

A criterion for the existence of a plane model with two inner Galois points for algebraic curves

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ABSTRACT. A criterion for the existence of a plane model with two non-smooth Galois points for algebraic curves is presented, which is a generalization of Fukasawa's criterion for two smooth Galois points. Multiplicities and order sequences at Galois points are also described in detail.

1. Introduction

Let k be an algebraically closed field of characteristic $p \geq 0$, and let $C \subset \mathbb{P}^2$ be an irreducible (possibly singular) plane curve of degree $d \geq 4$ over k . For points $P, Q \in \mathbb{P}^2$ with $P \neq Q$, the line passing through P, Q is denoted by \overline{PQ} . We consider the projection $\pi_P : C \dashrightarrow \mathbb{P}^1; Q \mapsto \overline{PQ}$ with the center $P \in \mathbb{P}^2$. If the field extension $k(C)/\pi_P^*k(\mathbb{P}^1)$ of function fields induced by π_P is Galois, then P is called a Galois point for C . This notion was introduced in 1996 by Yoshihara (see [3, 13, 16]). For a Galois point P , the associated Galois group $G_P = \text{Gal}(k(C)/\pi_P^*k(\mathbb{P}^1))$ is called a Galois group at P . Furthermore, a Galois point P is called a smooth Galois point (resp. a non-smooth Galois point, an inner Galois point, an outer Galois point) if P is a smooth point of C (resp. a singular point of C , a point contained in C , a point not contained in C), after [11, 12, 15]. Note that the definition of an inner Galois point in [5] is equivalent to the definition of a smooth Galois point in this article.

In 2016, Fukasawa [5] presented a criterion for the existence of a birational embedding of a smooth projective curve into a projective plane with two smooth Galois points and obtained new examples of plane curves with two smooth Galois points by using this criterion. On the other hand, there have been some known examples of plane curves with two or more non-smooth Galois points. For example, the Ballico-Hefez curve ([4, Theorem 1]), some self-dual curves ([9, Theorem 17]), the (plane model of) Giulietti-Korchmáros

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curve ([8, Theorem 2]), the (q^3, q^2) -Frobenius nonclassical curve ([1, Theorem 1]), and the Artin-Schreier-Mumford curve (proof of [6, Theorem 1]) are such curves. However, these examples are not intended to actively focus on non-smooth Galois points. Only few research studies have focused on non-smooth Galois points. Takahashi [15] studied plane quintic curves with a double point P and determined defining equations when P is a Galois point. As far as the author knows, this is the only study that focused on a non-smooth Galois point so far. In order to study non-smooth Galois points systematically, it is good to have a criterion for non-smooth Galois points.

In this article, we extend Fukasawa's criterion [5, Theorem 1] to all cases with two (possibly non-smooth) Galois points. Let X be a (reduced, irreducible) smooth projective curve over k , and let $k(X)$ be its function field. The full automorphism group of X is denoted by $\text{Aut}(X)$. For a finite subgroup $G \subset \text{Aut}(X)$ and a point $P \in X$, the stabilizer of P in G (resp. the orbit of P under G) is denoted by $G(P)$ (resp. $G \cdot P$). Furthermore, the quotient curve of X by G , that is, the smooth projective curve corresponding to the fixed field of $k(X)$ by G , is denoted by X/G . The following are our main theorems.

THEOREM 1. *Let G_1 and G_2 be finite subgroups of $\text{Aut}(X)$ and let P_1 and P_2 be different points of X . Then there exists a birational embedding $\varphi : X \rightarrow \mathbb{P}^2$ such that $\varphi(P_1)$ and $\varphi(P_2)$ are different inner Galois points, that $G_{\varphi(P_i)} = G_i$ for $i = 1, 2$, and that $L = \overline{\varphi(P_1)\varphi(P_2)}$ is not a tangent line at $\varphi(P_1)$, if and only if the following conditions are satisfied:*

- (a) $X/G_1 \cong \mathbb{P}^1$, $X/G_2 \cong \mathbb{P}^1$,
- (b) $G_1 \cap G_2 = \{1\}$, and
- (c) *one of the following holds:*
 - (c-i) $P_1 \notin G_1 \cdot P_2$, $P_2 \notin G_2 \cdot P_1$, $G_1 \cdot P_2 \cap G_2 \cdot P_1 \neq \emptyset$, and $|G_1(P_2)| = |G_2(P_1)|$.
 - (c-ii) $G_1 \cdot P_2 \cap G_2 \cdot P_1 = \emptyset$.
 - (c-iii) $P_1 \notin G_1 \cdot P_2$, $G_1 \cdot P_2 \cap G_2 \cdot P_1 \neq \emptyset$ and $|G_1(P_2)| > |G_2(P_1)|$.

Furthermore, for any φ as in the above, the following hold:

- (i) L is not a tangent line at $\varphi(P_2)$ with $L \cap \varphi(X) \supseteq \{\varphi(P_1), \varphi(P_2)\}$ if and only if condition (c-i) is satisfied.
- (ii) L is not a tangent line at $\varphi(P_2)$ with $L \cap \varphi(X) = \{\varphi(P_1), \varphi(P_2)\}$ if and only if condition (c-ii) is satisfied.
- (iii) L is a tangent line at $\varphi(P_2)$ if and only if condition (c-iii) is satisfied.

For the birational embedding φ in Theorem 1, we have the following.

