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Locally solvable subnormal and quasinormal subgroups in division rings

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ABSTRACT. In this paper, we show that every locally solvable subnormal subgroup or locally solvable quasinormal subgroup of the multiplicative group of a division ring is central.

1. Introduction

A subgroup N of a group G is said to be *subnormal* in G if there is a finite chain of subgroups

$$N = N_r \le N_{r-1} \le \cdots \le N_0 = G,$$

for which N_i is normal in N_{i-1} for all $1 \le i \le r$. Also, if Q is a subgroup of G such that the relation $QH = HQ$ holds for any subgroup H of G, then we say that Q is *quasinormal* (or *permutable*) in G. It is pointed out in [11, Chapter 7] that there are close relations between these types of subgroups. It was shown by S. E. Stonehewer that if G is a finitely generated group, then every quasinormal subgroup of G is subnormal ([12, Theorem B]). However, the converse does not hold up. As an example, let G be the dihedral group of order 8 generated by subgroups A and B which are of order 2. It follows that $AB \neq BA$ since $|AB| = 4$ and $G \neq AB$, implying that A and B are not quasinormal subgroups of G. On the other hand, the nilpotency of G implies that both A and B are subnormal. (Recall that every subgroup of a nilpotent group is subnormal.) In this paper, we study subnormal subgroups and quasinormal subgroups of the multiplicative group of a division ring. Relating to this, note that in [1] there is an example of a division ring which contains quasinormal subgroups that are not subnormal.

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In the literature, there are very rich results concerning the algebraic structure of multiplicative subgroups in a division ring (e.g., see [6]). As a direction of the study, in 1950's and 1960's, many authors paid attention to an interesting question of how far the multiplicative group D^* of a division ring D is from being abelian. In this direction, a well-known result of L. K. Hua says that if D^* is solvable, then D is a field. Motivated by this result, several other authors examined various aspects of subnormal subgroups of D^*, instead of the whole group D^*. For example, it was shown that every subnormal subgroup of D^* must be central in D if it is locally nilpotent, solvable, or n-Engel (see [13], [8], [10], respectively). Now, we consider the same problem in which the subnormal subgroup is assumed to be locally solvable. By definition, a group is called locally solvable if its every finitely generated subgroup is solvable. In [4], B. X. Hai and D. V. P. Ha proved that if D^* is locally solvable, then D is a field. Moreover, it was proved by A. E. Zalesskii in [17] that every locally solvable normal subgroup of D^* is contained in the center F of D. It is natural to ask whether every locally solvable subnormal subgroup, say G, of D^* is also contained in F. A positive answer to this question was given for some particular cases where D is supposed to be algebraic over F ([5]), or where the derived subgroup $G^{(i)}$ of G is assumed to be algebraic over F for some $i \geq 1$ ([9]). The first purpose of the present paper is to give the affirmative answer to the question in the general setting; that is, we shall show that every locally solvable subnormal subgroup of D^* is contained in F (Theorem 1). The second purpose is to prove that every locally solvable quasinormal subgroup of D^* is also central; and this goal will be achieved in Theorem 3.

Throughout this paper, the word "ring" always refers to a ring with an identity element $1 \neq 0$. For a ring R, the symbol R^* denotes the group of units in R. If D is a division ring with center F and S is a subset of D, then $F[S]$ (resp. $F(S)$) denotes the subring (resp. the division subring) of D generated by $F \cup S$. For a group G, the *Hirsch-Plotkin radical* of G is defined to be the subgroup generated by all locally nilpotent normal subgroups of G. If H and K are two subgroups of G, then the symbol $[H, K]$ stands for the subgroup of G generated by the set of all commutators $[a, b] = a^{-1}b^{-1}ab$, where $a \in H$ and $b \in K$. We say that G is radical over a subgroup Q if for each g in G, there is a positive integer n depending on g such that g^n belongs to Q.

2. Locally solvable subnormal subgroups

We begin with a group-theoretic lemma which, despite its apparent simplicity, will be frequently applied in the sequel.

Lemma 1. *Every group contains a unique maximal periodic normal subgroup. Moreover, such a subgroup is characteristic in the whole group.*

Proof. Our proof shall be obtained by mainly using Zorn's Lemma. First, we define a family of subgroups of a group G by taking

$$\mathscr{A} = \{H \mid H \text{ is a periodic normal subgroup of } G\}.$$

This family is obviously non-empty since the identity subgroup belongs to $\mathscr{A}$. Now, we consider an arbitrary chain $\{H_i\}$ of subgroups in $\mathscr{A}$. Our task, of course, is to show that $\bigcup H_i$ is again a member of $\mathscr{A}$; that is, to prove that $\bigcup H_i$ forms a periodic normal subgroup of G. For this purpose, pick any two elements $a, b \in \bigcup H_i$. Then, there exist indices i and j for which $a \in H_i$ and $b \in H_j$. Since the collection $\{H_i\}$ forms a chain, either $H_i \subseteq H_j$ or $H_j \subseteq H_i$. It is clear that we may assume that $H_i \subseteq H_j$ and so $a.b^{-1} \in H_j \subseteq \bigcup H_i$. This implies that $\bigcup H_i$ is a subgroup of G. The normality as well as the periodicity of $\bigcup H_i$ may be obtained by the same way. All of this shows that $\bigcup H_i$ is a member of $\mathscr{A}$, completing our task. Therefore, on the basic of Zorn's Lemma, the family $\mathscr{A}$ contains a maximal element M.

