$ denote the Hessian matrix

$$\begin{pmatrix} \left(\dfrac{\partial^2 f}{\partial x_1^2}\right)_p & \left(\dfrac{\partial^2 f}{\partial x_1 \partial x_2}\right)_p & \cdots & \left(\dfrac{\partial^2 f}{\partial x_1 \partial x_m}\right)_p \\ \left(\dfrac{\partial^2 f}{\partial x_2 \partial x_1}\right)_p & \left(\dfrac{\partial^2 f}{\partial x_2^2}\right)_p & \cdots & \left(\dfrac{\partial^2 f}{\partial x_2 \partial x_m}\right)_p \\ \vdots & \vdots & & \vdots \\ \left(\dfrac{\partial^2 f}{\partial x_m \partial x_1}\right)_p & \left(\dfrac{\partial^2 f}{\partial x_m \partial x_2}\right)_p & \cdots & \left(\dfrac{\partial^2 f}{\partial x_m^2}\right)_p \end{pmatrix}.$$

It is a symmetric matrix, and hence has real eigenvalues. Let μ denote the number of negative eigenvalues. By Morse theory (see [5]), p is a fold point of f if and only if $(\mathbf{H}_{x_1, x_2, \dots, x_m} f)_p$ has maximal rank. Moreover, its absolute index is equal to $\min\{\mu, m - \mu\}$.

We use the following criteria for recognizing stable singularities of surface-valued maps. Saji [6] gave general criteria for recognizing so-called Morin singularities, and the following are those in the special cases. Suppose that $m \geq n = 2$ and p is a singular point of f. Let U be a sufficiently small neighborhood of p in X, and let S denote the singular set of $f|_U$. Suppose that f has a local form: $f(x) = (f_1(x), f_2(x))$ for $x \in U$ such that $(df_1)_p \neq \mathbf{0}$ and $(df_2)_p = \mathbf{0}$. This implies that $\ker(df)_p$ has dimension $m - 1$. Let $\eta_2, \eta_3, \dots, \eta_m$ be C^∞ vector fields on U such that $\ker(df)_p = \langle (\eta_2)_p, (\eta_3)_p, \dots, (\eta_m)_p \rangle$, and let $(\mathbf{H}_{\eta_2, \eta_3, \dots, \eta_m} f_2)_p$ denote the matrix

$$\begin{pmatrix} (\eta_2 \eta_2 f_2)_p & (\eta_2 \eta_3 f_2)_p & \cdots & (\eta_2 \eta_m f_2)_p \\ (\eta_3 \eta_2 f_2)_p & (\eta_3 \eta_3 f_2)_p & \cdots & (\eta_3 \eta_m f_2)_p \\ \vdots & \vdots & & \vdots \\ (\eta_m \eta_2 f_2)_p & (\eta_m \eta_3 f_2)_p & \cdots & (\eta_m \eta_m f_2)_p \end{pmatrix}.$$

In fact, $(\eta_i \eta_j f_2)_p = (\eta_j \eta_i f_2)_p$ for $i, j \in \{2, 3, \dots, m\}$. That is to say, $(\mathbf{H}_{\eta_2, \eta_3, \dots, \eta_m} f_2)_p$ is a symmetric matrix, and hence has real eigenvalues. Let λ denote the number of negative eigenvalues. We regard $(\mathbf{H}_{\eta_2, \eta_3, \dots, \eta_m} f_2)_p$ as representing a linear transformation of $\ker(df)_p$ with respect to the basis $((\eta_2)_p, (\eta_3)_p, \dots, (\eta_m)_p)$, to treat $\ker(\mathbf{H}_{\eta_2, \eta_3, \dots, \eta_m} f_2)_p$ as a subspace of $\ker(df)_p$.

Theorem 1 (Saji). *The point p is a fold point of f if and only if $\ker(\mathbf{H}_{\eta_2, \eta_3, \dots, \eta_m} f_2)_p = \{\mathbf{0}\}$. Moreover, its absolute index is equal to* $\min\{\lambda, m - \lambda - 1\}$.

Theorem 2 (Saji). *The point p is a cusp point of f if there exists a C^∞ vector field θ on U such that*

- $\theta_p \neq \mathbf{0}$ *and* $\theta_s \in \ker(df)_s$ *for* $s \in S$,
- $\ker(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m} f_2)_p = \langle \theta_p \rangle$,
- $(d(\theta f_2))_p \neq \mathbf{0}$ *and* $(\theta\theta f_2)_p \neq 0$.

Moreover, its absolute index is equal to $\min\{\lambda, m - \lambda - 2\}$. *Conversely, p is not a cusp point of f if either* $\ker(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m} f_2)_p$ *is not* 1*-dimensional, or there exists a C^∞ vector field θ on U such that*

- $\theta_p \neq \mathbf{0}$ *and* $\theta_s \in \ker(df)_s$ *for* $s \in S$,
- $\ker(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m} f_2)_p = \langle \theta_p \rangle$,
- $(d(\theta f_2))_p = \mathbf{0}$ *or* $(\theta\theta f_2)_p = 0$.

3. Proof in cusp case

In this section, we give a proof of Proposition 7. We take this first because it is more straightforward than those of Propositions 6 and 8.

We begin with the following local forms of the relevant maps. Since p is a cusp point of f of absolute index λ, there exist local coordinate systems $(x_1, x_2, \dots, x_m)$ and (u, v, w) of X and Y, respectively, with respect to which $p = \mathbf{0}$ and

$$f(x_1, x_2, \dots, x_m) = (x_1, x_2, x_3^3 + x_1 x_3 - x_4^2 - \cdots - x_{\lambda+3}^2 + x_{\lambda+4}^2 + \cdots + x_m^2).$$

Since q is a regular point of π and Π, there exist a local coordinate y_1 of Y_1, and a local coordinate system (y_2, y_3) of $Y_{2,3}$, which give the local coordinate system (y_1, y_2, y_3) of Y, with respect to which $q = \mathbf{0}$ and $\pi(y_1, y_2, y_3) = y_1$ and $\Pi(y_1, y_2, y_3) = (y_2, y_3)$. Let f_2, f_3 denote the functions on U such that

$$f_{2,3}(x_1, x_2, \dots, x_m) = (f_2(x_1, x_2, \dots, x_m), f_3(x_1, x_2, \dots, x_m))$$

with respect to (y_2, y_3). Then f also has the local form:

$$f(x_1, x_2, \dots, x_m) = (f_1(x_1, x_2, \dots, x_m), f_2(x_1, x_2, \dots, x_m), f_3(x_1, x_2, \dots, x_m))$$

with respect to (y_1, y_2, y_3). Note that there is a coordinate transformation:

$$(u, v, w) \mapsto (y_1(u, v, w), y_2(u, v, w), y_3(u, v, w)).$$

Let S denote the singular set of $f|_U$. From the first local form of f, we can see that S has the local form:

$$\{(-x_3^2, x_2, x_3, 0, \dots, 0) \mid x_2, x_3 \in \mathbb{R}\}$$

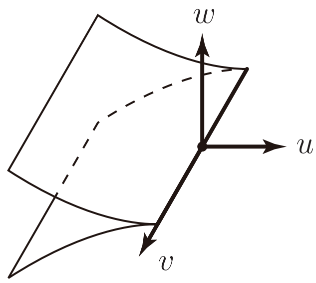

Fig. 2. The discriminant set D and the coordinate system (u, v, w).

with respect to $(x_1, x_2, \dots, x_m)$. We can regard (x_2, x_3) as a local coordinate system of S at p, and then $f|_S$ has the local form:

$$f|_S(x_2, x_3) = (-x_3^2, x_2, -2x_3^3)$$

with respect to (u, v, w). This shows that D is such a singular surface as described in Subsection 2.1, and is related with (u, v, w) as in Figure 2. The uv-plane is a tangent disk of D at $f(p)$, as well as the given one $\tilde{D}$. The v-axis coincides with the cuspidal edge E, and hence $\pi|_E$ and $\Pi|_E$ have the local forms: $\pi|_E(v) = y_1(0, v, 0)$ and $\Pi|_E(v) = (y_2(0, v, 0), y_3(0, v, 0))$.

We calculate partial derivatives as follows. By the chain rule, for example,

$$\begin{aligned}
\frac{\partial f_1}{\partial x_3} &= \frac{\partial u}{\partial x_3}\frac{\partial f_1}{\partial u} + \frac{\partial v}{\partial x_3}\frac{\partial f_1}{\partial v} + \frac{\partial w}{\partial x_3}\frac{\partial f_1}{\partial w} \\
&= \left(\frac{\partial}{\partial x_3}x_1\right)\frac{\partial y_1}{\partial u} + \left(\frac{\partial}{\partial x_3}x_2\right)\frac{\partial y_1}{\partial v} \\
&\quad + \left\{\frac{\partial}{\partial x_3}(x_3^3 + x_1x_3 - x_4^2 - \cdots - x_{\lambda+3}^2 + x_{\lambda+4}^2 + \cdots + x_m^2)\right\}\frac{\partial y_1}{\partial w} \\
&= (3x_3^2 + x_1)\frac{\partial y_1}{\partial w},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f_1}{\partial x_3^2} &= \frac{\partial}{\partial x_3}\left\{(3x_3^2 + x_1)\frac{\partial y_1}{\partial w}\right\} \\
&= \left\{\frac{\partial}{\partial x_3}(3x_3^2 + x_1)\right\}\frac{\partial y_1}{\partial w} + (3x_3^2 + x_1)\frac{\partial}{\partial x_3}\frac{\partial y_1}{\partial w}
\end{aligned}$$

$$= 6x_3 \frac{\partial y_1}{\partial w} + (3x_3^2 + x_1)\left\{(3x_3^2 + x_1)\frac{\partial}{\partial w}\right\}\frac{\partial y_1}{\partial w}$$

$$= 6x_3 \frac{\partial y_1}{\partial w} + (3x_3^2 + x_1)^2 \frac{\partial^2 y_1}{\partial w^2}.$$

By similar calculations, for each $k \in \{1,2,3\}$ and $i, j \in \{1,2,\ldots,m\}$,

$$\frac{\partial f_k}{\partial x_i} = \begin{cases} \dfrac{\partial y_k}{\partial u} + x_3 \dfrac{\partial y_k}{\partial w} & (i = 1) \\ \dfrac{\partial y_k}{\partial v} & (i = 2) \\ (3x_3^2 + x_1)\dfrac{\partial y_k}{\partial w} & (i = 3) \\ -2x_i \dfrac{\partial y_k}{\partial w} & (4 \le i \le \lambda + 3) \\ 2x_i \dfrac{\partial y_k}{\partial w} & (\lambda + 4 \le i \le m), \end{cases}$$

$$\frac{\partial^2 f_k}{\partial x_1 \partial x_j} = \begin{cases} \dfrac{\partial^2 y_k}{\partial u^2} + 2x_3 \dfrac{\partial^2 y_k}{\partial u \partial w} + x_3^2 \dfrac{\partial^2 y_k}{\partial w^2} & (j = 1) \\ \dfrac{\partial^2 y_k}{\partial u \partial v} + x_3 \dfrac{\partial^2 y_k}{\partial v \partial w} & (j = 2) \\ \dfrac{\partial y_k}{\partial w} + (3x_3^2 + x_1)\dfrac{\partial^2 y_k}{\partial u \partial w} + x_3(3x_3^2 + x_1)\dfrac{\partial^2 y_k}{\partial w^2} & (j = 3) \\ -2x_j \dfrac{\partial^2 y_k}{\partial u \partial w} - 2x_3 x_j \dfrac{\partial^2 y_k}{\partial w^2} & (4 \le j \le \lambda + 3) \\ 2x_j \dfrac{\partial^2 y_k}{\partial u \partial w} + 2x_3 x_j \dfrac{\partial^2 y_k}{\partial w^2} & (\lambda + 4 \le j \le m), \end{cases}$$

$$\frac{\partial^2 f_k}{\partial x_2 \partial x_j} = \begin{cases} \dfrac{\partial^2 y_k}{\partial v^2} & (j = 2) \\ (3x_3^2 + x_1)\dfrac{\partial^2 y_k}{\partial v \partial w} & (j = 3) \\ -2x_j \dfrac{\partial^2 y_k}{\partial v \partial w} & (4 \le j \le \lambda + 3) \\ 2x_j \dfrac{\partial^2 y_k}{\partial v \partial w} & (\lambda + 4 \le j \le m), \end{cases}$$

$$\frac{\partial^2 f_k}{\partial x_3 \partial x_j} = \begin{cases} 6x_3 \dfrac{\partial y_k}{\partial w} + (3x_3^2 + x_1)^2 \dfrac{\partial^2 y_k}{\partial w^2} & (j = 3) \\ -2x_j(3x_3^2 + x_1)\dfrac{\partial^2 y_k}{\partial w^2} & (4 \le j \le \lambda + 3) \\ 2x_j(3x_3^2 + x_1)\dfrac{\partial^2 y_k}{\partial w^2} & (\lambda + 4 \le j \le m), \end{cases}$$

$$\frac{\partial^2 f_k}{\partial x_i \partial x_j} = \begin{cases} -2\dfrac{\partial y_k}{\partial w} + 4x_i^2 \dfrac{\partial^2 y_k}{\partial w^2} & (4 \le i = j \le \lambda + 3) \\ 2\dfrac{\partial y_k}{\partial w} + 4x_i^2 \dfrac{\partial^2 y_k}{\partial w^2} & (\lambda + 4 \le i = j \le m) \\ 4x_i x_j \dfrac{\partial^2 y_k}{\partial w^2} & (4 \le i < j \le \lambda + 3 \ \text{ or } \ \lambda + 4 \le i < j \le m) \\ -4x_i x_j \dfrac{\partial^2 y_k}{\partial w^2} & (4 \le i \le \lambda + 3 < j \le m), \end{cases}$$

$$\frac{\partial^3 f_k}{\partial x_3^3} = 6\frac{\partial y_k}{\partial w} + 18x_3(3x_3^2 + x_1)\frac{\partial^2 y_k}{\partial w^2} + (3x_3^2 + x_1)^3 \frac{\partial^3 y_k}{\partial w^3}.$$

