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Notes on constructions of knots with the same trace

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Abstract. The m-trace of a knot is the 4-manifold obtained from $\mathbf{B}^4$ by attaching a 2-handle along the knot with m-framing. In 2015, Abe, Jong, Luecke and Osoinach introduced a technique to construct infinitely many knots with the same (diffeomorphic) m-trace, which is called the operation $(*m)$. In 2018, Miller and Piccirillo gave pairs of knots with diffeomorphic m-traces by utilizing Gompf and Miyazaki's dualizable pattern. In this paper, we clarify the relation between the two techniques. In particular, we prove that the "twistings" appearing in both techniques are corresponding. In addition, we show that the family of knots admitting the same 4-surgery given by Teragaito can be explained by the operation $(*m)$.

1. Introduction

For an integer m, the *m-trace* $X_K(m)$ of a knot K is the 4-manifold obtained from $\mathbf{B}^4$ by attaching a 2-handle along the knot with m-framing. On techniques to construct knots with the same trace, the following are known.

- Abe, Jong, Luecke and Osoinach [2] introduced a technique to construct infinitely many knots with the same (diffeomorphic) m-trace. The technique is based on "annulus presentation" and called *the operation* $(*m)$[1]. The operation is given by a composition of Osoinach's annular twisting technique [9] and twisting m times along a certain curve "γ". This twisting is denoted by T_m in this paper (for detail, see Section 2).
- Miller and Piccirillo [8] constructed a pair of knots with the same m-trace by utilizing dualizable patterns. In particular, such a pair is given by a dualizable pattern and twisting its dual m times along a meridian of the solid torus containing the dual. This twisting is denoted by τ_m (for dualizable patterns, see Section 3).

Miller and Piccirillo [8] pointed out that the construction by an annulus presentation can be regarded as that by a dualizable pattern. In fact they

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[1] In [2, Section 3.1.2], a notation $(*n)$ is used instead of $(*m)$.

constructed a dualizable pattern from the annulus presentation. This is the case $m = 0$, namely, untwisted case.

In this paper, we extend Miller and Piccirillo's work on a correspondence between annulus presentations and dualizable patterns to twisted cases. In particular, we find that the twisting T_m appearing in operation $(*m)$ corresponds to the twisting τ_m on the duals of dualizable patterns. As an application, we directly draw the duals to Miller and Piccirillo's dualizable patterns obtained from annulus presentations (Theorem 4.3 and Figure 5). In addition, we explain the family of knots admitting the same 4-surgery given by Teragaito [11] in terms of the operation $(*m)$ (Section 5). We also remark some observations in the final section. Throughout this paper,

- unless specifically mentioned, all knots and links are smooth and unoriented, and all other manifolds are smooth and oriented,
- for an n-component link $L_1 \cup \cdots \cup L_n$, we denote the 3-manifold obtained from $\mathbf{S}^3$ by m_i-surgery on the knot L_i for $i = 1, \dots, n$ by $M_{L_1 \cup \cdots \cup L_n}(m_1, \dots, m_n)$,
- we denote a tubular neighborhood of a knot K in a 3-manifold by $\nu(K)$, and
- we denote the unknot in $\mathbf{S}^3$ by U.

2. Annulus twist, annulus presentation and the operation $(*m)$

2.1. Annulus twist and annulus presentation. Let $A \subset \mathbf{S}^3$ be an embedded annulus with ordered boundaries $\partial A = c_1 \cup c_2$. An *$n$-fold annulus twist along A* is to apply $(\mathrm{lk}(c_1, c_2) + 1/n)$-surgery along c_1 and $(\mathrm{lk}(c_1, c_2) - 1/n)$-surgery along c_2, where $\mathrm{lk}(c_1, c_2)$ is the linking number of c_1 and c_2 and we give c_1 and c_2 parallel orientations. We see that the resulting manifold obtained by an annulus twist is also $\mathbf{S}^3$.

Let $A \subset \mathbf{S}^3$ be an embedded annulus with $\partial A = c_1 \cup c_2$. Take an embedding of a band $b : I \times I \to \mathbf{S}^3$ such that

- $b(I \times I) \cap \partial A = b(\partial I \times I)$,
- $b(I \times I) \cap \mathrm{Int}\, A$ consists of ribbon singularities, and
- $A \cup b(I \times I)$ is an immersion of an orientable surface,

where $I = [0, 1]$. If a knot $K \subset \mathbf{S}^3$ is isotopic to the knot $(\partial A \backslash b(\partial I \times I)) \cup b(I \times \partial I)$, then we call (A, b) an *annulus presentation* of K. An annulus presentation (A, b) is *special* if A is unknotted and $\mathrm{lk}(c_1, c_2) = \pm 1$ (that is, A is ± 1-full twisted). Let K be a knot with an annulus presentation (A, b). Let $A' \subset A$ be a shrunken annulus with $\partial A' = c_1' \cup c_2'$ which satisfies the following:

- $\overline{A \backslash A'}$ is a disjoint union of two annuli,
- each c_i' is isotopic to c_i in $\overline{A \backslash A'}$ for $i = 1, 2$, and
- $A \backslash (\partial A \cup A')$ does not intersect $b(I \times I)$.

Then, by $A^n(K)$, we denote the knot obtained from K by the n-fold annulus twist along A'. For simplicity, we denote $A^1(K)$ by $A(K)$ and $A^0(K)$ by K.

Remark 2.1. We find many examples of special annulus presentations in [1, 2, 4, 5, 10]. Remark that in [2, 5], our special annulus presentations are called "annulus presentations", simply. In this paper, for an annulus presentation (A, b), we often draw the attaching regions $A \cap b$ by bold arcs and we omit the band b.

