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The ring of modular forms for the even unimodular lattice of signature (2,18)

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ABSTRACT. We show that the ring of modular forms with characters for the even unimodular lattice of signature (2,18) is obtained from the invariant ring of $\mathrm{Sym}(\mathrm{Sym}^8(V) \oplus \mathrm{Sym}^{12}(V))$ with respect to the action of $\mathrm{SL}(V)$ by adding a Borcherds product of weight 132 with one relation of weight 264, where V is a 2-dimensional $\mathbb{C}$-vector space. The proof is based on the study of the moduli space of elliptic K3 surfaces with a section.

1. Introduction

Let $\mathbf{U}$ be the even unimodular hyperbolic lattice of rank 2. A $\mathbf{U}$-polarized K3 surface in the sense of [Nik79] is a pair (Y, j) of a K3 surface Y and a primitive lattice embedding $j : \mathbf{U} \hookrightarrow \mathrm{Pic}\, Y$. As explained, e.g., in [Huy], an elliptic K3 surface with a section corresponds naturally to a pseudo-ample $\mathbf{U}$-polarized K3 surface. Fix a primitive embedding of $\mathbf{U}$ to the K3 lattice $\Lambda = \mathbf{U} \perp \mathbf{U} \perp \mathbf{U} \perp \mathbf{E}_8 \perp \mathbf{E}_8$, which is unique up to the left action of $\mathrm{O}(\Lambda)$, and let $\mathbf{T} = \mathbf{U} \perp \mathbf{U} \perp \mathbf{E}_8 \perp \mathbf{E}_8$ be the orthogonal lattice. As explained in [Dol96, Section 3], the global Torelli theorem [PŠŠ71, BR75] and the surjectivity of the period map [Tod80] show that the period map gives an isomorphism from the coarse moduli scheme of pseudo-ample $\mathbf{U}$-polarized K3 surfaces to the quotient $M := \Gamma \backslash \mathscr{D}$ of the bounded Hermitian domain

$$\mathscr{D} := \{[\Omega] \in \mathbf{P}(\mathbf{T} \otimes \mathbb{C}) \mid (\Omega, \Omega) = 0,\ (\Omega, \bar{\Omega}) > 0\} \tag{1.1}$$

of type IV by $\Gamma := \mathrm{O}(\mathbf{T})$.

The moduli space of elliptic K3 surfaces with a section attracts much attention recently, not only from the point of view of modular compactification (see e.g. [AB, ABE] and references therein), but also because of the relation with tropical geometry and mirror symmetry [HU19, OO].

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A *modular form* on $\mathscr{D}$ with respect to Γ of weight $k \in \mathbb{Z}$ and character $\chi \in \mathrm{Char}(\Gamma) := \mathrm{Hom}(\Gamma, \mathbb{C}^\times)$ is a holomorphic function $f : \tilde{\mathscr{D}} \to \mathbb{C}$ on the total space

$$\tilde{\mathscr{D}} := \{\Omega \in \mathbf{T} \otimes \mathbb{C} \mid (\Omega, \Omega) = 0,\ (\Omega, \bar{\Omega}) > 0\} \tag{1.2}$$

of a principal $\mathbb{C}^\times$-bundle on $\mathscr{D}$ satisfying

(i) $f(\alpha z) = \alpha^{-k} f(z)$ for any $\alpha \in \mathbb{C}^\times$, and

(ii) $f(\gamma z) = \chi(\gamma) f(z)$ for any $\gamma \in \Gamma$.

The vector spaces $A_k(\Gamma, \chi)$ of modular forms constitute the ring

$$\tilde{A}(\Gamma) := \bigoplus_{k=0}^{\infty} \bigoplus_{\chi \in \mathrm{Char}(\Gamma)} A_k(\Gamma, \chi) \tag{1.3}$$

of modular forms. We also write the subring of modular forms without characters as

$$A(\Gamma) := \bigoplus_{k=0}^{\infty} A_k(\Gamma). \tag{1.4}$$

Let $V := \mathrm{Spec}\, \mathbb{C}[x, w]$ be a 2-dimensional affine space over $\mathbb{C}$. For $k \in \mathbb{N}$, we write the k-th symmetric product of V as $\mathrm{Sym}^k V$. The special linear group SL_2 acts naturally on $S := \mathrm{Sym}^8 V \times \mathrm{Sym}^{12} V$ considered as an affine variety, whose coordinate ring will be denoted by

$$\mathbb{C}[S] = \mathbb{C}[u_{8,0}, u_{7,1}, \ldots, u_{0,8}, u_{12,0}, u_{11,1}, \ldots, u_{0,12}]. \tag{1.5}$$

