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On meromorphic functions sharing three two-point sets CM

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Abstract. We show that if three meromorphic functions share three two-point sets CM, then there exist two of the meromorphic functions such that one of them is a Möbius transform of the other.

1. Introduction

For nonconstant meromorphic functions f and g on $\boldsymbol{C}$ and a finite set S in $\bar{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$, we say that f and g share S CM (counting multiplicities) if $f^{-1}(S) = g^{-1}(S)$ and if for each $z_0 \in f^{-1}(S)$ two functions $f - f(z_0)$ and $g - g(z_0)$ have the same multiplicity of zero at z_0, where we consider $1/f$ and $1/g$ for $f - f(z_0)$ and $g - g(z_0)$ if $f(z_0) = \infty$ and $g(z_0) = \infty$, respectively. Also, if $f^{-1}(S) = g^{-1}(S)$, then we say that f and g share S IM (ignoring multiplicities). In particular if S is a one-point set $\{a\}$, then we say also that f and g share a CM or IM.

In [C], H. Cartan showed the following theorem:

Theorem 1. *Let f, g and h be three nonconstant meromorphic functions on $\boldsymbol{C}$ and let a_1, a_2 and a_3 be three distinct points in $\bar{\boldsymbol{C}}$. If f, g and h share a_j CM for $j = 1, 2, 3$, then at least two of f, g and h are identical.*

On the other hand the author proved ([S3], see also [S2] and [ST]).

Theorem 2. *Let S_1, S_2, S_3, S_4 be four one-point or two-point sets in $\bar{\boldsymbol{C}}$. Suppose that S_1, S_2, S_3 and S_4 are pairwise disjoint. If two nonconstant meromorphic functions f and g on $\boldsymbol{C}$ share S_j CM for $j = 1, \dots, 4$, then f is a Möbius transform of g.*

Theorem 2 contains partially the result of Nevanlinna ([N1] and [N2]).

Theorem 3. *Let f and g be two distinct nonconstant meromorphic functions on $\boldsymbol{C}$ and let $a_1, \dots, a_4$ be four distinct points in $\bar{\boldsymbol{C}}$. If f and g share $a_1, \dots, a_4$ CM, then f is a Möbius transform of g, i.e., $f = (ag + b)/(cg + d)$*

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for some complex numbers a, b, c, d with $ad - bc \neq 0$. *Moreover, there exists a permutation* σ *of* $\{1,2,3,4\}$ *such that* $a_{\sigma(3)}$ *and* $a_{\sigma(4)}$ *are Picard exceptional values of* f *and* g *and the cross ratio* $(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)}) = -1$.

Theorems 1 and 2 raise the following problem:

PROBLEM. If three meromorphic functions on $\boldsymbol{C}$ share three pairwise disjoint two-point sets, then do there exist two in the three meromorphic functions such that one of them is a Möbius transform of the other?

In this paper we consider three meromorphic functions on $\boldsymbol{C}$ sharing three two-point sets in $\bar{\boldsymbol{C}}$ CM.

THEOREM 4. *Let* S_1, S_2, S_3 *be three two-point sets in* $\bar{\boldsymbol{C}}$. *Suppose that* S_1, S_2, S_3 *are pairwise disjoint. If three nonconstant meromorphic functions* f, g *and* h *on* $\boldsymbol{C}$ *share each of* S_1, S_2, S_3 *CM, then one of* f, g *and* h *is a Möbius transform of one of the others.*

For the proof of Theorem 4, by considering compositions of f, g, h and a suitable Möbius transformation, it is enough to prove the following theorem which assume that each S_j is in $\boldsymbol{C}$.

THEOREM 5. *Let* S_1, S_2, S_3 *be three two-point sets in* $\boldsymbol{C}$. *Suppose that* S_1, S_2, S_3 *are pairwise disjoint. If three nonconstant meromorphic functions* f, g *and* h *on* $\boldsymbol{C}$ *share each of* S_1, S_2, S_3 *CM, then one of* f, g *and* h *is a Möbius transform of one of the others.*

2. Representations of rank N and some lemmas

In this section we introduce the definition of representations of rank N. Let G be a torsion-free abelian multiplicative group, and consider a q-tuple $A = (a_1, \dots, a_q)$ of elements a_i in G.

DEFINITION 1. Let N be a positive integer. We call integers μ_j *representations of rank* N of a_j if

$$\prod_{j=1}^{q} a_j^{\varepsilon_j} = \prod_{j=1}^{q} a_j^{\varepsilon_j'}$$

and

$$\sum_{j=1}^{q} \varepsilon_j \mu_j = \sum_{j=1}^{q} \varepsilon_j' \mu_j$$

are equivalent for any integers ε_j, ε_j' with $\sum_{j=1}^{q} |\varepsilon_j| \leq N$ and $\sum_{j=1}^{q} |\varepsilon_j'| \leq N$.

For the existence of representations of rank N, see [S1].

For two entire function α and β without zeros we say that they are equivalent if α/β is constant. Then we denote $\alpha \sim \beta$. This relation "equivalent" is an equivalence relation.

We introduce following Borel's Lemma, whose proof can be found, for example, on p. 186 of [L].

Lemma 1. *If entire functions $\alpha_0, \alpha_1, \dots, \alpha_n$ without zeros satisfy*

$$\alpha_0 + \alpha_1 + \cdots + \alpha_n = 0,$$

then for each $j = 0, 1, \dots, n$ there exists some $k (\neq j)$ such that $\alpha_j \sim \alpha_k$, and the sum of all elements of each equivalence class in $\{\alpha_0, \dots, \alpha_n\}$ is zero.

Now we investigate the torsion-free abelian multiplicative group $G = \mathscr{E}/\mathscr{C}$, where $\mathscr{E}$ is the abelian group of entire functions without zeros and $\mathscr{C}$ is the subgroup of all non-zero constant functions. We represent by $[\alpha]$ the element of $\mathscr{E}/\mathscr{C}$ with the representative $\alpha \in \mathscr{E}$. Let $\alpha_1, \dots, \alpha_q$ be elements in $\mathscr{E}$. Take representations μ_j of rank N of $[\alpha_j]$. For $\alpha = \prod_{j=1}^{q} \alpha_j^{\varepsilon_j}$ we define its index $\mathrm{Ind}(\alpha)$ by $\sum_{j=1}^{q} \varepsilon_j \mu_j$. The indices depend only on $\left[\prod_{j=1}^{q} \alpha_j^{\varepsilon_j}\right]$ under the condition $\sum_{j=1}^{q} |\varepsilon_j| \le N$. Trivially $\mathrm{Ind}(1) = 0$, and hence $\mathrm{Ind}(\alpha) = 0$ and the constantness of α are equivalent, and $\mathrm{Ind}(\alpha) = \mathrm{Ind}(\alpha')$ is equivalent to that α/α' is constant, where $\alpha = \prod_{j=1}^{q} \alpha_j^{\varepsilon_j}$ and $\alpha' = \prod_{j=1}^{q} \alpha_j^{\varepsilon'_j}$ with $\sum_{j=1}^{q} |\varepsilon_j| \le N$ and $\sum_{j=1}^{q} |\varepsilon'_j| \le N$.

