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Some relations between complex structures on compact nilmanifolds and flag manifolds

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ABSTRACT. In this paper, we first consider relations between signatures of pseudo-Kähler metrics on a flag manifold and complex structures on a nilpotent Lie algebra corresponding to the flag manifold. On the nilpotent Lie algebra, we also consider complex structures which do not correspond to invariant complex structures on the flag manifold.

1. Introduction

In this paper, we first consider relations between signatures of pseudo-Kähler metrics on a flag manifold and complex structures on a nilpotent Lie algebra corresponding to the flag manifold. For the flag manifold $M = SU(3)/T^2$, there exist invariant pseudo-Kähler metrics of type $(6,0)$, $(0,6)$, $(4,2)$, and $(2,4)$ ([9]). On the other hand, for a 6-dimensional nilpotent Lie algebra $_{\mathbb{R}}(\mathfrak{h}(1)^{\mathbb{C}})$, where $_{\mathbb{R}}(\mathfrak{h}(1)^{\mathbb{C}})$ is the scalar restriction of 3-dimensional complex Heisenberg algebra $\mathfrak{h}(1)^{\mathbb{C}}$, there exist distinct 4 connected components of the modular space $\mathscr{C}(_{\mathbb{R}}(\mathfrak{h}(1)^{\mathbb{C}}))$ (see [8] for details). Let $T_o^{\mathbb{C}}M$ be the complexification of tangent space of the point $o = eT^2$, and $T^{1,0}M$ the complex eigendistribution of the complex structure J with an eigenvalue $\sqrt{-1}$. Then, $T_o^{1,0}M$ can be identified with a complex nilpotent Lie algebra $\mathfrak{h}(1)^{\mathbb{C}} = (_{\mathbb{R}}(\mathfrak{h}(1)^{\mathbb{C}}), J)$.

In previous papers, we considered signatures of pseudo-Kähler metrics on a flag manifold and complex structures on a nilpotent Lie algebra, separately. Let $\mathfrak{g}$ be a real Lie algebra, and $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ a direct sum decomposition such that $\mathfrak{a}$ and $\mathfrak{b}$ are Lie subalgebras of $\mathfrak{g}$. Then, we can construct an integrable complex structure $\tilde{J}$ on $_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ from the decomposition. Then, we studied relations between the decomposition and $\dim H^{s,t}_{\bar{\partial}_{\tilde{J}}}(_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}}))$ for investigating the complex structure $\tilde{J}$ (see e.g. [10, Theorems 3.2, 3.3]). On the other hand, in

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the paper [9], we considered the signatures of pseudo-Kähler metrics on the full flag manifolds.

More precisely, we now consider the case of root system A_2. Let $\{\alpha_1, \alpha_2\}$ be a basis of A_2 with natural manner ([4]). By using results of previous papers ([11, Section 4], [9]), we have the following relations among Weyl chambers of the root system A_2, left-invariant complex structures on ${}_{\mathbb{R}}(\mathfrak{h}(1)^{\mathbb{C}})$, and signatures of pseudo-Kähler metrics on $M = SU(3)/T^2$.

Weyl chamber	complex structure of nilpotent Lie group	signature
$C_0 = \{\alpha_1 > 0, \alpha_2 > 0\}$	$N_0 = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \middle\vert\, z_i \in \mathbb{C} \right\}$	$(6, 0)$
$C_1 = \{-\alpha_1 > 0, \alpha_1 + \alpha_2 > 0\}$	$N_1 = \left\{ \begin{pmatrix} 1 & \bar{z}_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \middle\vert\, z_i \in \mathbb{C} \right\}$	$(4, 2)$
$C_2 = \{-\alpha_2 > 0, \alpha_1 + \alpha_2 > 0\}$	$N_2 = \left\{ \begin{pmatrix} 1 & w_1 & w_3 + w_1\bar{w}_2 \\ 0 & 1 & \bar{w}_2 \\ 0 & 0 & 1 \end{pmatrix} \middle\vert\, w_i \in \mathbb{C} \right\}$	$(4, 2)$
$C_3 = \{\alpha_1 > 0, -\alpha_1 - \alpha_2 > 0\}$	$N_3 = \left\{ \begin{pmatrix} 1 & z_1 & \bar{z}_3 \\ 0 & 1 & \bar{z}_2 \\ 0 & 0 & 1 \end{pmatrix} \middle\vert\, z_i \in \mathbb{C} \right\}$	$(2, 4)$
$C_4 = \{\alpha_2 > 0, -\alpha_1 - \alpha_2 > 0\}$	$N_4 = \left\{ \begin{pmatrix} 1 & \bar{w}_1 & \bar{w}_3 + \bar{w}_1 w_2 \\ 0 & 1 & w_2 \\ 0 & 0 & 1 \end{pmatrix} \middle\vert\, w_i \in \mathbb{C} \right\}$	$(2, 4)$
$C_5 = \{-\alpha_1 > 0, -\alpha_2 > 0\}$	$N_5 = \left\{ \begin{pmatrix} 1 & \bar{z}_1 & \bar{z}_3 \\ 0 & 1 & \bar{z}_2 \\ 0 & 0 & 1 \end{pmatrix} \middle\vert\, z_i \in \mathbb{C} \right\}$	$(0, 6)$

There exist holomorphic isomorphisms $f_{12} : N_1 \to N_2$, $f_{34} : N_3 \to N_4$, and antiholomorphic isomorphisms $f_{05} : N_0 \to N_5$, $f_{13} : N_1 \to N_3$, $f_{24} : N_2 \to N_4$. Thus, $N_0, N_1 \cong N_2, N_3 \cong N_4$ and N_5 are not holomorphically isomorphic each other. On the other hand, we have a symmetry of the signatures of invariant pseudo-Kähler metrics. In this paper we generalize those relations.

On a real nilpotent Lie algebra given by the scalar restriction of a complex nilpotent Lie algebra $T_o^{1,0}M$ of a flag manifold M, we also consider complex structures $\tilde{J}$ which do not correspond to invariant complex structures on the flag manifold M. However, we use Weyl chambers for constructing complex structures on the real nilpotent Lie algebra (See Sections 5 and 6). For example, the nilpotent Lie group with a left-invariant complex structure defined by

$$\left\{ \begin{pmatrix} 1 & \bar{x}_1 & x_2 & z \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, x_1, x_2, y_1, y_2, z \in \mathbb{C} \right\}$$

does not correspond to an invariant complex structure on the flag manifold $SU(4)/T^2 \times SU(2)$.

Let ${}_{\mathbb{R}}(\mathfrak{n}^{\mathbb{C}})$ be a nilpotent Lie algebra constructed from a root system A_l. Then, we can construct complex structures $\tilde{J}_{1,k}$ and $\tilde{J}_{2,k}$ on ${}_{\mathbb{R}}(\mathfrak{n}^{\mathbb{C}})$ (for the details of ${}_{\mathbb{R}}(\mathfrak{n}^{\mathbb{C}})$, $\tilde{J}_{1,k}$ and $\tilde{J}_{2,k}$, see Sections 4 and 5). We denote $\dim H^{s,t}_{\bar{\partial}_J}({}_{\mathbb{R}}(\mathfrak{n}^{\mathbb{C}}))$ by $h^{s,t}(\mathfrak{n}_J)$. Then, we show the following result:

Theorem 8. *Let $\tilde{J}_{1,k}$, $\tilde{J}_{2,k}$ be complex structures on ${}_{\mathbb{R}}(\mathfrak{n}^{\mathbb{C}})$ corresponding to decompositions of roots induced by subsets $\Pi_{1,k}$, $\Pi_{2,k}$ of a basis of a root system A_l, respectively. Then,*

$$\sum_{s+t=r} h^{s,t}(\mathfrak{n}_{\tilde{J}_{1,k-1}}) = \sum_{s+t=r} h^{s,t}(\mathfrak{n}_{\tilde{J}_{2,k+1}})$$

for each k and r.

2. Preliminaries

In this section, we recall an integrability condition of an almost left-invariant complex structure on a Lie group, and relations between Dolbeault cohomology groups of a nilmanifold with a complex structure and cohomology groups of a nilpotent Lie algebra.

On the integrability condition, we have

Theorem 1 (Andrada-Salamon [2]). *Let $\mathfrak{g}$ be a Lie algebra with a complex structure J which satisfies $J[X,Y] = [JX,Y]$ for all $X, Y \in \mathfrak{g}$. Suppose there exists a splitting $\mathfrak{g} = \mathfrak{u}_1 + \mathfrak{u}_2$ with complex subalgebras $\mathfrak{u}_1$, $\mathfrak{u}_2$ of $\mathfrak{g}$. Then the linear endomorphism $\tilde{J}$ defined by*

$$\tilde{J}|_{\mathfrak{u}_1} = -J, \qquad \tilde{J}|_{\mathfrak{u}_2} = J$$

is a complex structure on $\mathfrak{g}$.

