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# Boundedness of maximal operator, Hardy operator and Sobolev's inequalities on non-homogeneous central Herz-Morrey-Musielak-Orlicz spaces

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**Abstract.** Our aim in this paper is to deal with the boundedness of the Hardy-Littlewood maximal operator and the Hardy operator on non-homogeneous central Herz-Morrey-Musielak-Orlicz spaces and to establish a generalization of Sobolev's inequalities for Riesz potentials of functions in such spaces.

## 1. Introduction

Let $\mathbf{R}^N$ be the Euclidean space and let $B(x,r)$ denote the open ball centered at $x \in \mathbf{R}^N$ with radius $r > 0$.

In harmonic analysis, the maximal operator is a classical tool when studying Sobolev functions and partial differential equations. This also plays a central role in the study of differentiation, singular integrals, smoothness of functions and so on (see [8, 28, 50], etc.). It is well known that the maximal operator is bounded on the Lebesgue space $L^p(\mathbf{R}^N)$ if $p > 1$ (see [50]). The boundedness of the maximal operator was studied on Morrey spaces in [11, 42], on Orlicz-Morrey spaces in [44], and also on non-homogeneous Herz spaces in [29]. For Morrey spaces, which were introduced to estimate solutions of partial differential equations, we refer to [40, 46].

One of the important applications of the boundedness of the maximal operator is Sobolev's inequality; in classical Lebesgue spaces, we know Sobolev's inequality:

$$\|I_\alpha f\|_{L^{p^*}(\mathbf{R}^N)} \le C\|f\|_{L^p(\mathbf{R}^N)}$$

for $f \in L^p(\mathbf{R}^N)$, $0 < \alpha < N$ and $1 < p < N/\alpha$, where $I_\alpha$ is the Riesz kernel of order $\alpha$ and $1/p^* = 1/p - \alpha/N$ (see, e.g. [2, Theorem 3.1.4]). Sobolev's inequal-

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ity for Morrey spaces was given by D. R. Adams [1] (also [11, 42]), and then the result was extended to Orlicz-Morrey spaces in [43]. See also [29] for non-homogeneous Herz spaces and [20] for non-homogeneous central Morrey spaces. For local Morrey-type spaces, we refer the reader to [9, 10] and so on.

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth condition. For a survey, see [15, 19]. The boundedness of the maximal and Riesz potential operators were studied for variable exponent Lebesgue spaces $L^{p(\cdot)}$ (see [16, 17, 18]), variable exponent Morrey spaces (see [4, 22, 23, 34, 39]), Herz spaces with variable exponents (see [3, 27, 47]), local variable exponent Morrey type spaces (see [23, 24]) and non-homogeneous central Morrey spaces of variable exponent (see [38]).

Recently, the boundedness of the maximal and Riesz potential operators were studied for Herz-Morrey spaces with variable exponents (see [35, 36]) and non-homogeneous central Herz-Morrey-Orlicz spaces in the constant exponent case (see [37]).

Let $\Omega$ be a measurable set in $\mathbf{R}^N$. Given a general function $\Phi(x,t)$ satisfying certain conditions, we consider the associated Musielak-Orlicz space (cf. [41])

$$L^{\Phi}(\Omega) = \left\{ f \in L^1_{loc}(\Omega);\ \int_{\Omega} \Phi(y, |f(y)|)dy < \infty \right\},$$

which is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(\Omega)} = \inf\left\{ \lambda > 0;\ \int_{\Omega} \bar{\Phi}(y, |f(y)|/\lambda)dy \le 1 \right\}$$

(see Section 2 for the definitions of $\Phi$ and $\bar{\Phi}$). For the recent development of the theory of PDEs in Musielak-Orlicz spaces and Herz spaces with variable exponents, we refer to [7, 12, 25, 48] and so on. Let $\omega(r) : (0,\infty) \to (0,\infty)$ be almost monotone on $(0,\infty)$ satisfying the doubling condition. Let $0 < q < \infty$. Given $\Phi(x,t)$ and $\omega(r)$, we denote by $\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$ the class of locally integrable functions $f$ on $\mathbf{R}^N$ satisfying

$$\|f\|_{\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)} = \|f\|_{L^{\Phi}(B(0,2))} + \left( \int_1^{\infty} (\omega(r)\|f\|_{L^{\Phi}(A(0,r))})^q \frac{dr}{r} \right)^{1/q} < \infty,$$

where $A(0,r) = B(0,2r)\backslash B(0,r)$. The space $\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$ is referred to as a non-homogeneous central Herz-Morrey-Musielak-Orlicz space (see Section 2).

Our first aim in this paper is to study the boundedness of the maximal operator on non-homogeneous central Herz-Morrey-Musielak-Orlicz spaces

$\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$ (Theorem 1 below), as an extension of [36, 37]. To this end, we apply the boundedness of the Hardy-Littlewood maximal operator on $L^{\Phi}$ given in [30]. The case when $q = \infty$ was treated in [45], as an extension of [35].

Next we study the boundedness of the Hardy operators $\hat{H}_{\beta}^{\infty}$ and $\hat{H}_{\beta}^{0}$ on $\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$ (Theorems 2 and 3 below). See Section 4 for the definitions of $\hat{H}_{\beta}^{\infty}$ and $\hat{H}_{\beta}^{0}$.

As an application of the boundedness of the maximal operator, we establish Sobolev's inequality for Riesz potentials $I_{\alpha}f$ of functions in $\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$ (Theorem 4 below), as an extension of [36, 37]. When $q = \infty$, we refer to [35, 45].

Further, we discuss Sobolev's inequality for generalized Riesz potentials $I_{\alpha,k}f$ of functions in $\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$ (Theorem 5 below), as an extension of [35, 36, 37]. See Section 6 for the definition of $I_{\alpha,k}f$.

In Section 7, in connection with the study in [21, 24], we investigate the space $\underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$ and its complementary space $\overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$.

In Section 8, we treat the case $q$ is variable.