By utilizing Osoinach's work [9, Theorem 2.3], for a knot K with an annulus presentation (A, b), we see that $M_K(0)$ and $M_{A^n(K)}(0)$ are orientation-preservingly homeomorphic for any $n \in \mathbf{Z}$. In particular, a homeomorphism $\phi_n : M_K(0) \to M_{A^n(K)}(0)$ is given as in Figure 1, which is explicitly given by Teragaito [11]. We call ϕ_n the n-th Osoinach-Teragaito's homeomorphism. Moreover, if (A, b) is special, by applying Abe, Jong, Omae and Takeuchi's result [1, Theorem 2.8] to the knot, we see that the homeomorphism ϕ_n extends to an orientation-preserving diffeomorphism $\Phi_n : X_K(0) \to X_{A^n(K)}(0)$ for any $n \in \mathbf{Z}$.

As a consequence, we obtain the following.

Theorem 2.2. *Let $K \subset \mathbf{S}^3$ be a knot with an annulus presentation (A, b). Then, there is an orientation-preservingly homeomorphism $\phi_n : M_K(0) \to$*

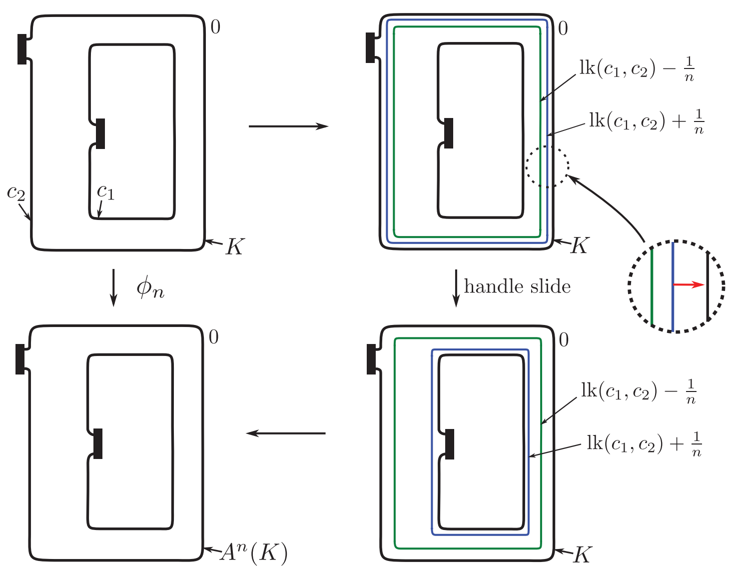

Fig. 1. (color online) Osoinach-Teragaito's homeomorphism ϕ_n. For simplicity we draw A as a flat annulus although A may be knotted and twisted.

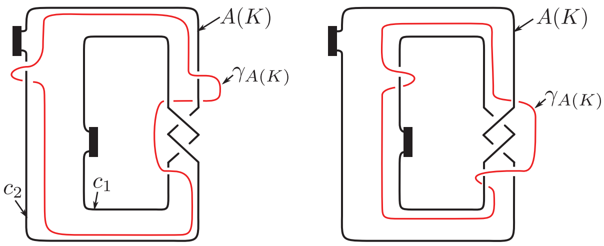

Fig. 2. (color online) The curve $\gamma_{A(K)}$.

$M_{A^n(K)}(0)$ for any $n \in \mathbf{Z}$. In particular, ϕ_n is given as in Figure 1. Moreover, if (A, b) is special, ϕ_n extends to an orientation-preserving diffeomorphism $\Phi_n : X_K(0) \to X_{A^n(K)}(0)$.

2.2. Operation $(*m)$. Let K be a knot with a special annulus presentation (A, b). Let $\gamma_{A(K)} \subset \mathbf{S}^3 \backslash \nu(A(K))$ be a curve depicted in Figure 2. Remark that the definition of $\gamma_{A(K)}$ depends on the twist of A. Denote the knot obtained from $A(K)$ by twisting m times along $\gamma_{A(K)}$ by $T_m(A(K))$. In [2, Section 3.1.2], the operation $K \mapsto T_m(A(K))$ is called *the operation $(*m)$*. Then, Abe, Jong, Luecke and Osoinach [2] proved the following theorem.

THEOREM 2.3 ([2, Theorem 3.7 and Theorem 3.10]). *Let K be a knot with a special annulus presentation (A, b). Then, there is an orientation-preservingly homeomorphism $\psi_m : M_K(m) \to M_{T_m(A(K))}(m)$ which extends to a diffeomorphism $\Psi_m : X_K(m) \to X_{T_m(A(K))}(m)$ for any $m \in \mathbf{Z}$.*

Concretely, ψ_m is given as in Figure 3 for the case A is $+1$ twisted. For the case A is -1 twisted, we can define ψ_m similarly (see also [3, Appendix]).

REMARK 2.4. Note that Osoinach-Teragaito's homeomorphism induces a homeomorphism $\phi_{+1} : (M_K(0), \alpha_K) \to (M_{A(K)}(0), \gamma_{A(K)})$, where $\alpha_K \subset \mathbf{S}^3 \backslash \nu(K)$ is a meridian of K and we regard α_K and $\gamma_{A(K)}$ as curves in $M_K(0)$ and $M_{A(K)}(0)$, respectively (see also the bottom arrow in Figure 3).

3. Relation between annulus presentation and dualizable pattern

3.1. Dualizable pattern. Here, we recall the definition of dualizable patterns, which is firstly given by Gompf and Miyazaki [7] and developed by Miller and Piccirillo [8] (see also [10]).