Since $p = \mathbf{0}$ and $f(p) = q$, for each $k \in \{1, 2, 3\}$ and $i, j \in \{1, 2, \ldots, m\}$ such that $i \le j$,

$$\left(\frac{\partial f_k}{\partial x_i}\right)_p = \begin{cases} \left(\dfrac{\partial y_k}{\partial u}\right)_q & (i = 1) \\ \left(\dfrac{\partial y_k}{\partial v}\right)_q & (i = 2) \\ 0 & (3 \le i \le m), \end{cases}$$

$$\left(\frac{\partial^2 f_k}{\partial x_i \partial x_j}\right)_p = \begin{cases} \left(\dfrac{\partial^2 y_k}{\partial u^2}\right)_q & (i = j = 1) \\ \left(\dfrac{\partial^2 y_k}{\partial u \partial v}\right)_q & (i = 1,\ j = 2) \\ \left(\dfrac{\partial y_k}{\partial w}\right)_q & (i = 1,\ j = 3) \\ \left(\dfrac{\partial^2 y_k}{\partial v^2}\right)_q & (i = j = 2) \\ -2\left(\dfrac{\partial y_k}{\partial w}\right)_q & (4 \le i = j \le \lambda + 3) \\ 2\left(\dfrac{\partial y_k}{\partial w}\right)_q & (\lambda + 4 \le i = j \le m) \\ 0 & (\text{otherwise}), \end{cases}$$

$$\left(\frac{\partial^3 f_k}{\partial x_3^3}\right)_p = 6\left(\frac{\partial y_k}{\partial w}\right)_q.$$

3.1. Function case. We first focus on the function f_1.

We consider whether p is a regular or singular point of f_1. By the results of partial derivatives,

$$(df_1)_p = \left(\left(\frac{\partial f_1}{\partial x_1} \right)_p, \left(\frac{\partial f_1}{\partial x_2} \right)_p, \ldots, \left(\frac{\partial f_1}{\partial x_m} \right)_p \right)$$
$$= \left(\left(\frac{\partial y_1}{\partial u} \right)_q, \left(\frac{\partial y_1}{\partial v} \right)_q, 0, \ldots, 0 \right).$$

As for the function $\pi|_{\tilde{D}}$, since the disk $\tilde{D}$ is tangent to the uv-plane at q,

$$(d(\pi|_{\tilde{D}}))_q = \left(\left(\frac{\partial \pi}{\partial u} \right)_q, \left(\frac{\partial \pi}{\partial v} \right)_q \right) = \left(\left(\frac{\partial y_1}{\partial u} \right)_q, \left(\frac{\partial y_1}{\partial v} \right)_q \right).$$

Hence, p is a regular point of f_1 if and only if q is a regular point of $\pi|_{\tilde{D}}$, which is the case if and only if

$$\left(\frac{\partial y_1}{\partial u} \right)_q \neq 0 \qquad \text{or} \qquad \left(\frac{\partial y_1}{\partial v} \right)_q \neq 0.$$

Supposing that p is a singular point of f_1, we consider what type it is. By the above result, q is a singular point of $\pi|_{\tilde{D}}$, and

$$\left(\frac{\partial y_1}{\partial u} \right)_q = \left(\frac{\partial y_1}{\partial v} \right)_q = 0.$$

By the regularity of the coordinate transformation: $(u, v, w) \mapsto (y_1, y_2, y_3)$,

$$\left(\frac{\partial y_1}{\partial w} \right)_q \neq 0.$$

By the results of partial derivatives, the Hessian matrix $(\mathbf{H}_{x_1, x_2, \ldots, x_m} f_1)_p$ is equal to

$$\begin{pmatrix} \left(\frac{\partial^2 y_1}{\partial u^2} \right)_q & \left(\frac{\partial^2 y_1}{\partial u \partial v} \right)_q & \left(\frac{\partial y_1}{\partial w} \right)_q & \mathbf{0} & \mathbf{0} \\ \left(\frac{\partial^2 y_1}{\partial u \partial v} \right)_q & \left(\frac{\partial^2 y_1}{\partial v^2} \right)_q & 0 & \mathbf{0} & \mathbf{0} \\ \left(\frac{\partial y_1}{\partial w} \right)_q & 0 & 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -2\left(\frac{\partial y_1}{\partial w} \right)_q \mathbf{I}_\lambda & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 2\left(\frac{\partial y_1}{\partial w} \right)_q \mathbf{I}_{m-\lambda-3} \end{pmatrix},$$

where $\mathbf{I}_n$ denotes the $n \times n$ identity submatrix for $n \in \mathbb{N}$. It shows that $(\mathbf{H}_{x_1, x_2, \dots, x_m} f_1)_p$ has maximal rank if and only if

$$\left(\frac{\partial^2 y_1}{\partial v^2}\right)_q \neq 0.$$

As for the function $\pi|_E$, by the local form,

$$(d(\pi|_E))_q = \left(\frac{\partial y_1}{\partial v}\right)_q = 0,$$
$$(\mathbf{H}_v(\pi|_E))_q = \left(\frac{\partial^2 y_1}{\partial v^2}\right)_q.$$

Hence, p is a fold point of f_1 if and only if q is a fold point of $\pi|_E$. The number of negative eigenvalues of $(\mathbf{H}_{x_1, x_2, \dots, x_m} f_1)_p$ is

$$\begin{cases} \lambda + 1 & \left(\left(\frac{\partial y_1}{\partial w}\right)_q > 0, \left(\frac{\partial^2 y_1}{\partial v^2}\right)_q > 0\right) \\ \lambda + 2 & \left(\left(\frac{\partial y_1}{\partial w}\right)_q > 0, \left(\frac{\partial^2 y_1}{\partial v^2}\right)_q < 0\right) \\ m - \lambda - 2 & \left(\left(\frac{\partial y_1}{\partial w}\right)_q < 0, \left(\frac{\partial^2 y_1}{\partial v^2}\right)_q > 0\right) \\ m - \lambda - 1 & \left(\left(\frac{\partial y_1}{\partial w}\right)_q < 0, \left(\frac{\partial^2 y_1}{\partial v^2}\right)_q < 0\right). \end{cases}$$

Hence, if p is a fold point of f_1, its absolute index is equal to either $\min\{\lambda + 1, m - \lambda - 1\}$ or $\min\{\lambda + 2, m - \lambda - 2\}$.

3.2. Surface-valued map case. We now focus on the map $f_{2,3}$.

We consider whether p is a regular or singular point of $f_{2,3}$. By the local form and the results of partial derivatives,

$$(df_{2,3})_p = \begin{pmatrix} \left(\frac{\partial f_2}{\partial x_1}\right)_p & \left(\frac{\partial f_2}{\partial x_2}\right)_p & \cdots & \left(\frac{\partial f_2}{\partial x_m}\right)_p \\ \left(\frac{\partial f_3}{\partial x_1}\right)_p & \left(\frac{\partial f_3}{\partial x_2}\right)_p & \cdots & \left(\frac{\partial f_3}{\partial x_m}\right)_p \end{pmatrix}$$
$$= \begin{pmatrix} \left(\frac{\partial y_2}{\partial u}\right)_q & \left(\frac{\partial y_2}{\partial v}\right)_q & 0 & \cdots & 0 \\ \left(\frac{\partial y_3}{\partial u}\right)_q & \left(\frac{\partial y_3}{\partial v}\right)_q & 0 & \cdots & 0 \end{pmatrix}.$$

As for the map $\Pi|_{\tilde{D}}$, since $\tilde{D}$ is tangent to the uv-plane at q,

$$(d(\Pi|_{\tilde{D}}))_q = \begin{pmatrix} \left(\dfrac{\partial y_2}{\partial u}\right)_q & \left(\dfrac{\partial y_2}{\partial v}\right)_q \\ \left(\dfrac{\partial y_3}{\partial u}\right)_q & \left(\dfrac{\partial y_3}{\partial v}\right)_q \end{pmatrix}.$$

Hence, p is a regular point of $f_{2,3}$ if and only if q is a regular point of $\Pi|_{\tilde{D}}$, which is the case if and only if

$$\left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial y_3}{\partial v}\right)_q - \left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial y_3}{\partial u}\right)_q \neq 0.$$

Supposing that p is a singular point of $f_{2,3}$, we consider what type it is, in the rest of this section. By the above result, q is a singular point of $\Pi|_{\tilde{D}}$, and

$$\left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial y_3}{\partial v}\right)_q - \left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial y_3}{\partial u}\right)_q = 0.$$

We have the following two subcases.

3.2.1. *Generic subcase.* We first deal with the subcase where q is a regular point of $\Pi|_E$. Since

$$(d(\Pi|_E))_q = \begin{pmatrix} \left(\dfrac{\partial y_2}{\partial v}\right)_q \\ \left(\dfrac{\partial y_3}{\partial v}\right)_q \end{pmatrix} \neq \mathbf{0},$$

and the coordinates y_2 and y_3 are symmetric so far, we may suppose that

$$\left(\frac{\partial y_2}{\partial v}\right)_q \neq 0$$

without loss of generality. By the regularity of the coordinate transformation: $(u, v, w) \mapsto (y_1, y_2, y_3)$,

$$\left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial y_3}{\partial w}\right)_q - \left(\frac{\partial y_2}{\partial w}\right)_q \left(\frac{\partial y_3}{\partial v}\right)_q \neq 0.$$

Let A denote the left-hand side of this inequality.

We modify the local form of $f_{2,3}$ as follows. Let $\widetilde{f_3}$ be the function on U defined as

$$\widetilde{f_3} = \left(\frac{\partial y_2}{\partial v}\right)_q f_3 - \left(\frac{\partial y_3}{\partial v}\right)_q f_2.$$

Noting that the coefficient of f_3 is non-zero, we obtain the local form:

$$f_{2,3}(x_1, x_2, \dots, x_m) = (f_2(x_1, x_2, \dots, x_m), \widetilde{f_3}(x_1, x_2, \dots, x_m)).$$

We have arranged that

$$\left(\frac{\partial f_2}{\partial x_2}\right)_p = \left(\frac{\partial y_2}{\partial v}\right)_q \neq 0,$$

$$\begin{aligned}\left(\frac{\partial \widetilde{f_3}}{\partial x_1}\right)_p &= \left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial f_3}{\partial x_1}\right)_p - \left(\frac{\partial y_3}{\partial v}\right)_q \left(\frac{\partial f_2}{\partial x_1}\right)_p \\ &= \left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial y_3}{\partial u}\right)_q - \left(\frac{\partial y_3}{\partial v}\right)_q \left(\frac{\partial y_2}{\partial u}\right)_q = 0,\end{aligned}$$

$$\begin{aligned}\left(\frac{\partial \widetilde{f_3}}{\partial x_2}\right)_p &= \left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial f_3}{\partial x_2}\right)_p - \left(\frac{\partial y_3}{\partial v}\right)_q \left(\frac{\partial f_2}{\partial x_2}\right)_p \\ &= \left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial y_3}{\partial v}\right)_q - \left(\frac{\partial y_3}{\partial v}\right)_q \left(\frac{\partial y_2}{\partial v}\right)_q = 0,\end{aligned}$$

$$\left(\frac{\partial \widetilde{f_3}}{\partial x_i}\right)_p = \left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial f_3}{\partial x_i}\right)_p - \left(\frac{\partial y_3}{\partial v}\right)_q \left(\frac{\partial f_2}{\partial x_i}\right)_p = 0 \qquad (3 \leq i \leq m),$$

to satisfy the conditions that $(df_2)_p \neq \mathbf{0}$ and $(d\widetilde{f_3})_p = \mathbf{0}$.