We use the following lemma in the proof of Theorem 5 which is an application of Lemma 1 (for the proof see Lemma 2.3 of [ST]).

Lemma 2. *Assume that there is a relation $\Psi(\alpha_1, \dots, \alpha_q) \equiv 0$ where $\Psi(X_1, \dots, X_q) \in \boldsymbol{C}[X_1, \dots, X_q]$ is a nonconstant polynomial of degree at most N of $X_1, \dots, X_q$. Then each term $aX_1^{\varepsilon_1} \cdots X_q^{\varepsilon_q}$ of $\Psi(X_1, \dots, X_q)$ has another term $bX_1^{\varepsilon'_1} \cdots X_q^{\varepsilon'_q}$ such that $\alpha_1^{\varepsilon_1} \cdots \alpha_q^{\varepsilon_q}$ and $\alpha_1^{\varepsilon'_1} \cdots \alpha_q^{\varepsilon'_q}$ have the same indices, where a and b are non-zero constants.*

3. A Lemma from the theory of general resultants

For the proof of Theorem 5 we preparate a result from the theory of general resultants in this section.

Let $d\ (\ge 2)$ be an integer and let $F_1, \dots, F_6$ be six homogeneous polynomials of degree d of six variables $X_0,\ X_1,\ Y_0,\ Y_1,\ Z_0,\ Z_1$. Denote their

Jacobian determinant by J:

$$J = \left| \frac{\partial F_j}{\partial X_0} \quad \frac{\partial F_j}{\partial X_1} \quad \frac{\partial F_j}{\partial Y_0} \quad \frac{\partial F_j}{\partial Y_1} \quad \frac{\partial F_j}{\partial Z_0} \quad \frac{\partial F_j}{\partial Z_1} \right|_{1 \le j \le 6}.$$

LEMMA 3. *Let P be a non-trivial common zero of $F_1, \dots, F_6$. Then* (i) *J is zero at P;* (ii) *all the partial derivatives* $\frac{\partial J}{\partial X_0}, \frac{\partial J}{\partial X_1}, \frac{\partial J}{\partial Y_0}, \frac{\partial J}{\partial Y_1}, \frac{\partial J}{\partial Z_0}, \frac{\partial J}{\partial Z_1}$ *are zero at P;* (iii) *if we assume that*

$$\text{(S)} \qquad \frac{\partial^2 F_j}{\partial X_k \partial Y_l} = \frac{\partial^2 F_j}{\partial Y_k \partial Z_l} = \frac{\partial^2 F_j}{\partial Z_k \partial X_l} = 0 \qquad (j = 1, \dots, 6;\ k, l = 0, 1)$$

and if plural components of P are not zero, then the second partial derivatives $\frac{\partial^2 J}{\partial X_j \partial Y_k}, \frac{\partial^2 J}{\partial Y_j \partial Z_k}, \frac{\partial^2 J}{\partial Z_j \partial X_k}$ *have zero at P for any $j, k = 0, 1$;* (iv) *under the assumption* (S), *if plural components of P are not zero, then the third partial derivative* $\frac{\partial^3 J}{\partial X_j \partial Y_k \partial Z_l}$ *has zero at P for any $j, k, l = 0, 1$.*

PROOF. Without loss of generality, we may assume that the X_0 component of P is not zero.

By Euler's relation we have

$$\begin{aligned} X_0 J &= \left| X_0 \frac{\partial F_j}{\partial X_0} \quad \frac{\partial F_j}{\partial X_1} \quad \cdots \quad \frac{\partial F_j}{\partial Z_1} \right|_{1 \le j \le 6} \\ &= d \left| F_j \quad \frac{\partial F_j}{\partial X_1} \quad \cdots \quad \frac{\partial F_j}{\partial Z_1} \right|_{1 \le j \le 6} \end{aligned} \tag{3.1}$$

and, hence we have $J(P) = 0$, which is (i).

By differentiating (3.1) by $X_0, X_1, \dots, Z_1$, respectively, we get

$$\begin{aligned} J + X_0 \frac{\partial J}{\partial X_0} = dJ &+ d \left| F_j \quad \frac{\partial^2 F_j}{\partial X_0 \partial X_1} \quad \frac{\partial F_j}{\partial Y_0} \quad \frac{\partial F_j}{\partial Y_1} \quad \frac{\partial F_j}{\partial Z_0} \quad \frac{\partial F_j}{\partial Z_1} \right|_{1 \le j \le 6} \\ &+ \cdots + d \left| F_j \quad \frac{\partial F_j}{\partial X_1} \quad \frac{\partial F_j}{\partial Y_0} \quad \frac{\partial F_j}{\partial Y_1} \quad \frac{\partial F_j}{\partial Z_0} \quad \frac{\partial^2 F_j}{\partial X_0 \partial Z_1} \right|_{1 \le j \le 6}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} X_0 \frac{\partial J}{\partial X_1} &= d \left| F_j \quad \frac{\partial^2 F_j}{\partial X_1^2} \quad \frac{\partial F_j}{\partial Y_0} \quad \frac{\partial F_j}{\partial Y_1} \quad \frac{\partial F_j}{\partial Z_0} \quad \frac{\partial F_j}{\partial Z_1} \right|_{1 \le j \le 6} \\ &+ \cdots + d \left| F_j \quad \frac{\partial F_j}{\partial X_1} \quad \frac{\partial F_j}{\partial Y_0} \quad \frac{\partial F_j}{\partial Y_1} \quad \frac{\partial F_j}{\partial Z_0} \quad \frac{\partial^2 F_j}{\partial X_1 \partial Z_1} \right|_{1 \le j \le 6}, \end{aligned} \tag{3.3}$$

$$\vdots$$

$$X_0\frac{\partial J}{\partial Z_1}=d\left|F_j\quad \frac{\partial^2 F_j}{\partial X_1\partial Z_1}\quad \frac{\partial F_j}{\partial Y_0}\quad \frac{\partial F_j}{\partial Y_1}\quad \frac{\partial F_j}{\partial Z_0}\quad \frac{\partial F_j}{\partial Z_1}\right|_{1\le j\le 6}$$

$$+\cdots+d\left|F_j\quad \frac{\partial F_j}{\partial X_1}\quad \frac{\partial F_j}{\partial Y_0}\quad \frac{\partial F_j}{\partial Y_1}\quad \frac{\partial F_j}{\partial Z_0}\quad \frac{\partial^2 F_j}{\partial Z_1^2}\right|_{1\le j\le 6}. \qquad (3.4)$$

Therefore we obtain (ii) since $F_j(P)=0$ $(j=1,\dots,6)$.