Let $P : \mathbf{S}^1 \to V$ be an oriented knot in a solid torus $V = \mathbf{S}^1 \times D^2$. Suppose that the image $P(\mathbf{S}^1)$ is not null-homologous in V. Such a P is called a

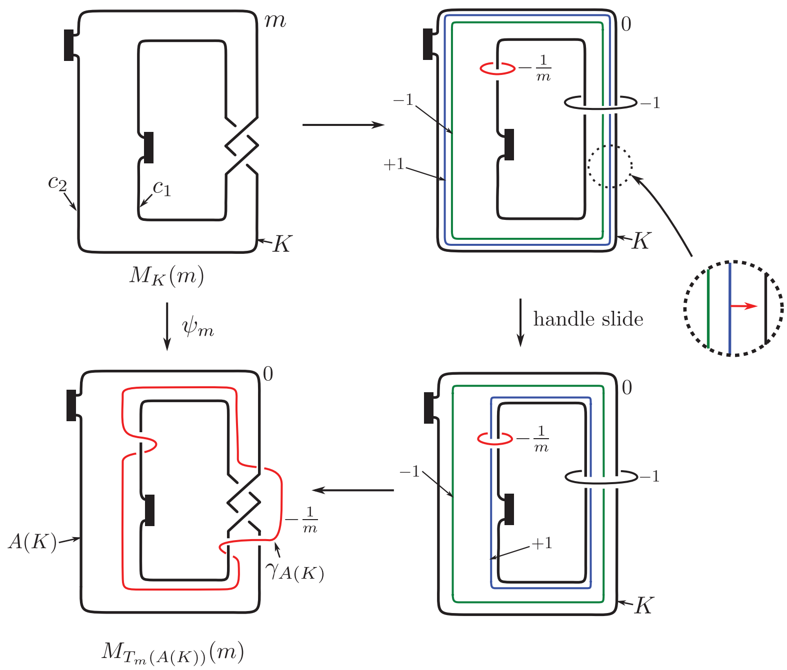

Fig. 3. (color online) The homeomorphism $\psi_m : M_K(m) \to M_{T_m(A(K))}(m)$ for the case A is $+1$ twisted.

pattern. By an abuse of notation, we use the notation P for both a map and its image. Define λ_V, μ_P, μ_V and λ_P as follows:

- put $\lambda_V = \mathbf{S}^1 \times \{x_0\} \subset \partial V \subset V$ for some $x_0 \in \partial D^2$ and orient λ_V so that P is homologous to $r\lambda_V$ in V for some positive $r \in \mathbf{Z}_{>0}$,
- define $\mu_P \subset V$ by a meridian of P and orient μ_P so that the linking number of P and μ_P is 1,
- put $\mu_V = \{x_1\} \times \partial D^2 \subset \partial V \subset V$ for some $x_1 \in \mathbf{S}^1$ and orient μ_V so that μ_V is homologous to $s\mu_P$ in $V \backslash \nu(P)$ for some positive $s \in \mathbf{Z}_{>0}$,
- define λ_P by a longitude of P which is homologous to $t\lambda_V$ in $V \backslash \nu(P)$ for some positive $t \in \mathbf{Z}_{>0}$.

For an oriented knot $K \subset \mathbf{S}^3$, let $\iota_K : V \to \mathbf{S}^3$ be an embedding which identifies V with $\overline{\nu(K)}$ and sends λ_V to an oriented curve on $\partial \overline{\nu(K)}$ which is null-homologous in $\mathbf{S}^3 \backslash \nu(K)$ and isotopic to K in $\mathbf{S}^3$. Then $\iota_K \circ P : \mathbf{S}^1 \to \mathbf{S}^3$ represents an oriented knot. The knot is called the *satellite* of K with pattern P and denoted by $P(K)$.

A pattern $P : \mathbf{S}^1 \to V$ is *dualizable* if there is a pattern $P^* : \mathbf{S}^1 \to V^*$ and an orientation-preserving homeomorphism $f : V \backslash \nu(P) \to V^* \backslash \nu(P^*)$ such that $f(\lambda_V) = \lambda_{P^*}$, $f(\lambda_P) = \lambda_{V^*}$, $f(\mu_V) = -\mu_{P^*}$ and $f(\mu_P) = -\mu_{V^*}$.

Miller and Piccirillo [8, Proposition 2.5] introduced a convenient technique to determine whether a given pattern is dualizable as follows (see also [7, Section 2]). Define $\Gamma : \mathbf{S}^1 \times D^2 \to \mathbf{S}^1 \times \mathbf{S}^2$ by $\Gamma(t, d) = (t, \gamma(d))$, where $\gamma : D^2 \to \mathbf{S}^2$ is an arbitrary orientation preserving embedding. For any curve $c : \mathbf{S}^1 \to \mathbf{S}^1 \times D^2$, define $\hat{c} = \Gamma \circ c : \mathbf{S}^1 \to \mathbf{S}^1 \times \mathbf{S}^2$. Then, we obtain the following proposition.

Proposition 3.1 ([8, Proposition 2.5]). *A pattern P in a solid torus V is dualizable if and only if $\hat{P}$ is isotopic to $\widehat{\lambda_V}$ in $\mathbf{S}^1 \times \mathbf{S}^2$.*

Related to knot traces, the following are known. Let $P \subset V$ be a pattern. Let $\tau_m : V \to V$ be a homeomorphism given by twisting m times along a meridian of V. It is known that if P is dualizable then $\tau_m(P)$ is also dualizable and its dual is given by $\tau_{-m}(P^*)$, where P^* is the dual to P (see [8, Theorem 3.6] and [10, Remark 4.6]). Moreover, we obtain the following.

Theorem 3.2 ([8, Theorem 3.6] and [10, Remark 4.6]). *Let P be a dualizable pattern and P^* be its dual. Then, we have $X_{P(U)}(m) \cong X_{\tau_m(P^*)(U)}(m)$ for any $m \in \mathbf{Z}$.*

Remark 3.3. For a dualizable pattern $P \subset V$, we see that $M_{P(U) \cup \mu_V}(0, 0) \cong \mathbf{S}^3$. Conversely, for a knot k in $\mathbf{S}^3$, if there exists an unknot c such that $M_{k \cup c}(0, 0) \cong \mathbf{S}^3$, we see that $k \subset \mathbf{S}^3 \backslash \nu(c)$ is a dualizable pattern after giving some orientation to k (for detail, see [6] and [10, Remarks 3.3 and 4.6]).