Throughout this paper, let $C$ denote various positive constants independent of the variables in question. The symbol $g \sim h$ means that $C^{-1}h \le g \le Ch$ for some constant $C > 0$.

## 2. Preliminaries

We consider a function

$$\Phi(x,t) = t\phi(x,t) : \mathbf{R}^N \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions $(\Phi 1)$–$(\Phi 4)$:

$(\Phi 1)$ $\phi(\cdot,t)$ is measurable on $\mathbf{R}^N$ for each $t \ge 0$ and $\phi(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in \mathbf{R}^N$;

$(\Phi 2)$ there exists a constant $A_1 \ge 1$ such that

$$A_1^{-1} \le \phi(x,1) \le A_1 \qquad \text{for all } x \in \mathbf{R}^N;$$

$(\Phi 3)$ $\phi(x,\cdot)$ is uniformly almost increasing; namely there exists a constant $A_2 \ge 1$ such that

$$\phi(x,t) \le A_2\phi(x,s) \qquad \text{for all } x \in \mathbf{R}^N \text{ whenever } 0 \le t < s;$$

$(\Phi 4)$ there exists a constant $A_3 \ge 1$ such that

$$\phi(x,2t) \le A_3\phi(x,t) \qquad \text{for all } x \in \mathbf{R}^N \text{ and } t > 0.$$

Note that ($\Phi$2), ($\Phi$3) and ($\Phi$4) imply

$$0 < \inf_{x \in \mathbf{R}^N} \phi(x,t) \le \sup_{x \in \mathbf{R}^N} \phi(x,t) < \infty$$

for each $t > 0$.

If $\Phi(x,\cdot)$ is convex for each $x \in \mathbf{R}^N$, then ($\Phi$3) holds with $A_2 = 1$; namely $\phi(x,\cdot)$ is non-decreasing for each $x \in \mathbf{R}^N$.

Let $\bar{\phi}(x,t) = \sup_{0 \le s \le t} \phi(x,s)$ and

$$\bar{\Phi}(x,t) = \int_0^t \bar{\phi}(x,r)dr$$

for $x \in \mathbf{R}^N$ and $t \ge 0$. Then $\bar{\Phi}(x,\cdot)$ is convex and

$$\frac{1}{2A_3}\Phi(x,t) \le \bar{\Phi}(x,t) \le A_2\Phi(x,t) \tag{1}$$

for all $x \in \mathbf{R}^N$ and $t \ge 0$.

By ($\Phi$3), we see that

$$\Phi(x,at)\begin{cases} \le A_2 a\Phi(x,t) & \text{if } 0 \le a \le 1, \\ \ge A_2^{-1} a\Phi(x,t) & \text{if } a \ge 1. \end{cases}$$

We shall also consider the following conditions for $\Phi(x,t)$: Let $p \ge 1$, $q \ge 1$, $\eta > 0$ and $\tau > 0$.

($\Phi3;0;p$) $t \mapsto t^{-p}\Phi(x,t)$ is uniformly almost increasing on $(0,1]$, namely there exists a constant $A_{2,0,p} \ge 1$ such that

$$t_1^{-p}\Phi(x,t_1) \le A_{2,0,p}t_2^{-p}\Phi(x,t_2) \qquad \text{for all } x \in \mathbf{R}^N \text{ whenever } 0 < t_1 < t_2 \le 1;$$

($\Phi3;\infty;q$) $t \mapsto t^{-q}\Phi(x,t)$ is uniformly almost increasing on $[1,\infty)$, namely there exists a constant $A_{2,\infty,q} \ge 1$ such that

$$t_1^{-q}\Phi(x,t_1) \le A_{2,\infty,q}t_2^{-q}\Phi(x,t_2) \qquad \text{for all } x \in \mathbf{R}^N \text{ whenever } 1 \le t_1 < t_2;$$

($\Phi5;\eta$) for every $\gamma > 0$, there exists a constant $B_{\gamma,\eta} \ge 1$ such that

$$\Phi(x,t) \le B_{\gamma,\eta}\Phi(y,t)$$

whenever $x, y \in \mathbf{R}^N$, $|x-y| \le \gamma t^{-\eta}$ and $t \ge 1$;

($\Phi6;\tau$) there exist a function $g$ on $\mathbf{R}^N$ and a constant $B_\infty \ge 1$ such that $0 \le g(x) \le 1$ for all $x \in \mathbf{R}^N$, $g \in L^\tau(\mathbf{R}^N)$ and

$$B_\infty^{-1}\Phi(x,t) \le \Phi(x',t) \le B_\infty\Phi(x,t)$$

whenever $x, x' \in \mathbf{R}^N$, $|x'| \ge |x|$ and $g(x) \le t \le 1$.

Note that $(\Phi3;0;1)+(\Phi3;\infty;1)=(\Phi3)$. If $\Phi(x,t)$ satisfies $(\Phi3;0;p)$, then it satisfies $(\Phi3;0;p')$ for $1 \le p' \le p$; if $\Phi(x,t)$ satisfies $(\Phi3;\infty;q)$, then it satisfies $(\Phi3;\infty;q')$ for $1 \le q' \le q$.

If $\Phi(x,t)$ satisfies $(\Phi3;0;p)$, then

$$\Phi(x,t) \le A_1 A_{2,0,p} t^p \qquad \text{for } 0 \le t \le 1;$$

if $\Phi(x,t)$ satisfies $(\Phi3;\infty;q)$, then

$$\Phi(x,t) \ge (A_1 A_{2,\infty,q})^{-1} t^q \qquad \text{for } t \ge 1.$$

If $\Phi(x,t)$ satisfies $(\Phi5;\eta)$, then it satisfies $(\Phi5;\eta')$ for all $\eta' \ge \eta$; if $\Phi(x,t)$ satisfies $(\Phi6;\tau)$, then it satisfies $(\Phi6;\tau')$ for all $\tau' \ge \tau$.