THEOREM 2. *Let φ be as in Theorem 1, and let Λ be the linear system on X corresponding to the morphism φ . Let $(0, \alpha_P, \beta_P)$ denote the (Λ, P) -order sequence at a point $P \in X$. Then the following hold.*

(1) *The multiplicity $m_{\varphi(P_1)}$ of $\varphi(X)$ at $\varphi(P_1)$ is equal to*

$$|G_2(P_1)| \cdot |G_2 \cdot P_1 \setminus (G_1 \cdot P_2 \cap G_2 \cdot P_1)|.$$

(2) *The divisor $\sum_{P \in \varphi^{-1}(\varphi(P_1))} \alpha_P P$ is equal to*

$$\sum_{Q \in G_2 \cdot P_1 \setminus (G_1 \cdot P_2 \cap G_2 \cdot P_1)} |G_2(P_1)| Q.$$

(3) *The multiplicity $m_{\varphi(P_2)}$ of $\varphi(X)$ at $\varphi(P_2)$ is equal to*

$$|G_1(P_2)| \cdot |G_1 \cdot P_2 \setminus (G_1 \cdot P_2 \cap G_2 \cdot P_1)| + (|G_1(P_2)| - |G_2(P_1)|) \cdot |G_1 \cdot P_2 \cap G_2 \cdot P_1|.$$

(4) *The divisor $\sum_{P \in \varphi^{-1}(\varphi(P_2))} \alpha_P P$ is equal to*

$$\sum_{R \in G_1 \cdot P_2 \setminus (G_1 \cdot P_2 \cap G_2 \cdot P_1)} |G_1(P_2)| R + \sum_{S \in G_1 \cdot P_2 \cap G_2 \cdot P_1} (|G_1(P_2)| - |G_2(P_1)|) S.$$

(5) *In the case (iii) of Theorem 1, the equality $\beta_P = |G_1(P_2)|$ holds at each point $P \in G_1 \cdot P_2 \cap G_2 \cdot P_1$.*

(6) *The divisor $\varphi^* L$ is equal to*

$$\sum_{Q \in G_2 \cdot P_1 \setminus (G_1 \cdot P_2 \cap G_2 \cdot P_1)} |G_2(P_1)| Q + \sum_{R \in G_1 \cdot P_2} |G_1(P_2)| R.$$

To explain the usefulness of Theorems 1 and 2, we apply our criterion to rational curves.

THEOREM 3. *There exist the following birational embeddings $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^2$.*

(1) *$p = 3$, $\deg(\varphi(\mathbb{P}^1)) = 14$ and there exist two non-smooth Galois points $\varphi(P_1)$ and $\varphi(P_2) \in \varphi(\mathbb{P}^1)$ such that $m_{\varphi(P_1)} = 4$, $m_{\varphi(P_2)} = 8$, $G_{\varphi(P_1)} \cong \mathbf{D}_5$, $G_{\varphi(P_2)} \cong \mathbf{AGL}(1, \mathbb{F}_3)$, and $L = \overline{\varphi(P_1)\varphi(P_2)}$ is not a tangent line at $\varphi(P_1)$ and $\varphi(P_2)$. The second order is equal to 2 at each point contained in $\text{supp}(\varphi^* L)$.*

(2) *$p \neq 2, 5$, $\deg(\varphi(\mathbb{P}^1)) = 16$ and there exist two non-smooth Galois points $\varphi(P_1)$ and $\varphi(P_2) \in \varphi(\mathbb{P}^1)$ such that $m_{\varphi(P_1)} = 4$, $m_{\varphi(P_2)} = 11$, $G_{\varphi(P_1)} \cong \mathbf{A}_4$, $G_{\varphi(P_2)} \cong \mathbb{Z}/5\mathbb{Z}$, $L = \overline{\varphi(P_1)\varphi(P_2)}$ is not a tangent line at $\varphi(P_1)$, and L is a tangent line at $\varphi(P_2)$. The second order is equal to 2 (resp. 1) at each point $Q \in G_{\varphi(P_1)} \cdot P_2 \setminus \{P_2\}$ (resp. $Q \in G_{\varphi(P_2)} \cdot P_1$), and the third order is equal to 2 at P_2 .*

(3) *$p \neq 2, 5$, $\deg(\varphi(\mathbb{P}^1)) = 28$ and there exist two non-smooth Galois points $\varphi(P_1)$ and $\varphi(P_2) \in \varphi(\mathbb{P}^1)$ such that $m_{\varphi(P_1)} = 4$, $m_{\varphi(P_2)} = 23$, $G_{\varphi(P_1)} \cong \mathbf{S}_4$, $G_{\varphi(P_2)} \cong \mathbb{Z}/5\mathbb{Z}$, $L = \overline{\varphi(P_1)\varphi(P_2)}$ is not a tangent line at $\varphi(P_1)$, and L is a tangent line at $\varphi(P_2)$. The second order is equal to 4 (resp. 3, 1)*

at each point $Q \in G_{\varphi(P_1)} \cdot P_2 \setminus \{P_2\}$ (resp. at P_2 , at each point $Q \in G_{\varphi(P_2)} \cdot P_1 \setminus \{P_2\}$), and the third order is equal to 4 at P_2 .