3.2. From special annulus presentations to dualizable patterns. In this section, we recall Miller and Piccirillo's construction ([8, Section 5]) of dualizable patterns from a special annulus presentation (see also [10]).

Let $K \subset \mathbf{S}^3$ be a knot with a special annulus presentation (A, b). In Figure 4, the left knots represent K, and each right knot represents $A^{\pm 1}(K)$ for the corresponding left K. Then, for each case, take curves $\beta_K^{\pm} \subset \mathbf{S}^3 \backslash \nu(K)$ as in Figure 4.

Let P_+ (resp. P_-) be the pattern given by $K \subset V_+ = \mathbf{S}^3 \backslash \nu(\beta_K^+)$ (resp. $K \subset V_- = \mathbf{S}^3 \backslash \nu(\beta_K^-)$), where we give a parameter of $V_\pm$ so that $P_\pm(U) = K$. Moreover, we give an orientation of $P_\pm$ arbitrarily. Then, we can check that $P_\pm$ are dualizable patterns (for example, slide K along the 0-framing of $\beta_K^\pm$ in $M_{\beta_K^\pm}(0) \cong \mathbf{S}^1 \times \mathbf{S}^2$ and apply Proposition 3.1). These dualizable patterns satisfy the following.

Proposition 3.4 (e.g. [8, Proposition 5.3] and [10, Proposition 3.9]). *Let K be a knot with a special annulus presentation (A, b). Let P_+ and P_- be the dualizable patterns as above. Then we have $P_\pm(U) = K$ and $P_\pm^*(U) = A^{\pm 1}(K)$. Here, $P_\pm^* (= (P_\pm)^*)$ denotes the dual of $P_\pm$ for each sign.*

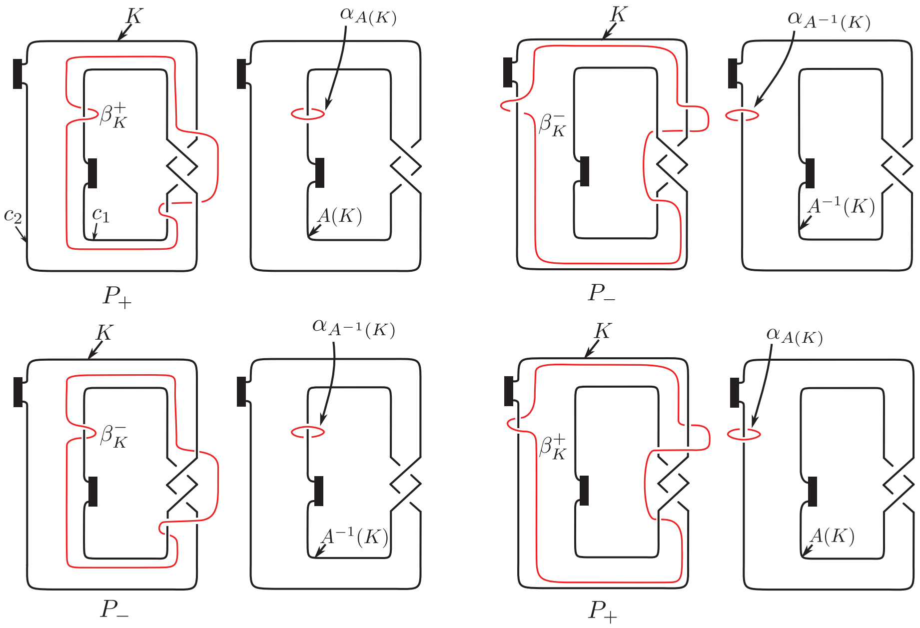

Fig. 4. (color online) From a special annulus presentation (A, b) of a knot K to dualizable patterns P_+ and P_- given by $K \subset \mathbf{S}^3 \backslash \nu(\beta_K^{\pm}) = V_{\pm}$.

Remark 3.5. The homeomorphisms given in Figure 1 induce homeomorphisms

$$\phi_{\pm 1} : (M_K(0), \beta_K^{\pm}) \to (M_{A^{\pm 1}(K)}(0), \alpha_{A^{\pm 1}(K)}),$$

where $\alpha_{A^{\pm 1}(K)} \subset \mathbf{S}^3 \backslash \nu(A^{\pm 1}(K))$ is a meridian of $A^{\pm 1}(K)$. Here we regard $\beta_K^{\pm}$ and $\alpha_{A^{\pm 1}(K)}$ as curves in $M_K(0)$ and $M_{A^{\pm 1}(K)}(0)$, respectively, under the identifications

$$\mathbf{S}^3 \backslash \nu(K) \cong M_K(0) \backslash \nu(L_K),$$

$$\mathbf{S}^3 \backslash \nu(A^{\pm 1}(K)) \cong M_{A^{\pm 1}(K)}(0) \backslash \nu(L_{A^{\pm 1}(K)}),$$

respectively, where L_K and $L_{A^{\pm 1}(K)}$ are the corresponding surgery duals.

4. Operation $(*m)$ and dualizable pattern

By Theorems 2.3 and 3.2, for a knot K with a special annulus presentation (A, b), we have

$$X_{\tau_m(P_+^*)(U)}(m) \cong X_{P_+(U)}(m) = X_K(m) \cong X_{T_m(A(K))}(m),$$

where P_+ is the dualizable pattern obtained from K as in Section 3.2. Hence, it is a natural question whether $\tau_m(P_+^*)(U)$ is isotopic to $T_m(A(K))$ or not. Proposition 3.4 implies that the answer is "yes" if $m = 0$. The following theorem gives the affirmative answer to this question for any $m \in \mathbf{Z}$.