In the following examples, we use the notation

$$f^- := \inf_{x \in \mathbf{R}^N} f(x) \qquad \text{and} \qquad f^+ := \sup_{x \in \mathbf{R}^N} f(x)$$

for a measurable function $f$ on $\mathbf{R}^N$.

Example 1. Let $p_i(\cdot)$, $i=1,2$ and $q_{i,j}(\cdot)$, $j=1,\dots,k_i$, be real valued measurable functions on $\mathbf{R}^N$ such that $p_i^- > 1$ and $q_{i,j}^- > -\infty$, $i=1,2$, $j=1,\dots,k_i$.

Set $L_c(t) = \log(c+t)$ for $c>1$ and $t \ge 0$, $L_c^{(1)}(t) = L_c(t)$, $L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))$. Let

$$\Phi(x,t) = \begin{cases} t^{p_1(x)} \prod_{j=1}^{k_1} (L_{e-1}^{(j)}(1/t))^{-q_{1,j}(x)} & \text{if } 0 \le t \le 1, \\ t^{p_2(x)} \prod_{j=1}^{k_2} (L_{e-1}^{(j)}(t))^{q_{2,j}(x)} & \text{if } t \ge 1. \end{cases}$$

Then, $\Phi(x,t)$ satisfies $(\Phi1)$, $(\Phi2)$ and $(\Phi3)$. It satisfies $(\Phi3;0;p)$ for $1 \le p < p_1^-$ in general and for $1 \le p \le p_1^-$ in case $q_{1,j}^- \ge 0$ for all $j=1,\dots,k_1$; it satisfies $(\Phi3;\infty;q)$ for $1 \le q < p_2^-$ in general and for $1 \le q \le p_2^-$ in case $q_{2,j}^- \ge 0$ for all $j=1,\dots,k_2$.

Moreover, we see that $\Phi(x,t)$ satisfies $(\Phi5;\eta)$ for every $\eta>0$ if $p_2(\cdot)$ is log-Hölder continuous, namely

$$|p_2(x) - p_2(y)| \le \frac{C_p}{L_e(1/|x-y|)} \qquad (x,y \in \mathbf{R}^N)$$

with a constant $C_p \ge 0$ and $q_{2,j}(\cdot)$ is $(j+1)$-log-Hölder continuous, namely

$$|q_{2,j}(x) - q_{2,j}(y)| \le \frac{C_j}{L_e^{(j+1)}(1/|x-y|)} \qquad (x,y \in \mathbf{R}^N)$$

with constants $C_j \ge 0$, for each $j=1,\dots,k_2$.

Finally, we see that $\Phi(x,t)$ satisfies $(\Phi 6;\tau)$ for every $\tau>0$ with $g(x)=(1+|x|)^{-(N+1)/\tau}$ if $p_1(\cdot)$ is log-Hölder continuous at $\infty$, namely

$$|p_1(x)-p_1(x')|\le \frac{C_{p,\infty}}{L_e(|x|)}$$

whenever $|x'|\ge|x|$ $(x,x'\in\mathbf{R}^N)$ with a constant $C_{p,\infty}\ge 0$, and $q_{1,j}(\cdot)$ is $(j+1)$-log-Hölder continuous at $\infty$, namely

$$|q_{1,j}(x)-q_{1,j}(x')|\le \frac{C_j'}{L_e^{(j+1)}(|x|)}$$

whenever $|x'|\ge|x|$ $(x,x'\in\mathbf{R}^N)$ with a constant $C_j'\ge 0$, for each $j=1,\dots,k_1$. In fact, if $(1+|x|)^{-(N+1)/\tau}<t\le 1$, then $t^{-|p_1(x)-p_1(x')|}\le e^{(N+1)C_{p,\infty}/\tau}$ for $|x'|\ge|x|$ and $(L_{e-1}^{(j)}(1/t))^{|q_{1,j}(x)-q_{1,j}(x')|}\le C(N,C_j')$ for $|x'|\ge|x|$.

The following example shows that our conditions are satisfied by the double phase functional with variable exponents.

EXAMPLE 2. Regarding regularity theory of differential equations, Baroni, Colombo and Mingione [5, 6, 7, 13, 14] have studied the double phase functional

$$\Phi(x,t)=t^p+a(x)t^q,$$

where $1<p<q$, $a(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta\in(0,1]$. In [31], we studied the double phase functional with variable exponents:

$$\Phi(x,t)=t^{p(x)}+a(x)t^{q(x)},\qquad x\in\mathbf{R}^N,\ t\ge 0,$$

where $p(\cdot)$ and $q(\cdot)$ are real valued functions on $\mathbf{R}^N$ such that $p(x)<q(x)$ for $x\in\mathbf{R}^N$, $a(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta\in(0,1]$. This $\Phi(x,t)$ satisfies $(\Phi 1)$, $(\Phi 2)$, $(\Phi 3)$, $(\Phi 3;0;p^-)$, $(\Phi 3;\infty;p^-)$ and $(\Phi 5;\eta)$ for $\eta\ge\sup_{\{x:a(x)>0\}}(q(x)-p(x))/\theta$ if $1\le p^-\le p^+<\infty$, $1\le q^-\le q^+<\infty$, $p(\cdot)$ and $q(\cdot)$ are log-Hölder continuous. Further it satisfies $(\Phi 6;\tau)$ with $g(x)=(1+|x|)^{-(N+1)/\tau}$ for every $\tau>0$ if $p(\cdot)$ is log-Hölder continuous at $\infty$. See [31] for details.

Let $\Omega$ be a measurable set in $\mathbf{R}^N$. From now on, we assume that $\Phi(x,t)$ satisfies $(\Phi 1)$, $(\Phi 2)$, $(\Phi 3)$ and $(\Phi 4)$. Then the associated Musielak-Orlicz space

$$L^{\Phi}(\Omega)=\left\{f\in L^1_{loc}(\Omega);\ \int_\Omega \Phi(y,|f(y)|)dy<\infty\right\}$$

is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(\Omega)} = \inf\left\{\lambda > 0; \int_{\Omega} \bar{\Phi}(y, |f(y)|/\lambda)dy \le 1\right\}$$

(cf. [41]).