Under the assumption (S), the equations (3.2), (3.3) and (3.4), and so on, become

$$J+X_0\frac{\partial J}{\partial X_0}=dJ+d\left|F_j\quad \frac{\partial^2 F_j}{\partial X_0\partial X_1}\quad \frac{\partial F_j}{\partial Y_0}\quad \frac{\partial F_j}{\partial Y_1}\quad \frac{\partial F_j}{\partial Z_0}\quad \frac{\partial F_j}{\partial Z_1}\right|_{1\le j\le 6}, \qquad (3.5)$$

$$X_0\frac{\partial J}{\partial X_1}=d\left|F_j\quad \frac{\partial^2 F_j}{\partial X_1^2}\quad \frac{\partial F_j}{\partial Y_0}\quad \frac{\partial F_j}{\partial Y_1}\quad \frac{\partial F_j}{\partial Z_0}\quad \frac{\partial F_j}{\partial Z_1}\right|_{1\le j\le 6}, \qquad (3.6)$$

$$\vdots$$

$$X_0\frac{\partial J}{\partial Z_1}=d\left|F_j\quad \frac{\partial F_j}{\partial X_1}\quad \frac{\partial F_j}{\partial Y_0}\quad \frac{\partial F_j}{\partial Y_1}\quad \frac{\partial^2 F_j}{\partial Z_0\partial Z_1}\quad \frac{\partial F_j}{\partial Z_1}\right|_{1\le j\le 6}$$

$$+d\left|F_j\quad \frac{\partial F_j}{\partial X_1}\quad \frac{\partial F_j}{\partial Y_0}\quad \frac{\partial F_j}{\partial Y_1}\quad \frac{\partial F_j}{\partial Z_0}\quad \frac{\partial^2 F_j}{\partial Z_1^2}\right|_{1\le j\le 6}. \qquad (3.7)$$

By differentiating (3.5), (3.6), (3.7), and so on, by Y_0, we get

$$\frac{\partial J}{\partial Y_0}+X_0\frac{\partial^2 J}{\partial X_0\partial Y_0}=d\frac{\partial J}{\partial Y_0}+d\left|F_j\quad \frac{\partial^2 F_j}{\partial X_0\partial X_1}\quad \frac{\partial^2 F_j}{\partial Y_0^2}\quad \frac{\partial F_j}{\partial Y_1}\quad \frac{\partial F_j}{\partial Z_0}\quad \frac{\partial F_j}{\partial Z_1}\right|_{1\le j\le 6}$$

$$+d\left|F_j\quad \frac{\partial^2 F_j}{\partial X_0\partial X_1}\quad \frac{\partial F_j}{\partial Y_0}\quad \frac{\partial^2 F_j}{\partial Y_0\partial Y_1}\quad \frac{\partial F_j}{\partial Z_0}\quad \frac{\partial F_j}{\partial Z_1}\right|_{1\le j\le 6}, \qquad (3.8)$$

$$X_0\frac{\partial^2 J}{\partial X_1\partial Y_0}=d\left|F_j\quad \frac{\partial^2 F_j}{\partial X_1^2}\quad \frac{\partial^2 F_j}{\partial Y_0^2}\quad \frac{\partial F_j}{\partial Y_1}\quad \frac{\partial F_j}{\partial Z_0}\quad \frac{\partial F_j}{\partial Z_1}\right|_{1\le j\le 6}$$

$$+d\left|F_j\quad \frac{\partial^2 F_j}{\partial X_1^2}\quad \frac{\partial F_j}{\partial Y_0}\quad \frac{\partial^2 F_j}{\partial Y_0\partial Y_1}\quad \frac{\partial F_j}{\partial Z_0}\quad \frac{\partial F_j}{\partial Z_1}\right|_{1\le j\le 6}, \qquad (3.9)$$

$$\vdots$$

$$X_0\frac{\partial^2 J}{\partial Y_0\partial Z_1}=d\left|F_j\quad \frac{\partial F_j}{\partial X_1}\quad \frac{\partial^2 F_j}{\partial Y_0^2}\quad \frac{\partial F_j}{\partial Y_1}\quad \frac{\partial^2 F_j}{\partial Z_0\partial Z_1}\quad \frac{\partial F_j}{\partial Z_1}\right|_{1\le j\le 6}$$

$$+ d \left| F_j \quad \frac{\partial F_j}{\partial X_1} \quad \frac{\partial F_j}{\partial Y_0} \quad \frac{\partial^2 F_j}{\partial Y_0 \partial Y_1} \quad \frac{\partial^2 F_j}{\partial Z_0 \partial Z_1} \quad \frac{\partial F_j}{\partial Z_1} \right|_{1 \le j \le 6}$$

$$+ d \left| F_j \quad \frac{\partial F_j}{\partial X_1} \quad \frac{\partial^2 F_j}{\partial Y_0^2} \quad \frac{\partial F_j}{\partial Y_1} \quad \frac{\partial^2 F_j}{\partial Z_0 \partial Z_1} \quad \frac{\partial^2 F_j}{\partial Z_1^2} \right|_{1 \le j \le 6}$$

$$+ d \left| F_j \quad \frac{\partial F_j}{\partial X_1} \quad \frac{\partial F_j}{\partial Y_0} \quad \frac{\partial^2 F_j}{\partial Y_0 \partial Y_1} \quad \frac{\partial^2 F_j}{\partial Z_0 \partial Z_1} \quad \frac{\partial^2 F_j}{\partial Z_1^2} \right|_{1 \le j \le 6},$$

hence we obtain, with similar manners, (iii).

Differentiate (3.8) and (3.9) by Z_0, then we have

$$\frac{\partial^2 J}{\partial Y_0 \partial Z_0} + X_0 \frac{\partial^3 J}{\partial X_0 \partial Y_0 \partial Z_0}$$

$$= d \frac{\partial^2 J}{\partial Y_0 \partial Z_0} + d \left| F_j \quad \frac{\partial^2 F_j}{\partial X_0 \partial X_1} \quad \frac{\partial^2 F_j}{\partial Y_0^2} \quad \frac{\partial F_j}{\partial Y_1} \quad \frac{\partial^2 F_j}{\partial Z_0^2} \quad \frac{\partial F_j}{\partial Z_1} \right|_{1 \le j \le 6}$$

$$+ d \left| F_j \quad \frac{\partial^2 F_j}{\partial X_0 \partial X_1} \quad \frac{\partial^2 F_j}{\partial Y_0^2} \quad \frac{\partial F_j}{\partial Y_1} \quad \frac{\partial F_j}{\partial Z_0} \quad \frac{\partial^2 F_j}{\partial Z_0 \partial Z_1} \right|_{1 \le j \le 6}$$

$$+ d \left| F_j \quad \frac{\partial^2 F_j}{\partial X_0 \partial X_1} \quad \frac{\partial F_j}{\partial Y_0} \quad \frac{\partial^2 F_j}{\partial Y_0 \partial Y_1} \quad \frac{\partial^2 F_j}{\partial Z_0^2} \quad \frac{\partial F_j}{\partial Z_1} \right|_{1 \le j \le 6}$$

$$+ d \left| F_j \quad \frac{\partial^2 F_j}{\partial X_0 \partial X_1} \quad \frac{\partial F_j}{\partial Y_0} \quad \frac{\partial^2 F_j}{\partial Y_0 \partial Y_1} \quad \frac{\partial F_j}{\partial Z_0} \quad \frac{\partial^2 F_j}{\partial Z_0 \partial Z_1} \right|_{1 \le j \le 6},$$