Theorem 4.1. *Let K be a knot with a special annulus presentation (A, b). Let P_+ be the dualizable pattern obtained from K as in Section 3.2. Then, we obtain $\tau_m(P_+^*)(U) = T_m(A(K))$ for any $m \in \mathbf{Z}$.*

Miller and Piccirillo [8, Proposition 5.3] proved Theorem 4.1 for $m = 0$. We can prove Theorem 4.1 by extending Miller and Piccirillo's proof as follows.

Proof. Let $L^{(m)}_{T_m(A(K))} \subset M_{T_m(A(K))}(m)$ be the surgery dual to $T_m(A(K))$. Let $\alpha_{T_m(A(K))} \subset \mathbf{S}^3 \backslash \nu(T_m(A(K)))$ be a meridian of $T_m(A(K))$. Then, we can regard $\alpha_{T_m(A(K))}$ as a curve in $M_{T_m(A(K))}(m)$ by using the following identification

$$\mathbf{S}^3 \backslash \nu(T_m(A(K))) = M_{T_m(A(K))}(m) \backslash \nu(L^{(m)}_{T_m(A(K))}). \tag{1}$$

Since $\alpha_{T_m(A(K))}$ is isotopic to $L^{(m)}_{T_m(A(K))}$ in $M_{T_m(A(K))}(m)$, we have

$$M_{T_m(A(K))}(m) \backslash \nu(L^{(m)}_{T_m(A(K))}) \cong M_{T_m(A(K))}(m) \backslash \nu(\alpha_{T_m(A(K))}). \tag{2}$$

Let $\beta_K^+ \subset \mathbf{S}^3 \backslash \nu(K)$ be the curve given in Section 3.2 (see also Figure 4). We can also regard β_K^+ as a curve in $M_K(m)$ under the identification $\mathbf{S}^3 \backslash \nu(K) \cong M_K(m) \backslash \nu(L_K^{(m)})$, where $L_K^{(m)}$ is the surgery dual to K. Then, we can check that $\psi_m(\beta_K^+) = \alpha_{T_m(A(K))}$, where $\psi_m : M_K(m) \to M_{T_m(A(K))}(m)$ is given in Figure 3. Hence, we obtain

$$\begin{aligned} M_{T_m(A(K))}(m) \backslash \nu(\alpha_{T_m(A(K))}) &\cong M_K(m) \backslash \nu(\beta_K^+) \\ &\cong M_{K \cup \alpha_K}(0, -1/m) \backslash \nu(\beta_K^+) \\ &\cong \mathbf{S}^3 \backslash \nu(K \cup \alpha_K \cup \beta_K^+) \cup \bigsqcup_{i=0,1} (S_i^1 \times D_i^2), \end{aligned} \tag{3}$$

where the last (small) union is given by identifying ∂D_0^2 with 0-framing of K and ∂D_1^2 with $-1/m$-framing of α_K.

Recall that the solid torus V_+ containing P_+ is given by $V_+ = \mathbf{S}^3 \backslash \nu(\beta_K^+)$. Since, the 0-framing of K is viewed as λ_{P_+} and α_K is viewed as μ_{P_+} in V_+, we have

$$\begin{aligned} &\mathbf{S}^3 \backslash \nu(K \cup \alpha_K \cup \beta_K^+) \cup \bigsqcup_{i=0,1} (S_i^1 \times D_i^2) \\ &\quad \cong ((V_+ \backslash \nu(P_+)) \backslash \nu(\mu_{P_+})) \cup \bigsqcup_{i=0,1} (S_i^1 \times D_i^2), \end{aligned} \tag{4}$$

where the last (small) union is given by identifying ∂D_0^2 with λ_{P_+} and ∂D_1^2 with $-1/m$-framing of μ_{P_+}. By the dualizability of P_+, we obtain

$$\begin{aligned}
&((V_+ \backslash \nu(P_+)) \backslash \nu(\mu_{P_+})) \cup \bigsqcup_{i=0,1} (S_i^1 \times D_i^2) \\
&\quad \cong ((V_+^* \backslash \nu(P_+^*)) \backslash \nu(\mu_{V_+^*})) \cup \bigsqcup_{i=0,1} (S_i^1 \times D_i^2) \\
&\quad \cong (V_+^* \backslash \nu(\tau_m(P_+^*))) \cup (S_0^1 \times D_0^2) \\
&\quad \cong \mathbf{S}^3 \backslash \nu(\tau_m(P_+^*)(U)),
\end{aligned} \tag{5}$$

where the last union is given by identifying ∂D_0^2 with $\lambda_{V_+^*}$. By (1)–(5) and the Knot Complement Theorem, we obtain $\tau_m(P_+^*)(U) = T_m(A(K))$.

Remark 4.2. Let K be a knot with a special annulus presentation (A, b). Let $\bar{K}$ be the mirror image of K and $(\bar{A}, \bar{b})$ be the special annulus presentation of $\bar{K}$ obtained from (A, b) by taking mirror image. Let $\gamma_{A^{-1}(K)} \subset \mathbf{S}^3 \backslash \nu(A^{-1}(K))$ be the mirror image of $\gamma_{\dot{\bar{A}}(\bar{K})} \subset \mathbf{S}^3 \backslash \nu(\bar{A}(\bar{K}))$ (see also Figure 5). Denote the knot obtained from $A^{-1}(K)$ by twisting m times along $\gamma_{A^{-1}(K)}$ by $T_m(A^{-1}(K))$. Then, by the similar discussion to Theorem 4.1, we see that $\tau_m(P_-^*)(U) = T_m(A^{-1}(K))$ for any $m \in \mathbf{Z}$.

We see that $A(K) \subset \mathbf{S}^3 \backslash \nu(\gamma_{A(K)}) = V_+'$ also gives a dualizable pattern, where the parameter of $V_+' \cong \mathbf{S}^1 \times D^2$ is given by the standard way. Denote it by P_+'. It is easy to see that $\tau_m(P_+')(U) = T_m(A(K)) = \tau_m(P_+^*)(U)$ for any $m \in \mathbf{Z}$. So we can consider the question which asks whether P_+' is equal to P_+^* as a pattern. We can give the affirmative answer to the question as follows.