2. Preliminaries

We recall some notation and facts. Let $\varphi : X \rightarrow \mathbb{P}^2$ be a morphism, which is birational onto its image. Such a morphism φ is called a birational embedding of X to \mathbb{P}^2 . Assume that $\varphi(X)$ is not a line. First, we recall the notion of order sequences (see [10, Chapter 7]). For a line $L \subset \mathbb{P}^2$, the intersection divisor of $\varphi(X)$ and L on X is denoted by φ^*L . Note that

$$\mathcal{A} = \{\varphi^*L \mid L \text{ is a line contained in } \mathbb{P}^2\}$$

is the linear system on X corresponding to the morphism φ . The support of the divisor φ^*L is denoted by $\text{supp}(\varphi^*L)$. For a point $P \in X$, the order of φ^*L at P is denoted by $\text{ord}_P(\varphi^*L)$. We put

$$\alpha_P = \min\{\text{ord}_P(\varphi^*L) \mid \varphi^*L \in \mathcal{A}, P \in \text{supp}(\varphi^*L)\}.$$

Then there exists a unique line \tilde{L} such that $\beta_P = \text{ord}_P(\varphi^*\tilde{L}) > \alpha_P$. We call the line \tilde{L} the osculating line at P , and we call the sequence $(0, \alpha_P, \beta_P)$ the (\mathcal{A}, P) -order sequence at P . A line \tilde{L} passing through $\varphi(P)$ is called a tangent line at $\varphi(P)$ if \tilde{L} is the osculating line at a point contained in $\varphi^{-1}(\varphi(P))$. Note that a line \tilde{L} is a tangent line at $\varphi(P)$ if and only if $m_{\varphi(P)} < I_{\varphi(P)}(\varphi(X), \tilde{L})$, where $I_{\varphi(P)}(\varphi(X), \tilde{L})$ is the intersection multiplicity of $\varphi(X)$ and \tilde{L} at $\varphi(P)$.

Next, we consider the projection $\pi_{\varphi(P)}$, and we put $\hat{\pi}_{\varphi(P)} = \pi_{\varphi(P)} \circ \varphi : X \rightarrow \mathbb{P}^1$. We recall some properties of a ramification index of $\hat{\pi}_{\varphi(P)}$. We put $\varphi^{-1}(\varphi(P)) = \{P_1, \dots, P_n\}$. Let $(0, \alpha_{P_i}, \beta_{P_i})$ be the (\mathcal{A}, P_i) -order sequence for $i = 1, \dots, n$. The ramification index of $\hat{\pi}_{\varphi(P)}$ at a point $Q \in X$ is denoted by $e_Q(\hat{\pi}_{\varphi(P)})$. Then the following fact is well-known.

FACT 4. Let $Q \in X \setminus \{P_1, \dots, P_n\}$.

- (1) The equality $e_Q(\hat{\pi}_{\varphi(P)}) = \text{ord}_Q(\overline{\varphi^*(P)\varphi(Q)})$ holds.
- (2) The equality $e_{P_i}(\hat{\pi}_{\varphi(P)}) = \beta_{P_i} - \alpha_{P_i}$ holds for $i = 1, \dots, n$.

Finally, we recall some properties of a Galois covering for the proof of our main theorems (see [14, III. 7.1, 7.2 and 8.2]).

FACT 5. Let $\theta : X \rightarrow Y$ be a surjective morphism of smooth projective curves, and assume that the field extension $k(X)/\theta^*k(Y)$ is a Galois extension with the Galois group G . Then the following hold.

- (1) If $P, Q \in X$ and $\theta(P) = \theta(Q)$, then there exists an element $\sigma \in G$ such that $\sigma(P) = Q$.

- (2) If $P, Q \in X$ and $\theta(P) = \theta(Q)$, then $e_P(\theta) = e_Q(\theta)$.
- (3) For each point $P \in X$, the order $|G(P)|$ is equal to $e_P(\theta)$.

3. Proofs of Theorems 1 and 2

The same notation is used as in the previous section. The following lemma shows that Theorem 1 describes all cases with two inner Galois points.

LEMMA 6. *Let $P_1, P_2 \in X$, and assume that $\varphi(P_1)$ and $\varphi(P_2)$ are different inner Galois points. We put $L = \overline{\varphi(P_1)\varphi(P_2)}$. Then either $m_{\varphi(P_1)} = I_{\varphi(P_1)}(\varphi(X), L)$ or $m_{\varphi(P_2)} = I_{\varphi(P_2)}(\varphi(X), L)$ holds.*

PROOF. We put

$$\begin{aligned} \varphi^{-1}(\varphi(P_1)) &= \{P_{11} = P_1, P_{12}, \dots, P_{1m_1}\}, \\ \varphi^{-1}(\varphi(P_2)) &= \{P_{21} = P_2, P_{22}, \dots, P_{2m_2}\}. \end{aligned}$$