$$X_0 \frac{\partial^3 J}{\partial X_1 \partial Y_0 \partial Z_0} = d \left| F_j \quad \frac{\partial^2 F_j}{\partial X_1^2} \quad \frac{\partial^2 F_j}{\partial Y_0^2} \quad \frac{\partial F_j}{\partial Y_1} \quad \frac{\partial^2 F_j}{\partial Z_0^2} \quad \frac{\partial F_j}{\partial Z_1} \right|_{1 \le j \le 6}$$

$$+ d \left| F_j \quad \frac{\partial^2 F_j}{\partial X_1^2} \quad \frac{\partial^2 F_j}{\partial Y_0^2} \quad \frac{\partial F_j}{\partial Y_1} \quad \frac{\partial F_j}{\partial Z_0} \quad \frac{\partial^2 F_j}{\partial Z_0 \partial Z_1} \right|_{1 \le j \le 6}$$

$$+ d \left| F_j \quad \frac{\partial^2 F_j}{\partial X_1^2} \quad \frac{\partial F_j}{\partial Y_0} \quad \frac{\partial^2 F_j}{\partial Y_0 \partial Y_1} \quad \frac{\partial^2 F_j}{\partial Z_0^2} \quad \frac{\partial F_j}{\partial Z_1} \right|_{1 \le j \le 6}$$

$$+ d \left| F_j \quad \frac{\partial^2 F_j}{\partial X_1^2} \quad \frac{\partial F_j}{\partial Y_0} \quad \frac{\partial^2 F_j}{\partial Y_0 \partial Y_1} \quad \frac{\partial F_j}{\partial Z_0} \quad \frac{\partial^2 F_j}{\partial Z_0 \partial Z_1} \right|_{1 \le j \le 6}.$$

Hence we have $\frac{\partial^3 F_j}{\partial X_0 \partial Y_0 \partial Z_0}(P) = \frac{\partial^3 F_j}{\partial X_1 \partial Y_0 \partial Z_0}(P) = 0$, and by the similar ways, we get (iv). □

Let

$$F_j(X_0, X_1, Y_0, Y_1, Z_0, Z_1) = \sum_{k=0}^{2} (a_{jk} X_0^{2-k} X_1^k + b_{jk} Y_0^{2-k} Y_1^k + c_{jk} Z_0^{2-k} Z_1^k)$$

$(j = 1, \dots, 6)$ be six quadratic homogeneous polynomials satisfying the assumption (S). Then the first derivatives are

$$\frac{\partial F_j}{\partial X_0} = 2a_{j0} X_0 + a_{j1} X_1, \qquad \frac{\partial F_j}{\partial X_1} = a_{j1} X_0 + 2a_{j2} X_1,$$

$$\frac{\partial F_j}{\partial Y_0} = 2b_{j0} Y_0 + b_{j1} Y_1, \qquad \frac{\partial F_j}{\partial Y_1} = b_{j1} Y_0 + 2b_{j2} Y_1,$$

$$\frac{\partial F_j}{\partial Z_0} = 2c_{j0} Z_0 + c_{j1} Z_1, \qquad \frac{\partial F_j}{\partial Z_1} = c_{j1} Z_0 + 2c_{j2} Z_1.$$

Since J is the determinant of the matrix DX, where

$$D = (a_{\mu 0} \quad a_{\mu 1} \quad a_{\mu 2} \quad b_{\mu 0} \quad b_{\mu 1} \quad b_{\mu 2} \quad c_{\mu 0} \quad c_{\mu 1} \quad c_{\mu 2})_{1 \le \mu \le 6} \tag{3.10}$$

and

$$X = \begin{pmatrix} 2X_0 & 0 & 0 & 0 & 0 & 0 \\ X_1 & X_0 & 0 & 0 & 0 & 0 \\ 0 & 2X_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2Y_0 & 0 & 0 & 0 \\ 0 & 0 & Y_1 & Y_0 & 0 & 0 \\ 0 & 0 & 0 & 2Y_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2Z_0 & 0 \\ 0 & 0 & 0 & 0 & Z_1 & Z_0 \\ 0 & 0 & 0 & 0 & 0 & 2Z_1 \end{pmatrix},$$

we see, by the formula of determinant of product of $m \times n$ matrix and $n \times m$ matrix with $1 \le m < n$,

$$J = 8 \sum_{0 \le j,k,l \le 2} 2^{j(2-j)+k(2-k)+l(2-l)} D_{jkl} X_0^j X_1^{2-j} Y_0^k Y_1^{2-k} Z_0^l Z_1^{2-l},$$

where D_{jkl} is the determinant of the 6×6 matrix obtained from D by excluding three columns $(a_{\mu j})_{1 \le \mu \le 6}$, $(b_{\mu k})_{1 \le \mu \le 6}$ and $(c_{\mu l})_{1 \le \mu \le 6}$.

By differentiating J, we have

$$\begin{aligned} \frac{1}{64} \frac{\partial^3 J}{\partial X_0 \partial Y_0 \partial Z_0} &= D_{222} X_0 Y_0 Z_0 + D_{221} X_0 Y_0 Z_1 + D_{212} X_0 Y_1 Z_0 \\ &\quad + D_{211} X_0 Y_1 Z_1 + D_{122} X_1 Y_0 Z_0 + D_{121} X_1 Y_0 Z_1 \\ &\quad + D_{112} X_1 Y_1 Z_0 + D_{111} X_1 Y_1 Z_1, \end{aligned}$$

$$\frac{1}{64}\frac{\partial^3 J}{\partial X_0 \partial Y_0 \partial Z_1} = D_{221}X_0Y_0Z_0 + D_{220}X_0Y_0Z_1 + D_{211}X_0Y_1Z_0 + D_{210}X_0Y_1Z_1 + D_{121}X_1Y_0Z_0 + D_{120}X_1Y_0Z_1 + D_{111}X_1Y_1Z_0 + D_{110}X_1Y_1Z_1,$$

$$\frac{1}{64}\frac{\partial^3 J}{\partial X_0 \partial Y_1 \partial Z_0} = D_{212}X_0Y_0Z_0 + D_{211}X_0Y_0Z_1 + D_{202}X_0Y_1Z_0 + D_{201}X_0Y_1Z_1 + D_{112}X_1Y_0Z_0 + D_{111}X_1Y_0Z_1 + D_{102}X_1Y_1Z_0 + D_{101}X_1Y_1Z_1,$$

$$\frac{1}{64}\frac{\partial^3 J}{\partial X_0 \partial Y_1 \partial Z_1} = D_{211}X_0Y_0Z_0 + D_{210}X_0Y_0Z_1 + D_{201}X_0Y_1Z_0 + D_{200}X_0Y_1Z_1 + D_{111}X_1Y_0Z_0 + D_{110}X_1Y_0Z_1 + D_{101}X_1Y_1Z_0 + D_{100}X_1Y_1Z_1,$$