REMARK 1. The Musielak-Orlicz spaces $L^{\Phi}(\Omega)$ include the following spaces:

- Orlicz spaces defined by Young functions satisfying the doubling condition;
- variable exponent Lebesgue spaces.

REMARK 2. The dominated convergence theorem and ($\Phi4$) yield

$$\int_{\Omega} \bar{\Phi}\left(y, \frac{|f(y)|}{\|f\|_{L^{\Phi}(\Omega)}}\right)dy = 1.$$

We consider a function $\omega(r) : (0, \infty) \to (0, \infty)$ satisfying the following conditions ($\omega1$) and ($\omega2$):

($\omega1$) $\omega(\cdot)$ is almost monotone on $(0, \infty)$; that is, $\omega(\cdot)$ is almost increasing on $(0, \infty)$ or $\omega(\cdot)$ is almost decreasing on $(0, \infty)$; namely there exists a constant $c_1 > 0$ such that

$$\omega(r) \le c_1\omega(s) \qquad \text{for all } 0 < r < s$$

or

$$\omega(s) \le c_1\omega(r) \qquad \text{for all } 0 < r < s$$

respectively;

($\omega2$) $\omega(\cdot)$ is doubling on $(0, \infty)$; that is, there exists a constant $c_2 > 1$ such that

$$c_2^{-1}\omega(r) \le \omega(2r) \le c_2\omega(r) \qquad \text{for all } r > 0.$$

Let $0 < q < \infty$. Given $\Phi(x, t)$ and $\omega(r)$ as above, we denote by $\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$ the class of locally integrable functions $f$ on $\mathbf{R}^N$ satisfying

$$\|f\|_{\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)} = \|f\|_{L^{\Phi}(B(0,2))} + \left(\int_1^{\infty} (\omega(r)\|f\|_{L^{\Phi}(A(0,r))})^q \frac{dr}{r}\right)^{1/q} < \infty,$$

where $A(0,r) = B(0,2r)\backslash B(0,r)$. The space $\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$ is referred to as a non-homogeneous central Herz-Morrey-Musielak-Orlicz space.

REMARK 3. The non-homogeneous central Herz-Morrey-Musielak-Orlicz spaces $\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$ include the following spaces:

- non-homogeneous Herz spaces introduced in [26];
- local Morrey-type spaces introduced in [9];
- non-homogeneous central Herz-Morrey-Orlicz spaces introduced in [37] where $\Phi(x,t) = \Phi(t)$;
- non-homogeneous central Herz-Morrey spaces with variable exponents introduced in [36] where $\Phi(x,t) = t^{p(x)}$.

LEMMA 1. *For $1/2 < a < 1 < b < 2$ with $2a \geq b$, there exists a constant $C > 0$ such that*

$$\int_{at}^{bt} (\omega(r)\|f\|_{L^\Phi(A(0,r))})^q \frac{dr}{r} \geq C(\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \tag{2}$$

*for all $t > 0$.*

PROOF. For $1/2 < a < 1 < b < 2$ with $2a \geq b$, we have

$$\int_{at}^{t} (\omega(r)\|f\|_{L^\Phi(A(0,r))})^q \frac{dr}{r} \geq C(\omega(t)\|f\|_{L^\Phi(B(0,2at)\setminus B(0,t))})^q$$

and

$$\int_{t}^{bt} (\omega(r)\|f\|_{L^\Phi(A(0,r))})^q \frac{dr}{r} \geq C(\omega(t)\|f\|_{L^\Phi(B(0,2t)\setminus B(0,bt))})^q,$$

so that we obtain

$$\begin{aligned}
&(\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \\
&\quad \leq (\omega(t)\|f\|_{L^\Phi(B(0,2t)\setminus B(0,bt))} + \omega(t)\|f\|_{L^\Phi(B(0,2at)\setminus B(0,t))})^q \\
&\quad \leq C\left\{\int_{t}^{bt} (\omega(r)\|f\|_{L^\Phi(A(0,r))})^q \frac{dr}{r} + \int_{at}^{t} (\omega(r)\|f\|_{L^\Phi(A(0,r))})^q \frac{dr}{r}\right\} \\
&\quad = C\int_{at}^{bt} (\omega(r)\|f\|_{L^\Phi(A(0,r))})^q \frac{dr}{r}.
\end{aligned}$$

□

LEMMA 2. *For a bounded measurable set $\Omega$, there exist constants $C_\Omega$ and $C'_\Omega$ such that*

$$\int_\Omega |f(x)|dx \leq C_\Omega \|f\|_{L^\Phi(\Omega)} \leq C'_\Omega \|f\|_{\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)} \tag{3}$$

*for all $f \in \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$.*

PROOF. If $\|f\|_{L^\Phi(\Omega)} \leq 1$, then

$$\int_\Omega |f(x)|dx \leq |\Omega| + 2A_1A_3 \int_\Omega \bar{\Phi}(x,|f(x)|)dx \leq |\Omega| + 2A_1A_3$$

by ($\Phi 2$), convexity of $\bar{\Phi}(x,\cdot)$ and (1), where $|\Omega|$ denotes the Lebesgue measure of $\Omega$. This shows the first inequality in (3).