We let $\mathbb{G}_{\mathrm{m}}$ act on S in such a way that $u_{i,j}$ has weight $(i+j)/2$. This $\mathbb{G}_{\mathrm{m}}$-action commutes with the SL_2-action, so that the invariant subring $\mathbb{C}[S]^{\mathrm{SL}_2}$ has an induced $\mathbb{G}_{\mathrm{m}}$-action.

Building on [Mir81], it is shown in [OO, Theorem 7.9] that the period map induces an isomorphism from $\mathrm{Proj}\, \mathbb{C}[S]^{\mathrm{SL}_2}$ to the Satake–Baily–Borel compactification of $\Gamma \backslash \mathscr{D}$. As we explain in Section 2, the period map also gives an isomorphism

$$A(\Gamma) \cong \mathbb{C}[S]^{\mathrm{SL}_2} \tag{1.6}$$

of graded rings.

Note that we have $\mathrm{Char}(\Gamma) = \{\mathrm{id}, \det\}$ (cf. e.g. [GHS09, Corollary 1.8]). The main result of this paper is the following:

THEOREM 1. *One has*

$$\tilde{A}(\Gamma) \cong (\mathbb{C}[S]^{\mathrm{SL}_2})[s_{132}]/(s_{132}^2 - \Delta_{264}), \tag{1.7}$$

where s_{132} *is an element of weight* 132 *and* $\Delta_{264} \in \mathbf{C}[S]^{\mathrm{SL}_2}$ *is an element of weight* 264.

The proof is based on the construction of an algebraic stack which is isomorphic to the orbifold quotient $[\mathrm{O}(\mathbf{T})\backslash\mathscr{D}]$ in codimension 1. The same strategy has been used in [HU] and [NU] to determine the rings of modular forms with characters for the lattices $\mathbf{U}\perp\mathbf{U}\perp\mathbf{E}_8$ and $\mathbf{U}\perp\mathbf{U}\perp\mathbf{A}_1\perp\mathbf{A}_1$, respectively.

The modular form s_{132} is constructed in [FSM07, Lemma 5.1]. It can also be obtained either as the quasi pull-back [GHS13, Theorem 8.2] of the Borcherds form Φ_{12} associated with the even unimodular lattice of signature $(2,26)$ [Bor95, Section 10, Example 2], or by applying [Bor95, Theorem 10.1] to the nearly holomorphic modular form

$$\frac{1728E_4}{E_4^3 - E_6^2} = \frac{1}{q} + 264 + 8244q + 139520q^2 + \cdots, \tag{1.8}$$

where

$$E_4 = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} = 1 + 240q + 2160q^2 + \cdots, \tag{1.9}$$

$$E_6 = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n} = 1 - 504q - 6632q^2 + \cdots. \tag{1.10}$$

In particular, it is a cusp form with character det admitting an infinite product expansion. See also [DKW19, Section 5] and references therein for the case of the even unimodular lattice of signature (2,10).

Since SL_2 is reductive, the invariant ring $\mathbf{C}[S]^{\mathrm{SL}_2}$ is finitely generated, and there exists an algorithm for computing a finite generating set (see e.g. [Stu08] and references therein). The element Δ_{264} can also be computed algorithmically, and it is an interesting problem to describe them explicitly.