$$\frac{1}{64}\frac{\partial^3 J}{\partial X_1 \partial Y_0 \partial Z_0} = D_{122}X_0Y_0Z_0 + D_{121}X_0Y_0Z_1 + D_{112}X_0Y_1Z_0 + D_{111}X_0Y_1Z_1 + D_{022}X_1Y_0Z_0 + D_{021}X_1Y_0Z_1 + D_{012}X_1Y_1Z_0 + D_{011}X_1Y_1Z_1,$$

$$\frac{1}{64}\frac{\partial^3 J}{\partial X_1 \partial Y_0 \partial Z_1} = D_{121}X_0Y_0Z_0 + D_{120}X_0Y_0Z_1 + D_{111}X_0Y_1Z_0 + D_{110}X_0Y_1Z_1 + D_{021}X_1Y_0Z_0 + D_{020}X_1Y_0Z_1 + D_{011}X_1Y_1Z_0 + D_{010}X_1Y_1Z_1,$$

$$\frac{1}{64}\frac{\partial^3 J}{\partial X_1 \partial Y_1 \partial Z_0} = D_{112}X_0Y_0Z_0 + D_{111}X_0Y_0Z_1 + D_{102}X_0Y_1Z_0 + D_{101}X_0Y_1Z_1 + D_{012}X_1Y_0Z_0 + D_{011}X_1Y_0Z_1 + D_{002}X_1Y_1Z_0 + D_{001}X_1Y_1Z_1,$$

$$\frac{1}{64}\frac{\partial^3 J}{\partial X_1 \partial Y_1 \partial Z_1} = D_{111}X_0Y_0Z_0 + D_{110}X_0Y_0Z_1 + D_{101}X_0Y_1Z_0 + D_{100}X_0Y_1Z_1 + D_{011}X_1Y_0Z_0 + D_{010}X_1Y_0Z_1 + D_{001}X_1Y_1Z_0 + D_{000}X_1Y_1Z_1.$$

If $P(x_0, x_1, y_0, y_1, z_0, z_1)$ is a common zero of $F_1, \dots, F_6$ such that some $x_j y_k z_l \neq 0$, then

$$\Delta := \begin{vmatrix} D_{222} & D_{221} & D_{212} & D_{211} & D_{122} & D_{121} & D_{112} & D_{111} \\ D_{221} & D_{220} & D_{211} & D_{210} & D_{121} & D_{120} & D_{111} & D_{110} \\ D_{212} & D_{211} & D_{202} & D_{201} & D_{112} & D_{111} & D_{102} & D_{101} \\ D_{211} & D_{210} & D_{201} & D_{200} & D_{111} & D_{110} & D_{101} & D_{100} \\ D_{122} & D_{121} & D_{112} & D_{111} & D_{022} & D_{021} & D_{012} & D_{011} \\ D_{121} & D_{120} & D_{111} & D_{110} & D_{021} & D_{020} & D_{011} & D_{010} \\ D_{112} & D_{111} & D_{102} & D_{101} & D_{012} & D_{011} & D_{002} & D_{001} \\ D_{111} & D_{110} & D_{101} & D_{100} & D_{011} & D_{010} & D_{001} & D_{000} \end{vmatrix} = 0 \qquad (3.11)$$

at P since all of the above derivatives are zero at P by (iii) of Lemma 3.

4. The key theorem and the proof of Theorem 5

By the following theorem we can prove Theorem 5 easily.

THEOREM 6. *Let $f = f_1/f_0$, $g = g_1/g_0$ and $h = h_1/h_0$ be nonconstant meromorphic functions on $\boldsymbol{C}$, where f_0 and f_1 are entire functions without common zero and so are g_0 and g_1, and h_0 and h_1. Let $P_j(z) = z^2 + a_j z + b_j$ $(j = 1, 2, 3)$ be polynomials such that $P_j(z)$ and $P_k(z)$ have no common zero for distinct j, k. Assume that there exist entire functions α_j, β_j without zeros such that*

$$\alpha_j(f_1^2 + a_j f_1 f_0 + b_j f_0^2) = g_1^2 + a_j g_1 g_0 + b_j g_0^2 \qquad (4.1)$$

and

$$\beta_j(f_1^2 + a_j f_1 f_0 + b_j f_0^2) = h_1^2 + a_j h_1 h_0 + b_j h_0^2 \qquad (4.2)$$

for $j = 1, 2, 3$. Then one of the followings holds: (A) *both α_1/α_2 and α_1/α_3 are constant;* (B) *both β_1/β_2 and β_1/β_3 are constant;* (C) *both $(\alpha_1/\beta_1)/(\alpha_2/\beta_2)$ and $(\alpha_1/\beta_1)/(\alpha_3/\beta_3)$ are constant;* (D) *both α_j/α_k and β_j/β_k are constant for some $1 \le j < k \le 3$.*

PROOF. Take $z \in \boldsymbol{C}$. Then $(f_0(z), f_1(z), g_0(z), g_1(z), h_0(z), h_1(z))$ is a common zero of

$$\alpha_j(z)(b_j X_0^2 + a_j X_0 X_1 + X_1^2) - (b_j Y_0^2 + a_j Y_0 Y_1 + Y_1^2)$$

and

$$\beta_j(z)(b_j X_0^2 + a_j X_0 X_1 + X_1^2) - (b_j Z_0^2 + a_j Z_0 Z_1 + Z_1^2)$$

for $j = 1, 2, 3$. Under this situation the matrix D of (3.10) is

$$D = \begin{pmatrix} b_1\alpha_1 & a_1\alpha_1 & \alpha_1 & -b_1 & -a_1 & -1 & 0 & 0 & 0 \\ b_2\alpha_2 & a_2\alpha_2 & \alpha_2 & -b_2 & -a_2 & -1 & 0 & 0 & 0 \\ b_3\alpha_3 & a_3\alpha_3 & \alpha_3 & -b_3 & -a_3 & -1 & 0 & 0 & 0 \\ b_1\beta_1 & a_1\beta_1 & \beta_1 & 0 & 0 & 0 & -b_1 & -a_1 & -1 \\ b_2\beta_2 & a_1\beta_2 & \beta_2 & 0 & 0 & 0 & -b_2 & -a_2 & -1 \\ b_3\beta_3 & a_1\beta_3 & \beta_3 & 0 & 0 & 0 & -b_3 & -a_3 & -1 \end{pmatrix}.$$

Since some $f_j(z)g_k(z)h_l(z) \neq 0$, by (3.11), we have $\Delta(z) = 0$, and hence $\Delta \equiv 0$. Put

$$D_0^{(\mu\nu)} = \begin{vmatrix} a_\mu & 1 \\ a_\nu & 1 \end{vmatrix}, \qquad D_1^{(\mu\nu)} = \begin{vmatrix} b_\mu & 1 \\ b_\nu & 1 \end{vmatrix}, \qquad D_2^{(\mu\nu)} = \begin{vmatrix} b_\mu & a_\mu \\ b_\nu & a_\nu \end{vmatrix}$$

for $\mu, \nu = 1, 2, 3$, and $A_j^{(1)} = D_j^{(23)}$, $A_j^{(2)} = D_j^{(13)}$, $A_j^{(3)} = D_j^{(12)}$ for $j = 0, 1, 2$. Then

$$D_{jkl} = \sum_{1 \leq \mu, \nu \leq 3} (-1)^{\mu+\nu} D_j^{(\mu\nu)} A_k^{(\mu)} A_l^{(\nu)} \alpha_\mu \beta_\nu. \tag{4.3}$$