We calculate some derivatives with respect to appropriate vector fields as follows. Let $\eta_2, \eta_3, \dots, \eta_m$ be the C^∞ vector fields on U defined as

$$\eta_2 = \left(\frac{\partial y_2}{\partial u}\right)_q \frac{\partial}{\partial x_2} - \left(\frac{\partial y_2}{\partial v}\right)_q \frac{\partial}{\partial x_1},$$

$$\eta_i = \frac{\partial}{\partial x_i} \qquad (3 \leq i \leq m).$$

We have arranged that $(\eta_2)_p, (\eta_3)_p, \dots, (\eta_m)_p$ are linearly independent, and

$$\begin{aligned}(\eta_2 f_2)_p &= \left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial f_2}{\partial x_2}\right)_p - \left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial f_2}{\partial x_1}\right)_p \\ &= \left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial y_2}{\partial v}\right)_q - \left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial y_2}{\partial u}\right)_q = 0,\end{aligned}$$

$$(\eta_i f_2)_p = \left(\frac{\partial f_2}{\partial x_i}\right)_p = 0 \qquad (3 \leq i \leq m),$$

$$(\eta_2 f_3)_p = \left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial f_3}{\partial x_2}\right)_p - \left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial f_3}{\partial x_1}\right)_p$$

$$= \left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial y_3}{\partial v}\right)_q - \left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial y_3}{\partial u}\right)_q = 0,$$

$$(\eta_i f_3)_p = \left(\frac{\partial f_3}{\partial x_i}\right)_p = 0 \qquad (3 \leq i \leq m),$$

to satisfy the condition that $\ker(df_{2,3})_p = \langle (\eta_2)_p, (\eta_3)_p, \ldots, (\eta_m)_p \rangle$. We have that, for example,

$$\eta_2\eta_3\widetilde{f_3} = \left\{ \left(\frac{\partial y_2}{\partial u}\right)_q \frac{\partial}{\partial x_2} - \left(\frac{\partial y_2}{\partial v}\right)_q \frac{\partial}{\partial x_1} \right\} \frac{\partial}{\partial x_3} \left\{ \left(\frac{\partial y_2}{\partial v}\right)_q f_3 - \left(\frac{\partial y_3}{\partial v}\right)_q f_2 \right\}$$

$$= \left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial y_2}{\partial v}\right)_q \frac{\partial^2 f_3}{\partial x_2 \partial x_3} - \left(\frac{\partial y_2}{\partial v}\right)_q^2 \frac{\partial^2 f_3}{\partial x_1 \partial x_3}$$

$$- \left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial y_3}{\partial v}\right)_q \frac{\partial^2 f_2}{\partial x_2 \partial x_3} + \left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial y_3}{\partial v}\right)_q \frac{\partial^2 f_2}{\partial x_1 \partial x_3},$$

$$(\eta_2\eta_3\widetilde{f_3})_p = \left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial^2 f_3}{\partial x_2 \partial x_3}\right)_p - \left(\frac{\partial y_2}{\partial v}\right)_q^2 \left(\frac{\partial^2 f_3}{\partial x_1 \partial x_3}\right)_p$$

$$- \left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial y_3}{\partial v}\right)_q \left(\frac{\partial^2 f_2}{\partial x_2 \partial x_3}\right)_p + \left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial y_3}{\partial v}\right)_q \left(\frac{\partial^2 f_2}{\partial x_1 \partial x_3}\right)_p$$

$$= -\left(\frac{\partial y_2}{\partial v}\right)_q^2 \left(\frac{\partial y_3}{\partial w}\right)_q + \left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial y_3}{\partial v}\right)_q \left(\frac{\partial y_2}{\partial w}\right)_q$$

$$= -\left(\frac{\partial y_2}{\partial v}\right)_q A \neq 0.$$

By similar calculations, we obtain that the matrix

$$\begin{pmatrix} (\eta_2\eta_2\widetilde{f_3})_p & (\eta_2\eta_3\widetilde{f_3})_p & \cdots & (\eta_2\eta_m\widetilde{f_3})_p \\ (\eta_3\eta_2\widetilde{f_3})_p & (\eta_3\eta_3\widetilde{f_3})_p & \cdots & (\eta_3\eta_m\widetilde{f_3})_p \\ \vdots & \vdots & & \vdots \\ (\eta_m\eta_2\widetilde{f_3})_p & (\eta_m\eta_3\widetilde{f_3})_p & \cdots & (\eta_m\eta_m\widetilde{f_3})_p \end{pmatrix},$$

denoted by $(\mathbf{H}_{\eta_2,\eta_3,\ldots,\eta_m}\widetilde{f_3})_p$, is equal to

$$\begin{pmatrix} (\eta_2\eta_2\widetilde{f_3})_p & -\left(\dfrac{\partial y_2}{\partial v}\right)_q A & \mathbf{0} & \mathbf{0} \\ -\left(\dfrac{\partial y_2}{\partial v}\right)_q A & 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -2A\mathbf{I}_\lambda & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2A\mathbf{I}_{m-\lambda-3} \end{pmatrix}.$$

We apply Theorem 1 to $f_{2,3}$. The above form of the matrix shows that $\ker(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m}\widetilde{f_3})_p = \{\mathbf{0}\}$. It follows that p is a fold point of $f_{2,3}$. The number of negative eigenvalues of $(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m}\widetilde{f_3})_p$ is

$$\begin{cases} \lambda+1 & (A>0) \\ m-\lambda-2 & (A<0). \end{cases}$$

It follows that the absolute index of the fold point p of $f_{2,3}$ is equal to $\min\{\lambda+1, m-\lambda-2\}$.

3.2.2. *Exceptional subcase.* We now deal with the subcase where q is a singular point of $\Pi|_E$. Since $(d(\Pi|_E))_q = \mathbf{0}$,

$$\left(\frac{\partial y_2}{\partial v}\right)_q = \left(\frac{\partial y_3}{\partial v}\right)_q = 0.$$

By the regularity of the coordinate transformation: $(u, v, w) \mapsto (y_1, y_2, y_3)$,

$$\left(\frac{\partial y_1}{\partial v}\right)_q \neq 0, \qquad \text{and either} \qquad \left(\frac{\partial y_2}{\partial u}\right)_q \neq 0 \qquad \text{or} \qquad \left(\frac{\partial y_3}{\partial u}\right)_q \neq 0.$$

Since y_2 and y_3 are symmetric so far, we may suppose that

$$\left(\frac{\partial y_2}{\partial u}\right)_q \neq 0$$

without loss of generality. Again by the regularity of the coordinate transformation,

$$\left(\frac{\partial y_2}{\partial u}\right)_q\left(\frac{\partial y_3}{\partial w}\right)_q - \left(\frac{\partial y_2}{\partial w}\right)_q\left(\frac{\partial y_3}{\partial u}\right)_q \neq 0.$$

Let B denote the left-hand side of this inequality.

We modify the local form of $f_{2,3}$ as follows. Let $\widetilde{f_3}$ be the function on U defined as

$$\widetilde{f_3} = \left(\frac{\partial y_2}{\partial u}\right)_q f_3 - \left(\frac{\partial y_3}{\partial u}\right)_q f_2.$$

Noting that the coefficient of f_3 is non-zero, we obtain the local form:

$$f_{2,3}(x_1, x_2, \dots, x_m) = (f_2(x_1, x_2, \dots, x_m), \widetilde{f_3}(x_1, x_2, \dots, x_m)).$$

We have arranged that

$$\left(\frac{\partial f_2}{\partial x_1}\right)_p = \left(\frac{\partial y_2}{\partial u}\right)_q \neq 0,$$

$$\begin{aligned}
\left(\frac{\partial \widetilde{f_3}}{\partial x_1}\right)_p &= \left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial f_3}{\partial x_1}\right)_p - \left(\frac{\partial y_3}{\partial u}\right)_q \left(\frac{\partial f_2}{\partial x_1}\right)_p \\
&= \left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial y_3}{\partial u}\right)_q - \left(\frac{\partial y_3}{\partial u}\right)_q \left(\frac{\partial y_2}{\partial u}\right)_q = 0, \\
\left(\frac{\partial \widetilde{f_3}}{\partial x_2}\right)_p &= \left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial f_3}{\partial x_2}\right)_p - \left(\frac{\partial y_3}{\partial u}\right)_q \left(\frac{\partial f_2}{\partial x_2}\right)_p \\
&= \left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial y_3}{\partial v}\right)_q - \left(\frac{\partial y_3}{\partial u}\right)_q \left(\frac{\partial y_2}{\partial v}\right)_q = 0,
\end{aligned}$$

$$\left(\frac{\partial \widetilde{f_3}}{\partial x_i}\right)_p = \left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial f_3}{\partial x_i}\right)_p - \left(\frac{\partial y_3}{\partial u}\right)_q \left(\frac{\partial f_2}{\partial x_i}\right)_p = 0 \qquad (3 \leq i \leq m),$$

to satisfy the conditions that $(df_2)_p \neq \mathbf{0}$ and $(d\widetilde{f_3})_p = \mathbf{0}$.

We calculate some derivatives with respect to appropriate vector fields as follows. Let $\eta_1, \eta_2, \dots, \eta_m$ be the C^∞ vector fields on U defined as

$$\eta_1 = \frac{\partial}{\partial x_1},$$

$$\eta_2 = \frac{\partial}{\partial x_2},$$

$$\eta_3 = \frac{\partial f_2}{\partial x_1}\frac{\partial}{\partial x_3} - \frac{\partial f_2}{\partial x_3}\frac{\partial}{\partial x_1},$$

$$\eta_i = \frac{\partial}{\partial x_i} \qquad (4 \leq i \leq m).$$

Noting that

$$\left(\frac{\partial f_2}{\partial x_1}\right)_p = \left(\frac{\partial y_2}{\partial u}\right)_q \neq 0,$$

we have that the vectors $(\eta_1)_p, (\eta_2)_p, \dots, (\eta_m)_p$ are linearly independent. We have arranged that

$$(\eta_1 f_2)_p = \left(\frac{\partial f_2}{\partial x_1}\right)_p = \left(\frac{\partial y_2}{\partial u}\right)_q \neq 0,$$

$$(\eta_2 f_2)_p = \left(\frac{\partial f_2}{\partial x_2}\right)_p = \left(\frac{\partial y_2}{\partial v}\right)_q = 0,$$

$$\eta_3 f_2 = \frac{\partial f_2}{\partial x_1}\frac{\partial f_2}{\partial x_3} - \frac{\partial f_2}{\partial x_3}\frac{\partial f_2}{\partial x_1} = 0,$$

$$(\eta_i f_2)_p = \left(\frac{\partial f_2}{\partial x_i}\right)_p = 0 \qquad (4 \leq i \leq m).$$