Next, suppose $f \in \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$ and $\Omega \subset B(0,2^k)$ $(k \geq 1)$. Then

$$\|f\|_{L^{\Phi}(\Omega)} \leq \|f\|_{L^{\Phi}(B(0,2))} + \sum_{j=1}^{k-1} \|f\|_{L^{\Phi}(A(0,2^j))}$$

$$\leq \|f\|_{L^{\Phi}(B(0,2))} + C_k \sum_{j=1}^{k-1} \omega(2^j)\|f\|_{L^{\Phi}(A(0,2^j))},$$

where $C_k^{-1} = \inf_{2 \leq r \leq 2^{k-1}} \omega(r) > 0$. Then, using Lemma 1, we obtain the second inequality in (3). □

Lemma 3 (cf. [30, Lemma 5.1]). *Let $F(x,t)$ be a positive function on $\mathbf{R}^N \times (0,\infty)$ satisfying the following conditions:*

(F1) *$F(x,\cdot)$ is strictly increasing and continuous on $(0,\infty)$ for each $x \in \mathbf{R}^N$;*

(F2) *there exists a constant $K_1 \geq 1$ such that*

$$K_1^{-1} \leq F(x,1) \leq K_1 \qquad \text{for all } x \in \mathbf{R}^N;$$

(F3) *$t \mapsto t^{-\varepsilon}F(x,t)$ is uniformly almost increasing for $\varepsilon > 0$; namely there exists a constant $K_2 \geq 1$ such that*

$$t^{-\varepsilon}F(x,t) \leq K_2 s^{-\varepsilon}F(x,s) \qquad \text{for all } x \in \mathbf{R}^N \text{ whenever } 0 < t < s;$$

(F4) *there exists a constant $K_3 > 1$ such that*

$$F(x,2t) \leq K_3 F(x,t) \qquad \text{for all } x \in \mathbf{R}^N \text{ and } t > 0.$$

*Let $F^{-1}(x,\cdot)$ be the inverse function of $F(x,\cdot)$. Then:*

(1) *$F^{-1}(x,\cdot)$ is strictly increasing.*

(2)

$$F^{-1}(x,\lambda t) \leq (K_2\lambda)^{1/\varepsilon}F^{-1}(x,t)$$

*for all $x \in \mathbf{R}^N$, $t > 0$ and $\lambda \geq 1$.*

(3)

$$F^{-1}(x,\lambda t) \leq 2\lambda^{1/\log_2 K_3}F^{-1}(x,t)$$

*for all $x \in \mathbf{R}^N$, $t > 0$ and $0 < \lambda < 1$.*

(4)

$$\min\left\{1, \left(\frac{t}{K_1K_2}\right)^{1/\varepsilon}\right\} \leq F^{-1}(x,t) \leq \max\{1, (K_1K_2t)^{1/\varepsilon}\}$$

*for all $x \in \mathbf{R}^N$ and $t > 0$.*

REMARK 4. $F(x,t) = \bar{\Phi}(x,t)$ satisfies (F1), (F2), (F3) and (F4) with $K_1 = A_1 \max\{A_2, 2A_3\}$, $K_2 = 1$, $K_3 = 2A_3$ and $\varepsilon = 1$.

We also consider a convex function $\Phi_\infty(t) = t\phi_\infty(t) : [0,\infty) \to [0,\infty)$ such that $\phi_\infty(t) > 0$ for $t > 0$, $\phi_\infty(t)$ is increasing on $[0,\infty)$ and satisfies the doubling condition and

($\Phi_\infty 1$) there exists a constant $Q \ge 1$ such that

$$Q^{-1}\Phi(x,t) \le \Phi_\infty(t) \le Q\Phi(x,t) \qquad \text{whenever } g(x) \le t \le 1$$

for $g$ in condition ($\Phi 6; \tau$).

REMARK 5. Note from ($\Phi_\infty 1$) that for $c_1, c_2 > 0$, there exists a constant $Q \ge 1$ such that

$$Q^{-1}\Phi(x,t) \le \Phi_\infty(t) \le Q\Phi(x,t) \qquad \text{whenever } c_1 g(x) \le t \le c_2$$

for $g$ in condition ($\Phi 6; \tau$).

REMARK 6. Suppose $\Phi(x,t)$ satisfies ($\Phi 6; \tau$). Set

$$\Phi_\infty(t) = \limsup_{|x|\to\infty} \bar{\Phi}(x,t) \qquad \text{and} \qquad \phi_\infty(t) = \Phi_\infty(t)/t.$$

Then note that $\phi_\infty(t) > 0$ for $t > 0$, $\phi_\infty(t)$ is increasing on $[0,\infty)$ and satisfies the doubling condition. Further, by ($\Phi 6; \tau$), we find that $\Phi_\infty(t)$ satisfies ($\Phi_\infty 1$).

We denote by $\chi_E$ the characteristic function of $E$ and by $\Phi_\infty^{-1}(t)$ the inverse of $\Phi_\infty(t)$.

LEMMA 4. *Assume*

($\Phi_\infty 2$) *there exists a constant* $Q \ge 1$ *such that*

$$\Phi_\infty(g(x)) \le Q(1+|x|)^{-N}$$

*for all* $x \in \mathbf{R}^N$.

*Then there is a constant* $C > 0$ *such that*

$$\|\chi_{B(0,r)}\|_{L^\Phi(\mathbf{R}^N)} \le C\{\Phi_\infty^{-1}(r^{-N})\}^{-1}$$

*for all* $r \ge 1$.

PROOF. Note from ($\Phi_\infty 2$) and Lemma 3 (2) that

$$g(x) \le C\Phi_\infty^{-1}((1+|x|)^{-N}) \le C\Phi_\infty^{-1}(1)$$

for all $x \in \mathbf{R}^N$.

Let $R \ge 1/2$. We have by ($\Phi 3$), ($\Phi 4$), Lemma 3 (2) and ($\Phi_\infty 1$)

$$\begin{aligned}\int_{A(0,R)} \Phi(y, \Phi_\infty^{-1}(R^{-N}))dy &\le C \int_{A(0,R)} \Phi(y, \Phi_\infty^{-1}((1+|y|)^{-N}))dy \\ &\le C \int_{A(0,R)} \Phi_\infty(\Phi_\infty^{-1}((1+|y|)^{-N}))dy \\ &\le C \int_{A(0,R)} (1+|y|)^{-N} dy \\ &\le C.\end{aligned}$$

Hence we obtain

$$\|\chi_{A(0,R)}\|_{L^\Phi(\mathbf{R}^N)} \le C\{\Phi_\infty^{-1}(R^{-N})\}^{-1}$$

for all $R \ge 1/2$.