Theorem 4.3. *Let K be a knot with a special annulus presentation (A, b). Let $P_+' \subset V_+'$ be the dualizable pattern as above, and let $P_+^* \subset V_+^*$ be the dualizable pattern obtained from K as in Section 3.2. Then, for any $m \in \mathbf{Z}$, there is an orientation-preserving homeomorphism $h : V_+' \to V_+^*$ such that*

- *$h(\tau_m(P_+')) = \tau_m(P_+^*)$, and*
- *$h(\lambda_{V_+'}) = \lambda_{V_+^*}$ and $h(\mu_{V_+'}) = \mu_{V_+^*}$.*

Namely, $P_+' = P_+^*$ as patterns.

Proof. By the definition of the operation $(*m)$, we see that

$$\begin{aligned}
&M_{T_m(A(K))}(m) \backslash \nu(\alpha_{T_m(A(K))}) \\
&\quad \cong M_{A(K) \cup \gamma_{A(K)}}(0, -1/m) \backslash \nu(\alpha_{A(K)}) \\
&\quad \cong \mathbf{S}^3 \backslash \nu(A(K) \cup \gamma_{A(K)} \cup \alpha_{A(K)}) \cup \bigsqcup_{i=0,1} (S_i^1 \times D_i^2),
\end{aligned} \tag{6}$$

where the last (small) union is given by identifying ∂D_0^2 with 0-framing of $A(K)$ and ∂D_1^2 with $-1/m$-framing of $\gamma_{(A(K))}$. Since $\alpha_{A(K)}$ is isotopic to the surgery dual to $A(K)$, we have

$$\begin{aligned}
\mathbf{S}^3 \backslash \nu(A(K) \cup \gamma_{A(K)} \cup \alpha_{A(K)}) &\cup \bigsqcup_{i=0,1} (S_i^1 \times D_i^2) \\
&\cong \mathbf{S}^3 \backslash \nu(A(K) \cup \gamma_{A(K)}) \cup (S_1^1 \times D_1^2) \\
&= (V'_+ \backslash \nu(\tau_m(P'_+))) \cup (S_2^1 \times D_2^2), \qquad (7)
\end{aligned}$$

where the last union is given by identifying ∂D_2^2 with $\lambda_{V'_+}$. By considering the composition of (7), (6), (3), (4) and (5), we obtain an orientation-preserving homeomorphism

$$\bar{h} : (V'_+ \backslash \nu(\tau_m(P'_+))) \cup (S_2^1 \times D_2^2) \to (V^*_+ \backslash \nu(\tau_m(P^*_+))) \cup (S_0^1 \times D_0^2).$$

Then, we can check that
- $\bar{h}(\lambda_{\tau_m(P'_+)}) = \lambda_{\tau_m(P^*_+)}$,
- $\bar{h}(\lambda_{V'_+}) = \lambda_{V^*_+}$ and $\bar{h}(\mu_{V'_+}) = \mu_{V^*_+}$, and
- $\bar{h}(S_2^1 \times D_2^2) = S_0^1 \times D_0^2$.

Hence, $\bar{h}$ induces a desired homeomorphism.

REMARK 4.4. Similarly, we can define P'_- as $A^{-1}(K) \subset \mathbf{S}^3 \backslash \nu(\gamma_{A^{-1}(K)}) = V'_-$ (see also Remark 4.2). By the same discussion as the proof of Theorem 4.3, we see that there is an orientation-preserving homeomorphism $h : V'_- \to V^*_-$ which satisfies $h(\tau_m(P'_-)) = \tau_m(P^*_-)$, $h(\lambda_{V'_-}) = \lambda_{V^*_-}$ and $h(\mu_{V'_-}) = \mu_{V^*_-}$.

REMARK 4.5. We see that Theorem 4.3 induces Theorem 4.1 since $\tau_m(P^*_+)(U) = \tau_m(P'_+)(U)$ by Theorem 4.3 and $\tau_m(P'_+)(U) = T_m(A(K))$ by the definition of P'_+.

By Theorem 4.3 and Remark 4.4, we can draw the duals $P^*_\pm$ to $P_\pm$ as in Figure 5, where $P_\pm$ are the dualizable patterns obtained from a knot K with a special annulus presentation (A, b) as in Section 3.2.

5. Flipped annulus twist and operation $(*m)$ with $m = \pm 4$

In [11], Teragaito gave the first example of a Seifert fibered manifold which is represented by the same integral surgery on infinitely many hyperbolic knots. In the work, Teragaito used a presentation of 9_{42}, which is almost the same as a special annulus presentation but does not satisfy the last condition: $A \cup b$ is an immersion of an orientable surface. In fact, in the examples, the surface is non-orientable, see (the final figure in) Figure 6. Teragaito explained that, for a knot with such a presentation, we obtain a family of knots admitting the