Note that $(df_2)_p$ and $(df_3)_p$ are linearly dependent, since p is a singular point of $f_{2,3}$. It follows that $(\eta_i f_3)_p = 0$ as well as $(\eta_i f_2)_p = 0$ for each $i \in \{2, 3, \dots, m\}$, to satisfy the condition that $\ker(df_{2,3})_p = \langle (\eta_2)_p, (\eta_3)_p, \dots, (\eta_m)_p \rangle$. Note also that $(\eta_1 f_2)_s \neq 0$ for any point s sufficiently close to p, and that $(df_2)_s$ and $(df_3)_s$ are linearly dependent for any singular point s of $f_{2,3}$. It follows that $(\eta_3 f_3)_s = 0$ as well as $(\eta_3 f_2)_s = 0$ for any singular point s of $f_{2,3}$ sufficiently close to p, to satisfy the condition that $(\eta_3)_s \in \ker(df_{2,3})_s$. We have that, for example,

$$\begin{aligned}
\eta_2\eta_2\widetilde{f_3} &= \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2}\left\{\left(\frac{\partial y_2}{\partial u}\right)_q f_3 - \left(\frac{\partial y_3}{\partial u}\right)_q f_2\right\} \\
&= \left(\frac{\partial y_2}{\partial u}\right)_q \frac{\partial^2 f_3}{\partial x_2^2} - \left(\frac{\partial y_3}{\partial u}\right)_q \frac{\partial^2 f_2}{\partial x_2^2},
\end{aligned}$$

$$(\eta_2\eta_2\widetilde{f_3})_p = \left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial^2 y_3}{\partial v^2}\right)_q - \left(\frac{\partial y_3}{\partial u}\right)_q \left(\frac{\partial^2 y_2}{\partial v^2}\right)_q,$$

$$\begin{aligned}
\eta_1\eta_3\widetilde{f_3} &= \frac{\partial}{\partial x_1}\left(\frac{\partial f_2}{\partial x_1}\frac{\partial}{\partial x_3} - \frac{\partial f_2}{\partial x_3}\frac{\partial}{\partial x_1}\right)\left\{\left(\frac{\partial y_2}{\partial u}\right)_q f_3 - \left(\frac{\partial y_3}{\partial u}\right)_q f_2\right\} \\
&= \frac{\partial}{\partial x_1}\left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial f_2}{\partial x_1}\frac{\partial f_3}{\partial x_3} - \frac{\partial f_2}{\partial x_3}\frac{\partial f_3}{\partial x_1}\right) \\
&= \left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial^2 f_2}{\partial x_1^2}\frac{\partial f_3}{\partial x_3} + \frac{\partial f_2}{\partial x_1}\frac{\partial^2 f_3}{\partial x_1 \partial x_3} - \frac{\partial^2 f_2}{\partial x_1 \partial x_3}\frac{\partial f_3}{\partial x_1} - \frac{\partial f_2}{\partial x_3}\frac{\partial^2 f_3}{\partial x_1^2}\right),
\end{aligned}$$

$$(\eta_1\eta_3\widetilde{f_3})_p = \left(\frac{\partial y_2}{\partial u}\right)_q B \neq 0,$$

$$\eta_3\eta_3\widetilde{f_3} = \left(\frac{\partial f_2}{\partial x_1}\frac{\partial}{\partial x_3} - \frac{\partial f_2}{\partial x_3}\frac{\partial}{\partial x_1}\right)\left(\frac{\partial f_2}{\partial x_1}\frac{\partial}{\partial x_3} - \frac{\partial f_2}{\partial x_3}\frac{\partial}{\partial x_1}\right)\left\{\left(\frac{\partial y_2}{\partial u}\right)_q f_3 - \left(\frac{\partial y_3}{\partial u}\right)_q f_2\right\}$$
$$= \left(\frac{\partial f_2}{\partial x_1}\frac{\partial}{\partial x_3} - \frac{\partial f_2}{\partial x_3}\frac{\partial}{\partial x_1}\right)\left(\frac{\partial y_2}{\partial u}\right)_q\left(\frac{\partial f_2}{\partial x_1}\frac{\partial f_3}{\partial x_3} - \frac{\partial f_2}{\partial x_3}\frac{\partial f_3}{\partial x_1}\right)$$
$$= \left(\frac{\partial y_2}{\partial u}\right)_q\frac{\partial f_2}{\partial x_1}\left(\frac{\partial^2 f_2}{\partial x_1\partial x_3}\frac{\partial f_3}{\partial x_3} + \frac{\partial f_2}{\partial x_1}\frac{\partial^2 f_3}{\partial x_3^2} - \frac{\partial^2 f_2}{\partial x_3^2}\frac{\partial f_3}{\partial x_1} - \frac{\partial f_2}{\partial x_3}\frac{\partial^2 f_3}{\partial x_1\partial x_3}\right)$$
$$- \left(\frac{\partial y_2}{\partial u}\right)_q\frac{\partial f_2}{\partial x_3}\left(\frac{\partial^2 f_2}{\partial x_1^2}\frac{\partial f_3}{\partial x_3} + \frac{\partial f_2}{\partial x_1}\frac{\partial^2 f_3}{\partial x_1\partial x_3} - \frac{\partial^2 f_2}{\partial x_1\partial x_3}\frac{\partial f_3}{\partial x_1} - \frac{\partial f_2}{\partial x_3}\frac{\partial^2 f_3}{\partial x_1^2}\right),$$

and $(\eta_3\eta_3\widetilde{f_3})_p = 0$. By similar calculations, we obtain that the matrix $(\mathbf{H}_{\eta_2,\eta_3,\ldots,\eta_m}\widetilde{f_3})_p$ is equal to

$$\begin{pmatrix} (\eta_2\eta_2\widetilde{f_3})_p & 0 & \mathbf{0} & \mathbf{0} \\ 0 & 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -2B\mathbf{I}_\lambda & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2B\mathbf{I}_{m-\lambda-3} \end{pmatrix}.$$

By similar but more complicated calculations,

$$\eta_3\eta_3\eta_3\widetilde{f_3} = \left(\frac{\partial y_2}{\partial u}\right)_q\left\{\frac{\partial f_2}{\partial x_1}\left(\frac{\partial^2 f_2}{\partial x_1\partial x_3}\right)^2\frac{\partial f_3}{\partial x_3} + \left(\frac{\partial f_2}{\partial x_1}\right)^2\frac{\partial^3 f_2}{\partial x_1\partial x_3^2}\frac{\partial f_3}{\partial x_3}\right.$$
$$+ 3\left(\frac{\partial f_2}{\partial x_1}\right)^2\frac{\partial^2 f_2}{\partial x_1\partial x_3}\frac{\partial^2 f_3}{\partial x_3^2} + \left(\frac{\partial f_2}{\partial x_1}\right)^3\frac{\partial^3 f_3}{\partial x_3^3} - \left(\frac{\partial f_2}{\partial x_1}\right)^2\frac{\partial^3 f_2}{\partial x_3^3}\frac{\partial f_3}{\partial x_1}$$
$$- 3\left(\frac{\partial f_2}{\partial x_1}\right)^2\frac{\partial^2 f_2}{\partial x_3^2}\frac{\partial^2 f_3}{\partial x_1\partial x_3} - 3\left(\frac{\partial f_2}{\partial x_1}\right)^2\frac{\partial f_2}{\partial x_3}\frac{\partial^3 f_3}{\partial x_1\partial x_3^2} - \frac{\partial f_2}{\partial x_1}\frac{\partial^2 f_2}{\partial x_3^2}\frac{\partial^2 f_2}{\partial x_1^2}\frac{\partial f_3}{\partial x_3}$$
$$- 2\frac{\partial f_2}{\partial x_1}\frac{\partial f_2}{\partial x_3}\frac{\partial^3 f_2}{\partial x_1^2\partial x_3}\frac{\partial f_3}{\partial x_3} - 3\frac{\partial f_2}{\partial x_1}\frac{\partial f_2}{\partial x_3}\frac{\partial^2 f_2}{\partial x_1^2}\frac{\partial^2 f_3}{\partial x_3^2} + 2\frac{\partial f_2}{\partial x_1}\frac{\partial f_2}{\partial x_3}\frac{\partial^3 f_2}{\partial x_1\partial x_3^2}\frac{\partial f_3}{\partial x_1}$$
$$+ 3\frac{\partial f_2}{\partial x_1}\frac{\partial f_2}{\partial x_3}\frac{\partial^2 f_2}{\partial x_3^2}\frac{\partial^2 f_3}{\partial x_1^2} + 3\frac{\partial f_2}{\partial x_1}\left(\frac{\partial f_2}{\partial x_3}\right)^2\frac{\partial^3 f_3}{\partial x_1^2\partial x_3} + \frac{\partial f_2}{\partial x_3}\frac{\partial^2 f_2}{\partial x_1^2}\frac{\partial^2 f_2}{\partial x_3^2}\frac{\partial f_3}{\partial x_1}$$
$$+ 3\left(\frac{\partial f_2}{\partial x_3}\right)^2\frac{\partial^2 f_2}{\partial x_1^2}\frac{\partial^2 f_3}{\partial x_1\partial x_3} + \left(\frac{\partial f_2}{\partial x_3}\right)^2\frac{\partial^3 f_2}{\partial x_1^3}\frac{\partial f_3}{\partial x_3} - \frac{\partial f_2}{\partial x_3}\left(\frac{\partial^2 f_2}{\partial x_1\partial x_3}\right)^2\frac{\partial f_3}{\partial x_1}$$
$$\left. - \left(\frac{\partial f_2}{\partial x_3}\right)^2\frac{\partial^3 f_2}{\partial x_1^2\partial x_3}\frac{\partial f_3}{\partial x_1} - 3\left(\frac{\partial f_2}{\partial x_3}\right)^2\frac{\partial^2 f_2}{\partial x_1\partial x_3}\frac{\partial^2 f_3}{\partial x_1^2} - \left(\frac{\partial f_2}{\partial x_3}\right)^3\frac{\partial^3 f_3}{\partial x_1^3}\right\},$$

$$(\eta_3\eta_3\eta_3\widetilde{f_3})_p = 6\left(\frac{\partial y_2}{\partial u}\right)_q^3 B \neq 0.$$

We apply Theorems 1 and 2 to $f_{2,3}$. The above form of the matrix shows that $\ker(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m}\widetilde{f_3})_p \neq \{\mathbf{0}\}$. It follows that p is not a fold point of $f_{2,3}$. Note that $\ker(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m}\widetilde{f_3})_p$ is 1-dimensional if and only if $(\eta_2\eta_2\widetilde{f_3})_p \neq 0$. If so, then $\ker(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m}\widetilde{f_3})_p = \langle(\eta_3)_p\rangle$. Since $(\eta_1\eta_3\widetilde{f_3})_p \neq 0$, we can see that $(d(\eta_3\widetilde{f_3}))_p \neq \mathbf{0}$. From this and the result that $(\eta_3\eta_3\eta_3\widetilde{f_3})_p \neq 0$, it follows that p is a cusp point of $f_{2,3}$ if and only if $(\eta_2\eta_2\widetilde{f_3})_p \neq 0$. The number of negative eigenvalues of $(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m}\widetilde{f_3})_p$ is

$$\begin{cases} \lambda & (B > 0,\ (\eta_2\eta_2\widetilde{f_3})_p > 0) \\ \lambda + 1 & (B > 0,\ (\eta_2\eta_2\widetilde{f_3})_p < 0) \\ m - \lambda - 3 & (B < 0,\ (\eta_2\eta_2\widetilde{f_3})_p > 0) \\ m - \lambda - 2 & (B < 0,\ (\eta_2\eta_2\widetilde{f_3})_p < 0). \end{cases}$$

It follows that, if p is a cusp point of $f_{2,3}$, its absolute index is equal to either $\min\{\lambda, m-\lambda-2\}$ or $\min\{\lambda+1, m-\lambda-3\}$. Note that $\min\{\lambda, m-\lambda-2\} = \lambda$ since $\lambda \leq m - \lambda - 3$.

We consider when the limiting normal curvature of D at q vanishes. Recall that

$$f|_S(x_2, x_3) = (-x_3^2, x_2, -2x_3^3)$$

with respect to (u, v, w). Note that

$$f|_S(x_2, x_3) = (f_1|_S(x_2, x_3), f_2|_S(x_2, x_3), f_3|_S(x_2, x_3))$$

with respect to (y_1, y_2, y_3), and

$$f_k|_S(x_2, x_3) = y_k(-x_3^2, x_2, -2x_3^3)$$

for each $k \in \{1, 2, 3\}$. By the chain rule, for example,

$$\frac{\partial(f_1|_S)}{\partial x_2} = \left\{\frac{\partial}{\partial x_2}(-x_3^2)\right\}\frac{\partial y_1}{\partial u} + \left(\frac{\partial}{\partial x_2}x_2\right)\frac{\partial y_1}{\partial v} + \left\{\frac{\partial}{\partial x_2}(-2x_3^3)\right\}\frac{\partial y_1}{\partial w} = \frac{\partial y_1}{\partial v},$$

$$\left(\frac{\partial(f_1|_S)}{\partial x_2}\right)_p = \left(\frac{\partial y_1}{\partial v}\right)_q \neq 0.$$

By similar calculations,

$$(d(f|_S))\left(\frac{\partial}{\partial x_2}\right) = \sum_{k=1}^{3}\frac{\partial(f_k|_S)}{\partial x_2}\frac{\partial}{\partial y_k} = \sum_{k=1}^{3}\frac{\partial y_k}{\partial v}\frac{\partial}{\partial y_k},$$

$$(d(f|_S))_p\left(\frac{\partial}{\partial x_2}\right) = \sum_{k=1}^{3}\left(\frac{\partial y_k}{\partial v}\right)_q \frac{\partial}{\partial y_k} = \left(\frac{\partial y_1}{\partial v}\right)_q \frac{\partial}{\partial y_1} \neq \mathbf{0},$$

$$(d(f|_S))_p\left(\frac{\partial}{\partial x_3}\right) = \mathbf{0}.$$

Let g denote the Riemannian metric of Y, and ∇ denote the Levi-Civita connection. Let $g_{i,j}$ denote the function

$$g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right)$$

on Y, and $\Gamma_{i,j}^k$ be the Christoffel's symbol, that is to say,

$$\nabla_{\partial/(\partial y_i)} \frac{\partial}{\partial y_j} = \sum_{k=1}^{3} \Gamma_{i,j}^k \frac{\partial}{\partial y_k}.$$