Let $(0, \alpha_{P_{ij}}, \beta_{P_{ij}})$ be the (A, P_{ij}) -order sequence for i, j . Assume by contradiction that

$$m_{\varphi(P_1)} < I_{\varphi(P_1)}(\varphi(X), L) \quad \text{and} \quad m_{\varphi(P_2)} < I_{\varphi(P_2)}(\varphi(X), L)$$

hold. By Fact 5, the ramification index of $\hat{\pi}_{\varphi(P_1)}$ (resp. $\hat{\pi}_{\varphi(P_2)}$) at each point contained in $\varphi^{-1}(\varphi(P_2))$ (resp. $\varphi^{-1}(\varphi(P_1))$) coincides with $|G_{\varphi(P_1)}(P_2)|$ (resp. $|G_{\varphi(P_2)}(P_1)|$). By Fact 4 (1) and Fact 5, $|G_{\varphi(P_1)}(P_2)|$ (resp. $|G_{\varphi(P_2)}(P_1)|$) coincides with $\text{ord}_{P_{2j}}(\varphi^*L)$ for each j (resp. $\text{ord}_{P_{1i}}(\varphi^*L)$ for each i). Since L is a tangent line at $\varphi(P_1)$ (resp. $\varphi(P_2)$), there exists i_0 (resp. j_0) such that

$$\beta_{P_{1i_0}} = \text{ord}_{P_{1i_0}}(\varphi^*L) \quad (\text{resp. } \beta_{P_{2j_0}} = \text{ord}_{P_{2j_0}}(\varphi^*L)).$$

By Fact 4 (2) and Fact 5,

$$|G_{\varphi(P_1)}(P_2)| = \beta_{P_{1i_0}} - \alpha_{P_{1i_0}} \quad (\text{resp. } |G_{\varphi(P_2)}(P_1)| = \beta_{P_{2j_0}} - \alpha_{P_{2j_0}})$$

holds. Therefore, we have a contradiction as follows:

$$\begin{aligned} |G_{\varphi(P_2)}(P_1)| &< |G_{\varphi(P_2)}(P_1)| + \alpha_{P_{2j_0}} = \beta_{P_{2j_0}} = \text{ord}_{P_{2j_0}}(\varphi^*L) = |G_{\varphi(P_1)}(P_2)| \\ &< |G_{\varphi(P_1)}(P_2)| + \alpha_{P_{1i_0}} = \beta_{P_{1i_0}} = \text{ord}_{P_{1i_0}}(\varphi^*L) = |G_{\varphi(P_2)}(P_1)|. \end{aligned}$$

PROOF (Proof of Theorem 1). We consider the ‘if’ part. Assume that conditions (a), (b), and (c) in Theorem 1 are satisfied. Let $f, g \in k(X)$ be the generators of $k(X)^{G_1}$ and $k(X)^{G_2}$ such that

$$(f)_\infty = \sum_{\sigma \in G_1} \sigma(P_2), \quad (g)_\infty = \sum_{\tau \in G_2} \tau(P_1),$$

which exist by condition (a), where $(f)_\infty$ (resp. $(g)_\infty$) is the pole divisor of f (resp. g). We consider the morphism $\varphi = (f : g : 1) : X \rightarrow \mathbb{P}^2$. First, we show that the equality $\varphi(P_1) = (0 : 1 : 0)$ holds. We put $n_g = \text{ord}_{P_1}((g)_\infty)$. Note that n_g is equal to $|G_2(P_1)|$. Let t_{P_1} be a local parameter at P_1 . Since $P_1 \notin G_1 \cdot P_2 = \text{supp}((f)_\infty)$ by condition (c),

$$\text{ord}_{P_1}(t_{P_1}^{n_g} f) = n_g + \text{ord}_{P_1}(f) \geq n_g > 0$$

hold. Therefore, we have the equality $\varphi(P_1) = (0 : 1 : 0)$. We also show that the equality $\varphi(P_2) = (1 : 0 : 0)$ holds. We put $n_f = \text{ord}_{P_2}((f)_\infty)$. Note that n_f is equal to $|G_1(P_2)|$. Let t_{P_2} be a local parameter at P_2 . If $P_2 \notin G_2 \cdot P_1 = \text{supp}((g)_\infty)$, we have

$$\text{ord}_{P_2}(t_{P_2}^{n_f} g) = n_f + \text{ord}_{P_2}(g) \geq n_f > 0.$$

If $P_2 \in G_2 \cdot P_1$, then condition (c-iii) is satisfied, and we have

$$\text{ord}_{P_2}(t_{P_2}^{n_f} g) = n_f + \text{ord}_{P_2}(g) = |G_1(P_2)| - |G_2(P_1)| > 0.$$

Therefore, the equality $\varphi(P_2) = (1 : 0 : 0)$ holds. By a method similar to the proof of [5, Proposition 1], by condition (b), we can show that the morphism φ is birational onto its image. The morphism $(f : 1)$ (resp. $(g : 1)$) coincides with the projection from the point

$$\varphi(P_1) = (0 : 1 : 0) \quad (\text{resp. } \varphi(P_2) = (1 : 0 : 0)).$$

Therefore $\varphi(P_1)$ and $\varphi(P_2)$ are different inner Galois points, and $G_{\varphi(P_i)} = G_i$ for $i = 1, 2$. We show that $L = \overline{\varphi(P_1)\varphi(P_2)}$ is not a tangent line at $\varphi(P_1)$. Assume by contradiction that L is a tangent line at $\varphi(P_1)$. Then there exists a point $Q \in \varphi^{-1}(\varphi(P_1))$ such that $Q \in G_1 \cdot P_2$. Let \mathcal{A} be the linear system on X corresponding to the morphism φ , and let $(0, \alpha_Q, \beta_Q)$ be the (\mathcal{A}, Q) -order sequence at Q . Since L is the osculating line at Q , we have

$$|G_2(P_1)| = \text{ord}_Q(\varphi^* L) = \beta_Q$$

by Fact 4 (1) and Fact 5. On the other hand, by Fact 4 (2) and Fact 5, the equality

$$|G_1(P_2)| = \beta_Q - \alpha_Q$$

holds. Therefore, we have $G_1 \cdot P_2 \cap G_2 \cdot P_1 \neq \emptyset$ and $|G_1(P_2)| < |G_2(P_1)|$. This is a contradiction to condition (c). Therefore, L is not a tangent line at $\varphi(P_1)$.