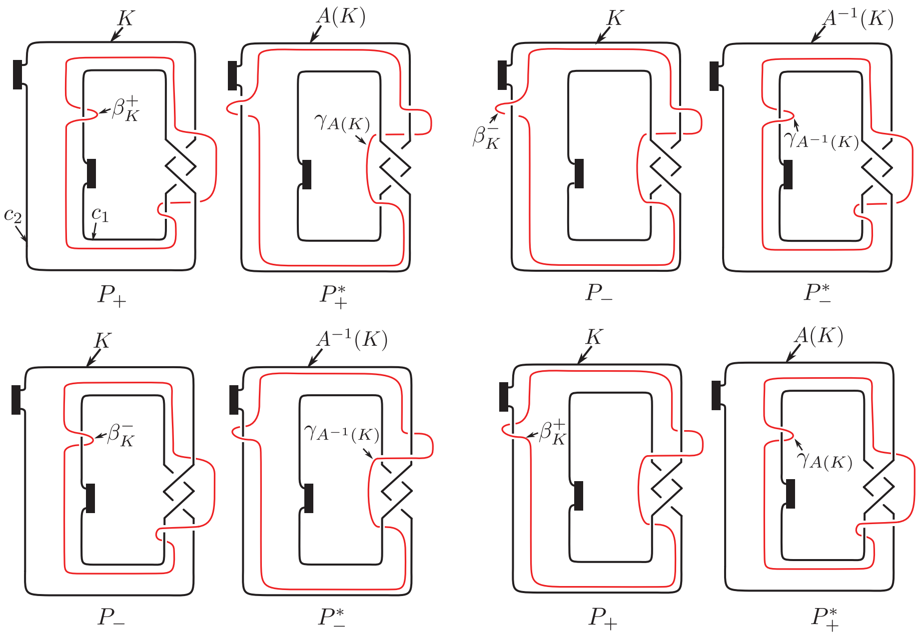

Fig. 5. (color online) The dualizable patterns $P_\pm \subset V_\pm = \mathbf{S}^3 \backslash \nu(\beta_K^\pm)$ and $P_\pm^* \subset V_\pm^* = \mathbf{S}^3 \backslash \nu(\gamma_{A^{\pm 1}(K)})$.

same 4-surgery (not 0-surgery) by annulus twists along (a shrunken annulus of) the annulus. It has been known that such knots have the same 4-trace (see [1, Theorem 2.8]).

In this section, we prove that the above phenomenon can be explained in terms of the operation $(*m)$ with $m = 4$.

5.1. Flipped annulus twist. Let $A \subset \mathbf{S}^3$ be an embedded annulus with ordered boundary $\partial A = c_1 \cup c_2$. We suppose that A is unknottend and $\mathrm{lk}(c_1, c_2) = \pm 1$, where we give c_1 and c_2 parallel orientations. Then, an *n-fold flipped annulus twist along* A is to apply $(-\mathrm{lk}(c_1, c_2) + 1/n)$-surgery along c_1 and $(-\mathrm{lk}(c_1, c_2) - 1/n)$-surgery along c_2 (compare with Section 2.1).

Let K be a knot with a special annulus presentation (A, b). Then, by $A_f^n(K)$, we denote the knot obtained from K by the n-fold flipped annulus twist along A', where A' is a shrunken annulus given in Section 2.1. For simplicity, we also denote $A_f^1(K)$ by $A_f(K)$. We also see $A_f^n(K)$ as follows: After "flipping" c_1 (or c_2) as in Figure 6, we find another annulus A_f. Then, by using [3, Lemma 7.15], we see that $A_f^n(K)$ is obtained from K by applying the n-fold annulus twist along A_f', where A_f' is a shrunken annulus of A_f. Remark that (A_f, b) is not an annulus presentation any more since $A_f \cup b$ is an immersion of a non-orientable surface.

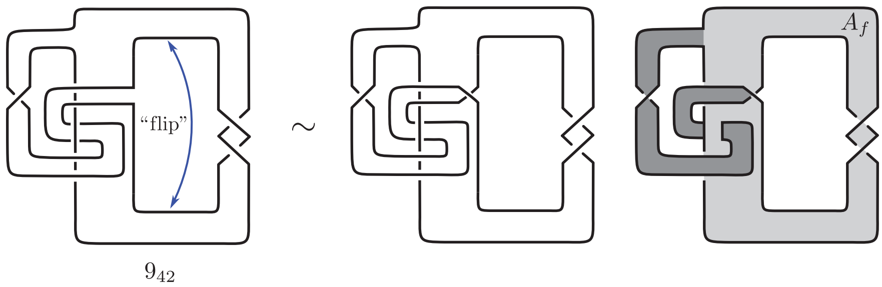

Fig. 6. (color online) An annulus presentation of 9_{42} (left). After “flipping” c_1, we find a new annulus A_f.

5.2. Relation to the operation $(*m)$ **with** $m = \pm 4$. Teragaito [11, Proposition 2.1] proved that there is an orientation-preserving homeomorphism $M_K(r) \to M_{A_f^n(K)}(r)$, where $r = -4\,\mathrm{lk}(c_1, c_2) \in \{\pm 4\}$. Denote this homeomorphism by