By the product structure, we have that $g_{i,j} = 0$ and $\Gamma_{i,j}^k = 0$ unless either $i = j = k = 1$ or $i, j, k \in \{2, 3\}$. It follows that

$$\begin{aligned}
&\nabla_{(d(f|_S))(\partial/(\partial x_2))}(d(f|_S))\left(\frac{\partial}{\partial x_2}\right) \\
&\quad = \nabla_{\sum_{i=1}^{3}(\partial y_i/\partial v)(\partial/\partial y_i)} \sum_{j=1}^{3} \frac{\partial y_j}{\partial v}\frac{\partial}{\partial y_j} \\
&\quad = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial y_i}{\partial v}\frac{\partial y_j}{\partial v} \nabla_{\partial/(\partial y_i)} \frac{\partial}{\partial y_j} + \sum_{j=1}^{3}\left\{\left(\sum_{i=1}^{3}\frac{\partial y_i}{\partial v}\frac{\partial}{\partial y_i}\right)\frac{\partial y_j}{\partial v}\right\}\frac{\partial}{\partial y_j} \\
&\quad = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \frac{\partial y_i}{\partial v}\frac{\partial y_j}{\partial v}\Gamma_{i,j}^k \frac{\partial}{\partial y_k} + \sum_{j=1}^{3}\frac{\partial^2 y_j}{\partial v^2}\frac{\partial}{\partial y_j},
\end{aligned}$$

$$\begin{aligned}
&\left(\nabla_{(d(f|_S))(\partial/(\partial x_2))}(d(f|_S))\left(\frac{\partial}{\partial x_2}\right)\right)_p \\
&\quad = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\left(\frac{\partial y_i}{\partial v}\right)_q\left(\frac{\partial y_j}{\partial v}\right)_q (\Gamma_{i,j}^k)_q \frac{\partial}{\partial y_k} + \sum_{j=1}^{3}\left(\frac{\partial^2 y_j}{\partial v^2}\right)_q \frac{\partial}{\partial y_j} \\
&\quad = \left(\frac{\partial y_1}{\partial v}\right)_q^2 (\Gamma_{1,1}^1)_q \frac{\partial}{\partial y_1} + \sum_{j=1}^{3}\left(\frac{\partial^2 y_j}{\partial v^2}\right)_q \frac{\partial}{\partial y_j}.
\end{aligned}$$

It is well-known that D is a so-called frontal, that is to say, $f|_S$ has a unit normal vector field $\nu : U \to TY$. Recall that the uv-plane is a tangent disk of

D at q, and note that the v-axis is parallel to the y_1-axis in T_qY. It follows that $v(q)$ is obtained by normalizing

$$\pm\left\{g_{2,3}(q)\left(\frac{\partial y_2}{\partial u}\right)_q + g_{3,3}(q)\left(\frac{\partial y_3}{\partial u}\right)_q\right\}\frac{\partial}{\partial y_2}$$

$$\mp\left\{g_{2,2}(q)\left(\frac{\partial y_2}{\partial u}\right)_q + g_{2,3}(q)\left(\frac{\partial y_3}{\partial u}\right)_q\right\}\frac{\partial}{\partial y_3}.$$

By substituting them, the limiting normal curvature

$$\frac{g\left(\left(\nabla_{(d(f|_S))(\partial/(\partial x_2))}(d(f|_S))\left(\frac{\partial}{\partial x_2}\right)\right)_p, v(p)\right)}{g\left((d(f|_S))_p\left(\frac{\partial}{\partial x_2}\right), (d(f|_S))_p\left(\frac{\partial}{\partial x_2}\right)\right)}$$

is equal to

$$\mp\{g_{2,2}(q)g_{3,3}(q) - (g_{2,3}(q))^2\}\left\{\left(\frac{\partial y_2}{\partial u}\right)_q\left(\frac{\partial^2 y_3}{\partial v^2}\right)_q - \left(\frac{\partial y_3}{\partial u}\right)_q\left(\frac{\partial^2 y_2}{\partial v^2}\right)_q\right\}$$

multiplied by a certain non-zero constant. By the regularity of the metric, it is non-zero if and only if $(\eta_2\eta_2\widetilde{f_3})_p \neq 0$. Combining this with the result in the previous paragraph, we complete our proof of Proposition 7. □

4. Proof in swallow-tail case

In this section, we give a proof of Proposition 8. We take this here because it is similar to that in the previous section.

We begin with the following local forms of the relevant maps. There exist local coordinate systems $(x_1, x_2, \dots, x_m)$ and (u, v, w) of X and Y, respectively, with respect to which $p = \mathbf{0}$ and

$$f(x_1, x_2, \dots, x_m) = (x_1, x_2, x_3^4 + x_1x_3^2 + x_2x_3 - x_4^2 - \cdots - x_{\lambda+3}^2 + x_{\lambda+4}^2 + \cdots + x_m^2).$$

There also exist a local coordinate y_1 of Y_1, and a local coordinate system (y_2, y_3) of $Y_{2,3}$ with respect to which $q = \mathbf{0}$ and $\pi(y_1, y_2, y_3) = y_1$ and $\Pi(y_1, y_2, y_3) = (y_2, y_3)$. Let f_2, f_3 denote the functions on U such that

$$f_{2,3}(x_1, x_2, \dots, x_m) = (f_2(x_1, x_2, \dots, x_m), f_3(x_1, x_2, \dots, x_m))$$

with respect to (y_2, y_3), and

$$f(x_1, x_2, \dots, x_m) = (f_1(x_1, x_2, \dots, x_m), f_2(x_1, x_2, \dots, x_m), f_3(x_1, x_2, \dots, x_m))$$

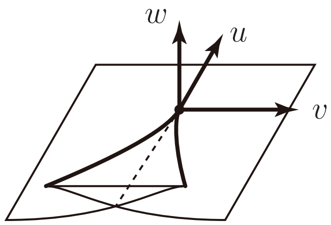

Fig. 3. The discriminant set D and the coordinate system (u, v, w).

with respect to (y_1, y_2, y_3). Note that there is a coordinate transformation:

$$(u, v, w) \mapsto (y_1(u, v, w), y_2(u, v, w), y_3(u, v, w)).$$

Let S denote the singular set of $f|_U$. We can see that S has the local form:

$$\{(x_1, -4x_3^3 - 2x_1x_3, x_3, 0, \dots, 0) \mid x_1, x_3 \in \mathbb{R}\}$$

with respect to $(x_1, x_2, \dots, x_m)$. We can regard (x_1, x_3) as a local coordinate system of S at p, and then $f|_S$ has the local form:

$$f|_S(x_1, x_3) = (x_1, -4x_3^3 - 2x_1x_3, -3x_3^4 - x_1x_3^2)$$

with respect to (u, v, w). This shows that D is such a singular surface as described in Subsection 2.1, and is related with (u, v, w) as in Figure 3. The uv-plane is a tangent disk of D at $f(p)$, as well as $\tilde{D}$. The u-axis is a tangent arc of E at $f(p)$ in D, as well as $\tilde{E}$.

We calculate partial derivatives similarly to those in Section 3. Then we obtain that, for each $k \in \{1, 2, 3\}$ and $i, j \in \{1, 2, \dots, m\}$,

$$\frac{\partial f_k}{\partial x_i} = \begin{cases} \dfrac{\partial y_k}{\partial u} + x_3^2 \dfrac{\partial y_k}{\partial w} & (i = 1) \\ \dfrac{\partial y_k}{\partial v} + x_3 \dfrac{\partial y_k}{\partial w} & (i = 2) \\ (4x_3^3 + 2x_1x_3 + x_2) \dfrac{\partial y_k}{\partial w} & (i = 3) \\ -2x_i \dfrac{\partial y_k}{\partial w} & (4 \le i \le \lambda + 3) \\ 2x_i \dfrac{\partial y_k}{\partial w} & (\lambda + 4 \le i \le m), \end{cases}$$

$$\frac{\partial^2 f_k}{\partial x_1 \partial x_j} = \begin{cases} \dfrac{\partial^2 y_k}{\partial u^2} + 2x_3^2 \dfrac{\partial^2 y_k}{\partial u \partial w} + x_3^4 \dfrac{\partial^2 y_k}{\partial w^2} & (j = 1) \\ \dfrac{\partial^2 y_k}{\partial u \partial v} + x_3 \dfrac{\partial^2 y_k}{\partial u \partial w} + x_3^2 \dfrac{\partial^2 y_k}{\partial v \partial w} + x_3^3 \dfrac{\partial^2 y_k}{\partial w^2} & (j = 2) \end{cases}$$

$$\frac{\partial^2 f_k}{\partial x_1 \partial x_j} = \begin{cases} 2x_3 \dfrac{\partial y_k}{\partial w} + (4x_3^3 + 2x_1x_3 + x_2) \dfrac{\partial^2 y_k}{\partial u \partial w} & \\ \qquad + x_3^2(4x_3^3 + 2x_1x_3 + x_2) \dfrac{\partial^2 y_k}{\partial w^2} & (j = 3) \\ -2x_j \dfrac{\partial^2 y_k}{\partial u \partial w} - 2x_3^2 x_j \dfrac{\partial^2 y_k}{\partial w^2} & (4 \le j \le \lambda + 3) \\ 2x_j \dfrac{\partial^2 y_k}{\partial u \partial w} + 2x_3^2 x_j \dfrac{\partial^2 y_k}{\partial w^2} & (\lambda + 4 \le j \le m), \end{cases}$$

$$\frac{\partial^2 f_k}{\partial x_2 \partial x_j} = \begin{cases} \dfrac{\partial^2 y_k}{\partial v^2} + 2x_3 \dfrac{\partial^2 y_k}{\partial v \partial w} + x_3^2 \dfrac{\partial^2 y_k}{\partial w^2} & (j = 2) \\ \dfrac{\partial y_k}{\partial w} + (4x_3^3 + 2x_1x_3 + x_2) \dfrac{\partial^2 y_k}{\partial v \partial w} & \\ \qquad + x_3(4x_3^3 + 2x_1x_3 + x_2) \dfrac{\partial^2 y_k}{\partial w^2} & (j = 3) \\ -2x_j \dfrac{\partial^2 y_k}{\partial v \partial w} - 2x_3 x_j \dfrac{\partial^2 y_k}{\partial w^2} & (4 \le j \le \lambda + 3) \\ 2x_j \dfrac{\partial^2 y_k}{\partial v \partial w} + 2x_3 x_j \dfrac{\partial^2 y_k}{\partial w^2} & (\lambda + 4 \le j \le m), \end{cases}$$

$$\frac{\partial^2 f_k}{\partial x_3 \partial x_j} = \begin{cases} (12x_3^2 + 2x_1) \dfrac{\partial y_k}{\partial w} + (4x_3^3 + 2x_1x_3 + x_2)^2 \dfrac{\partial^2 y_k}{\partial w^2} & (j = 3) \\ -2x_j(4x_3^3 + 2x_1x_3 + x_2) \dfrac{\partial^2 y_k}{\partial w^2} & (4 \le j \le \lambda + 3) \\ 2x_j(4x_3^3 + 2x_1x_3 + x_2) \dfrac{\partial^2 y_k}{\partial w^2} & (\lambda + 4 \le j \le m), \end{cases}$$

$$\frac{\partial^2 f_k}{\partial x_i \partial x_j} = \begin{cases} -2 \dfrac{\partial y_k}{\partial w} + 4x_i^2 \dfrac{\partial^2 y_k}{\partial w^2} & (4 \le i = j \le \lambda + 3) \\ 2 \dfrac{\partial y_k}{\partial w} + 4x_i^2 \dfrac{\partial^2 y_k}{\partial w^2} & (\lambda + 4 \le i = j \le m) \\ 4x_i x_j \dfrac{\partial^2 y_k}{\partial w^2} & (4 \le i < j \le \lambda + 3 \ \text{ or } \ \lambda + 4 \le i < j \le m) \\ -4x_i x_j \dfrac{\partial^2 y_k}{\partial w^2} & (4 \le i \le \lambda + 3 < j \le m), \end{cases}$$

$$\frac{\partial^3 f_k}{\partial x_3^3} = 24x_3 \frac{\partial y_k}{\partial w} + 3(12x_3^2 + 2x_1)(4x_3^3 + 2x_1x_3 + x_2) \frac{\partial^2 y_k}{\partial w^2}$$
$$+ (4x_3^3 + 2x_1x_3 + x_2)^3 \frac{\partial^3 y_k}{\partial w^3}.$$