We consider the ‘only if’ part. Assume that there exists a birational embedding $\varphi : X \rightarrow \mathbb{P}^2$ such that $\varphi(P_1)$ and $\varphi(P_2)$ are different inner Galois

points, $G_{\varphi(P_i)} = G_i$ for $i = 1, 2$, and $L = \overline{\varphi(P_1)\varphi(P_2)}$ is not a tangent line at $\varphi(P_1)$. Since

$$k(X)^{G_i} = (\hat{\pi}_{\varphi(P_i)})^*(k(\mathbb{P}^1)) \cong k(\mathbb{P}^1)$$

for $i = 1, 2$, condition (a) is satisfied. By a method similar to the proof of [5, Theorem 1], condition (b) is satisfied. Since L is not a tangent line at $\varphi(P_1)$, we have $P_1 \notin G_1 \cdot P_2$. We show condition (c) by dividing into the following three cases (I), (II), and (III).

(I) Assume that L is not a tangent line at $\varphi(P_2)$ and

$$L \cap \varphi(X) \supseteq \{\varphi(P_1), \varphi(P_2)\}.$$

We show that condition (c-i) is satisfied. Since L is not a tangent line at $\varphi(P_2)$ and $(L \cap \varphi(X)) \setminus \{\varphi(P_1), \varphi(P_2)\} \neq \emptyset$, we have $P_2 \notin G_2 \cdot P_1$ and $G_1 \cdot P_2 \cap G_2 \cdot P_1 \neq \emptyset$. We take a point

$$Q \in \varphi^{-1}((L \cap \varphi(X)) \setminus \{\varphi(P_1), \varphi(P_2)\}).$$

By Fact 4 (1) and Fact 5, we have the equalities

$$|G_1(P_2)| = \text{ord}_Q(\varphi^*L) = |G_2(P_1)|.$$

Therefore, condition (c-i) is satisfied.

(II) Assume that L is not a tangent line at $\varphi(P_2)$ and

$$L \cap \varphi(X) = \{\varphi(P_1), \varphi(P_2)\}.$$

Then $G_1 \cdot P_2 = \varphi^{-1}(\varphi(P_2))$ and $G_2 \cdot P_1 = \varphi^{-1}(\varphi(P_1))$ hold, and we have

$$G_1 \cdot P_2 \cap G_2 \cdot P_1 = \emptyset.$$

Therefore, condition (c-ii) is satisfied.

(III) Assume that L is a tangent line at $\varphi(P_2)$. We show that condition (c-iii) is satisfied. Since L is a tangent line at $\varphi(P_2)$, there exists a point $Q \in \varphi^{-1}(\varphi(P_2))$ such that $Q \in G_2 \cdot P_1$. Since $G_1 \cdot P_2 \supset \varphi^{-1}(\varphi(P_2))$, we have $G_1 \cdot P_2 \cap G_2 \cdot P_1 \neq \emptyset$. Let \mathcal{A} be the linear system on X corresponding to the morphism φ , and let $(0, \alpha_Q, \beta_Q)$ be the (\mathcal{A}, Q) -order sequence at Q . Since L is the osculating line at Q , we have

$$|G_1(P_2)| = \text{ord}_Q(\varphi^*L) = \beta_Q$$

by Fact 4 (1) and Fact 5. On the other hand, by Fact 4 (2) and Fact 5, the equality

$$|G_2(P_1)| = \beta_Q - \alpha_Q$$

holds, and we have $|G_1(P_2)| > |G_2(P_1)|$. Therefore, condition (c-iii) is satisfied.

Finally, we show (i), (ii), and (iii) in Theorem 1. Let φ be as in Theorem 1. Then condition (c-i), (c-ii), or (c-iii) is satisfied. Since these conditions are mutually exclusive, it is enough to show ‘only if’ part of (i), (ii), and (iii) in Theorem 1. This task has been already done above.

PROOF (Proof of Theorem 2). Let φ be as in Theorem 1, and let \mathcal{A} be the linear system on X corresponding to the morphism φ . We put

$$\begin{aligned}\varphi^{-1}(\varphi(P_1)) &= \{P_{11} = P_1, P_{12}, \dots, P_{1n_1}\}, \\ \varphi^{-1}(\varphi(P_2)) &= \{P_{21} = P_2, P_{22}, \dots, P_{2n_2}\}.\end{aligned}$$

Let $(0, \alpha_{P_{ij}}, \beta_{P_{ij}})$ be the (\mathcal{A}, P_{ij}) -order sequence for i, j .