Here note from Lemma 3 (4) that

$$\Phi_\infty^{-1}(R^{-N}) \le \max\{1, CR^{-N}\} \le C,$$

so that

$$\max\{\{\Phi_\infty^{-1}(R^{-N})\}^{-1}, 1\} \le C\{\Phi_\infty^{-1}(R^{-N})\}^{-1}$$

for all $R \ge 1/2$.

Fix $r \ge 1$. Let $j_0$ be the largest integer such that $2^{-j_0+1} r \ge 1$. Now we see from Lemma 3 (3) that $t \mapsto t^{-\varepsilon}\{\Phi_\infty^{-1}(t^{-N})\}^{-1}$ is almost increasing on $(0, \infty)$ for some constant $\varepsilon > 0$, so that

$$\begin{aligned}\|\chi_{B(0,r)}\|_{L^\Phi(\mathbf{R}^N)} &\le \sum_{j=1}^{j_0} \|\chi_{A(0,2^{-j}r)}\|_{L^\Phi(\mathbf{R}^N)} + \|\chi_{B(0,1)}\|_{L^\Phi(\mathbf{R}^N)} \\ &\le C\left\{\sum_{j=1}^{j_0} \{\Phi_\infty^{-1}((2^{-j}r)^{-N})\}^{-1} + 1\right\} \\ &\le C\left\{r^{-\varepsilon}\{\Phi_\infty^{-1}(r^{-N})\}^{-1} \sum_{j=1}^{j_0} (2^{-j}r)^{\varepsilon} + 1\right\} \\ &\le C\{\Phi_\infty^{-1}(r^{-N})\}^{-1},\end{aligned}$$

as required. □

REMARK 7. If $g(x) \le C(1+|x|)^{-N}$, then ($\Phi_\infty 2$) holds by convexity of $\Phi_\infty$.

REMARK 8. Let $\Phi(x,t)$ and $g(x)$ be as in Example 1. Then there exist constants $p_1(\infty) > 1$ and $q_{1,j}(\infty) \in \mathbf{R}$ for $j = 1, \dots, k_1$ such that

$$\lim_{|x|\to\infty} p_1(x) = p_1(\infty) \qquad \text{and} \qquad \lim_{|x|\to\infty} q_{1,j}(x) = q_{1,j}(\infty).$$

Set $\bar{\phi}_\infty(t) = \sup_{0 \le s \le t}\{s^{p_1(\infty)-1}\prod_{j=1}^{k_1}(L_{e-1}^{(j)}(1/s))^{-q_{1,j}(\infty)}\}$ and

$$\Phi_\infty(t) = \int_0^t \bar{\phi}_\infty(r)\frac{dr}{r}.$$

Then $\Phi_\infty(t)$ satisfies $(\Phi_\infty 1)$ and $(\Phi_\infty 2)$ for $0 < \tau < (N+1)p_1(\infty)/N$.

LEMMA 5. *Suppose that $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$. Then there is a constant $C > 0$ such that*

$$\frac{1}{|A(0,r)|}\int_{A(0,r)} |f(y)|dy \le C\Phi_\infty^{-1}(r^{-N})\|f\|_{L^\Phi(A(0,r))}$$

*when $r \ge 1$ and $\|f\|_{L^\Phi(A(0,r))} < \infty$.*

PROOF. Fix $r \ge 1$. Let $f$ be a nonnegative measurable function on $A(0,r)$ satisfying $\|f\|_{L^\Phi(A(0,r))} \le 1$. Then we have by $(\Phi 3)$

$$\begin{aligned}
&\frac{1}{|A(0,r)|}\int_{A(0,r)} f(y)dy \\
&\quad \le \Phi_\infty^{-1}(r^{-N}) + \frac{A_2}{|A(0,r)|}\int_{A(0,r)} f(y)\frac{\varphi(y,f(y))}{\varphi(y,\Phi_\infty^{-1}(r^{-N}))}\,dy \\
&\quad = \Phi_\infty^{-1}(r^{-N}) + \frac{A_2\Phi_\infty^{-1}(r^{-N})}{|A(0,r)|}\int_{A(0,r)} \Phi(y,f(y))\{\Phi(y,\Phi_\infty^{-1}(r^{-N}))\}^{-1}dy.
\end{aligned}$$

Since

$$g(y) \le C\Phi_\infty^{-1}((1+|y|)^{-N}) \le C\Phi_\infty^{-1}(r^{-N}) \le C\Phi_\infty^{-1}(1)$$

for all $y \in A(0,r)$ by $(\Phi_\infty 2)$, we have by $(\Phi_\infty 1)$

$$\Phi(y,\Phi_\infty^{-1}(r^{-N})) \ge Cr^{-N}$$

for all $y \in A(0,r)$. Hence we obtain

$$\begin{aligned}
\frac{1}{|A(0,r)|}\int_{A(0,r)} f(y)dy &\le \Phi_\infty^{-1}(r^{-N}) + C\Phi_\infty^{-1}(r^{-N})\int_{A(0,r)} \Phi(y,f(y))dy \\
&\le C\Phi_\infty^{-1}(r^{-N}),
\end{aligned}$$

as required. □

## 3. Boundedness of the maximal operator

For a locally integrable function $f$ on $\mathbf{R}^N$, the Hardy-Littlewood maximal function $Mf$ is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy.$$

The mapping $f \mapsto Mf$ is called the maximal operator.

By [31, Theorem 3.1], we have the following result.