$$\phi_n^f : M_K(r) \to M_{A_f^n(K)}(r).$$

For a sketch of the proof, see Figure 7. Then, we notice that

$$\phi_{\pm 1}^f(\beta_K^{\mp}) = \alpha_{A_f^{\pm 1}(K)}, \tag{8}$$

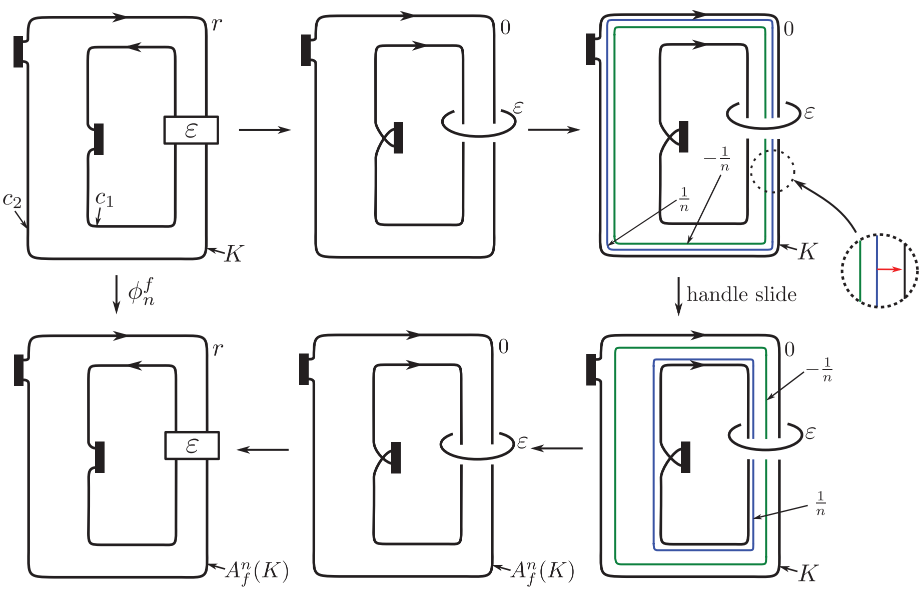

Fig. 7. (color online). The homeomorphism $\phi_n^f : M_K(r) \to M_{A_f^n(K)}(r)$, where $\varepsilon \in \{\pm 1\}$ and $r = -4\varepsilon$. The box with ε represents ε-full-twist. For convenience, we draw an orientation of the knot (not c_1 and c_2).

where $\alpha_{A_f^{\pm 1}(K)}$ is a meridian of $A_f^{\pm 1}(K)$ and we regard $\beta_K^{\pm}$ and $\alpha_{A_f^{\pm 1}(K)}$ as curves in $M_K(r)$ and $M_{A_f^n(K)}(r)$, respectively (by using the same discussion in Remark 3.5). We have seen that $M_{T_r(A(K))}(r) \cong M_K(r) \cong M_{A_f^{-1}(K)}(r)$. Moreover, we can prove that

$$T_r(A^{\mp 1}(K)) = A_f^{\pm 1}(K). \tag{9}$$

In fact, by replacing ψ_m with $\phi_{\pm 1}^f$ and β_K^+ with $\beta_K^{\mp}$ in the proof of Theorem 4.1, we see that $\mathbf{S}^3 \backslash \nu(A_f^{\pm 1}(K)) \cong \mathbf{S}^3 \backslash \nu(\tau_r(P_{\mp}^*)(U)) \cong \mathbf{S}^3 \backslash \nu(T_r(A^{\mp 1}(K)))$. By the Knot Complement Theorem, we obtain Equation (9). As a consequence, we obtain the following.

Theorem 5.1. *Let K be a knot with a special annulus presentation (A, b) with $\partial A = c_1 \cup c_2$. Then we obtain*

$$T_r(A^{\mp 1}(K)) = A_f^{\pm 1}(K),$$

where $r = -4\,\mathrm{lk}(c_1, c_2)$, and we give c_1 and c_2 parallel orientations.

Remark 5.2. In private communication, Tetsuya Abe commented that $T_4(A(9_{42}))$ and $A_f^{-1}(9_{42})$ may be equivalent because of computational calculations. Theorem 5.1 is inspired by the comment.

6. Discussions

6.1. Naturality. Let K be a knot with a special annulus presentation (A, b). Then, we obtain a dualizable pattern P_+ as in Section 3.2. Put $\check{K} = A(K)$ and give the natural annulus presentation $(\check{A}, \check{b})$ of $\check{K}$ from (A, b). Then we obtain another dualizable pattern $\check{P}_-$ from $\check{K}$ as in Section 3.2. We see that these patterns satisfy $P_+(U) = K$, $P_+^*(U) = A(K)$, $\check{P}_-(U) = A(K)$ and $\check{P}_-^*(U) = K$. More strongly, Theorem 4.3 and Figure 5 imply that $P_+ = \check{P}_-^*$ and $P_+^* = \check{P}_-$.

Let SAP be the set of special annulus presentations and DP be the set of unoriented patterns which are dualizable after giving some orientation. Then, by the above discussion, we obtain the following commutative diagram:

$$\begin{tikzcd} SAP \arrow[r, "\pm"] \arrow[d, "A^{\pm 1}"'] & DP \arrow[d, "*"] \\ SAP \arrow[r, "\mp"] & DP \end{tikzcd}$$

where

- $A^{\pm 1} : SAP \to SAP$ is the map induced by ± 1-fold annulus twist,
- $\pm : SAP \to DP$ is given by $(A, b) \mapsto P_{\pm}$ as in Section 3.2, and
- $* : DP \to DP$ is given by $P \mapsto P^*$.

6.2. Generalization. For more general setting, we obtain the following result by the proof of Theorem 4.1. Remark that, in Theorem 6.1 below, we regard curves β and α_{K_2} as curves in $M_{K_1}(m)$ and $M_{K_2}(m)$, respectively, under the identification $\mathbf{S}^3 \backslash \nu(K_i) = M_{K_i}(m) \backslash \nu(L_{K_i})$, where L_{K_i} is the surgery dual.

THEOREM 6.1. *Let K_1 and K_2 be knots in $\mathbf{S}^3$. Let $\beta \subset \mathbf{S}^3 \backslash \nu(K_1)$ be an unknot. Let $\alpha_{K_2} \subset \mathbf{S}^3 \backslash \nu(K_2)$ be a meridian of K_2. Suppose that $P = K_1 \subset \mathbf{S}^3 \backslash \nu(\beta)$ gives a dualizable pattern. Then, if there is an orientation-preserving homeomorphism $\phi : M_{K_1}(m) \to M_{K_2}(m)$ such that $\phi(\beta) = \alpha_{K_2}$, we have $\tau_m(P^*)(U) = K_2$. Moreover ϕ extends to a diffeomorphism $\Phi : X_{K_1}(m) \to X_{K_2}(m)$.*

The last claim follows from the same discussion as Theorem 3.2.

QUESTION 6.2. *Let $\phi : M_{K_1}(m) \to M_{K_2}(m)$ be an orientation-preserving homeomorphism. Then, when is there an unknot $\beta \subset \mathbf{S}^3 \backslash \nu(K_1)$ which satisfies the condition of Theorem 6.1? Moreover, if exists, is such β unique up to isotopy in $\mathbf{S}^3 \backslash \nu(K_1)$?*

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