Let N be a simply connected real nilpotent Lie group. It is well known that there exists a lattice in N if and only if there exists a rational Lie subalgebra $\mathfrak{n}_{\mathbb{Q}}$ such that $\mathfrak{n} \cong \mathfrak{n}_{\mathbb{Q}} \otimes \mathbb{R}$ (cf. [6]). Then, a complex structure J

on $\mathfrak{n}$ is said to be *rational* if $J(\mathfrak{n}_{\mathbf{Q}}) \subset \mathfrak{n}_{\mathbf{Q}}$ ([5]). Then, we have the following results.

Theorem 2 (Console-Fino [5]). *Let N be a simply connected nilpotent Lie group, and Γ a lattice in N. If J is a Γ-rational complex structure on $\mathfrak{n}$, then*

$$H_{\bar{\partial}}^{s,t}(\Gamma\backslash N) \cong H_{\bar{\partial}}^{s,t}(\mathfrak{n}^{\mathbf{C}})$$

for each s, t.

Theorem 3 (Console-Fino [5]). *For any small deformation of a Γ-rational complex structure, the isomorphism*

$$H_{\bar{\partial}}^{s,t}(\Gamma\backslash N) \cong H_{\bar{\partial}}^{s,t}(\mathfrak{n}^{\mathbf{C}})$$

holds for each s, t.

Theorem 4 (Sakane [7]). *Let N be a simply connected complex nilpotent Lie group, and Γ a lattice in N. Then,*

$$H_{\bar{\partial}}^{s,t}(\Gamma\backslash N) \cong H_{\bar{\partial}}^{0,t}(\mathfrak{n}^{-}) \otimes \bigwedge^{s}(\mathfrak{n}^{+})^{*} \cong H^{t}(\mathfrak{n}^{-}) \otimes \bigwedge^{s}(\mathfrak{n}^{+})^{*}$$

for each s, t.

Thus, results on $H_{\bar{\partial}}^{s,t}(\mathfrak{n}^{\mathbf{C}})$ of a nilpotent Lie algebra with good complex structures yield results on $H_{\bar{\partial}}^{s,t}(\Gamma\backslash N)$ of a compact nilmanifold with invariant complex structures.

3. Flag manifolds and Nilpotent Lie algebras

In this section, we first consider relations between signatures of pseudo-Kähler metrics on a flag manifold and complex structures on a nilpotent Lie algebra corresponding to the flag manifold. For details of notions of T-root systems, see [1], [3].

Let G be a compact semi-simple Lie group, $\mathfrak{g}$ the Lie algebra of G, and $\mathfrak{h}$ a maximal abelian subalgebra. We identify an element of the root system R of $\mathfrak{g}^{\mathbf{C}}$ relative to the Cartan subalgebra $\mathfrak{h}^{\mathbf{C}}$ with an element of $\mathfrak{h}_0 = \sqrt{-1}\mathfrak{h}$ by the duality defined by the Killing form of $\mathfrak{g}^{\mathbf{C}}$. We consider the following root system decomposition relative to $\mathfrak{h}^{\mathbf{C}}$:

$$\mathfrak{g}^{\mathbf{C}} = \mathfrak{h}^{\mathbf{C}} + \sum_{\alpha \in R} \mathfrak{g}_{\alpha}^{\mathbf{C}}.$$

Definition 1. (1) A subset $Q \subset R$ is said to be *closed* if for each $\alpha, \beta \in Q$ with $\alpha + \beta \in R$, it holds $\alpha + \beta \in Q$.

(2) A subset $Q \subset R$ is said to be *asymmetric* if $Q \cap (-Q) = \emptyset$.

Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be a basis of the root system R. We denote by R^+ the set of all positive roots relative to Π. Let Π_0 be a subset of Π and $\Pi - \Pi_0 = \{\alpha_{i_1}, \dots, \alpha_{i_r}\}$, where $1 \leq i_1 < \cdots < i_r \leq l$. We put $\mathfrak{t} = \{H \in \mathfrak{h}_0 \mid (H, \Pi_0) = 0\}$. Then,

$$\mathfrak{n}^{\mathbf{C}} = \sum_{\alpha \in R^+ - [\Pi_0]} \mathfrak{g}_\alpha^{\mathbf{C}}$$

is a nilpotent Lie algebra, where $[\Pi_0] = R \cap \{\Pi_0\}_{\mathbf{Z}}$. We put $R_{\mathfrak{m}} = R - [\Pi_0]$ and $R_{\mathfrak{m}}^+ = R_{\mathfrak{m}} \cap R^+$. Take a Weyl basis $E_\alpha \in \mathfrak{g}_\alpha^{\mathbf{C}}$ ($\alpha \in R$). Then, the structure constants $N_{\alpha,\beta}$ satisfy $N_{\alpha,\beta} = N_{-\alpha,-\beta} \in \mathbf{R}$, where $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$ if $\alpha, \beta, \alpha+\beta \in R$. Let $\mathfrak{g}_\alpha = \mathbf{R} E_\alpha$, and $\mathfrak{n} = \sum_{\alpha \in R_{\mathfrak{m}}^+} \mathfrak{g}_\alpha$.

We consider the restriction map

$$\kappa : \mathfrak{h}_0 \to \mathfrak{t}^* \qquad \alpha \mapsto \alpha|_{\mathfrak{t}}$$

and set $R_T = \kappa(R)$. The elements of R_T are called *T-roots*. The collection of hyperplanes $\{\kappa(\alpha) = 0\}$ corresponding to T-roots decomposes the space $\mathfrak{t}$ into a finite number of cones, which are called *T-chambers*. We denote by $B(C)$ a basis of $\mathfrak{t}^*$ corresponding to a T-chamber C. We denote by C_0 a chamber $\{\kappa(\alpha_{i_1}) > 0, \dots, \kappa(\alpha_{i_r}) > 0\}$. We also denote by $R_T^+(C)$ the set of the positive T-roots corresponding to a T-chamber C.

Let $G^{\mathbf{C}}$ be a simply connected complex semi-simple Lie group and U the parabolic subgroup of $G^{\mathbf{C}}$. Then the homogeneous complex manifold $G^{\mathbf{C}}/U$ is compact and simply connected, and G acts transitively on $G^{\mathbf{C}}/U$. Note that $K = G \cap U$ is a connected closed subgroup of G, and $G^{\mathbf{C}}/U = G/K = M$ as C^∞-manifolds. Let $\mathfrak{m}$ be the orthogonal complement of the Lie algebra $\mathfrak{k}$ of K with respect to the negative of the Killing form of $\mathfrak{g}$.

There exists a one-to-one correspondence between T-roots ξ and irreducible submodules $\mathfrak{m}_\xi$ of the $Ad_G(K)$-module $\mathfrak{m}^{\mathbf{C}}$ given by

$$R_T \ni \xi \mapsto \mathfrak{m}_\xi = \sum_{\kappa(\alpha) = \xi} \mathfrak{g}_\alpha^{\mathbf{C}}.$$

Thus, we have a decomposition of the $Ad_G(K)$-module $\mathfrak{m}^{\mathbf{C}}$:

$$\mathfrak{m}^{\mathbf{C}} = \sum_{\xi \in R_T} \mathfrak{m}_\xi.$$

We will identify $\mathfrak{m}^{\mathbf{C}}$ with the complexification $T_o^{\mathbf{C}} M$ of tangent space $T_o M$ at the point $o = eK$.

Theorem 5 (cf. [1]). *There exist natural one-to-one correspondences between*

(1) *T-bases $\Pi_T = \{\xi_1, \dots, \xi_r\}$;*
(2) *T-chambers $C = \{\xi_1 > 0, \dots, \xi_r > 0\}$;*
(3) *systems $\Pi = \{(\xi_1)_-, \dots, (\xi_r)_-\} \cup \Pi_0$ of simple roots of R which contain fixed system Π_0, where $(\xi_i)_-$ is the lowest weight of irreducible $Ad_G(K)$-module $\mathfrak{m}_{\xi_i}$ for each i;*
(4) *decomposition $R_{\mathfrak{m}} = R_+ \cup R_-$ into disjoint union asymmetric closed subsets R_+ and $R_- = -R_+$;*
(5) *invariant complex structures on the flag manifold G/K (up to a sign).*

In particular, for a decomposition $R_{\mathfrak{m}} = R_+ \cup R_-$, we define a decomposition of the complexified tangent space $T_o^{\mathbf{C}} M = \mathfrak{m}^{\mathbf{C}} = \mathfrak{m}^{1,0} + \mathfrak{m}^{0,1}$, where $\mathfrak{m}^{1,0} = \sum_{\alpha \in R_+} \mathfrak{g}_\alpha^{\mathbf{C}}$, $\mathfrak{m}^{0,1} = \sum_{\alpha \in R_-} \mathfrak{g}_\alpha^{\mathbf{C}}$. Since the subspaces $\mathfrak{m}^{1,0}$, $\mathfrak{m}^{0,1}$ are $Ad_G(K)$-invariant, they can be extended to two complex invariant distributions $T^{1,0}M$ and $T^{0,1}M$. We define an invariant complex structure J on M such that $T^{1,0}M$ and $T^{0,1}M$ are eigendistributions of J with eigenvalues $+\sqrt{-1}$ and $-\sqrt{-1}$, respectively. Since $\mathfrak{k}^{\mathbf{C}} + \mathfrak{m}^{1,0}$ is a subalgebra of $\mathfrak{g}^{\mathbf{C}}$, we have J is integrable, where $\mathfrak{k}^{\mathbf{C}} = \mathfrak{h}^{\mathbf{C}} + \sum_{\alpha \in [\Pi_0]} \mathfrak{g}_\alpha^{\mathbf{C}}$.