Next, we shall prove that M is maximal with respect to being periodic and normal. Let N be a periodic normal subgroup of G for which $M \subseteq N$. Since M is a maximal element of $\mathscr{A}$ and $N \in \mathscr{A}$, we must have $M = N$, which implies the maximality of M.

To see the uniqueness of M, take any periodic normal subgroup A of G. The normality of M and A in G permits us to form the product subgroup AM, which is obviously a periodic normal subgroup of G. But then, the maximality of M reveals that $AM = M$, or $A \subseteq M$. This argument shows that every periodic normal subgroup of G is contained in M, proving the uniqueness of M.

It remains only to show that M is characteristic in G. For this purpose, we pick $\varphi \in \mathrm{Aut}(G)$, then $\varphi(M)$ is certainly a periodic normal subgroup of G. The uniqueness of M implies that $\varphi(M) = M$. Our proof is finally finished.

For any group G, let us denote by $\tau(G)$ the unique maximal periodic normal subgroup of G and by $B(G)$ the subgroup of G such that $B(G)/\tau(G)$ is the Hirsch-Plotkin radical of $G/\tau(G)$. It is easy to see that $B(G)$ is a normal subgroup of G.

Proposition 1. *Let D be a division ring with center F. If G is a subnormal subgroup of D^*, then $B(G)$ is contained in F.*

Proof. Being a normal subgroup of G, the subgroup $\tau(G)$ is a periodic subnormal subgroup of D^*. With reference to [7, Theorem 8], we conclude that $\tau(G)$ is contained in F.

Our next step is to assert that $B(G)$ is indeed a locally nilpotent group. For this purpose, we take an arbitrary finitely generated subgroup H of $B(G)$, and our aim is to show that this is a nilpotent group. It is a simple matter to see that $H\tau(G)/\tau(G)$ is a finitely generated subgroup of $B(G)/\tau(G)$. Accordingly, the local nilpotence of $B(G)/\tau(G)$ implies that $H\tau(G)/\tau(G)$ is nilpotent. We set

$$H_1 = [H, H], \qquad H_2 = [H_1, H],$$
$$H_3 = [H_2, H], \ldots$$

Now, as $H\tau(G)/\tau(G)$ is nilpotent, we can find an integer n for which $H_n \subseteq \tau(G) \subseteq F$. This fact says that any element of H_n commutes element-wise with H and, in consequence, we have $H_{n+1} = [H_n, H] = 1$, from which it follows that H is nilpotent. In other words, we obtain that $B(G)$ is locally nilpotent, as asserted.

As we have pointed out before, $B(G)$ is a normal subgroup of G. This assures us to conclude that $B(G)$ is a locally nilpotent subnormal subgroup of D^*. By virtue of Huzurbazar's result ([8]), we finally obtain that $B(G) \subseteq F$. Our proof is finished.

The following lemma, which provides the key to later success, gives us a way to calculate the normalizer of a locally solvable subgroup in a division ring.

Lemma 2 ([15, Point 20]). *Let $R = F[G]$ be an algebra over the field F that is a domain. If G is a locally solvable, then R is an Ore domain. Moreover, if we assume that D is the skew field of fractions of R and that $B(G) = F^* \cap G$, then $N_{D^*}(G) = GF^*$.*

Lemma 3. *Let D be a division ring with center F. If G is a locally solvable non-central subnormal subgroup of D^*, then $F(G) = D$ and $N_{D^*}(G)$ is locally solvable.*

Proof. With reference to the previous lemma, the local solvability of G assures us to conclude that $R = F[G]$ is an Ore domain. Accordingly, its skew field of fractions is exactly $F(G)$, the division subring of D generated by G over F. Since $F(G)$ contains G which is assumed to be non-central, in the light of Stuth's Theorem ([13, Theorem 1]), we obtain that $F(G) = D$.

Next, we argue that $B(G) = F^* \cap G$. First, it follows directly from Proposition 1 that $B(G) \subseteq F^* \cap G$, which implies that $B(G)/\tau(G) \subseteq (F^* \cap G)/\tau(G)$. In regard to the reverse inclusion, we note that, being the Hirsch-Plotkin radical of $G/\tau(G)$, the factor group $B(G)/\tau(G)$ is the largest locally nilpotent normal subgroup of $G/\tau(G)$. On the other hand, it is clear that $(F^* \cap G)/\tau(G)$

is an abelian normal subgroup of $G/\tau(G)$, which yields that $(F^* \cap G)/\tau(G) \subseteq B(G)/\tau(G)$. In other words, we must have $(F^* \cap G)/\tau(G) = B(G)/\tau(G)$, from which it follows that $B(G) = F^* \cap G$. Our argument is now finished. Finally, the last assertion follows immediately from the proceeding lemma.