2. The coarse moduli space of U-polarized K3 surfaces

As is well known (cf. e.g. [SS10, Section 4]), a **U**-polarized K3 surface admits a Weierstrass model of the form

$$z^2 = y^3 + g_2(x,w;u)y + g_3(x,w;u) \tag{2.1}$$

in $\mathbf{P}(1,4,6,1)$, where

$$g_2(x,w;u) = \sum_{i=0}^{8} u_{8-i,i}x^{8-i}w^i \tag{2.2}$$

$$= u_{8,0}x^8 + u_{7,1}x^7u + \cdots + u_{0,8}w^8, \tag{2.3}$$

$$g_3(x, w; u) = \sum_{i=0}^{12} u_{12-i,i} x^{12-i} w^i \tag{2.4}$$

$$= u_{12,0}x^{12} + u_{11,1}x^{11}u + \cdots + u_{0,12}w^{12} \tag{2.5}$$

for

$$u = ((u_{8,0}, \dots, u_{0,8}), (u_{12,0}, \dots, u_{0,12})) \in S. \tag{2.6}$$

The hypersurface in $\mathbf{P}(1,4,6,1)$ defined by (2.1) has a singularity worse than rational double points on the fiber at $a \in \mathbf{P}^1$ if and only if $\mathrm{ord}_a(g_2) \geq 4$ and $\mathrm{ord}_a(g_3) \geq 6$ (see e.g. [Mir89, Proposition III.3.2]). Let $U \subset S$ be the open subscheme parametrizing hypersurfaces with at worst rational double points.

The parameter u describing a given $\mathbf{U}$-polarized K3 surface is unique up to the action of $\mathrm{SL}_2 \times \mathbb{G}_\mathrm{m}$, where $\mathbb{G}_\mathrm{m}$ acts on $\mathbf{P}(1,4,6,1) \times \mathrm{Sym}^8 V \times \mathrm{Sym}^{12} V$ by

$$\mathbb{G}_\mathrm{m} \ni \lambda : ((x, y, z, w), (u_{i,j})_{i,j}) \mapsto ((x, \lambda^2 y, \lambda^3 z, w), (\lambda^{(i+j)/2} u_{i,j})_{i,j}) \tag{2.7}$$

rescaling the holomorphic volume form

$$\Omega = \mathrm{Res} \frac{w\, dx \wedge dy \wedge dz}{z^2 - y^3 - g_2(x, w; u) y - g_3(x, w; u)} \tag{2.8}$$

as

$$\Omega_{\lambda u} = \mathrm{Res} \frac{w\, dx \wedge d(\lambda^2 y) \wedge d(\lambda^3 z)}{(\lambda^3 z)^2 - (\lambda^2 y)^3 - g_2(x, w; \lambda \cdot u)(\lambda^2 y) - g_3(x, w; \lambda \cdot u)} = \lambda^{-1} \Omega_u. \tag{2.9}$$

The categorical quotient $T := U/\mathrm{SL}_2$ is the coarse moduli scheme of pairs (Y, Ω) consisting of a $\mathbf{U}$-polarized K3 surface Y and a holomorphic volume form $\Omega \in H^0(\omega_Y)$ on Y. The fact that the codimension of $S \backslash U$ is greater than 2 implies an isomorphism

$$\mathbb{C}[S]^{\mathrm{SL}_2} \cong \mathbb{C}[T] \tag{2.10}$$

of graded rings. Since the character of $\mathbb{C}[S]$ as a $\mathrm{SL}_2 \times \mathbb{G}_\mathrm{m}$-module is given by

$$\prod_{i=0}^{8} (1 - q^{2i-8} t^4)^{-1} \prod_{i=0}^{12} (1 - q^{2i-12} t^6)^{-1}, \tag{2.11}$$

the Hilbert series of the invariant ring is given by

$$\sum_{i=0}^{\infty} \dim(\mathbb{C}[S]^{\mathrm{SL}_2})_i t^i$$

$$= \mathrm{Res}_{q=0}\left((q^{-1}-q)\prod_{i=0}^{8}(1-q^{2i-8}t^4)^{-1}\prod_{i=0}^{12}(1-q^{2i-12}t^6)^{-1}\right) \tag{2.12}$$

as explained, e.g., in [Muk03, Section 4.4]. It follows from the global Torelli theorem and the surjectivity of the period map that the period map induces a ring isomorphism

$$A(\Gamma) \xrightarrow{\sim} \mathbb{C}[T], \tag{2.13}$$

which preserves the grading by (2.9). The isomorphism (1.6) follows from (2.10) and (2.13).