Since each D_{jkl} is a quadratic homogeneous polynomial of α_1, α_2, α_3, β_1, β_2, β_3 which consists of terms $\alpha_k\beta_l$ $(k \neq l)$, by (3.11) Δ is a homogeneous polynomial of degree sixteen of them whose terms are $\prod_{m=1}^{8} \alpha_{j_m}\beta_{k_m}$, where $j_m \neq k_m$, $m = 1, \ldots, 8$, with complex coefficients. Fix μ, ν such that $1 \leq \mu, \nu \leq 3$ and $\mu \neq \nu$. For simplicity, we write D_j for $D_j^{(\mu\nu)}$, A_j for $A_j^{(\mu)}$ and B_j for $A_j^{(\nu)}$. Then, in the expansion of Δ, from (3.11) and (4.3) the term $(-1)^{\mu+\nu}(\alpha_\mu\beta_\nu)^8$ has the coefficient

$$\begin{vmatrix} D_2A_2B_2 & D_2A_2B_1 & D_2A_1B_2 & D_2A_1B_1 & D_1A_2B_2 & D_1A_2B_1 & D_1A_1B_2 & D_1A_1B_1 \\ D_2A_2B_1 & D_2A_2B_0 & D_2A_1B_1 & D_2A_1B_0 & D_1A_2B_1 & D_1A_2B_0 & D_1A_1B_1 & D_1A_1B_0 \\ D_2A_1B_2 & D_2A_1B_1 & D_2A_0B_2 & D_2A_0B_1 & D_1A_1B_2 & D_1A_1B_1 & D_1A_0B_2 & D_1A_0B_1 \\ D_2A_1B_1 & D_2A_1B_0 & D_2A_0B_1 & D_2A_0B_0 & D_1A_1B_1 & D_1A_1B_0 & D_1A_0B_1 & D_1A_0B_0 \\ D_1A_2B_2 & D_1A_2B_1 & D_1A_1B_2 & D_1A_1B_1 & D_0A_2B_2 & D_0A_2B_1 & D_0A_1B_2 & D_0A_1B_1 \\ D_1A_2B_1 & D_1A_2B_0 & D_1A_1B_1 & D_1A_1B_0 & D_0A_2B_1 & D_0A_2B_0 & D_0A_1B_1 & D_0A_1B_0 \\ D_1A_1B_2 & D_1A_1B_1 & D_1A_0B_2 & D_1A_0B_1 & D_0A_1B_2 & D_0A_1B_1 & D_0A_0B_2 & D_0A_0B_1 \\ D_1A_1B_1 & D_1A_1B_0 & D_1A_0B_1 & D_1A_0B_0 & D_0A_1B_1 & D_0A_1D_0 & D_0A_0B_1 & D_0A_0B_0 \end{vmatrix}$$

$$= \begin{vmatrix} D_2E_4 & D_1E_4 \\ D_1E_4 & D_0E_4 \end{vmatrix}$$

$$
\times \begin{vmatrix}
A_2B_2 & A_2B_1 & A_1B_2 & A_1B_1 & & & & \\
A_2B_1 & A_2B_0 & A_1B_1 & A_1B_0 & & & & \\
A_1B_2 & A_1B_1 & A_0B_2 & A_0B_1 & & & & \\
A_1B_1 & A_1B_0 & A_0B_1 & A_0B_0 & & & & \\
 & & & & A_2B_2 & A_2B_1 & A_1B_2 & A_1B_1 \\
 & & & & A_2B_1 & A_2B_0 & A_1B_1 & A_1B_0 \\
 & & & & A_1B_2 & A_1B_1 & A_0B_2 & A_0B_1 \\
 & & & & A_1B_1 & A_1B_0 & A_0B_1 & A_0B_0
\end{vmatrix}
$$

$$
= \begin{vmatrix} D_2E_4 & D_1E_4 \\ D_1E_4 & D_0E_4 \end{vmatrix} \cdot \begin{vmatrix}
A_2E_2 & A_1E_2 & & \\
A_1E_2 & A_0E_2 & & \\
 & & A_2E_2 & A_1E_2 \\
 & & A_1E_2 & A_0E_2
\end{vmatrix}
$$

$$
\times \begin{vmatrix}
B_2 & B_1 & & & & & & \\
B_1 & B_0 & & & & & & \\
 & & B_2 & B_1 & & & & \\
 & & B_1 & B_0 & & & & \\
 & & & & B_2 & B_1 & & \\
 & & & & B_1 & B_0 & & \\
 & & & & & & B_2 & B_1 \\
 & & & & & & B_1 & B_0
\end{vmatrix}
$$

$$
= (D_0D_2 - D_1^2)^4(A_0A_2 - A_1^2)^4(B_0B_2 - B_1^2)^4
$$

$$
= \{R(P_\mu, P_v)R(P_\lambda, P_v)R(P_\lambda, P_\mu)\}^4,
$$

where void elements represent 0, and E_n is the unit matrix of size n and $R(P, Q)$ is the resultant of two polynomials $P(z)$ and $Q(z)$, and $\{\lambda, \mu, v\} = \{1, 2, 3\}$. Since $R(P_j, P_k) \neq 0$ for $j \neq k$, every term $(\alpha_\mu \beta_v)^8$ really appears in the expansion of Δ for $\mu \neq v$.

Now take representations μ_j, v_j of $[\alpha_j]$, $[\beta_j]$ of rank 16. Let λ_1, λ_2, λ_3, λ_4, λ_5, λ_6 be the indices $\mu_j + v_k$ of $\alpha_j\beta_k$ $(j \neq k)$, which are arranged as $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5 \geq \lambda_6$. If $\lambda_1 > \lambda_2$, then there is no term in the expansion of Δ with the index $8\lambda_1$ except one, which contradicts Lemma 2. Hence $\lambda_1 = \lambda_2$, and similarly, $\lambda_5 = \lambda_6$.

Without loss of generality, we may assume that $\mu_1 \geq \mu_2 \geq \mu_3$. Note that (A), (B), (C) and (D) are equivalent to the followings, respectively: (a) $\mu_1 = \mu_2 = \mu_3$; (b) $v_1 = v_2 = v_3$; (c) $\mu_1 - v_1 = \mu_2 - v_2 = \mu_3 - v_3$; (d) $\mu_j = \mu_k$, $v_j = v_k$ for some $1 \leq j < k \leq 3$.