Before presenting the main theorem, we need a result of Zalesskii:

Lemma 4 ([17]). *Let D be a division ring with center F. If G is locally solvable normal subgroup of D^*, then G is contained in F.*

Here now is the main results of this section.

Theorem 1. *Let D be a division ring with center F. If G is a locally solvable subnormal subgroup of D^*, then G is contained in F.*

Proof. There is nothing to prove if D is commutative. Therefore, we may suppose that D is non-commutative. Assume that G is not contained in F. Since G is a subnormal subgroup of D^*, there exists a finite chain of subgroups

$$G = G_r \leq G_{r-1} \leq \cdots \leq G_0 = D^*,$$

in which G_i is normal in G_{i-1} for $0 \leq i \leq r$. By virtue of Lemma 3, we conclude that $N_{D^*}(G)$, the normalizer of G in D^*, is a locally solvable group. The normality of G_r in G_{r-1} implies that G_{r-1} is contained in $N_{D^*}(G)$ and, in consequence, the subgroup G_{r-1} is locally solvable and non-central.

Repeat this procedure, now starting with G_{r-1}, we obtain that G_{r-2} is locally solvable, too. This process must eventually terminate after finite steps, and at the final stage, we have the fact that D^* is locally solvable. It follows immediately from Lemma 4 that D is commutative, which is a contradiction. Our proof is finally completed.

3. Locally solvable quasinormal subgroups

We prepare the way by first establishing a few results concerning groups which are radical over subgroups.

Lemma 5 ([3, Theorem 2]). *Let G be a group and Q a quasinormal subgroup of G. If C is an infinite cyclic subgroup of G such that $Q \cap C = 1$, then Q is a normal subgroup of Q^G and Q/Q_G is abelian.*

Lemma 6. *Let G be a group. If Q is a quasinormal subgroup of G, then either G is radical over Q or Q is subnormal in G of defect at most 2.*

Proof. To start, we assume that G is not radical over Q. As such, we can find an element $g \in G$ such that g^n does not belong to Q for every integer number n. Let C be the cyclic subgroup of G generated by the element g. Then, the fact that $g^n \notin Q$ for any choice of n ensures that $Q \cap C = 1$. By virtue of the above lemma, we obtain that Q is normal in Q^G, which is a normal subgroup of G. Phrased in another way, Q is a subnormal subgroup of G with the correspondent series $Q \trianglelefteq Q^G \trianglelefteq G$. This completes our proof.

The next lemma, which is an interesting result of C. Faith, provides the key to establish the main result of this section.

Lemma 7 ([2, Theorem B]). *Every division ring which is radical over a proper subring is a field.*

By an analogy with C. Faith's result, a ring which is radical over a subgroup may be characterized in the following manner.

Proposition 2. *Let R be a ring and G a subgroup of R^*. If $R \backslash \{0\}$ is radical over G, then R is a division ring.*

Proof. To prove that R is a division ring, it suffices to show that each nonzero element of R is right invertible. For this purpose, we take x to be an arbitrary nonzero element of R. The radicality over G of x permits us to find an integer $n \geq 1$ for which $x^n \in G$. As G is a group, we can find an element $g \in G$ such that $x^n g = 1$. Or, equivalently, we have $x(x^{n-1}g) = 1$. This relation shows that x is right invertible with the right inverse $x^{-1} = x^{n-1}g$. Therefore, the ring R is indeed a division ring and our proposition is proved.

Lemma 8. *Let D be a division ring, and G a non-abelian subgroup of D^*. Assume that D^* is radical over G. Then, every subring of D containing G is coincided with D.*

Proof. For a proof by contradiction, we assume that E is a proper subring of D containing G. It is a fairly simple matter to see that $E = E[G]$. The assumption on D^* assures us to deduce that D is radical over E and so D is a field by Lemma 7. But this contrasts to the fact that G is assume to be non-abelian.

The following theorem illustrates how the multiplicative group of a division ring is affected by certain subgroups over which it is radical.

Theorem 2. *Let D be a division ring, and G a locally solvable subgroup of D^*. If D^* is radical over G, then D is a field.*

Proof. Suppose, to the contrary, that D is non-commutative. If G is abelian, then $F(G)$ is a proper subfield over which D is radical. It follows from previous lemma that D is a field, which violates our supposition. We may therefore assume that G is non-abelian. In the light of Lemma 8, we obtain that $F[G] = D$ and so G possesses an abelian normal subgroup A for which G/A is locally finite ([14, Point 3]). This last fact ensures that G is radical over A and, in consequence, so is D^*. As a result, the division ring D is radical over the subfield $F(A)$, from which it follows that D is a field. Again, we arrive at a desired contradiction, proving our theorem.

This may be a good place to give the main result of this section.

Theorem 3. *Let D be a division ring with center F. If Q is a locally solvable quasinormal subgroup of D^*, then Q is contained in F.*

Proof. With reference to Lemma 6, we have either Q is subnormal in D^* or D^* is radical over Q. In the first event, our result follows immediately from Theorem 1. It remains to examine the case where D^* is radical over Q. In this case, previous theorem says that D is commutative, and our result certainly holds. Our proof is now completed.

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