3. Modular forms with characters

The coarse moduli space M of $\mathbf{U}$-polarized K3 surfaces is an open subvariety of its Satake–Baily–Borel compactification Proj $A(\Gamma) \cong \mathbf{P}(4^9, 6^{13})/\!\!/\mathrm{SL}_2$. Although $M = \Gamma\backslash\mathscr{D}$ and the orbifold quotient $\mathbb{M} := [\Gamma\backslash\mathscr{D}]$ are closely related, the canonical morphism $\mathbb{M} \to M$ is not an isomorphism even in codimension 1. In order to obtain an orbifold which is isomorphic to $\mathbb{M}$ in codimension 1 (so that the total coordinate rings are isomorphic), consider the stack

$$\mathbb{P} := [\mathbb{P}(4^9, 6^{13})/\mathrm{SL}_2], \tag{3.1}$$

defined as the quotient of $\mathbb{C}^{22}\backslash\mathbf{0}$ by the action of $\mathrm{SL}_2 \times \mathbb{G}_\mathrm{m}$. The morphism $\mathbb{M} \to M$ lifts to a morphism $\mathbb{M} \to \mathbb{P}$, which is an isomorphism in codimension 0, since the generic stabilizers are $\{\pm\mathrm{id}\}$ on both sides.

Stabilizers of $\mathbb{M}$ along divisors come from reflections. One divisor with a generic stabilizer comes from the reflection with respect to a (-2)-vector whose reflection hyperplane corresponds to the locus where the Picard lattice contains $\mathbf{U}\perp\mathbf{A}_1$. In order to describe this locus, first consider the discriminant

$$h(x, w; u) := 4g_2(x, w; u)^3 + 27g_3(x, w; u)^2 \tag{3.2}$$

of $y^3 + g_2(x, w; u)y + g_3(x, w; u)$ as a polynomial in y, which is homogeneous of degree 24 in (x, w) and degree 12 in u. Note that the discriminant of a polynomial $\sum_{i=0}^{n} a_i x^i w^{n-i}$ with respect to (x, w) is homogeneous of degree $2(n-1)$ in $\mathbb{Z}[a_0, \dots, a_n]$ if $\deg a_0 = \cdots = \deg a_n = 1$. It follows that the discriminant $k_{552}(u)$ of $h(x, w; u)$ with respect to (x, w) is a homogeneous polynomial of degree $2 \cdot 23 \cdot 12 = 552$ in u. A general point on the divisor $\mathbb{D}_{552}$ of $\mathbb{P}$ defined by $k_{552}(u)$ corresponds to the locus where two fibers of Kodaira type I_1 collapse into one fiber. This divisor has two components; a general point on one component corresponds to the case when there exists a point $p = [x : w]$ on $\mathbf{P}^1$ such that neither g_2 nor g_3 vanishes at p, and a general point on the other component corresponds to the case when both g_2 and g_3 vanishes at p. In the former case, the resulting singular fiber is of Kodaira type I_2, and the surface acquires an A_1-singularity. In the latter case, the resulting singular fiber is of Kodaira type II, and the surface does not acquire any new singularity. The defining equation of the latter component is the resultant of g_2 and g_3. It is given as the determinant

$$r_{96}(u) = \begin{vmatrix} u_{8,0} & u_{7,1} & \cdots & u_{0,8} & & & & \\ & u_{8,0} & \cdots & u_{1,7} & u_{0,8} & & & \\ & & \ddots & & \ddots & \ddots & & \\ & & & \ddots & & \ddots & \ddots & \\ & & & & u_{8,0} & \cdots & u_{1,7} & u_{0,8} \\ u_{12,0} & u_{11,1} & \cdots & \cdots & u_{1,11} & u_{0,12} & & \\ & \ddots & & & & \ddots & \ddots & \\ & & u_{12,0} & u_{11,1} & \cdots & \cdots & u_{1,11} & u_{0,12} \end{vmatrix} \tag{3.3}$$

of the Sylvester matrix, which is homogeneous of degree

$$12 \times 4 + 8 \times 6 = 96. \tag{3.4}$$

As shown in [HU, Lemma 6.1], the polynomial $k_{552}(u)$ is divisible by $r_{96}(u)^3$, and the quotient

$$\Delta_{264}(u) := k_{552}(u)/r_{96}(u)^3 \tag{3.5}$$

defines the reflection hyperplane along a (-2)-vector.