Since $p = \mathbf{0}$ and $f(p) = q$, for each $k \in \{1, 2, 3\}$ and $i, j \in \{1, 2, \dots, m\}$ such that $i \le j$,

$$\left(\frac{\partial f_k}{\partial x_i}\right)_p = \begin{cases} \left(\dfrac{\partial y_k}{\partial u}\right)_q & (i=1) \\ \left(\dfrac{\partial y_k}{\partial v}\right)_q & (i=2) \\ 0 & (3 \le i \le m), \end{cases}$$

$$\left(\frac{\partial^2 f_k}{\partial x_i \partial x_j}\right)_p = \begin{cases} \left(\dfrac{\partial^2 y_k}{\partial u^2}\right)_q & (i=j=1) \\ \left(\dfrac{\partial^2 y_k}{\partial u \partial v}\right)_q & (i=1,\ j=2) \\ \left(\dfrac{\partial^2 y_k}{\partial v^2}\right)_q & (i=j=2) \\ \left(\dfrac{\partial y_k}{\partial w}\right)_q & (i=2,\ j=3) \\ -2\left(\dfrac{\partial y_k}{\partial w}\right)_q & (4 \le i = j \le \lambda+3) \\ 2\left(\dfrac{\partial y_k}{\partial w}\right)_q & (\lambda+4 \le i = j \le m) \\ 0 & (\text{otherwise}), \end{cases}$$

$$\left(\frac{\partial^3 f_k}{\partial x_3^3}\right)_p = 0.$$

4.1. Function case. We first focus on the function f_1.

We consider whether p is a regular or singular point of f_1. The same argument as in Subsection 3.1 shows that p is a regular point of f_1 if and only if q is a regular point of $\pi|_{\tilde{D}}$, which is the case if and only if

$$\left(\frac{\partial y_1}{\partial u}\right)_q \neq 0 \qquad \text{or} \qquad \left(\frac{\partial y_1}{\partial v}\right)_q \neq 0.$$

Supposing that p is a singular point of f_1, we consider what type it is. By the above result, q is a singular point of $\pi|_{\tilde{D}}$, and

$$\left(\frac{\partial y_1}{\partial u}\right)_q = \left(\frac{\partial y_1}{\partial v}\right)_q = 0.$$

By the regularity of the coordinate transformation: $(u, v, w) \mapsto (y_1, y_2, y_3)$,

$$\left(\frac{\partial y_1}{\partial w}\right)_q \neq 0, \qquad \text{and either} \qquad \left(\frac{\partial y_2}{\partial u}\right)_q \neq 0 \qquad \text{or} \qquad \left(\frac{\partial y_3}{\partial u}\right)_q \neq 0.$$

The Hessian matrix $(\mathbf{H}_{x_1,x_2,\dots,x_m} f_1)_p$ is equal to

$$\begin{pmatrix} \left(\frac{\partial^2 y_1}{\partial u^2}\right)_q & \left(\frac{\partial^2 y_1}{\partial u \partial v}\right)_q & 0 & \mathbf{0} & \mathbf{0} \\ \left(\frac{\partial^2 y_1}{\partial u \partial v}\right)_q & \left(\frac{\partial^2 y_1}{\partial v^2}\right)_q & \left(\frac{\partial y_1}{\partial w}\right)_q & \mathbf{0} & \mathbf{0} \\ 0 & \left(\frac{\partial y_1}{\partial w}\right)_q & 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -2\left(\frac{\partial y_1}{\partial w}\right)_q \mathbf{I}_\lambda & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 2\left(\frac{\partial y_1}{\partial w}\right)_q \mathbf{I}_{m-\lambda-3} \end{pmatrix},$$

which has maximal rank if and only if

$$\left(\frac{\partial^2 y_1}{\partial u^2}\right)_q \neq 0.$$

Hence, p is a fold point of f_1 if and only if this inequality holds. The number of negative eigenvalues of $(\mathbf{H}_{x_1,x_2,\dots,x_m} f_1)_p$ is

$$\begin{cases} \lambda + 1 & \left(\left(\frac{\partial y_1}{\partial w}\right)_q > 0, \left(\frac{\partial^2 y_1}{\partial u^2}\right)_q > 0\right) \\ \lambda + 2 & \left(\left(\frac{\partial y_1}{\partial w}\right)_q > 0, \left(\frac{\partial^2 y_1}{\partial u^2}\right)_q < 0\right) \\ m - \lambda - 2 & \left(\left(\frac{\partial y_1}{\partial w}\right)_q < 0, \left(\frac{\partial^2 y_1}{\partial u^2}\right)_q > 0\right) \\ m - \lambda - 1 & \left(\left(\frac{\partial y_1}{\partial w}\right)_q < 0, \left(\frac{\partial^2 y_1}{\partial u^2}\right)_q < 0\right). \end{cases}$$

Hence, if p is a fold point of f_1, its absolute index is equal to either $\min\{\lambda + 1, m - \lambda - 1\}$ or $\min\{\lambda + 2, m - \lambda - 2\}$.

We consider when the limiting normal curvature of D at q vanishes. Note that

$$f|_S(x_1, x_3) = (f_1|_S(x_1, x_3), f_2|_S(x_1, x_3), f_3|_S(x_1, x_3))$$

with respect to (y_1, y_2, y_3), and

$$f_k|_S(x_1, x_3) = y_k(x_1, -4x_3^3 - 2x_1x_3, -3x_3^4 - x_1x_3^2)$$

for each $k \in \{1,2,3\}$. By calculations similar to those in Subsubsection 3.2.2,

$$(d(f|_S))\left(\frac{\partial}{\partial x_1}\right) = \sum_{k=1}^{3}\left(\frac{\partial y_k}{\partial u} - 2x_3\frac{\partial y_k}{\partial v} - x_3^2\frac{\partial y_k}{\partial w}\right)\frac{\partial}{\partial y_k},$$

$$(d(f|_S))_p\left(\frac{\partial}{\partial x_1}\right) = \left(\frac{\partial y_2}{\partial u}\right)_q\frac{\partial}{\partial y_2} + \left(\frac{\partial y_3}{\partial u}\right)_q\frac{\partial}{\partial y_3} \neq \mathbf{0},$$

$$(d(f|_S))_p\left(\frac{\partial}{\partial x_3}\right) = \mathbf{0},$$

$$\left(\nabla_{(d(f|_S))(\partial/(\partial x_1))}(d(f|_S))\left(\frac{\partial}{\partial x_1}\right)\right)_p$$
$$= \sum_{i=2}^{3}\sum_{j=2}^{3}\sum_{k=2}^{3}\left(\frac{\partial y_i}{\partial u}\right)_q\left(\frac{\partial y_j}{\partial u}\right)_q(\Gamma_{i,j}^k)_q\frac{\partial}{\partial y_k} + \sum_{j=1}^{3}\left(\frac{\partial^2 y_j}{\partial u^2}\right)_q\frac{\partial}{\partial y_j},$$

where ∇ denotes the Levi-Civita connection for the product metric of Y, and $\Gamma_{i,j}^k$ is the Christoffel's symbol. It is well-known that $f|_S$ has a unit normal vector field v. Since the $y_2 y_3$-plane is a tangent disk of D at q, the vector $v(q)$ is parallel to the y_1-axis in $T_q Y$. It follows that the limiting normal curvature is equal to

$$\left(\frac{\partial^2 y_1}{\partial u^2}\right)_q$$

multiplied by a certain non-zero constant. Combining this with the result in the previous paragraph, we complete our proof of Proposition 8 in the function case.

4.2. Surface-valued map case. We now focus on the map $f_{2,3}$.

We consider whether p is a regular or singular point of $f_{2,3}$. The same argument as in Subsection 3.2 shows that p is a regular point of $f_{2,3}$ if and only if q is a regular point of $\Pi|_{\tilde{D}}$, which is the case if and only if

$$\left(\frac{\partial y_2}{\partial u}\right)_q\left(\frac{\partial y_3}{\partial v}\right)_q - \left(\frac{\partial y_2}{\partial v}\right)_q\left(\frac{\partial y_3}{\partial u}\right)_q \neq 0.$$

Supposing that p is a singular point of $f_{2,3}$, we consider what type it is, in the rest of this section. By the above result, q is a singular point of $\Pi|_{\tilde{D}}$, and

$$\left(\frac{\partial y_2}{\partial u}\right)_q\left(\frac{\partial y_3}{\partial v}\right)_q - \left(\frac{\partial y_2}{\partial v}\right)_q\left(\frac{\partial y_3}{\partial u}\right)_q = 0.$$

We have the following two subcases.

4.2.1. *Generic subcase.* We first deal with the subcase where q is a regular point of $\Pi|_{\tilde{E}}$. Since the arc $\tilde{E}$ is tangent to the u-axis at q,

$$(d(\Pi|_{\tilde{E}}))_q = \begin{pmatrix} \left(\dfrac{\partial y_2}{\partial u}\right)_q \\ \left(\dfrac{\partial y_3}{\partial u}\right)_q \end{pmatrix}.$$

Since $(d(g|_{\tilde{E}}))_q \neq \mathbf{0}$ and the coordinates y_2 and y_3 are symmetric so far, we may suppose that

$$\left(\frac{\partial y_2}{\partial u}\right)_q \neq 0$$

without loss of generality. By the regularity of the coordinate transformation: $(u, v, w) \mapsto (y_1, y_2, y_3)$,

$$\left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial y_3}{\partial w}\right)_q - \left(\frac{\partial y_2}{\partial w}\right)_q \left(\frac{\partial y_3}{\partial u}\right)_q \neq 0.$$

Let B denote the left-hand side of this inequality.

We modify the local form of $f_{2,3}$ as follows. Let $\widetilde{f_3}$ be the function on U defined as

$$\widetilde{f_3} = \left(\frac{\partial y_2}{\partial u}\right)_q f_3 - \left(\frac{\partial y_3}{\partial u}\right)_q f_2.$$

Noting that the coefficient of f_3 is non-zero, we obtain the local form:

$$f_{2,3}(x_1, x_2, \ldots, x_m) = (f_2(x_1, x_2, \ldots, x_m), \widetilde{f_3}(x_1, x_2, \ldots, x_m)).$$

We have arranged that

$$\left(\frac{\partial f_2}{\partial x_1}\right)_p = \left(\frac{\partial y_2}{\partial u}\right)_q \neq 0,$$

$$\left(\frac{\partial \widetilde{f_3}}{\partial x_i}\right)_p = \left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial f_3}{\partial x_i}\right)_p - \left(\frac{\partial y_3}{\partial u}\right)_q \left(\frac{\partial f_2}{\partial x_i}\right)_p = 0 \qquad (1 \leq i \leq m),$$

to satisfy the conditions that $(df_2)_p \neq \mathbf{0}$ and $(d\widetilde{f_3})_p = \mathbf{0}$.

We calculate some derivatives with respect to appropriate vector fields as follows. Let $\eta_2, \eta_3, \ldots, \eta_m$ be the C^∞ vector fields on U defined as

$$\eta_2 = \left(\frac{\partial y_2}{\partial u}\right)_q \frac{\partial}{\partial x_2} - \left(\frac{\partial y_2}{\partial v}\right)_q \frac{\partial}{\partial x_1},$$

$$\eta_i = \frac{\partial}{\partial x_i} \qquad (3 \leq i \leq m).$$

We have arranged that $(\eta_2)_p, (\eta_3)_p, \dots, (\eta_m)_p$ are linearly independent, and $(\eta_i f_2)_p = (\eta_i f_3)_p = 0$ for $i \in \{2, 3, \dots m\}$, to satisfy the condition that $\ker(df_{2,3})_p = \langle (\eta_2)_p, (\eta_3)_p, \dots, (\eta_m)_p \rangle$. We have that, for example,

$$\begin{aligned}
\eta_2\eta_3\widetilde{f_3} &= \left\{ \left(\frac{\partial y_2}{\partial u}\right)_q \frac{\partial}{\partial x_2} - \left(\frac{\partial y_2}{\partial v}\right)_q \frac{\partial}{\partial x_1} \right\} \frac{\partial}{\partial x_3} \left\{ \left(\frac{\partial y_2}{\partial u}\right)_q f_3 - \left(\frac{\partial y_3}{\partial u}\right)_q f_2 \right\} \\
&= \left(\frac{\partial y_2}{\partial u}\right)_q^2 \frac{\partial^2 f_3}{\partial x_2 \partial x_3} - \left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial y_2}{\partial u}\right)_q \frac{\partial^2 f_3}{\partial x_1 \partial x_3} \\
&\quad - \left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial y_3}{\partial u}\right)_q \frac{\partial^2 f_2}{\partial x_2 \partial x_3} + \left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial y_3}{\partial u}\right)_q \frac{\partial^2 f_2}{\partial x_1 \partial x_3},
\end{aligned}$$

$$(\eta_2\eta_3\widetilde{f_3})_p = \left(\frac{\partial y_2}{\partial u}\right)_q B \neq 0.$$

By similar calculations, we obtain that the matrix $(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m}\widetilde{f_3})_p$ is equal to

$$\begin{pmatrix}
(\eta_2\eta_2\widetilde{f_3})_p & \left(\frac{\partial y_2}{\partial u}\right)_q B & \mathbf{0} & \mathbf{0} \\
\left(\frac{\partial y_2}{\partial u}\right)_q B & 0 & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & -2B\mathbf{I}_\lambda & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & 2B\mathbf{I}_{m-\lambda-3}
\end{pmatrix}.$$

We apply Theorem 1 to $f_{2,3}$. The above form of the matrix shows that $\ker(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m}\widetilde{f_3})_p = \{\mathbf{0}\}$. It follows that p is a fold point of $f_{2,3}$. The number of negative eigenvalues of $(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m}\widetilde{f_3})_p$ is

$$\begin{cases} \lambda + 1 & (B > 0) \\ m - \lambda - 2 & (B < 0). \end{cases}$$

It follows that the absolute index of the fold point p of $f_{2,3}$ is equal to $\min\{\lambda + 1, m - \lambda - 2\}$.