Lemma 6. *Suppose that $\Phi(x,t)$ satisfies $(\Phi3;0;p)$, $(\Phi3;\infty;q)$, $(\Phi5;\eta)$ and $(\Phi6;\tau)$ for $p>1$, $q>1$, $\eta>0$ and $\tau>0$ satisfying $\eta \le q/N$ and $\tau \le p$. Then the maximal operator $M$ is bounded from $L^{\Phi}(\mathbf{R}^N)$ into itself; namely, there is a constant $C>0$ such that*

$$\|Mf\|_{L^{\Phi}(\mathbf{R}^N)} \le C\|f\|_{L^{\Phi}(\mathbf{R}^N)}$$

*for all $f \in L^{\Phi}(\mathbf{R}^N)$.*

For a nonnegative function $f \in L^1_{loc}(\mathbf{R}^N)$ and a real number $\beta$, set

$$H^{\infty}_{\beta} f(r) = r^{\beta} \int_{\mathbf{R}^N \setminus B(0,2r)} |y|^{-N-\beta} f(y)dy.$$

Lemma 7. *For a real number $\beta$, suppose that $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty\omega 1;\beta)$ $\quad t^{\varepsilon_1-\beta}\omega(t)^{-1}\Phi_\infty^{-1}(t^{-N})$ is almost decreasing in $[1,\infty)$ for some $\varepsilon_1>0$. If $0<\varepsilon<\varepsilon_1$, then there exists a constant $C>0$ such that*

$$H^{\infty}_{\beta} f(r) \le Cr^{\varepsilon}\omega(r)^{-1}\Phi_\infty^{-1}(r^{-N}) \left( \int_r^{\infty} (t^{-\varepsilon}\omega(t)\|f\|_{L^{\Phi}(A(0,t))})^q \frac{dt}{t} \right)^{1/q}$$

*for all $r \ge 1$ and nonnegative functions $f \in L^1_{loc}(\mathbf{R}^N)$.*

Proof. Let $f \in L^1_{loc}(\mathbf{R}^N)$ be a nonnegative function on $\mathbf{R}^N$. Let $r \ge 1$ and $0<\varepsilon<\varepsilon_1$.

First we consider the case $1<q<\infty$. Then we have by Lemma 5 and Hölder's inequality

$$\begin{aligned} H^{\infty}_{\beta} f(r) &= r^{\beta} \sum_{j=1}^{\infty} \int_{A(0,2^j r)} |y|^{-N-\beta} f(y)dy \\ &\le Cr^{\beta} \sum_{j=1}^{\infty} (2^j r)^{-\beta} \frac{1}{|A(0,2^j r)|} \int_{A(0,2^j r)} f(y)dy \\ &\le Cr^{\beta} \sum_{j=1}^{\infty} (2^j r)^{-\beta} \Phi_\infty^{-1}((2^j r)^{-N}) \|f\|_{L^{\Phi}(A(0,2^j r))} \end{aligned}$$

$$\leq Cr^{\beta}\left(\sum_{j=1}^{\infty}((2^j r)^{\varepsilon-\beta}\omega(2^j r)^{-1}\Phi_{\infty}^{-1}((2^j r)^{-N}))^{q'}\right)^{1/q'}$$

$$\times\left(\sum_{j=1}^{\infty}((2^j r)^{-\varepsilon}\omega(2^j r)\|f\|_{L^{\Phi}(A(0,2^j r))})^{q}\right)^{1/q}.$$

Here note from $(\Phi_{\infty}\omega 1;\beta)$ that

$$\left(\sum_{j=1}^{\infty}((2^j r)^{\varepsilon-\beta}\omega(2^j r)^{-1}\Phi_{\infty}^{-1}((2^j r)^{-N}))^{q'}\right)^{1/q'}$$

$$\leq Cr^{\varepsilon_1-\beta}\omega(r)^{-1}\Phi_{\infty}^{-1}(r^{-N})\left(\sum_{j=1}^{\infty}(2^j r)^{(\varepsilon-\varepsilon_1)q'}\right)^{1/q'}$$

$$\leq Cr^{\varepsilon-\beta}\omega(r)^{-1}\Phi_{\infty}^{-1}(r^{-N}).$$

By (2), we have

$$\left(\sum_{j=1}^{\infty}((2^j r)^{-\varepsilon}\omega(2^j r)\|f\|_{L^{\Phi}(A(0,2^j r))})^{q}\right)^{1/q}$$

$$\leq C\left(\sum_{j=1}^{\infty}(2^j r)^{-\varepsilon}\int_{(2/3)2^j r}^{(4/3)2^j r}(\omega(t)\|f\|_{L^{\Phi}(A(0,t))})^{q}\frac{dt}{t}\right)^{1/q}$$

$$\leq C\left(\sum_{j=1}^{\infty}\int_{2^{j-1}r}^{2^{j+1}r}(t^{-\varepsilon}\omega(t)\|f\|_{L^{\Phi}(A(0,t))})^{q}\frac{dt}{t}\right)^{1/q}$$

$$\leq C\left(\int_{r}^{\infty}(t^{-\varepsilon}\omega(t)\|f\|_{L^{\Phi}(A(0,t))})^{q}\frac{dt}{t}\right)^{1/q}.$$

Hence

$$H_{\beta}^{\infty}f(r)\leq Cr^{\varepsilon}\omega(r)^{-1}\Phi_{\infty}^{-1}(r^{-N})\left(\int_{r}^{\infty}(t^{-\varepsilon}\omega(t)\|f\|_{L^{\Phi}(A(0,t))})^{q}\frac{dt}{t}\right)^{1/q}.$$

For the case $0<q\leq 1$, by the fact that $(a+b)^q\leq a^q+b^q$ for all $a,b\geq 0$ instead of Hölder's inequality, we obtain the required inequality. □

For a nonnegative function $f\in L^1_{loc}(\mathbf{R}^N)$ and a real number $\beta$, set

$$H_{\beta}^{0}f(r)=r^{\beta}\int_{B(0,r)\setminus B(0,1)}|y|^{-N-\beta}f(y)dy.$$

Lemma 8. *For a real number* $\beta$, *suppose that* $\Phi_\infty(t)$ *satisfies* $(\Phi_\infty 2)$ *and*

$(\Phi_\infty\omega 2;\beta)$ $t^{-\varepsilon_2-\beta}\omega(t)^{-1}\Phi_\infty^{-1}(t^{-N})$ *is almost increasing in* $[1,\infty)$ *for some* $\varepsilon_2 > 0$.