Recall from [AGV08, Cad07] that the *root construction* is an operation which adds a stabilizer along a divisor. Let $\mathbb{T}$ be the stack obtained from $\mathbb{P}$ by the root construction of order 2 along the divisor on $\mathbb{P}$ defined by $\Delta_{264}(t)$, which is the quotient of the double cover of $\mathbb{P}$ branched along $\Delta_{264}(t)$ by the group G of deck transformations. The Picard group of $\mathbb{T}$ (or the G-equivariant Picard group of $\mathbb{P}$) is generated by the pull-back $\mathcal{O}_{\mathbb{T}}(1) := p^*\mathcal{O}_{\mathbb{P}}(1)$ of the generator $\mathcal{O}_{\mathbb{P}}(1)$ of the Picard group of $\mathbb{P}$ by the structure morphism $p : \mathbb{T} \to \mathbb{P}$ and the line bundle $\mathcal{O}_{\mathbb{T}}(\mathbb{D}_{132})$ such that the space $H^0(\mathcal{O}_{\mathbb{T}}(\mathbb{D}_{132}))$ is generated by an element s_{132} satisfying $s_{132}^2 = \Delta_{264} \in H^0(\mathcal{O}_{\mathbb{T}}(264)) \cong H^0(\mathcal{O}_{\mathbb{P}}(264))$. Note also that $\omega_{\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}}(a)$, where

$$a = -\sum_{i=0}^{8} \deg u_{8-i,i} - \sum_{i=0}^{12} \deg u_{12-i,i} = -9 \times 4 - 13 \times 6 = -114. \tag{3.6}$$

The ramification formula for the canonical bundle gives

$$\omega_{\mathbb{T}} \cong p^*\omega_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{T}}(\mathbb{D}_{132}) \tag{3.7}$$

$$\cong \mathcal{O}_{\mathbb{T}}(-114) \otimes \mathcal{O}_{\mathbb{T}}(132 + (-132 + \mathbb{D}_{132})) \tag{3.8}$$

$$\cong \mathcal{O}_{\mathbb{T}}(18) \otimes \mathcal{O}_{\mathbb{T}}(-132 + \mathbb{D}_{132}). \tag{3.9}$$

Note that $\mathcal{O}_{\mathbb{T}}(-132 + \mathbb{D}_{132})$ is an element of order two in $\mathrm{Pic}\,\mathbb{T}$. By comparing (3.9) with

$$\omega_{\mathbb{M}} \cong \mathcal{O}_{\mathbb{M}}(\dim \mathbb{M}) \otimes \det = \mathcal{O}_{\mathbb{M}}(18) \otimes \det \tag{3.10}$$

which follows from (the proof of) [HU, Proposition 5.1], one concludes that $\mathbb{M}$ has no further stabilizer along a divisor, so that the lift $\mathbb{M} \to \mathbb{T}$ of $\mathbb{M} \to \mathbb{P}$ is an isomorphism in codimension 1. It follows that the injective map $\mathbb{Z} \times \mathrm{Char}(\Gamma) \to \mathrm{Pic}\,\mathbb{M}$, $(i, \chi) \mapsto \mathcal{O}_{\mathbb{M}}(i) \otimes \chi$ is surjective, and the total coordinate ring (also known as the Cox ring) of $\mathbb{M}$ is given by

$$\bigoplus_{\mathcal{L} \in \mathrm{Pic}\,\mathbb{M}} H^0(\mathcal{L}) \cong \bigoplus_{i=0}^{\infty} H^0(\mathcal{O}_{\mathbb{M}}(i))/(s_{132}^2 - \Delta_{264}(t)). \tag{3.11}$$

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