(I) The case where $v_1 \geq v_2 \geq v_3$. In this case $\mu_1 + v_2 \geq \mu_1 + v_3 \geq \mu_2 + v_3$, $\mu_2 + v_1 \geq \mu_3 + v_1 \geq \mu_3 + v_2$ and $\mu_2 + v_1 \geq \mu_2 + v_3$. When we consider the maximal index, following three cases arise: (i) $\mu_1 + \nu_2 > \mu_2 + \nu_1$, (ii) $\mu_1 + \nu_2 < \mu_2 + \nu_1$ and (iii) $\mu_1 + \nu_2 = \mu_2 + \nu_1$. If (i), then $\mu_1 + \nu_2 = \mu_1 + \nu_3$ are the maximal indices, and hence $\nu_2 = \nu_3$. If (ii), then $\mu_2 + \nu_1 = \mu_3 + \nu_1$ are the maximal incices, and hence $\mu_2 = \mu_3$. When we consider the minimal index, following three cases arise: (iv) $\mu_2 + \nu_3 > \mu_3 + \nu_2$, (v) $\mu_2 + \nu_3 < \mu_3 + \nu_2$ and (vi) $\mu_2 + \nu_3 = \mu_3 + \nu_2$. If (iv), then $\mu_3 + \nu_2 = \mu_3 + \nu_1$ are the minimal indices, and hence $\nu_2 = \nu_1$. If (v), then $\mu_2 + \nu_3 = \mu_1 + \nu_3$ are the minimal indices, and hence $\mu_1 = \mu_2$.

Furthermore we must consider nine cases by multipying the first three cases (i), (ii), (iii) and the secondary three cases (iv), (v), (vi). In the case where (i) and (iv), $\nu_1 = \nu_2 = \nu_3$, which is (b). In the case of (i) and (v), $0 = \mu_1 - \mu_2 > \nu_1 - \nu_2$, which contradicts $\nu_1 \geq \nu_2$. In the case where (i) and (vi), $\nu_2 = \nu_3$ and $\mu_2 = \mu_3$, which is (d). In the case where (ii) and (iv), $0 = \mu_2 - \mu_3 > \nu_2 - \nu_3 \geq 0$, which is a contradiction. In the case where (ii) and (v), $\mu_1 = \mu_2 = \mu_3$, which is (a). In the case where (ii) and (vi), $\mu_2 = \mu_3$ and $\nu_2 = \nu_3$. We get (d). If (iii) and (iv) hold, then $\nu_1 = \nu_2$ and $\mu_1 = \mu_2$, which is (d). Also, in the case where (iii) and (v), we have (d). If (iii) and (vi) hold, then $\mu_1 - \nu_1 = \mu_2 - \nu_2 = \mu_3 - \nu_3$, which is (c).

(II) The case where $\nu_1 \geq \nu_3 \geq \nu_2$. In this case $\mu_1 + \nu_3 \geq \mu_1 + \nu_2 \geq \mu_3 + \nu_2$, $\mu_2 + \nu_1 \geq \mu_3 + \nu_1 \geq \mu_3 + \nu_2$ and $\mu_1 + \nu_3, \mu_2 + \nu_1 \geq \mu_2 + \nu_3 \geq \mu_3 + \nu_2$. When we consider the minimal index, the following two subcases arises: (i) $\mu_2 + \nu_3 = \mu_3 + \nu_2$. Then $0 \leq \mu_2 - \mu_3 = \nu_2 - \nu_3 \leq 0$, and hecne, $\mu_2 = \mu_3$, $\mu_2 = \nu_3$, which is (d). (ii) $\mu_2 + \nu_3 > \mu_3 + \nu_2$. Then $\mu_3 + \nu_2 = \mu_1 + \nu_2$ or $\mu_3 + \nu_2 = \mu_3 + \nu_1$ holds. In the former case, we have $\mu_1 = \mu_3$, which implies (a). In the latter case, we have $\nu_1 = \nu_2$, which is (b).

(III) The case where $\nu_2 \geq \nu_1 \geq \nu_3$. In this case we have $\mu_1 + \nu_2 \geq \mu_1 + \nu_3 \geq \mu_1 + \nu_3$, $\mu_1 + \nu_2 \geq \mu_2 + \nu_1 \geq \mu_3 + \nu_1$ and $\mu_2 + \nu_1 \geq \mu_2 + \nu_3$, $\mu_1 + \nu_2 \geq \mu_3 + \nu_2$. When we consider the maximal index, we have following three subcases: (i) $\mu_1 + \nu_2 = \mu_3 + \nu_2$, and hence, $\mu_1 = \mu_3$, which is (a). (ii) $\mu_1 + \nu_2 = \mu_1 + \nu_3$, and hence, $\nu_2 = \nu_3$, which is (b). (iii) $\mu_1 + \nu_2 = \mu_2 + \nu_1$. In this case $0 \leq \mu_1 - \mu_2 = \nu_1 - \mu_2 \leq 0$, and hence, $\mu_1 = \mu_2$, $\nu_1 = \nu_2$, which is (d).

(IV) The case where $\nu_2 \geq \nu_3 \geq \nu_1$. In this case the inequalties $\mu_1 + \nu_2 \geq \mu_1 + \nu_3 \geq \mu_2 + \mu_3 \geq \mu_2 + \nu_1 \geq \mu_3 + \nu_1$ and $\mu_1 + \nu_2 \geq \mu_3 + \nu_2 \geq \mu_3 + \nu_1$ hold. We see that $\mu_3 + \nu_1$ is the minimal index and that $\mu_3 + \nu_2$ or $\mu_2 + \nu_1$ equals it. If $\mu_3 + \nu_2 = \mu_3 + \nu_1$, then $\nu_2 = \nu_1$, which implies (b). If $\mu_2 + \nu_1 = \mu_3 + \nu_1$, then $\mu_2 = \mu_3$. On the other hand the maximal indices are $\mu_1 + \nu_2 = \mu_1 + \nu_3$ or $\mu_1 + \nu_2 = \mu_3 + \nu_2$. In the former, we obtain $\mu_2 = \nu_3$ with $\mu_2 = \mu_3$, which is (d). In the latter, we get (a).

(V) The case where $\nu_3 \geq \nu_1 \geq \nu_2$. In this case the inequalities $\mu_1 + \nu_3 \geq \mu_2 + \nu_3 \geq \mu_2 + \nu_1 \geq \mu_3 + \nu_1 \geq \mu_3 + \nu_2$ and $\mu_1 + \nu_3 \geq \mu_1 + \nu_2 \geq \mu_3 + \nu_2$ hold. The maximal indices are $\mu_1 + \nu_3 = \mu_1 + \nu_2$ or $\mu_1 + \nu_3 = \mu_2 + \nu_3$. In the former

$\nu_2 = \nu_3$, which implies (b). In the latter, we have $\mu_1 = \mu_2$. The minimal indices are $\mu_3 + \nu_2 = \mu_1 + \nu_2$ or $\mu_3 + \nu_2 = \mu_3 + \nu_1$. In the former $\mu_1 = \mu_3$, which is (a). In the latter, we have $\nu_1 = \nu_2$. Hence in any cases, we get one of (a), (b) and (d).