Conversely, any invariant complex structure J of $M = G/K$ defines a decomposition

$$\mathfrak{m}^{\mathbf{C}} = \mathfrak{m}^{1,0} + \mathfrak{m}^{0,1}, \qquad \tau\mathfrak{m}^{1,0} = \mathfrak{m}^{0,1},$$

where τ is the complex conjugation of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\mathfrak{g}$. We denote by J_C the complex structure on $\mathfrak{m}^{\mathbf{C}}$ corresponding to a T-chamber C. Note that $J_C = +\sqrt{-1}\,\mathrm{id}$ on $\mathfrak{m}^{1,0}$, and $J_C = -\sqrt{-1}\,\mathrm{id}$ on $\mathfrak{m}^{0,1}$.

Let C be a T-chamber. Put

$$R_{\mathfrak{m}}^A = \{\alpha \in R_{\mathfrak{m}}^+ \mid \kappa(\alpha) \in R_T^-(C)\}, \qquad R_{\mathfrak{m}}^B = \{\alpha \in R_{\mathfrak{m}}^+ \mid \kappa(\alpha) \in R_T^+(C)\}.$$

Then, $R_{\mathfrak{m}}^A$ and $R_{\mathfrak{m}}^B$ are closed because $\kappa(\alpha) + \kappa(\beta) = \kappa(\alpha + \beta)$ for $\alpha, \beta \in R_{\mathfrak{m}}^+$. Thus we have a direct sum decomposition $\mathfrak{n}^{\mathbf{C}} = \mathfrak{a}^{\mathbf{C}} + \mathfrak{b}^{\mathbf{C}}$, where $\mathfrak{a}^{\mathbf{C}}$ and $\mathfrak{b}^{\mathbf{C}}$ are complex Lie subalgebras defined by

$$\mathfrak{a}^{\mathbf{C}} = \sum_{\alpha \in R_{\mathfrak{m}}^A} \mathfrak{g}_\alpha^{\mathbf{C}}, \qquad \mathfrak{b}^{\mathbf{C}} = \sum_{\alpha \in R_{\mathfrak{m}}^B} \mathfrak{g}_\alpha^{\mathbf{C}}.$$

Thus, we can consider a complex structure $\tilde{J}$ on ${}_{\mathbf{R}}(\mathfrak{n}^{\mathbf{C}})$ defined by

$$\tilde{J} = \begin{cases} -J & \text{on } {}_{\mathbf{R}}(\mathfrak{a}^{\mathbf{C}}) \\ J & \text{on } {}_{\mathbf{R}}(\mathfrak{b}^{\mathbf{C}}). \end{cases}$$

Because $\mathfrak{a}^{\mathbf{C}} = \mathfrak{n}^{\mathbf{C}} \cap \mathfrak{m}^{0,1}$, and $\mathfrak{b}^{\mathbf{C}} = \mathfrak{n}^{\mathbf{C}} \cap \mathfrak{m}^{1,0}$, where $\mathfrak{m}^{\mathbf{C}} = \mathfrak{m}^{1,0} + \mathfrak{m}^{0,1}$ is the decomposition corresponding to J_C, we have the following.

Theorem 6. *Let C be a T-chamber. Then,*

$$\tilde{J} = J_C|_{\mathfrak{n}^{\mathbf{C}}}.$$

Proof. This follows from the fact that, if Z is an eigenvector of J with the eigenvalue $+\sqrt{-1}$, then Z is an eigenvector of $-J$ with the eigenvalue $-\sqrt{-1}$. □

Therefore, we can use classifications of invariant complex structures of a flag manifold for classifications of left-invariant complex structures of a nilpotent Lie group corresponding to the flag manifold.

Proposition 1. *There exists a Weyl chamber C' such that*

$$\mathfrak{a}^{\mathbf{C}} = \sum_{\alpha \in R_{\mathfrak{m}}^{A}} \mathfrak{g}_{\alpha}^{\mathbf{C}} = \sum_{\alpha \in R_{\mathfrak{m}}^{+} \cap R^{-}(C')} \mathfrak{g}_{\alpha}^{\mathbf{C}}, \qquad \mathfrak{b}^{\mathbf{C}} = \sum_{\alpha \in R_{\mathfrak{m}}^{B}} \mathfrak{g}_{\alpha}^{\mathbf{C}} = \sum_{\alpha \in R_{\mathfrak{m}}^{+} \cap R^{+}(C')} \mathfrak{g}_{\alpha}^{\mathbf{C}}.$$

Proof. Put

$$R_{+} = R_{\mathfrak{m}}^{B} \cup [\Pi_0]^{+} \cup (-R_{\mathfrak{m}}^{A}), \qquad R_{-} = -R_{+},$$

where $[\Pi_0]^{+} = [\Pi_0] \cap R^{+}$. Then, we see $R = R_{+} \cup R_{-}$, $R_{+} \cap R_{-} = \varnothing$. Let $\beta \in R_{\mathfrak{m}}^{B}$, $-\alpha \in -R_{\mathfrak{m}}^{A}$, and suppose that $\beta + (-\alpha) \in R$. Since $\kappa(\beta - \alpha) \in R_{T}^{+}(C)$ and $R = R_{+} \cup R_{-}$, we have $\beta - \alpha \in R_{\mathfrak{m}}^{B}$ or $\beta - \alpha \in -R_{\mathfrak{m}}^{A}$. The other cases are trivial. Thus, R_{+} is closed. Since R_{+} is closed, there exists a Weyl chamber C' which satisfies $R_{+} = R^{+}(C')$ and $R_{-} = R^{-}(C')$ ([4]; Chapter 6, Corollary 1 of Proposition 20). Then,

$$R_{\mathfrak{m}}^{A} = R_{\mathfrak{m}}^{+} \cap R_{-} = R_{\mathfrak{m}}^{+} \cap R^{-}(C'), \qquad R_{\mathfrak{m}}^{B} = R_{\mathfrak{m}}^{+} \cap R_{+} = R_{\mathfrak{m}}^{+} \cap R^{+}(C'). \qquad \square$$

For integers $j_1, \dots, j_r$ with $(j_1, \dots, j_r) \neq (0, \dots, 0)$, we put

$$R(j_1, \dots, j_r) = \left\{ \sum_{j=1}^{l} m_j \alpha_j \in R^{+} \;\middle|\; m_{i_1} = j_1, \dots, m_{i_r} = j_r \right\}.$$

Note that

$$R_{\mathfrak{m}}^{+} = R^{+} - [\Pi_0] = \bigcup_{j_1, \dots, j_r} R(j_1, \dots, j_r).$$

We denote $m(j_1, \dots, j_r) = \sharp R(j_1, \dots, j_r)$, where $\sharp R(j_1, \dots, j_r)$ means the number of elements of $R(j_1, \dots, j_r)$.

We denote by ω_{α} ($\alpha \in R$) the complex linear forms on $\mathfrak{g}^{\mathbf{C}}$ dual to the basis vectors E_{α}:

$$\omega_{\alpha}(E_{\beta}) = \delta_{\alpha\beta}, \qquad \omega_{\alpha}(\mathfrak{h}^{\mathbf{C}}) = \{0\}.$$

There exists a natural isomorphism $\mathfrak{t}^* \to H^2(G/K, \mathbb{R})$ given by the formula

$$\mathfrak{t}^* \ni \lambda \to \eta(\lambda) = -\frac{1}{2\pi\sqrt{-1}}\, d\lambda = -\frac{\sqrt{-1}}{2\pi} \sum_{\alpha \in R_{\mathfrak{m}}^+} (\lambda, \alpha)\omega_{-\alpha} \wedge \bar{\omega}_{-\alpha},$$

where we consider λ as a complex linear form on $\mathfrak{g}$ by extending.