First, we show Theorem 2 (1) and (2). Since the linear system corresponding to the morphism $\hat{\pi}_{\varphi(P_1)}$ is

$$\left\{ E - \sum_{i=1}^{n_1} \alpha_{P_{1i}} P_{1i} \mid E \in \mathcal{A}, E \geq \sum_{i=1}^{n_1} \alpha_{P_{1i}} P_{1i} \right\}$$

and $\hat{\pi}_{\varphi(P_1)}$ is a Galois covering, the following equalities of divisors hold:

$$\varphi^* L - \sum_{i=1}^{n_1} \alpha_{P_{1i}} P_{1i} = (\hat{\pi}_{\varphi(P_1)})^*([L]) = \sum_{\sigma \in G_1} \sigma(P_2),$$

where $[L]$ represents the divisor of the point $[L] \in \mathbb{P}^1$ corresponding to the line L . By Fact 4 (1) and Fact 5, the equality $|G_2(P_1)| = \text{ord}_{P_{1i}}(\varphi^* L)$ holds for all i . Since L is not a tangent line at $\varphi(P_1)$, the equality $\alpha_{P_{1i}} = |G_2(P_1)|$ holds for all i . It is not difficult to check that

$$(\varphi^{-1}(\varphi(P_1))) \cup (G_1 \cdot P_2) = \text{supp}(\varphi^* L) = (G_2 \cdot P_1) \cup (G_1 \cdot P_2)$$

hold. Since the intersection of the two sets $\varphi^{-1}(\varphi(P_1))$ and $G_1 \cdot P_2$ is the empty set, we have

$$\begin{aligned}\varphi^{-1}(\varphi(P_1)) &= ((\varphi^{-1}(\varphi(P_1))) \cup (G_1 \cdot P_2)) \setminus (G_1 \cdot P_2) \\ &= G_2 \cdot P_1 \setminus (G_1 \cdot P_2 \cap G_2 \cdot P_1).\end{aligned}$$

Therefore, the equality

$$\sum_{i=1}^{n_1} \alpha_{P_{1i}} P_{1i} = \sum_{Q \in G_2 \cdot P_1 \setminus (G_1 \cdot P_2 \cap G_2 \cdot P_1)} |G_2(P_1)| Q$$

of divisors holds, and we have Theorem 2 (2). Since

$$m_{\varphi(P_1)} = \sum_{i=1}^{n_1} \alpha_{P_{1i}},$$

we have Theorem 2 (1).

Next, we show Theorem 2 (6). By the above, the equality

$$\varphi^* L = \sum_{Q \in G_2 \cdot P_1 \setminus (G_1 \cdot P_2 \cap G_2 \cdot P_1)} |G_2(P_1)|Q + \sum_{\sigma \in G_1} \sigma(P_2)$$

holds. Since

$$\sum_{\sigma \in G_1} \sigma(P_2) = \sum_{R \in G_1 \cdot P_2} |G_1(P_2)|R,$$

we have Theorem 2 (6).

Finally, we show Theorem 2 (3), (4), and (5). Since

$$\sum_{\tau \in G_2} \tau(P_1) = \sum_{S \in G_1 \cdot P_2 \cap G_2 \cdot P_1} |G_2(P_1)|S + \sum_{Q \in G_2 \cdot P_1 \setminus (G_1 \cdot P_2 \cap G_2 \cdot P_1)} |G_2(P_1)|Q,$$

the following equalities of divisors hold:

$$\begin{aligned} & \sum_{R \in G_1 \cdot P_2 \setminus (G_1 \cdot P_2 \cap G_2 \cdot P_1)} |G_1(P_2)|R \\ & + \sum_{S \in G_1 \cdot P_2 \cap G_2 \cdot P_1} (|G_1(P_2)| - |G_2(P_1)|)S + \sum_{\tau \in G_2} \tau(P_1) \\ & = \left(\sum_{R \in G_1 \cdot P_2 \setminus (G_1 \cdot P_2 \cap G_2 \cdot P_1)} |G_1(P_2)|R + \sum_{S \in G_1 \cdot P_2 \cap G_2 \cdot P_1} |G_1(P_2)|S \right) \\ & + \sum_{Q \in G_2 \cdot P_1 \setminus (G_1 \cdot P_2 \cap G_2 \cdot P_1)} |G_2(P_1)|Q \\ & = \sum_{Q \in G_2 \cdot P_1 \setminus (G_1 \cdot P_2 \cap G_2 \cdot P_1)} |G_2(P_1)|Q + \sum_{R \in G_1 \cdot P_2} |G_1(P_2)|R \\ & = \varphi^* L, \end{aligned}$$

where the last equality comes from Theorem 2 (6). Therefore, the equality

$$\begin{aligned} \varphi^* L - \sum_{\tau \in G_2} \tau(P_1) & = \sum_{R \in G_1 \cdot P_2 \setminus (G_1 \cdot P_2 \cap G_2 \cdot P_1)} |G_1(P_2)|R \\ & + \sum_{S \in G_1 \cdot P_2 \cap G_2 \cdot P_1} (|G_1(P_2)| - |G_2(P_1)|)S \end{aligned}$$

of divisors holds. On the other hand, since the linear system corresponding to the morphism $\hat{\pi}_{\varphi(P_2)}$ is

$$\left\{ E - \sum_{j=1}^{n_2} \alpha_{P_{2j}} P_{2j} \mid E \in \mathcal{A}, E \geq \sum_{j=1}^{n_2} \alpha_{P_{2j}} P_{2j} \right\}$$

and $\hat{\pi}_{\varphi(P_2)}$ is a Galois covering, the following equalities of divisors hold:

$$\varphi^* L - \sum_{j=1}^{n_2} \alpha_{P_{2j}} P_{2j} = (\hat{\pi}_{\varphi(P_2)})^*([L]) = \sum_{\tau \in G_2} \tau(P_1).$$