(VI) The case where $\nu_3 \geq \nu_2 \geq \nu_1$. In this case $\mu_1 + \nu_3 \geq \mu_1 + \nu_2 \geq \mu_2 + \nu_1 \geq \mu_3 + \nu_1$, $\mu_1 + \nu_3 \geq \mu_2 + \nu_3 \geq \mu_2 + \nu_1$ and $\mu_2 + \nu_3 \geq \mu_3 + \nu_2 \geq \mu_3 + \nu_1$. When we consider the maximal index, we have two cases: (i) $\mu_1 + \nu_3 = \mu_1 + \nu_2$, and hence, $\nu_2 = \nu_3$; (ii) $\mu_1 + \nu_3 = \mu_2 + \nu_3$, and hence, $\mu_1 = \mu_2$. When we consider the minimal index, we have two cases: (iii) $\mu_3 + \nu_1 = \mu_2 + \nu_1$, which implies $\mu_2 = \mu_3$; (iv) $\mu_3 + \nu_1 = \mu_3 + \nu_2$, which implies $\nu_1 = \nu_2$. If (i) and (iii) hold, then we have (d). In the case where (i) and (iv), we have (b). In the case where (ii) and (iii), we have (a). If (ii) and (iv) hold, then we have (d).

We have completed the proof. □

Remark. Note that we did not assume that P_j have no double zeros in the above proof.

Now, we start the proof of Theorem 5.

Let

$$S_j = \{\xi_j, \eta_j\} = \{z; z^2 + a_j z + b_j = 0\} \qquad (j = 1, 2, 3)$$

be pairwise disjoint two-point sets in $\mathbf{C}$ and let f, g, h be nonconstant meromorphic functions on $\mathbf{C}$ sharing each S_j CM. Then we can take $P_j(z) = z^2 + a_j z + b_j$ in Theorem 6 and there exist some entire functions α_j without zeros satisfying (4.1) and (4.2) for $j = 1, 2, 3$, where f_0, f_1, g_0, g_1, h_0, h_1 are as in Theorem 6. By Theorem 6, one of (A), (B), (C) and (D) holds.

First we consider the case where (A) holds. If $\{z : f(z) = g(z) \in S_j\} = \emptyset$ $(j = 1, 2)$, then $f^{-1}(\xi_j) = g^{-1}(\eta_j)$ and $f^{-1}(\eta_j) = g^{-1}(\xi_j)$ for $j = 1, 2$. We can take a Möbius transformation T such that $T(\xi_j) = \eta_j$, $T(\eta_j) = \xi_j$ $(j = 1, 2)$. Then f and $T \circ g$ share four values ξ_1, η_1, ξ_2 and η_2 CM, and we get the conclusion by Nevanlinna's four-value theorem (Theorem 3). So, we may assume there exists $z_0 \in \boldsymbol{C}$ such that $f(z_0) = g(z_0) = \xi_1$, without loss of generality. Now, $c := \alpha_2/\alpha_3$ is a nonzero constant and

$$c\frac{f^2 + a_2 f + b_2}{f^2 + a_3 f + b_3} = \frac{g^2 + a_2 g + b_2}{g^2 + a_3 g + b_3}$$

holds. This equality at z_0 induces $c = 1$, and hence, we get the conclusion.

Similarly, we get the conclusion in each case (B) and (C).

Now, we consider the case (D). Without loss of generality, we may assume that $\mu_1 = \mu_2$, $\nu_1 = \nu_2$. Then

$$c\frac{f^2 + a_1 f + b_1}{f^2 + a_2 f + b_2} = \frac{g^2 + a_1 g + b_1}{g^2 + a_2 g + b_2}$$

and

$$c'\frac{f^2 + a_1 f + b_1}{f^2 + a_2 f + b_2} = \frac{h^2 + a_1 h + b_1}{h^2 + a_2 h + b_2}$$

hold, where $c := \alpha_1/\alpha_2$, $c' := \beta_1/\beta_2$ are nonzero constants. If $c = 1$ or $c' = 1$ or $c = c'$, then we get the conclusion. Now assume that $c \neq 1$, $c' \neq 1$ and $c \neq c'$. Then there is no $z \in \boldsymbol{C}$ such that $f(z) = g(z) \in S_3$ or $f(z) = h(z) \in S_3$ or $g(z) = h(z) \in S_3$. This fact implies that f, g and h omit two values ξ_3 and η_3, and hence, f, g and h share S_1, S_2, $\{\xi_3\}$ and $\{\eta_3\}$ CM, and we get the conclusion by Theorem 2.

We have completed the proof.

5. Proof of Theorem 1

Though proofs of Theorem 1 are given by H. Cartan in §56 of [C] and by R. Nevanlinna in p. 125 of [N2], we prove it, again, by using Theorem 6.

Let f, g and h be nonconstant meromorphic functions on $\boldsymbol{C}$ and let ξ_1, ξ_2, ξ_3 be distinct points in $\overline{\mathbf{C}}$. Assume that f, g and h share each ξ_j CM. Then, we prove that two of f, g and h are identical.

By considering compositions of each of f, g, h and a suitable Möbius transformation, we may assume that $\xi_j \in \mathbf{C}$ $(j = 1, 2, 3)$. Put $P_j(z) = (z - \xi_j)^2$. Then, by Theorem 6, one of (A), (B), (C) and (D) holds.

In the case (A), we have

$$c_1(f - \xi_1)/(f - \xi_3) = (g - \xi_1)/(g - \xi_3),$$
$$c_2(f - \xi_2)/(f - \xi_3) = (g - \xi_2)/(g - \xi_3),$$

where $c_j^2 = \alpha_j/\alpha_3$ $(j = 1, 2)$ are nonzero constants. Since f and g are nonconstant, we obtain $f = g$ from these identities.

Similarly, we get $f = h$ in the case (B) and $g = h$ in the case (C).

Consider the case (D). We may assume that $\mu_1 = \mu_2$ and $\nu_1 = \nu_2$. Then we have

$$c(f - \xi_1)/(f - \xi_2) = (g - \xi_1)/(g - \xi_2),$$
$$c'(f - \xi_1)/(f - \xi_2) = (h - \xi_1)/(h - \xi_2),$$

where c and c' are nonzero constants. We get $f = g$, $f = h$ and $g = h$ if $c = 1$, $c' = 1$ and $c = c'$, respectively.

Assume that $c \neq 1$, $c' \neq 1$ and $c \neq c'$. Then f, g and h must omit ξ_3. Since from the above identities

$$f = \frac{(\xi_2 - c\xi_1)g - (1 - c)\xi_1\xi_2}{(1 - c)g - (\xi_1 - c\xi_2)}$$

and

$$f = \frac{(\xi_2 - c'\xi_1)h - (1 - c')\xi_1\xi_2}{(1 - c')h - (\xi_1 - c'\xi_2)}$$

hold, f omit also two values

$$\frac{(\xi_2 - c\xi_1)\xi_3 - (1 - c)\xi_1\xi_2}{(1 - c)\xi_3 - (\xi_1 - c\xi_2)}$$

and

$$\frac{(\xi_2 - c'\xi_1)\xi_3 - (1 - c')\xi_1\xi_2}{(1 - c')\xi_3 - (\xi_1 - c'\xi_2)}.$$

It follows from $c \neq 1$, $c' \neq 1$, $c \neq c'$ and distinctness of ξ_1, ξ_2, ξ_3 that three exceptional values of f are distinct, which is a contradiction.

Hence we have proved Theorem 1.

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