Let $\lambda \in \mathfrak{t}^*$, and C the T-chamber corresponding to λ. Then,

$$\begin{aligned}\eta(\lambda) &= -\frac{\sqrt{-1}}{2\pi} \sum_{\alpha \in R_{\mathfrak{m}}^+} (\lambda, \alpha)\omega_{-\alpha} \wedge \bar{\omega}_{-\alpha} \\ &= -\frac{\sqrt{-1}}{2\pi}\left(\sum_{\substack{\alpha \in R_{\mathfrak{m}}^+ \\ \kappa(\alpha) \in R_T^+(C)}} (\lambda, \alpha)\omega_{-\alpha} \wedge \bar{\omega}_{-\alpha} + \sum_{\substack{\alpha \in R_{\mathfrak{m}}^+ \\ \kappa(\alpha) \in R_T^-(C)}} (\lambda, \alpha)\omega_{-\alpha} \wedge \bar{\omega}_{-\alpha} \right).\end{aligned}$$

Note that $(\lambda, \cdot)$ is constant on $R(j_1, \dots, j_r)$. Thus, the signature of $\eta(\lambda)$ can be written as

$$\begin{aligned}&2(\sharp\{\alpha \in R_{\mathfrak{m}}^+ \mid \kappa(\alpha) \in R_T^+(C)\}, \sharp\{\alpha \in R_{\mathfrak{m}}^+ \mid \kappa(\alpha) \in R_T^-(C)\}) \\ &\quad = 2\left(\sum_{\xi \in R_T^+(C_0) \cap R_T^+(C)} \dim_{\mathbb{C}} \mathfrak{m}_\xi, \sum_{\xi \in R_T^+(C_0) \cap R_T^-(C)} \dim_{\mathbb{C}} \mathfrak{m}_\xi \right).\end{aligned}$$

We have the following:

PROPOSITION 2. *Let C be a T-chamber, and $\lambda \in C$. Then, the signature of $\eta(\lambda)$ can be written as*

$$2\left(\sum_{\xi \in R_T^+(C_0) \cap R_T^+(C)} \dim_{\mathbb{C}} \mathfrak{m}_\xi, \sum_{\xi \in R_T^+(C_0) \cap R_T^-(C)} \dim_{\mathbb{C}} \mathfrak{m}_\xi \right) = 2(\dim_{\mathbb{C}} \mathfrak{b}^{\mathbb{C}}, \dim_{\mathbb{C}} \mathfrak{a}^{\mathbb{C}}).$$

COROLLARY 1. *Let C_1, C_2 be T-chambers, and $\lambda_1 \in C_1$, $\lambda_2 \in C_2$. Let $\mathfrak{n}^{\mathbb{C}} = \mathfrak{a}_1^{\mathbb{C}} + \mathfrak{b}_1^{\mathbb{C}}$, $\mathfrak{n}^{\mathbb{C}} = \mathfrak{a}_2^{\mathbb{C}} + \mathfrak{b}_2^{\mathbb{C}}$ be decompositions corresponding to T-chambers C_1 and C_2, respectively. If signatures of $\eta(\lambda_1)$ and $\eta(\lambda_2)$ are different, then there exist no linear mapping $f_* : \mathfrak{n}^{\mathbb{C}} \to \mathfrak{n}^{\mathbb{C}}$ such that $f_*(\mathfrak{a}_1^{\mathbb{C}}) = \mathfrak{a}_2^{\mathbb{C}}$ and $f_*(\mathfrak{b}_1^{\mathbb{C}}) = \mathfrak{b}_2^{\mathbb{C}}$.*

We consider the case where R is of A_l-type. Note that $\kappa(R)$ is also a root system A_r. Let ε be the automorphism of $R = A_l$ that transforms α_i to α_{l+1-i}.

Assume that $\varepsilon(\Pi_0) = \Pi_0$. Then,

$$\varepsilon(R(j_1, \dots, j_r)) = R(j_r, \dots, j_1),$$

which implies $m(j_1, \dots, j_r) = m(j_r, \dots, j_1)$. Thus, we have the following:

Proposition 3. *Suppose that R is of A_l-type. Assume that $\varepsilon(\Pi_0) = \Pi_0$. Let C_1, C_2 be T-chambers, and $\lambda_1 \in C_1$, $\lambda_2 \in C_2$. If $\varepsilon(C_1) = C_2$, then the signatures of $\eta(\lambda_1)$ and $\eta(\lambda_2)$ are equal.*

Proof. By assumption, we have $\sharp R_T^+(C_0) \cap R_T^\pm(C_1) = \sharp R_T^+(C_0) \cap R_T^\pm(C_2)$. Since $\dim_{\mathbb{C}} \mathfrak{m}_\xi = \dim_{\mathbb{C}} \mathfrak{m}_{\varepsilon(\xi)}$, we have our proposition. □

4. The case of root systems A_l, D_l, and E_6

In this section, we consider complex structures on a nilpotent Lie algebra given by the scalar restriction of $T_0^{1,0}M$ of a flag manifold M. We mainly consider cases of root systems A_l, D_l, and E_6. From now on, we take a Chevalley basis $E_\alpha \in \mathfrak{g}_\alpha^{\mathbb{C}}$ $(\alpha \in R)$. Then, structure constants $N_{\alpha,\beta}$ satisfy $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$, and $N_{\alpha,\beta} \in \mathbb{Z}$, where $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$ if $\alpha, \beta, \alpha+\beta \in R$.

Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be a basis of a root system R. Let Π_0 be a subset of Π. Put $\mathfrak{n}_0 = \sum_{\alpha \in R^+} \mathfrak{g}_\alpha$. Let C be the Weyl chamber corresponding to $\{\alpha_1, \dots, \alpha_l\}$. Let C' be a Weyl chamber. Then, we have a decomposition $R = R^+(C') \cup R^-(C')$. Put $R_A = R^+(C) \cap R^-(C')$, $R_B = R^+(C) \cap R^+(C')$. Let

$$\mathfrak{a} = \sum_{\alpha \in R_A} \mathfrak{g}_\alpha, \qquad \mathfrak{b} = \sum_{\alpha \in R_B} \mathfrak{g}_\alpha.$$

Then, we have

Proposition 4. *The sets $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras of $\mathfrak{n}$ which satisfy $\mathfrak{n} = \mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b} = \{\mathbf{0}\}$.*

Proof. We have that R_A and R_B are closed because $R^+(C)$, $R^+(C')$ and $R^-(C')$ are closed. □

Thus, we can consider a complex structure $\tilde{J}$ on ${}_{\mathbb{R}}(\mathfrak{n}^{\mathbb{C}})$ corresponding to the decomposition $\mathfrak{n} = \mathfrak{a} + \mathfrak{b}$.

Put $\mathfrak{a}(\mathfrak{g}^{\mathbb{C}}) = \{\sigma \in \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}}) \mid \sigma(\mathfrak{h}^{\mathbb{C}}) = \mathfrak{h}^{\mathbb{C}}\}$, and $A(R) = \{\hat{\sigma} \in \mathrm{Aut}(\mathfrak{h}^*) \mid \hat{\sigma}(R) = R\}$. Let $\mathfrak{i}(\mathfrak{g}^{\mathbb{C}}) \subset \mathfrak{a}(\mathfrak{g}^{\mathbb{C}})$ be the set of the inner automorphisms, and $W(R)$ the Weyl group of R. Then, the following isomorphism are well-known:

$$\mathfrak{a}(\mathfrak{g}^{\mathbb{C}})/\mathfrak{i}(\mathfrak{g}^{\mathbb{C}}) \cong A(R)/W(R) \cong \{\varphi \in A(R) \mid \varphi(\Pi) = \Pi\}.$$

For root systems A_l, D_l, and E_6, there exists a non-indentity map $\varepsilon \in \{\varphi \in A(R) \mid \varphi(\Pi) = \Pi\}$. Let σ be an element of $\mathfrak{a}(\mathfrak{g}^{\mathbf{C}})$ which induces ε by the above isomorphisms. Then, σ induces an isomorphism $f_* = \sigma|_{\mathfrak{n}_0^{\mathbf{C}}} : \mathfrak{n}_0^{\mathbf{C}} \to \mathfrak{n}_0^{\mathbf{C}}$ because $\varepsilon(\Pi) = \Pi$ implies $\varepsilon(R^+) = R^+$.

Lemma 1. *Let C_1 and C_2 be Weyl chambers. If $\varepsilon(R^+(C_1)) = R^+(C_2)$, then $\varepsilon(R^-(C_1)) = R^-(C_2)$.*

Proof. Since $\varepsilon(R) = R$, we have $\varepsilon(R^+(C_1)) \cup \varepsilon(R^-(C_1)) = R^+(C_2) \cup R^-(C_2)$. □

Let C_1 and C_2 be Weyl chambers. Put

$$\mathfrak{a}_i^{\mathbf{C}} = \sum_{\alpha \in R_{\mathfrak{m}}^+ \cap R^-(C_i)} \mathfrak{g}_\alpha^{\mathbf{C}}, \qquad \mathfrak{b}_i^{\mathbf{C}} = \sum_{\alpha \in R_{\mathfrak{m}}^+ \cap R^+(C_i)} \mathfrak{g}_\alpha^{\mathbf{C}}$$

for $i = 1, 2$. Then, by the above lemma, we have

Corollary 2. *Assume that $\Pi_0 = \emptyset$. Let C_1, C_2 be Weyl chambers, and $\lambda_1 \in C_1$, $\lambda_2 \in C_2$. If $\varepsilon(C_1) = C_2$, then the signatures of $\eta(\lambda_1)$ and $\eta(\lambda_2)$ are equal.*

Lemma 2. *Assume that $\varepsilon(\Pi_0) = \Pi_0$, and $\varepsilon(C_1) = C_2$. Then, $f_*(\mathfrak{a}_1^{\mathbf{C}}) = \mathfrak{a}_2^{\mathbf{C}}$, and $f_*(\mathfrak{b}_1^{\mathbf{C}}) = \mathfrak{b}_2^{\mathbf{C}}$.*