Therefore, the equalities

$$\begin{aligned} \sum_{j=1}^{n_2} \alpha_{P_{2j}} P_{2j} &= \varphi^* L - \sum_{\tau \in G_2} \tau(P_1) \\ &= \sum_{R \in G_1 \cdot P_2 \setminus (G_1 \cdot P_2 \cap G_2 \cdot P_1)} |G_1(P_2)| R \\ &\quad + \sum_{S \in G_1 \cdot P_2 \cap G_2 \cdot P_1} (|G_1(P_2)| - |G_2(P_1)|) S \end{aligned}$$

of divisors hold, and we have Theorem 2 (4). Since

$$m_{\varphi(P_2)} = \sum_{j=1}^{n_2} \alpha_{P_{2j}},$$

we have Theorem 2 (3). Assume that the condition (c-iii) in Theorem 1 is satisfied. Then

$$0 < |G_1(P_2)| - |G_2(P_1)| < |G_1(P_2)|$$

hold. By Theorem 2 (6), the equality $|G_1(P_2)| = \text{ord}_P(\varphi^* L)$ holds at each point $P \in G_1 \cdot P_2$. By Theorem 2 (4), the second (\mathcal{A}, P) -order coincides with $|G_1(P_2)| - |G_2(P_1)|$ at each point $P \in G_1 \cdot P_2 \cap G_2 \cdot P_1$. Therefore, the third (\mathcal{A}, P) -order coincides with $|G_1(P_2)|$ at each point $P \in G_1 \cdot P_2 \cap G_2 \cdot P_1$, and Theorem 2 (5) holds.

REMARK 7. In [7], Fukasawa presented a criterion for the existence of a birational embedding with a pair of Galois points consisting of a smooth Galois point and an outer Galois point. By a method similar to the proof of Theorems 1 and 2, we can extend the criterion to non-smooth and outer Galois points. The necessary and sufficient conditions for the existence of a birational embedding with inner and outer Galois points are that $X/G_i \cong \mathbb{P}^1$ for $i = 1, 2$,

$G_1 \cap G_2 = \{1\}$, and there exist $\eta \in G_2$ and $P \in X$ such that

$$|G_2(P)| \sum_{Q \in (G_2 \cdot P) - (G_1 \cdot \eta(P))} Q + (|G_2(P)| - |G_1(\eta(P))|) \sum_{R \in G_1 \cdot \eta(P)} R \geq P.$$

4. Proof of Theorem 3

We apply Theorems 1 and 2 to rational curves. In this case, condition (a) in Theorem 1 is always satisfied, by Lüroth’s theorem. We identify $\text{Aut}(\mathbb{P}^1)$ with the projective linear group $\text{PGL}(2, k)$. We put $Q_\infty = (1 : 0)$ and $Q_a = (a : 1) \in \mathbb{P}^1$ for any $a \in k$.

PROOF (Proof of Theorem 3). Let $p \neq 2, 5$, let $i \in k$ be a root of the polynomial $T^2 + 1 \in k[T]$, and let ξ be a primitive fifth root of unity.

(1) Let $p = 3$, and let $P_1 = Q_0$ and $P_2 = Q_\xi$. We consider two sets:

$$G_1 = \left\langle \begin{bmatrix} \xi & 0 \\ 0 & 1 \end{bmatrix} \right\rangle \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle \quad \text{and} \quad G_2 = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle.$$

It is known that

$$G_1 = \left\langle \begin{bmatrix} \xi & 0 \\ 0 & 1 \end{bmatrix} \right\rangle \rtimes \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle \cong \mathbf{D}_5,$$

where \mathbf{D}_5 is the dihedral group of degree 5 (see [2, Theorem C]). It is not difficult to check that

$$G_2 = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle \rtimes \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle \cong \text{AGL}(1, \mathbb{F}_3),$$

where $\text{AGL}(1, \mathbb{F}_3)$ is the general affine group of degree 1 over \mathbb{F}_3 . By direct computations, we have the equalities

$$\begin{aligned} G_1 \cap G_2 &= \{1\}, \\ G_1 \cdot P_2 &= \{Q_1, Q_\xi, Q_{\xi^2}, Q_{\xi^3}, Q_{\xi^4}\}, \\ G_2 \cdot P_1 &= \{Q_{-1}, Q_0, Q_1\}, \\ G_1 \cdot P_2 \cap G_2 \cdot P_1 &= \{Q_1\}, \\ G_1(P_2) &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \xi^2 \\ 1 & 0 \end{bmatrix} \right\}, \quad \text{and} \\ G_2(P_1) &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}. \end{aligned}$$

Therefore, conditions (b) and (c-i) in Theorem 1 are satisfied, and there exists a birational embedding $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ such that $\varphi(P_1)$ and $\varphi(P_2)$ are different inner Galois points, $G_{\varphi(P_1)} \cong \mathbf{D}_5$, $G_{\varphi(P_2)} \cong \text{AGL}(1, \mathbb{F}_3)$, and $L = \overline{\varphi(P_1)\varphi(P_2)}$ is not a tangent line at $\varphi(P_1)$, $\varphi(P_2)$. By Theorem 2 (1), (3) and (6), $m_{\varphi(P_1)} = 4$, $m_{\varphi(P_2)} = 8$ and $\deg(\varphi(\mathbb{P}^1)) = 14$. By Theorem 2 (6), the second order is equal to 2 at each point contained in $\text{supp}(\varphi^*L)$.