4.2.2. *Exceptional subcase.* We now deal with the subcase where q is a singular point of $\Pi|_{\tilde{E}}$. Since $(d(\Pi|_{\tilde{E}}))_q = \mathbf{0}$,

$$\left(\frac{\partial y_2}{\partial u}\right)_q = \left(\frac{\partial y_3}{\partial u}\right)_q = 0.$$

By the regularity of the coordinate transformation: $(u, v, w) \mapsto (y_1, y_2, y_3)$,

$$\left(\frac{\partial y_2}{\partial v}\right)_q \neq 0 \qquad \text{or} \qquad \left(\frac{\partial y_3}{\partial v}\right)_q \neq 0.$$

Since y_2 and y_3 are symmetric so far, we may suppose that

$$\left(\frac{\partial y_2}{\partial v}\right)_q \neq 0$$

without loss of generality.

We modify the local form of $f_{2,3}$ as follows. Let $\widetilde{f_3}$ be the function on U defined as

$$\widetilde{f_3} = \left(\frac{\partial y_2}{\partial v}\right)_q f_3 - \left(\frac{\partial y_3}{\partial v}\right)_q f_2.$$

Noting that the coefficient of f_3 is non-zero, we obtain the local form:

$$f_{2,3}(x_1, x_2, \dots, x_m) = (f_2(x_1, x_2, \dots, x_m), \widetilde{f_3}(x_1, x_2, \dots, x_m)).$$

We have arranged that

$$\left(\frac{\partial f_2}{\partial x_2}\right)_p = \left(\frac{\partial y_2}{\partial v}\right)_q \neq 0,$$

$$\left(\frac{\partial \widetilde{f_3}}{\partial x_i}\right)_p = \left(\frac{\partial y_2}{\partial v}\right)_q \left(\frac{\partial f_3}{\partial x_i}\right)_p - \left(\frac{\partial y_3}{\partial v}\right)_q \left(\frac{\partial f_2}{\partial x_i}\right)_p = 0 \qquad (1 \leq i \leq m),$$

to satisfy the conditions that $(df_2)_p \neq \mathbf{0}$ and $(d\widetilde{f_3})_p = \mathbf{0}$.

We calculate some derivatives with respect to appropriate vector fields as follows. Let $\eta_1, \eta_2, \dots, \eta_m$ be the C^∞ vector fields on U defined as

$$\eta_1 = \frac{\partial}{\partial x_2},$$

$$\eta_2 = \frac{\partial}{\partial x_1},$$

$$\eta_3 = \frac{\partial f_2}{\partial x_2}\frac{\partial}{\partial x_3} - \frac{\partial f_2}{\partial x_3}\frac{\partial}{\partial x_2},$$

$$\eta_i = \frac{\partial}{\partial x_i} \qquad (4 \leq i \leq m).$$

Noting that

$$\left(\frac{\partial f_2}{\partial x_2}\right)_p = \left(\frac{\partial y_2}{\partial v}\right)_q \neq 0,$$

we have that the vectors $(\eta_1)_p, (\eta_2)_p, \dots, (\eta_m)_p$ are linearly independent. We have arranged that $(\eta_1 f_2)_p \neq 0$ and $(\eta_i f_2)_p = 0$ for $i \in \{2, 3, \dots m\}$, and $\eta_3 f_2 =$

0. By a similar argument to that in Subsubsection 3.2.2, they satisfy the conditions that $\ker(df_{2,3})_p = \langle(\eta_2)_p, (\eta_3)_p, \dots, (\eta_m)_p\rangle$ and $(\eta_3)_s \in \ker(df_{2,3})_s$ for any singular point s of $f_{2,3}$ sufficiently close to p. We have that, for example,

$$\begin{aligned}
\eta_3\eta_3\widetilde{f_3} &= \left(\frac{\partial f_2}{\partial x_2}\frac{\partial}{\partial x_3} - \frac{\partial f_2}{\partial x_3}\frac{\partial}{\partial x_2}\right)\left(\frac{\partial f_2}{\partial x_2}\frac{\partial}{\partial x_3} - \frac{\partial f_2}{\partial x_3}\frac{\partial}{\partial x_2}\right)\left\{\left(\frac{\partial y_2}{\partial v}\right)_q f_3 - \left(\frac{\partial y_3}{\partial v}\right)_q f_2\right\} \\
&= \left(\frac{\partial y_2}{\partial v}\right)_q \frac{\partial f_2}{\partial x_2}\left(\frac{\partial^2 f_2}{\partial x_2 \partial x_3}\frac{\partial f_3}{\partial x_3} + \frac{\partial f_2}{\partial x_2}\frac{\partial^2 f_3}{\partial x_3^2} - \frac{\partial^2 f_2}{\partial x_3^2}\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\frac{\partial^2 f_3}{\partial x_2 \partial x_3}\right) \\
&\quad - \left(\frac{\partial y_2}{\partial v}\right)_q \frac{\partial f_2}{\partial x_3}\left(\frac{\partial^2 f_2}{\partial x_2^2}\frac{\partial f_3}{\partial x_3} + \frac{\partial f_2}{\partial x_2}\frac{\partial^2 f_3}{\partial x_2 \partial x_3} - \frac{\partial^2 f_2}{\partial x_2 \partial x_3}\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\frac{\partial^2 f_3}{\partial x_2^2}\right),
\end{aligned}$$

and $(\eta_3\eta_3\widetilde{f_3})_p = 0$. By similar calculations, the second column of the matrix $(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m}\widetilde{f_3})_p$ turns out to vanish. By similar but more complicated calculations, we obtain that $(\eta_3\eta_3\eta_3\widetilde{f_3})_p = 0$.

We apply Theorems 1 and 2 to $f_{2,3}$. Since $\ker(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m}\widetilde{f_3})_p \neq \{\mathbf{0}\}$, it follows that p is not a fold point of $f_{2,3}$. If $\ker(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m}\widetilde{f_3})_p$ is 1-dimensional, then $\ker(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m}\widetilde{f_3})_p = \langle(\eta_3)_p\rangle$. Since $(\eta_3\eta_3\eta_3\widetilde{f_3})_p = 0$, it follows that p is not a cusp point of $f_{2,3}$. □

5. Proof in fold case

In this section, we give a proof of Proposition 6.

We begin with the following local forms of the relevant maps. There exist local coordinate systems $(x_1, x_2, \dots, x_m)$ and (u, v, w) of X and Y, respectively, with respect to which $p = \mathbf{0}$ and

$$f(x_1, x_2, \dots, x_m) = (x_1, x_2, -x_3^2 - \cdots - x_{\lambda+2}^2 + x_{\lambda+3}^2 + \cdots + x_m^2).$$

There also exist a local coordinate y_1 of Y_1, and a local coordinate system (y_2, y_3) of $Y_{2,3}$ with respect to which $q = \mathbf{0}$ and $\pi(y_1, y_2, y_3) = y_1$ and $\Pi(y_1, y_2, y_3) = (y_2, y_3)$. Let f_2, f_3 denote the functions on U such that

$$f_{2,3}(x_1, x_2, \dots, x_m) = (f_2(x_1, x_2, \dots, x_m), f_3(x_1, x_2, \dots, x_m))$$

with respect to (y_2, y_3), and

$$f(x_1, x_2, \dots, x_m) = (f_1(x_1, x_2, \dots, x_m), f_2(x_1, x_2, \dots, x_m), f_3(x_1, x_2, \dots, x_m))$$

with respect to (y_1, y_2, y_3). Note that there is a coordinate transformation:

$$(u, v, w) \mapsto (y_1(u, v, w), y_2(u, v, w), y_3(u, v, w)).$$

Let S denote the singular set of $f|_U$. We can see that S coincides with the x_1x_2-plane, and $f|_S$ is an embedding, and its image D coincides with the uv-plane. Hence, $\pi|_D$ and $\Pi|_D$ have the local forms: $\pi|_D(u,v) = y_1(u,v,0)$ and $\Pi|_D(u,v) = (y_2(u,v,0), y_3(u,v,0))$.

We calculate partial derivatives similarly to those in Section 3. Then we obtain that, for each $k \in \{1,2,3\}$ and $i,j \in \{1,2,\dots,m\}$ such that $i \le j$,

$$\left(\frac{\partial f_k}{\partial x_i}\right)_p = \begin{cases} \left(\dfrac{\partial y_k}{\partial u}\right)_q & (i=1) \\ \left(\dfrac{\partial y_k}{\partial v}\right)_q & (i=2) \\ 0 & (3 \le i \le m), \end{cases}$$

$$\left(\frac{\partial^2 f_k}{\partial x_i \partial x_j}\right)_p = \begin{cases} \left(\dfrac{\partial^2 y_k}{\partial u^2}\right)_q & (i=j=1) \\ \left(\dfrac{\partial^2 y_k}{\partial u \partial v}\right)_q & (i=1,\ j=2) \\ \left(\dfrac{\partial^2 y_k}{\partial v^2}\right)_q & (i=j=2) \\ -2\left(\dfrac{\partial y_k}{\partial w}\right)_q & (3 \le i=j \le \lambda+2) \\ 2\left(\dfrac{\partial y_k}{\partial w}\right)_q & (\lambda+3 \le i=j \le m) \\ 0 & (\text{otherwise}). \end{cases}$$

5.1. Function case. We first focus on the function f_1.

We consider whether p is a regular or singular point of f_1. By the local forms and the results of partial derivatives,

$$(df_1)_p = \left(\left(\frac{\partial y_1}{\partial u}\right)_q, \left(\frac{\partial y_1}{\partial v}\right)_q, 0, \dots, 0\right),$$

$$(d(\pi|_D))_q = \left(\left(\frac{\partial y_1}{\partial u}\right)_q, \left(\frac{\partial y_1}{\partial v}\right)_q\right).$$

Hence, p is a regular point of f_1 if and only if q is a regular point of $\pi|_D$, which is the case if and only if

$$\left(\frac{\partial y_1}{\partial u}\right)_q \neq 0 \qquad \text{or} \qquad \left(\frac{\partial y_1}{\partial v}\right)_q \neq 0.$$

Supposing that p is a singular point of f_1, we consider what type it is. By the above result, q is a singular point of $\pi|_D$, and

$$\left(\frac{\partial y_1}{\partial u}\right)_q = \left(\frac{\partial y_1}{\partial v}\right)_q = 0.$$

By the regularity of the coordinate transformation: $(u, v, w) \mapsto (y_1, y_2, y_3)$,

$$\left(\frac{\partial y_1}{\partial w}\right)_q \neq 0.$$

The Hessian matrix $(\mathbf{H}_{x_1, x_2, \ldots, x_m} f_1)_p$ is equal to

$$\begin{pmatrix} \left(\frac{\partial^2 y_1}{\partial u^2}\right)_q & \left(\frac{\partial^2 y_1}{\partial u \partial v}\right)_q & \mathbf{0} & \mathbf{0} \\ \left(\frac{\partial^2 y_1}{\partial u \partial v}\right)_q & \left(\frac{\partial^2 y_1}{\partial v^2}\right)_q & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -2\left(\frac{\partial y_1}{\partial w}\right)_q \mathbf{I}_\lambda & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2\left(\frac{\partial y_1}{\partial w}\right)_q \mathbf{I}_{m-\lambda-2} \end{pmatrix},$$

and the Hessian matrix $(\mathbf{H}_{u,v}(\pi|_D))_q$ is equal to

$$\begin{pmatrix} \left(\frac{\partial^2 y_1}{\partial u^2}\right)_q & \left(\frac{\partial^2 y_1}{\partial u \partial v}\right)_q \\ \left(\frac{\partial^2 y_1}{\partial u \partial v}\right)_q & \left(\frac{\partial^2 y_1}{\partial v^2}\right)_q \end{pmatrix}.$$

These show that $(\mathbf{H}_{x_1, x_2, \ldots, x_m} f_1)_p$ has maximal rank if and only if $(\mathbf{H}_{u,v}(\pi|_D))_q$ does. Hence, p is a fold point of f_1 if and only if q is a fold point of $\pi|_D$. The number of negative eigenvalues of $(\mathbf{H}_{x_1, x_2, \ldots, x_m} f_1)_p$ is

$$\begin{cases} \lambda + \bar{\mu} & \left(\left(\frac{\partial y_1}{\partial w}\right)_q > 0\right) \\ (m - \lambda - 2) + \bar{\mu} & \left(\left(\frac{\partial y_1}{\partial w}\right)_q < 0\right), \end{cases}$$

where $\bar{\mu}$ is the number of negative eigenvalues of $(\mathbf{H}_{u,v}(\pi|_D))_q$. Suppose that q is a fold point of $\pi|_D$, and let μ denote its absolute index. Note that $\mu = \min\{\bar{\mu}, 2 - \bar{\mu}\}$, and hence $\bar{\mu}$ is equal to either μ or $2 - \mu$. The number of negative eigenvalues of $(\mathbf{H}_{x_1,x_2,\dots,x_m} f_1)_p$ is, therefore, equal to

$$\begin{cases} \lambda+\mu & \left(\left(\dfrac{\partial y_1}{\partial w}\right)_q > 0,\ \bar{\mu}=\mu\right) \\ \lambda-\mu+2 & \left(\left(\dfrac{\partial y_1}{\partial w}\right)_q > 0,\ \bar{\mu}=2-\mu\right) \\ m-\lambda+\mu-2 & \left(\left(\dfrac{\partial y_1}{\partial w}\right)_q < 0,\ \bar{\mu}=\mu\right) \\ m-\lambda-\mu & \left(\left(\dfrac{\partial y_1}{\partial w}\right)_q < 0,\ \bar{\mu}=2-\mu\right). \end{cases}$$

Thus, the absolute index of the fold point p of f_1 is equal to either $\min\{\lambda+\mu, m-\lambda-\mu\}$ or $\min\{\lambda-\mu+2, m-\lambda+\mu-2\}$.