Note that $f_* : \mathfrak{n}^{\mathbf{C}} \to \mathfrak{n}^{\mathbf{C}}$ induces $f_* : {}_{\mathbf{R}}(\mathfrak{n}^{\mathbf{C}}) \to {}_{\mathbf{R}}(\mathfrak{n}^{\mathbf{C}})$. Let $\tilde{J}_{C_i}$ be a complex structure on ${}_{\mathbf{R}}(\mathfrak{n}^{\mathbf{C}})$ corresponding to a decomposition $\mathfrak{n}^{\mathbf{C}} = \mathfrak{a}_i^{\mathbf{C}} + \mathfrak{b}_i^{\mathbf{C}}$ for each $i = 1, 2$. Then, we have the following:

Proposition 5. *Assume that $\varepsilon(\Pi_0) = \Pi_0$, and $\varepsilon(C_1) = C_2$. Then, $f_* : ({}_{\mathbf{R}}(\mathfrak{n}^{\mathbf{C}}), \tilde{J}_{C_1}) \to ({}_{\mathbf{R}}(\mathfrak{n}^{\mathbf{C}}), \tilde{J}_{C_2})$ satisfies $f_* \circ \tilde{J}_{C_1} = \tilde{J}_{C_2} \circ f_*$.*

Proof. Since $\tilde{J}_{C_i} = -J$ on ${}_{\mathbf{R}}(\mathfrak{a}_i^{\mathbf{C}})$, and $\tilde{J}_{C_i} = J$ on ${}_{\mathbf{R}}(\mathfrak{b}_i^{\mathbf{C}})$ for $i = 1, 2$, we have our proposition by Lemma 2. □

Now, we consider the case of a root system A_l. Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be a basis of $R = A_l$ with natural manner. Let us consider a map $f_* : \mathfrak{n}_0 \to \mathfrak{n}_0$ defined by

$$E_{\alpha_i + \cdots + \alpha_j} \mapsto (-1)^{j-i} E_{\varepsilon(\alpha_i + \cdots + \alpha_j)}$$

for each $1 \le i < j \le l$, where $\varepsilon \in A(R)$ satisfies $\varepsilon(\alpha_i) = \alpha_{l-i+1}$ for $i = 1, \dots, l$.

Lemma 3. *The map f_* is an isomorphism of $\mathfrak{n}_0$.*

Proof. It is obvious that f_* is bijective by definition. Recall that

$$N_{\alpha_i+\cdots+\alpha_j,\,\alpha_k+\cdots+\alpha_h} = \begin{cases} 1 & k = j+1 \\ 0 & \text{otherwise,} \end{cases}$$

where $i \le j$, $k \le h$, and $i \le k$. Thus, we consider indices of top terms and last terms of roots for investigating $N_{\varepsilon(\alpha_i+\cdots+\alpha_j),\,\varepsilon(\alpha_{j+1}+\cdots+\alpha_k)}$. Note that $\varepsilon(\alpha_i) = \alpha_{l-i+1}$. Then, for $i \le j$, $j+1 \le k \le l$, we have

$$l-k+1 \le l-(j+1)+1 = l-j < l-j+1 \le l-i+1.$$

Thus, we have

$$N_{\varepsilon(\alpha_i+\cdots+\alpha_j),\,\varepsilon(\alpha_{j+1}+\cdots+\alpha_k)} = -N_{\alpha_i+\cdots+\alpha_j,\,\alpha_{j+1}+\cdots+\alpha_k}.$$

Moreover,

$$\begin{aligned}
&[(-1)^{j-i}E_{\varepsilon(\alpha_i+\cdots+\alpha_j)}, (-1)^{k-j-1}E_{\varepsilon(\alpha_{j+1}+\cdots+\alpha_k)}] \\
&\quad = (-1)^{k-i-1}N_{\varepsilon(\alpha_i+\cdots+\alpha_j),\,\varepsilon(\alpha_{j+1}+\cdots+\alpha_k)}E_{\varepsilon(\alpha_i+\cdots+\alpha_k)} \\
&\quad = (-1)^{k-i-1}(-N_{\alpha_i+\cdots+\alpha_j,\,\alpha_{j+1}+\cdots+\alpha_k})E_{\varepsilon(\alpha_i+\cdots+\alpha_k)} \\
&\quad = (-1)^{k-i}N_{\alpha_i+\cdots+\alpha_j,\,\alpha_{j+1}+\cdots+\alpha_k}E_{\varepsilon(\alpha_i+\cdots+\alpha_k)}.
\end{aligned}$$

The other cases are trivial. Thus, f_* is an isomorphism. □

Example 1. *Let $l = 2$, i.e., $\Pi = \{\alpha_1, \alpha_2\}$, and $\Pi_0 = \emptyset$. Let $C_1 = \{-\alpha_1 > 0, \alpha_1 + \alpha_2 > 0\}$, and $C_2 = \{-\alpha_2 > 0, \alpha_1 + \alpha_2 > 0\}$. Then, we see*

$$f_*(E_{\alpha_1}) = E_{\alpha_2}, \qquad f_*(E_{\alpha_2}) = E_{\alpha_1}, \qquad f_*(E_{\alpha_1+\alpha_2}) = -E_{\alpha_1+\alpha_2}.$$

Then, nilpotent Lie groups with complex structures $\tilde{J}_{C_1}$ and $\tilde{J}_{C_2}$ corresponding to C_1 and C_2 are

$$N_1 = \left\{ \begin{pmatrix} 1 & \bar{z}_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, z_i \in \mathbb{C} \right\}, \qquad N_2 = \left\{ \begin{pmatrix} 1 & w_1 & w_3 + w_1\overline{w}_2 \\ 0 & 1 & \overline{w}_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, w_i \in \mathbb{C} \right\},$$

respectively. We have that a holomorphic and homomorphic map $f : N_1 \to N_2$ is given by

$$w_1(z_1, z_2, z_3) = z_2, \qquad w_2(z_1, z_2, z_3) = z_1, \qquad w_3(z_1, z_2, z_3) = -z_3.$$

Indeed,

$$\begin{pmatrix} 1 & \bar{z}_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} = \exp(z_2 E_{\alpha_2} + z_3 E_{\alpha_1+\alpha_2}) \exp(\bar{z}_1 E_{\alpha_1})$$

$$\xrightarrow{f} \exp(z_2 E_{\alpha_1} - z_3 E_{\alpha_1+\alpha_2}) \exp(\bar{z}_1 E_{\alpha_2}) = \begin{pmatrix} 1 & z_2 & -z_3 + \bar{z}_1 z_2 \\ 0 & 1 & \bar{z}_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

REMARK 1. *Except for A_l, D_l and E_6, if $\sigma \in \mathfrak{a}(\mathfrak{g}^{\mathbf{C}})$ induces a homomorphism $\sigma : \mathfrak{n}^{\mathbf{C}} \to \mathfrak{n}^{\mathbf{C}}$, then $\sigma(\mathfrak{g}_\alpha^{\mathbf{C}}) = \mathfrak{g}_\alpha^{\mathbf{C}}$ for each $\alpha \in R$ because $\hat{\sigma}(R^+) = R^+$, where $\hat{\sigma}(\alpha)(H) = \alpha(\sigma^{-1}(H))$. Conversely, except for A_l, D_l and E_6, if $\varphi \in \{\varphi \in \mathrm{Aut}(\mathfrak{h}^*) \mid \varphi(R) = R\}$ satisfies $\varphi(R^+) = R^+$, then $\sigma \in \mathfrak{a}(\mathfrak{g}^{\mathbf{C}})$ such that $\hat{\sigma} = \varphi$ satisfies $\sigma(\mathfrak{g}_\alpha^{\mathbf{C}}) = \mathfrak{g}_\alpha^{\mathbf{C}}$ for each $\alpha \in R$.*

5. Construction of nilpotent Lie algebras with a decomposition

In this section, we construct nilpotent Lie algebras $\mathfrak{n}$ with a decomposition $\mathfrak{n} = \mathfrak{a} + \mathfrak{b}$ by root systems, and see a relation between Weyl chambers and decompositions.

Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be a basis of root system R. Let Π_0, Π_1 be subsets of Π. Put $\mathfrak{n} = \sum_{\alpha \in R_{\mathfrak{m}}^+} \mathfrak{g}_\alpha$. Then we have the following results:

PROPOSITION 6. *Let Q be a subset of R^+ such that Q and $R^+ \backslash Q$ are closed. Put*

$$\mathfrak{a} = \sum_{\alpha \in R_{\mathfrak{m}}^+ \cap Q} \mathfrak{g}_\alpha, \qquad \mathfrak{b} = \sum_{\alpha \in R_{\mathfrak{m}}^+ - Q} \mathfrak{g}_\alpha.$$

Then, $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras of $\mathfrak{n}$ which satisfy $\mathfrak{n} = \mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b} = \{\mathbf{0}\}$. Moreover, there exists a Weyl chamber C' such that $\mathfrak{a} = \sum_{\alpha \in R_A} \mathfrak{g}_\alpha$, $\mathfrak{b} = \sum_{\alpha \in R_B} \mathfrak{g}_\alpha$, where $R_A = R^+(C) \cap R^-(C')$, $R_B = R^+(C) \cap R^+(C')$. Conversely, let C' be a Weyl chamber. Then, $Q = R^+ \cap R^-(C')$ is a subset of R^+ such that Q and $R^+ \backslash Q$ are closed.