(2) Let $P_1 = Q_\xi$ and $P_2 = Q_1$. We consider

$$G_1 = \left\langle \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \right\rangle \left\langle \left[\begin{array}{cc} 1 & i \\ 1 & -i \end{array} \right] \right\rangle \quad \text{and} \quad G_2 = \left\langle \left[\begin{array}{cc} \xi & 0 \\ 0 & 1 \end{array} \right] \right\rangle.$$

Obviously, $G_2 \cong \mathbb{Z}/5\mathbb{Z}$, and the following fact is known.

$$G_1 = \left\langle \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \right\rangle \rtimes \left\langle \left[\begin{array}{cc} 1 & i \\ 1 & -i \end{array} \right] \right\rangle \cong \mathbf{A}_4,$$

where \mathbf{A}_4 is the alternating group of degree 4 (see [2, Theorem C]). Since 5 and 12 are coprime, condition (b) in Theorem 1 is satisfied. By direct computations, we have the following equalities:

$$G_1 \cdot P_2 = \{Q_{-i}, Q_{-1}, Q_0, Q_1, Q_i, Q_\infty\},$$

$$G_2 \cdot P_1 = \{Q_1, Q_\xi, Q_{\xi^2}, Q_{\xi^3}, Q_{\xi^4}\},$$

$$G_1 \cdot P_2 \cap G_2 \cdot P_1 = \{Q_1 = P_2\},$$

$$G_1(P_2) = \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \right\}, \quad \text{and}$$

$$G_2(P_1) = \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \right\}.$$

Therefore, condition (c-iii) in Theorem 1 is satisfied, and there exists a birational embedding $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ such that $\varphi(P_1)$ and $\varphi(P_2)$ are different inner Galois points, $G_{\varphi(P_1)} \cong \mathbf{A}_4$, $G_{\varphi(P_2)} \cong \mathbb{Z}/5\mathbb{Z}$, $L = \overline{\varphi(P_1)\varphi(P_2)}$ is not a tangent line at $\varphi(P_1)$, and L is a tangent line at $\varphi(P_2)$. By Theorem 2 (1), (3) and (6), $m_{\varphi(P_1)} = 4$, $m_{\varphi(P_2)} = 11$ and $\deg(\varphi(\mathbb{P}^1)) = 16$. By Theorem 2 (2), (4), and (5), the second order is equal to 2 (resp. 1) at each point $Q \in G_1 \cdot P_2 \setminus \{P_2\}$ (resp. $Q \in G_2 \cdot P_1$), and the third order is equal to 2 at P_2 .

(3) Let $P_1 = Q_\xi$ and $P_2 = Q_1$. We consider two groups:

$$G_1 = \left\langle \left\langle \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \right\rangle \rtimes \left\langle \left[\begin{array}{cc} 1 & i \\ 1 & -i \end{array} \right] \right\rangle, \left\langle \left[\begin{array}{cc} i & 0 \\ 0 & 1 \end{array} \right] \right\rangle \right\rangle,$$

$$G_2 = \left\langle \left[\begin{array}{cc} \xi & 0 \\ 0 & 1 \end{array} \right] \right\rangle.$$

Obviously, $G_2 \cong \mathbb{Z}/5\mathbb{Z}$, and the following fact is known:

$$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle \rtimes \left\langle \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \right\rangle \triangleleft G_1 \cong \mathbf{S}_4,$$

where \mathbf{S}_4 is the symmetric group of degree 4 (see [2, Theorem C]). Since 5 and 24 are coprime, condition (b) in Theorem 1 is satisfied. By direct computations, we have the following equalities:

$$\begin{aligned} G_1 \cdot P_2 &= \{Q_{-i}, Q_{-1}, Q_0, Q_1, Q_i, Q_\infty\}, \\ G_2 \cdot P_1 &= \{Q_1, Q_\xi, Q_{\xi^2}, Q_{\xi^3}, Q_{\xi^4}\}, \\ G_1 \cdot P_2 \cap G_2 \cdot P_1 &= \{Q_1 = P_2\}, \\ G_1(P_2) &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}, \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \right\}, \quad \text{and} \\ G_2(P_1) &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}. \end{aligned}$$

Therefore, condition (c-iii) in Theorem 1 is satisfied, and there exists a birational embedding $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ such that $\varphi(P_1)$ and $\varphi(P_2)$ are different inner Galois points, $G_{\varphi(P_1)} \cong \mathbf{S}_4$, $G_{\varphi(P_2)} \cong \mathbb{Z}/5\mathbb{Z}$, $L = \varphi(P_1)\varphi(P_2)$ is not a tangent line at $\varphi(P_1)$, and L is a tangent line at $\varphi(P_2)$. By Theorem 2 (1), (3) and (6), $m_{\varphi(P_1)} = 4$, $m_{\varphi(P_2)} = 23$ and $\deg(\varphi(\mathbb{P}^1)) = 28$. By Theorem 2 (2), (4), and (5) the second order is equal to 4 (resp. 3, 1) at each point $Q \in G_1 \cdot P_2 \setminus \{P_2\}$ (resp. at P_2 , at each point $Q \in G_2 \cdot P_1 \setminus \{P_2\}$), and the third order is equal to 4 at P_2 .

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