5.2. Surface-valued map case. We now focus on the map $f_{2,3}$.

We consider whether p is a regular or singular point of $f_{2,3}$. By the local forms and the results of partial derivatives,

$$(df_{2,3})_p = \begin{pmatrix} \left(\dfrac{\partial y_2}{\partial u}\right)_q & \left(\dfrac{\partial y_2}{\partial v}\right)_q & 0 & \cdots & 0 \\ \left(\dfrac{\partial y_3}{\partial u}\right)_q & \left(\dfrac{\partial y_3}{\partial v}\right)_q & 0 & \cdots & 0 \end{pmatrix},$$

$$(d(\Pi|_D))_q = \begin{pmatrix} \left(\dfrac{\partial y_2}{\partial u}\right)_q & \left(\dfrac{\partial y_2}{\partial v}\right)_q \\ \left(\dfrac{\partial y_3}{\partial u}\right)_q & \left(\dfrac{\partial y_3}{\partial v}\right)_q \end{pmatrix}.$$

Hence, p is a regular point of $f_{2,3}$ if and only if q is a regular point of $\Pi|_D$, which is the case if and only if

$$\left(\frac{\partial y_2}{\partial u}\right)_q\left(\frac{\partial y_3}{\partial v}\right)_q - \left(\frac{\partial y_2}{\partial v}\right)_q\left(\frac{\partial y_3}{\partial u}\right)_q \neq 0.$$

Supposing that p is a singular point of $f_{2,3}$, we consider what type it is, in the rest of this section. By the above result, q is a singular point of $\Pi|_D$, and

$$\left(\frac{\partial y_2}{\partial u}\right)_q\left(\frac{\partial y_3}{\partial v}\right)_q - \left(\frac{\partial y_2}{\partial v}\right)_q\left(\frac{\partial y_3}{\partial u}\right)_q = 0.$$

By the regularity of the coordinate transformation: $(u, v, w) \mapsto (y_1, y_2, y_3)$,

$$\begin{pmatrix} \left(\dfrac{\partial y_2}{\partial u}\right)_q & \left(\dfrac{\partial y_2}{\partial v}\right)_q \\ \left(\dfrac{\partial y_3}{\partial u}\right)_q & \left(\dfrac{\partial y_3}{\partial v}\right)_q \end{pmatrix} \neq \mathbf{0}.$$

By the symmetries of y_2 and y_3, and of u and v, we may suppose that

$$\left(\frac{\partial y_2}{\partial u}\right)_q \neq 0$$

without loss of generality. Again by the regularity,

$$\left(\frac{\partial y_2}{\partial u}\right)_q \left(\frac{\partial y_3}{\partial w}\right)_q - \left(\frac{\partial y_2}{\partial w}\right)_q \left(\frac{\partial y_3}{\partial u}\right)_q \neq 0.$$

Let B denote the left-hand side of this inequality.

We modify the local form of $f_{2,3}$ as follows. Let $\widetilde{f_3}$ be the function on U defined as

$$\widetilde{f_3} = \left(\frac{\partial y_2}{\partial u}\right)_q f_3 - \left(\frac{\partial y_3}{\partial u}\right)_q f_2,$$

to give $f_{2,3}$ the local form:

$$f_{2,3}(x_1, x_2, \ldots, x_m) = (f_2(x_1, x_2, \ldots, x_m), \widetilde{f_3}(x_1, x_2, \ldots, x_m)).$$

We have arranged the conditions that $(df_2)_p \neq \mathbf{0}$ and $(d\widetilde{f_3})_p = \mathbf{0}$.

We calculate some derivatives with respect to appropriate vector fields as follows. Let $\eta_1, \eta_2, \ldots, \eta_m$ be the C^∞ vector fields on U defined as

$$\eta_1 = \frac{\partial}{\partial x_1},$$

$$\eta_2 = \frac{\partial f_2}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial f_2}{\partial x_2}\frac{\partial}{\partial x_1},$$

$$\eta_i = \frac{\partial}{\partial x_i} \qquad (3 \leq i \leq m).$$

We have arranged that $(\eta_1)_p, (\eta_2)_p, \ldots, (\eta_m)_p$ are linearly independent, and $(\eta_1 f_2)_p \neq 0$ and $(\eta_i f_2)_p = 0$ for $i \in \{2, 3, \ldots m\}$, and $\eta_2 f_2 = 0$. By a similar argument to that in Subsubsection 3.2.2, they satisfy the conditions that $\ker(df_{2,3})_p = \langle (\eta_2)_p, (\eta_3)_p, \ldots, (\eta_m)_p \rangle$ and $(\eta_2)_s \in \ker(df_{2,3})_s$ for any singular point s of $f_{2,3}$ sufficiently close to p. By calculations similar to those in the previous sections, we obtain that the matrix $(\mathbf{H}_{\eta_2, \eta_3, \ldots, \eta_m} \widetilde{f_3})_p$ is equal to

$$\begin{pmatrix} (\eta_2\eta_2\widetilde{f_3})_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -2B\mathbf{I}_\lambda & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2B\mathbf{I}_{m-\lambda-2} \end{pmatrix}.$$

We apply Theorems 1 and 2 to $f_{2,3}$. The above form of the matrix shows that $\ker(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m}\widetilde{f_3})_p = \{\mathbf{0}\}$ if and only if $(\eta_2\eta_2\widetilde{f_3})_p \neq 0$. It follows that p is a fold point of $f_{2,3}$ if and only if $(\eta_2\eta_2\widetilde{f_3})_p \neq 0$. Note that $\ker(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m}\widetilde{f_3})_p$ is 1-dimensional if and only if $(\eta_2\eta_2\widetilde{f_3})_p = 0$. If so, then $\ker(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m}\widetilde{f_3})_p = \langle(\eta_2)_p\rangle$. It follows that p is a cusp point of $f_{2,3}$ if and only if $(\eta_2\eta_2\widetilde{f_3})_p = 0$ and $(d(\eta_2\widetilde{f_3}))_p \neq \mathbf{0}$ and $(\eta_2\eta_2\eta_2\widetilde{f_3})_p \neq 0$. Here, the condition that $(d(\eta_2\widetilde{f_3}))_p \neq \mathbf{0}$ holds if and only if $(\eta_1\eta_2\widetilde{f_3})_p \neq 0$, since $(\eta_i\eta_2\widetilde{f_3})_p = 0$ for $i \in \{2, 3, \dots, m\}$. The number of negative eigenvalues of $(\mathbf{H}_{\eta_2,\eta_3,\dots,\eta_m}\widetilde{f_3})_p$ is

$$\begin{cases} \lambda & (B > 0,\ (\eta_2\eta_2\widetilde{f_3})_p \geq 0) \\ \lambda + 1 & (B > 0,\ (\eta_2\eta_2\widetilde{f_3})_p < 0) \\ m - \lambda - 2 & (B < 0,\ (\eta_2\eta_2\widetilde{f_3})_p \geq 0) \\ m - \lambda - 1 & (B < 0,\ (\eta_2\eta_2\widetilde{f_3})_p < 0). \end{cases}$$

It follows that, if p is a fold point of $f_{2,3}$, its absolute index is equal to either $\min\{\lambda, m - \lambda - 1\}$ or $\min\{\lambda + 1, m - \lambda - 2\}$, and that, if p is a cusp point of $f_{2,3}$, its absolute index is equal to $\min\{\lambda, m - \lambda - 2\}$. Note that $\min\{\lambda, m - \lambda - 1\} = \min\{\lambda, m - \lambda - 2\} = \lambda$ since $\lambda \leq m - \lambda - 2$.

We also apply Theorems 1 and 2 to the restriction of $f_{2,3}$ to the singular set S of f. Since S coincides with the x_1x_2-plane, we have the local forms: $f_{2,3}|_S(x_1, x_2) = (f_2|_S(x_1, x_2), \widetilde{f_3}|_S(x_1, x_2))$, and $f_2|_S(x_1, x_2) = f_2(x_1, x_2, 0, \dots, 0)$ and $\widetilde{f_3}|_S(x_1, x_2) = \widetilde{f_3}(x_1, x_2, 0, \dots, 0)$. We can see that $(d(f_2|_S))_p \neq \mathbf{0}$, and $(d(\widetilde{f_3}|_S))_p = \mathbf{0}$. Since η_2 is the sum of only derivations with respect to x_1 and x_2 multiplied by functions, and hence $\eta_2\eta_2$ and $\eta_2\eta_2\eta_2$ are also, we may regard them as defined on S. Then we have that $\ker(d(f_{2,3}|_S))_p = \langle(\eta_2)_p\rangle$, and that $(\eta_2)_s \in \ker(d(f_{2,3}|_S))_s$ for any singular point s of $f_{2,3}|_S$ sufficiently close to p, and that $(\mathbf{H}_{\eta_2}(\widetilde{f_3}|_S))_p = (\eta_2\eta_2(\widetilde{f_3}|_S))_p$. It follows that p is a fold point of $f_{2,3}|_S$ if and only if $(\eta_2\eta_2(\widetilde{f_3}|_S))_p \neq 0$. Note that $\ker(\mathbf{H}_{\eta_2}(\widetilde{f_3}|_S))_p = \langle(\eta_2)_p\rangle$ if $(\eta_2\eta_2(\widetilde{f_3}|_S))_p = 0$. It follows that p is a cusp point of $f_{2,3}|_S$ if and only if $(\eta_2\eta_2(\widetilde{f_3}|_S))_p = 0$ and $(d(\eta_2(\widetilde{f_3}|_S)))_p \neq \mathbf{0}$ and $(\eta_2\eta_2\eta_2(\widetilde{f_3}|_S))_p \neq 0$. Here, the condition that $(d(\eta_2(\widetilde{f_3}|_S)))_p \neq \mathbf{0}$ holds if and only if $(\eta_1\eta_2(\widetilde{f_3}|_S))_p \neq 0$, since $(\eta_2\eta_2(\widetilde{f_3}|_S))_p = 0$.

We compare the results of the above two paragraphs. Note that $(\eta_2\eta_2\widetilde{f_3})_p = (\eta_2\eta_2(\widetilde{f_3}|_S))_p$ and $(\eta_1\eta_2\widetilde{f_3})_p = (\eta_1\eta_2(\widetilde{f_3}|_S))_p$ and $(\eta_2\eta_2\eta_2\widetilde{f_3})_p = (\eta_2\eta_2\eta_2(\widetilde{f_3}|_S))_p$. It follows that p is a fold point (resp. cusp point) of $f_{2,3}$ if and only if p is a fold point (resp. cusp point) of $f_{2,3}|_S$. Since $f|_S$ is an embedding from S to

D, the point p is a fold point (resp. cusp point) of $f_{2,3}|_S$ if and only if q is a fold point (resp. cusp point) of $\Pi|_D$. Thus, we conclude that p is a fold point (resp. cusp point) of $f_{2,3}$ if and only if q is a fold point (resp. cusp point) of $\Pi|_D$. □

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