PROOF. Because Q and $R^+ \backslash Q$ are closed, $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras of $\mathfrak{n}$ which satisfy $\mathfrak{n} = \mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b} = \{\mathbf{0}\}$. Put $Q_1 = Q$ and $Q_2 = R^+ \backslash Q$. Moreover, put $R_+ = Q_1 \cup (-Q_2)$ and $R_- = Q_2 \cup (-Q_1)$. Since $R = R_+ \cup R_-$ and $R_+ \cap R_- = \varnothing$, we only have to prove that R_+ is closed to prove an existence of a corresponding Weyl chamber ([4]; Chapter 6, Corollary 1 of Proposition 20).

Suppose that $\alpha, \beta \in R_+$, and $\alpha + \beta \in R$. If $\alpha, \beta \in Q_1$, or $\alpha, \beta \in Q_2$, because Q_1, Q_2 are closed, then $\alpha + \beta \in Q_1$, or $\alpha + \beta \in Q_2$. Thus, suppose that $\alpha \in Q_1$, $\beta \in -Q_2$. Since $R = R_+ \cup R_-$, we see that $\alpha + \beta \in R_+$ or $\alpha + \beta \in R_-$. We shall prove $\alpha + \beta \notin R_-$ by contradiction. Suppose that $\alpha + \beta \in R_-$. Then, since $R_- = Q_2 \cup (-Q_1)$, we see that $\alpha + \beta \in Q_2$ or $\alpha + \beta \in -Q_1$. If $\alpha + \beta \in -Q_1$, then

$$-Q_1 \ni -\alpha + (\alpha + \beta) = \beta \in -Q_2,$$

which is contrary to $(-Q_1) \cap (-Q_2) = \emptyset$. So, $\alpha + \beta \in Q_2$. However, then,

$$-Q_1 \ni -\alpha = -(\alpha + \beta) + \beta \in -Q_2,$$

which is contrary to $(-Q_1) \cap (-Q_2) = \emptyset$. Hence, we have $\alpha + \beta \notin R_-$. $\square$

Similarly, we have the following:

Proposition 7. *Let Q_T be a subset of $R_T^+ \subset R_T = \kappa(R)$ such that Q_T and $R_T^+ \backslash Q_T$ are closed. Put*

$$\mathfrak{a} = \sum_{\alpha \in R_{\mathfrak{m}}^+, \kappa(\alpha) \in Q_T} \mathfrak{g}_\alpha, \qquad \mathfrak{b} = \sum_{\alpha \in R_{\mathfrak{m}}^+, \kappa(\alpha) \in R_T^+ \backslash Q_T} \mathfrak{g}_\alpha.$$

Then, $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras of $\mathfrak{n}$ which satisfy $\mathfrak{n} = \mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b} = \{\mathbf{0}\}$.

Let

$$Q(\alpha_t) = \left\{ \sum_{k=1}^{t} a_k \alpha_k \in R^+ \,\middle|\, a_t \neq 0 \right\},$$

and

$$Q(t_1, \dots, t_k) = \bigcup_{t \in \{t_1, \dots, t_k\}} Q(\alpha_t)$$

for $1 \leq t_1 < \cdots < t_k \leq l$.

Corollary 3. *Let $Q = Q(t_1, \dots, t_k) = \bigcup_{t \in \{t_1, \dots, t_k\}} Q(\alpha_t)$ for $1 \leq t_1 < \cdots < t_k \leq l$. Then, $\mathfrak{a} = \sum_{\alpha \in R_{\mathfrak{m}}^+ \cap Q} \mathfrak{g}_\alpha$ and $\mathfrak{b} = \sum_{\alpha \in R_{\mathfrak{m}}^+ - Q} \mathfrak{g}_\alpha$ are subalgebras of $\mathfrak{n}$ which satisfy $\mathfrak{n} = \mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b} = \{\mathbf{0}\}$.*

Proof. For $\xi = \sum_{k=1}^{t} a_k \alpha_k \in R^+$ such that $a_t \neq 0$, we say that $a_t \alpha_t$ is the last term of ξ. Let $\xi \in Q(t_1)$, $\eta \in Q(t_2)$, where $t_1 \leq t_2$. If $\xi + \eta \in R$, then by considering the last term of $\xi + \eta$ we have $\xi + \eta \in Q(t_2)$. Thus, Q is closed. Similarly, we have that $R^+ \backslash Q$ is closed. $\square$

Corollary 4 ([11]). *Put*

$$\mathfrak{a}(\Pi_0, \Pi_1) = \sum_{R_{\mathfrak{m}}^+ \cap [\Pi_1]} \mathfrak{g}_\alpha, \qquad \mathfrak{b}(\Pi_0, \Pi_1) = \sum_{R_{\mathfrak{m}}^+ - [\Pi_1]} \mathfrak{g}_\alpha.$$

Then, $\mathfrak{a}$ is a subalgebra of $\mathfrak{n}$, and $\mathfrak{b}$ is an ideal of $\mathfrak{n}$.

Example 2. *We consider the case where R is of A_7-type, and $\Pi_0 = \{\alpha_1, \alpha_3, \alpha_5, \alpha_7\}$. Then, R_T is of A_3-type. Thus, $R_T^+ = \{\sum_{h=i}^{j} \xi_h \mid 1 \le i \le j \le 3\}$. Let $Q_T = \{\xi_2, \xi_1 + \xi_2\} \subset R_T^+$. Then, we have the following nilpotent Lie group with a complex structure:*

$$\left\{ \left. \begin{pmatrix} 1 & 0 & z_{13} & z_{14} & \bar{z}_{15} & \bar{z}_{16} & z_{17} & z_{18} \\ 0 & 1 & z_{23} & z_{24} & \bar{z}_{25} & \bar{z}_{26} & z_{27} & z_{28} \\ 0 & 0 & 1 & 0 & \bar{z}_{35} & \bar{z}_{36} & z_{37} & z_{38} \\ 0 & 0 & 0 & 1 & \bar{z}_{45} & \bar{z}_{46} & z_{47} & z_{48} \\ 0 & 0 & 0 & 0 & 1 & 0 & z_{57} & z_{58} \\ 0 & 0 & 0 & 0 & 0 & 1 & z_{67} & z_{68} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| z_{ij} \in \mathbf{C} \right\}.$$

Example 3. *Let $\{\alpha_1, \dots, \alpha_l\}$ be a basis of the root system R of A_l-type. Then,*

$$Q(t_1, \dots, t_k) = \left\{ \left. \sum_{h=i}^{t_j} \alpha_h \right| 1 \le i \le t_j,\ j = 1, \dots, k \right\},$$

for $1 \le t_1 < \cdots < t_k \le l$.

We now consider the case where R is of A_3-type. Let (N, J_0) be the complex nilpotent Lie group defined by

$$(N, J_0) = \left\{ \left. \begin{pmatrix} 1 & z_{12} & z_{13} & z_{14} \\ 0 & 1 & z_{23} & z_{24} \\ 0 & 0 & 1 & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| z_{ij} \in \mathbf{C} \right\}$$

and

$$\Gamma = \left\{ \left. \begin{pmatrix} 1 & \mu_{12} & \mu_{13} & \mu_{14} \\ 0 & 1 & \mu_{23} & \mu_{24} \\ 0 & 0 & 1 & \mu_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| \mu_{ij} \in \mathbf{Z}[\sqrt{-1}] \right\}.$$

Then, $\Gamma \backslash N$ is a compact complex nilmanifold. We define inductively $C^k(N)$ by $C^0(N) = N$, $C^k(N) = [N, C^{k-1}(N)]$. Then, the complex nilmanifold $\Gamma \backslash N$ is the top of a tower of holomorphic bundles with complex torus fibers $\Gamma \backslash N \to C^2(N)\Gamma \backslash N \to C^1(N)\Gamma \backslash N = T_{\mathbf{C}}^3$, where the fibers of $\Gamma \backslash N \to C^2(N)\Gamma \backslash N$ and $C^2(N)\Gamma \backslash N \to C^1(N)\Gamma \backslash N$ are complex tori $T_{\mathbf{C}}^1$ and $T_{\mathbf{C}}^2$, respectively.

We now consider N as a real Lie group. Let $\mathscr{C}(\mathfrak{n})$ be the set of all integrable complex structures on the Lie algebra $\mathfrak{n}$ of N, i.e.,

$$\mathscr{C}(\mathfrak{n}) = \{J : \mathfrak{n} \to \mathfrak{n} \mid J^2 = -1, [JX, JY] = [X, Y] + J[JX, Y] + J[X, JY]\}.$$

By the above argument, we can consider the following left-invariant complex structures on the real Lie group N which lie in distinct connected components of the set $\mathscr{C}(\mathfrak{n})$*:*

$$\left\{\begin{pmatrix}1 & z_{12} & z_{13} & z_{14}\\ 0 & 1 & z_{23} & z_{24}\\ 0 & 0 & 1 & z_{34}\\ 0 & 0 & 0 & 1\end{pmatrix}\middle|\, z_{ij}\in\mathbb{C}\right\},$$

$$\left\{\begin{pmatrix}1 & \bar z_{12} & z_{13} & z_{14}\\ 0 & 1 & z_{23} & z_{24}\\ 0 & 0 & 1 & z_{34}\\ 0 & 0 & 0 & 1\end{pmatrix}\right\},\left\{\begin{pmatrix}1 & z_{12} & \bar z_{13} & z_{14}\\ 0 & 1 & \bar z_{23} & z_{24}\\ 0 & 0 & 1 & z_{34}\\ 0 & 0 & 0 & 1\end{pmatrix}\right\},\left\{\begin{pmatrix}1 & z_{12} & z_{13} & \bar z_{14}\\ 0 & 1 & z_{23} & \bar z_{24}\\ 0 & 0 & 1 & \bar z_{34}\\ 0 & 0 & 0 & 1\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1 & \bar z_{12} & \bar z_{13} & z_{14}\\ 0 & 1 & \bar z_{23} & z_{24}\\ 0 & 0 & 1 & z_{34}\\ 0 & 0 & 0 & 1\end{pmatrix}\right\},\left\{\begin{pmatrix}1 & \bar z_{12} & z_{13} & \bar z_{14}\\ 0 & 1 & z_{23} & \bar z_{24}\\ 0 & 0 & 1 & \bar z_{34}\\ 0 & 0 & 0 & 1\end{pmatrix}\right\},\left\{\begin{pmatrix}1 & z_{12} & \bar z_{13} & \bar z_{14}\\ 0 & 1 & \bar z_{23} & \bar z_{24}\\ 0 & 0 & 1 & \bar z_{34}\\ 0 & 0 & 0 & 1\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1 & \bar z_{12} & \bar z_{13} & \bar z_{14}\\ 0 & 1 & \bar z_{23} & \bar z_{24}\\ 0 & 0 & 1 & \bar z_{34}\\ 0 & 0 & 0 & 1\end{pmatrix}\right\}.$$

Indeed, consider $T^1_{\mathbb{C}}$ *and* $T^2_{\mathbb{C}}$ *and the base space* $T^3_{\mathbb{C}}$ *as real manifolds. Let us consider the orientations induced on real tori* $T^1_{\mathbb{C}}$, $T^2_{\mathbb{C}}$ *and* $T^3_{\mathbb{C}}$ *of the tower by the above invariant complex structures on* $\Gamma\backslash N$*. Note that* $\Gamma C^k(N) = C^k(N)\Gamma$*. Moreover, note that elements of* N *and* $C^1(N)$ *can be written by elements of* $C^1(N)$, $C^2(N)$*, and elements of special-type as follows:*

$$N \ni \begin{pmatrix}1 & z_{12} & z_{13} & z_{14}\\ 0 & 1 & z_{23} & z_{24}\\ 0 & 0 & 1 & z_{34}\\ 0 & 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & z_{13} & z_{14}-z_{13}z_{34}\\ 0 & 1 & 0 & z_{24}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & z_{12} & 0 & 0\\ 0 & 1 & z_{23} & 0\\ 0 & 0 & 1 & z_{34}\\ 0 & 0 & 0 & 1\end{pmatrix}$$

and

$$C^1(N) \ni \begin{pmatrix}1 & 0 & z_{13} & z_{14}\\ 0 & 1 & 0 & z_{24}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & z_{14}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & z_{13} & 0\\ 0 & 1 & 0 & z_{24}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Thus, we have that the above invariant complex structures lie in distinct connected components of the set $\mathscr{C}(\mathfrak{n})$.

6. Hodge numbers

In this section, we recall relations between the decomposition $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ and $h^{s,t}(\mathfrak{g}_{\tilde{J}}) = \dim H_{\bar{\partial}}^{s,t}(\mathfrak{g}_{\tilde{J}})$ in the previous paper [12].

Let $\mathfrak{g}$ be a real Lie algebra with a direct decomposition

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{b},$$

where $\mathfrak{a}$ and $\mathfrak{b}$ are Lie subalgebras of $\mathfrak{g}$. Take bases of the Lie subalgebras $\mathfrak{a}$ and $\mathfrak{b}$:

$$\mathfrak{a} = \mathrm{span}_{\mathbb{R}}\{U_1, \dots, U_p\},$$
$$\mathfrak{b} = \mathrm{span}_{\mathbb{R}}\{V_1, \dots, V_q\}.$$

Let $\{\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q\}$ be the dual basis of $\{U_1, \dots, U_p, V_1, \dots, V_q\}$. We can assume that

$$d\alpha_i = -\sum_{k,h} C_{kh}^i \alpha_k \wedge \alpha_h - \sum_{k,s} D_{ks}^i \alpha_k \wedge \beta_s, \qquad d\beta_j = -\sum_{s,t} E_{st}^j \beta_s \wedge \beta_t - \sum_{k,s} F_{ks}^j \alpha_k \wedge \beta_s$$

for each i, j, where $C_{kh}^i, D_{ks}^i, E_{st}^j, F_{ks}^j \in \mathbb{R}$.

Let $\mathfrak{g}_{\mathfrak{a}}$, $\mathfrak{g}_{\mathfrak{b}}$ be real Lie algebras such that $\mathfrak{g}_{\mathfrak{a}}^* = \mathrm{span}\{\mu_1^0, \dots, \mu_p^0, \nu_1^0, \dots, \nu_q^0\}$ and $\mathfrak{g}_{\mathfrak{b}}^* = \mathrm{span}\{\mu_1^1, \dots, \mu_p^1, \nu_1^1, \dots, \nu_q^1\}$ have the structure equations

$$d\mu_i^0 = -\sum_{k,s} D_{ks}^i \mu_k^0 \wedge \nu_s^0, \qquad d\nu_j^0 = -\sum_{s,t} E_{st}^j \nu_s^0 \wedge \nu_t^0, \tag{1}$$

$$d\mu_i^1 = -\sum_{k,h} C_{kh}^i \mu_k^1 \wedge \mu_h^1, \qquad d\nu_j^1 = -\sum_{k,s} F_{ks}^j \mu_k^1 \wedge \nu_s^1, \tag{2}$$

respectively. Since $\bar{\partial}^2 = 0$ on $\bigwedge_{\tilde{J}}^{*,*}(\mathfrak{g}^{\mathbb{C}})^*$, we have that $d^2 = 0$ on $\bigwedge^1 \mathfrak{g}_{\mathfrak{a}}^*$ and $\bigwedge^1 \mathfrak{g}_{\mathfrak{b}}^*$, which implies that $\mathfrak{g}_{\mathfrak{a}}$, $\mathfrak{g}_{\mathfrak{b}}$ are Lie algebras (cf. [12]). Then we have

THEOREM 7 ([12]). *For each r,*

$$\sum_{s+t=r} h^{s,t}(\mathfrak{g}_{\tilde{J}}) = \dim H^r(\mathfrak{g}_{\mathfrak{a}} \times \mathfrak{g}_{\mathfrak{b}}).$$

7. Symmetry of summations

In this section, we prove a property of $\sum_{s+t=r} h^{s,t}(\mathfrak{n}_{\tilde{J}})$ for the case of a root system A_l.

Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be a basis of the root system R of A_l-type, and Π_0 a subset of Π. Let

$$\Pi_{1,k} = \{\alpha_1, \dots, \alpha_k\}, \qquad \Pi_{2,k} = \{\alpha_k, \dots, \alpha_l\}.$$

Then, by the equations (1), (2), we have

$$\mathfrak{g}_{\mathfrak{a}(\Pi_0,\Pi_{1,k-1})} = \sum_{\alpha \in R_{\mathfrak{m}}^+ - [\Pi_{1,k-1}]} \mathfrak{g}_\alpha \times \mathbb{R}^{\sharp(R_{\mathfrak{m}} \cap [\Pi_{1,k-1}])},$$

$$\mathfrak{g}_{\mathfrak{b}(\Pi_0,\Pi_{1,k-1})} = \sum_{\alpha \in R_{\mathfrak{m}}^+ - [\Pi_{2,k+1}]} \mathfrak{g}_\alpha \times \mathbb{R}^{\sharp(R_{\mathfrak{m}} \cap [\Pi_{2,k+1}])},$$

$$\mathfrak{g}_{\mathfrak{a}(\Pi_0,\Pi_{2,k+1})} = \sum_{\alpha \in R_{\mathfrak{m}}^+ - [\Pi_{2,k+1}]} \mathfrak{g}_\alpha \times \mathbb{R}^{\sharp(R_{\mathfrak{m}} \cap [\Pi_{2,k+1}])},$$

$$\mathfrak{g}_{\mathfrak{b}(\Pi_0,\Pi_{2,k+1})} = \sum_{\alpha \in R_{\mathfrak{m}}^+ - [\Pi_{1,k-1}]} \mathfrak{g}_\alpha \times \mathbb{R}^{\sharp(R_{\mathfrak{m}} \cap [\Pi_{1,k-1}])}.$$

Indeed, let us consider second equations $dv_j^1 = -\sum_{k,s} F_{ks}^j \mu_k^1 \wedge v_s^1$ of (2). Let $i \le k \le j$. Then, we see

$$\sum_{s=i}^{j} \alpha_s \in (R^+ \backslash [\Pi_{1,k-1}]) \cap (R^+ \backslash [\Pi_{2,k+1}]).$$

Consider elements

$$\sum_{s=m}^{i-1} \alpha_s \in [\Pi_{1,k-1}] \cap (R^+ \backslash [\Pi_{2,k+1}]),$$

where $m \le i-1$. Then, we have

$$\sum_{s=m}^{i-1} \alpha_s + \sum_{s=i}^{j} \alpha_s = \sum_{s=m}^{j} \alpha_s \in (R^+ \backslash [\Pi_{1,k-1}]) \cap (R^+ \backslash [\Pi_{2,k+1}]).$$

Then, by the equation (2), we have

$$\mathfrak{g}_{\mathfrak{b}(\Pi_0,\Pi_{1,k-1})} = \sum_{\alpha \in R_{\mathfrak{m}}^+ - [\Pi_{2,k+1}]} \mathfrak{g}_\alpha.$$

Similarly, we have

$$\mathfrak{g}_{\mathfrak{b}(\Pi_0,\Pi_{2,k+1})} = \sum_{\alpha \in R_{\mathfrak{m}}^+ - [\Pi_{1,k-1}]} \mathfrak{g}_\alpha \times \mathbb{R}^{\sharp(R_{\mathfrak{m}} \cap [\Pi_{1,k-1}])}.$$

Thus, we have

$$\mathfrak{g}_{\mathfrak{a}(\Pi_0,\Pi_{1,k-1})} = \mathfrak{g}_{\mathfrak{b}(\Pi_0,\Pi_{2,k+1})}, \mathfrak{g}_{\mathfrak{b}(\Pi_0,\Pi_{1,k-1})} = \mathfrak{g}_{\mathfrak{a}(\Pi_0,\Pi_{2,k+1})}.$$

Hence, we have

$$\mathfrak{g}_{\mathfrak{a}(\Pi_0,\Pi_{1,k-1})} \times \mathfrak{g}_{\mathfrak{b}(\Pi_0,\Pi_{1,k-1})} \cong \mathfrak{g}_{\mathfrak{a}(\Pi_0,\Pi_{2,k+1})} \times \mathfrak{g}_{\mathfrak{b}(\Pi_0,\Pi_{2,k+1})}.$$

Thus, we have

THEOREM 8. *Let $\tilde{J}_{1,k}$, $\tilde{J}_{2,k}$ be complex structures on ${}_{\mathbb{R}}(\mathfrak{n}^{\mathbb{C}})$ corresponding to decompositions induced by $\Pi_{1,k}$, $\Pi_{2,k}$, respectively. Then,*

$$\sum_{s+t=r} h^{s,t}(\mathfrak{n}_{\tilde{J}_{1,k-1}}) = \sum_{s+t=r} h^{s,t}(\mathfrak{n}_{\tilde{J}_{2,k+1}})$$

for each k and r.

As an application, we have the following:

PROPOSITION 8 ([11]). *Let Π_0 be a subset of Π such that $\varepsilon(\Pi_0) = \Pi_0$. Let $\tilde{J}_k$ be complex structures on ${}_{\mathbb{R}}(\mathfrak{n}^{\mathbb{C}})$ corresponding to decompositions induced by $\Pi_{1,k}$ for each $k = 0, \dots, l$. Then,*

$$\sum_{s+t=r} h^{s,t}(\mathfrak{n}_{\tilde{J}_k}) = \sum_{s+t=r} h^{s,t}(\mathfrak{n}_{\tilde{J}_{l-k-1}})$$

for each k and r.

PROOF. Since $\varepsilon(\Pi_0) = \Pi_0$, we have

$$\begin{aligned}\sum_{s+t=r} h^{s,t}(\mathfrak{n}_{\tilde{J}_k}) &= \sum_{s+t=r} h^{s,t}(\mathfrak{n}_{\tilde{J}_{1,k}}) = \sum_{s+t=r} h^{s,t}(\mathfrak{n}_{\tilde{J}_{2,k+2}}) \\ &= \sum_{s+t=r} h^{s,t}(\mathfrak{n}_{\tilde{J}_{1,l-k-1}}) = \sum_{s+t=r} h^{s,t}(\mathfrak{n}_{\tilde{J}_{l-k-1}})\end{aligned}$$

by considering a holomorphic isomorphism $f_* : ({}_{\mathbb{R}}(\mathfrak{n}^{\mathbb{C}}), \tilde{J}_{2,k+2}) \to ({}_{\mathbb{R}}(\mathfrak{n}^{\mathbb{C}}), \tilde{J}_{1,l-k-1})$ (see Proposition 5; note that $\varepsilon(\alpha_{k+2}) = \alpha_{l-k-1}$ and $\varepsilon(\alpha_l) = \alpha_1$). □

EXAMPLE 4. *Let $\Pi = \{\alpha_1, \dots, \alpha_6\}$, and $\Pi_0 = \{\alpha_4\}$. Then,* ${}_{\mathbb{R}}(\mathfrak{n}^{\mathbb{C}}) = \mathrm{span}\{E_\alpha, F_\alpha\}_{\alpha \in R_{\mathfrak{m}}^+}$ *is a real 40-dimensional nilpotent Lie algebra with the structure equations $[E_\alpha, E_\beta] = E_{\alpha+\beta}$, $[E_\alpha, F_\beta] = [F_\alpha, E_\beta] = F_{\alpha+\beta}$, $[F_\alpha, F_\beta] = -E_{\alpha+\beta}$ for $\alpha, \beta, \alpha+\beta \in R_{\mathfrak{m}}^+$, where $\alpha = \sum_{h=i_\alpha}^{j_\alpha} \alpha_h$ and $\beta = \sum_{h=i_\beta}^{j_\beta} \alpha_h$ with $i_\alpha < i_\beta$. Then, in the case of $k = 4$, $\tilde{J}_{1,3}$ satisfies $\tilde{J}_{1,3}E_\alpha = -F_\alpha$ ($\tilde{J}_{1,3}F_\alpha = E_\alpha$) for $\alpha \in \{\alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}$ and otherwise $\tilde{J}_{1,3}E_\alpha = F_\alpha$ ($\tilde{J}_{1,3}F_\alpha = -E_\alpha$). On the other hand, $\tilde{J}_{2,5}$ satisfies $\tilde{J}_{2,5}E_\alpha = -F_\alpha$ for $\alpha \in \{\alpha_5, \alpha_6, \alpha_5+\alpha_6\}$ and otherwise $\tilde{J}_{2,5}E_\alpha = F_\alpha$. Then,*

$$\sum_{s+t=r} h^{s,t}(\mathfrak{n}_{\tilde{J}_{1,3}}) = \sum_{s+t=r} h^{s,t}(\mathfrak{n}_{\tilde{J}_{2,5}}).$$

REMARK 2. *In general, $\mathfrak{g}_{\mathfrak{a}}$ and $\mathfrak{g}_{\mathfrak{b}}$ have different type. For example, let us consider the case of B_3. Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a basis of the root system R of B_3-type with natural manner. Let $\Pi_0 = \emptyset$, and $\Pi_1 = \{\alpha_1\}$. Then, we have*

$$\mathfrak{g}_{\mathfrak{a}} = \sum_{\alpha \in R^+ - [\Pi_1]} \mathfrak{g}_\alpha \times \mathbb{R}^1, \qquad \mathfrak{g}_{\mathfrak{b}} = \sum_{\alpha \in R^+_{A_3} - \{\alpha_3\}} \mathfrak{g}_\alpha \times \mathbb{R}^4,$$

where $R^+_{A_3} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$, and we define $[E_\alpha, E_\beta] = 0$ for $\alpha, \beta \in R^+_{A_3}$ such that $\alpha + \beta \notin R^+_{A_3}$.

EXAMPLE 5. *Let $\Pi_0 = \{\alpha_2, \dots, \alpha_{l-1}\}$. Then, $\varepsilon(\Pi_0) = \Pi_0$, and $\mathfrak{n}^{\mathbb{C}}$ is a complex $(2l-1)$-dimensional Heisenberg algebra. Then, $\sum_{s+t=r} h^{s,t}(\mathfrak{n}_{\tilde{J}_k}) = \sum_{s+t=r} h^{s,t}(\mathfrak{n}_{\tilde{J}_{l-k-1}})$ for each k and r.*

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