*If* $0 < \varepsilon < \varepsilon_2$, *then there exists a constant* $C > 0$ *such that*

$$H_\beta^0 f(r) \le C r^{-\varepsilon}\omega(r)^{-1}\Phi_\infty^{-1}(r^{-N})\left(\int_{1/2}^{r}(t^{\varepsilon}\omega(t)\|f\|_{L^{\Phi}(A(0,t))})^q\frac{dt}{t}\right)^{1/q}$$

*for all* $r \ge 1$ *and nonnegative functions* $f \in L^1_{loc}(\mathbf{R}^N)$.

Proof. We show only the case $1 < q < \infty$ since the remaining case is easily treated. Let $f \in L^1_{loc}(\mathbf{R}^N)$ be a nonnegative function on $\mathbf{R}^N$. Let $r \ge 1$ and $0 < \varepsilon < \varepsilon_2$. Let $j_0$ be the largest integer such that $2^{-j_0+1}r \ge 1$. We have by Lemma 5, Hölder's inequality, $(\Phi_\infty\omega 2;\beta)$ and (2)

$$\begin{aligned}
H_\beta^0 f(r) &= r^\beta \sum_{j=1}^{j_0}\int_{A(0,2^{-j}r)\setminus B(0,1)}|y|^{-N-\beta}f(y)dy\\
&\le C r^\beta\sum_{j=1}^{j_0}(2^{-j}r)^{-\beta}\frac{1}{|A(0,2^{-j}r)|}\int_{A(0,2^{-j}r)\setminus B(0,1)}f(y)dy\\
&\le C r^\beta\sum_{j=1}^{j_0}(2^{-j}r)^{-\beta}\Phi_\infty^{-1}((2^{-j}r)^{-N})\|f\chi_{\mathbf{R}^N\setminus B(0,1)}\|_{L^{\Phi}(A(0,2^{-j}r))}\\
&\le C r^\beta\left(\sum_{j=1}^{j_0}((2^{-j}r)^{-\varepsilon-\beta}\omega(2^{-j}r)^{-1}\Phi_\infty^{-1}((2^{-j}r)^{-N}))^{q'}\right)^{1/q'}\\
&\quad\times\left(\sum_{j=1}^{j_0}((2^{-j}r)^{\varepsilon}\omega(2^{-j}r)\|f\chi_{\mathbf{R}^N\setminus B(0,1)}\|_{L^{\Phi}(A(0,2^{-j}r))})^q\right)^{1/q}\\
&\le C r^{-\varepsilon_2}\omega(r)^{-1}\Phi_\infty^{-1}(r^{-N})\left(\sum_{j=1}^{j_0}(2^{-j}r)^{(\varepsilon_2-\varepsilon)q'}\right)^{1/q'}\\
&\quad\times\left(\sum_{j=1}^{j_0}((2^{-j}r)^{\varepsilon}\omega(2^{-j}r)\|f\chi_{\mathbf{R}^N\setminus B(0,1)}\|_{L^{\Phi}(A(0,2^{-j}r))})^q\right)^{1/q}\\
&\le C r^{-\varepsilon}\omega(r)^{-1}\Phi_\infty^{-1}(r^{-N})\\
&\quad\times\left(\sum_{j=1}^{j_0}((2^{-j}r)^{\varepsilon}\omega(2^{-j}r)\|f\chi_{\mathbf{R}^N\setminus B(0,1)}\|_{L^{\Phi}(A(0,2^{-j}r))})^q\right)^{1/q}
\end{aligned}$$

$$\le Cr^{-\varepsilon}\omega(r)^{-1}\Phi_\infty^{-1}(r^{-N})\left(\int_{1/4}^{r}(t^{\varepsilon}\omega(t)\|f\chi_{\mathbf{R}^N\setminus B(0,1)}\|_{L^{\Phi}(A(0,t))})^q\frac{dt}{t}\right)^{1/q}$$

$$\le Cr^{-\varepsilon}\omega(r)^{-1}\Phi_\infty^{-1}(r^{-N})\left(\int_{1/2}^{r}(t^{\varepsilon}\omega(t)\|f\|_{L^{\Phi}(A(0,t))})^q\frac{dt}{t}\right)^{1/q},$$

which gives the required result. □

We present the boundedness of the maximal operator in $\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$.

THEOREM 1. *Suppose that $\Phi(x,t)$ satisfies $(\Phi3;0;p)$, $(\Phi3;\infty;q)$, $(\Phi5;\eta)$ and $(\Phi6;\tau)$ for $p>1$, $q>1$, $\eta>0$ and $\tau>0$ satisfying $\eta\le q/N$ and $\tau\le p$. Assume that $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$, $(\Phi_\infty\omega 1;0)$ and $(\Phi_\infty\omega 2;-N)$. Then the maximal operator $M$ is bounded from $\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$ to itself, that is,*

$$\|Mf\|_{\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)}\le C\|f\|_{\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)}\qquad \text{for all } f\in\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N).$$

REMARK 9. Let $\Phi(x,t)$ be as in Example 1 and let $\Phi_\infty(t)$ be as in Remark 8. If $\omega(r)=r^{\nu}$, then $(\Phi_\infty\omega 1;0)$ and $(\Phi_\infty\omega 2;-N)$ hold when

$$-N/p_1(\infty)<\nu<N(1-1/p_1(\infty)).$$

PROOF (Proof of Theorem 1). Let $f$ be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)}\le 1$. First we show

$$\int_2^\infty(\omega(r)\|Mf\|_{L^{\Phi}(A(0,r))})^q\frac{dr}{r}\le C. \tag{4}$$

For $r\ge 2$, set

$$f=f\chi_{B(0,1)}+f\chi_{B(0,r/2)\setminus B(0,1)}+f\chi_{B(0,4r)\setminus B(0,r/2)}+f\chi_{\mathbf{R}^N\setminus B(0,4r)}$$
$$=f_0+f_{1,r}+f_{2,r}+f_{3,r}.$$

For $f_0$, by Lemma 2 we have

$$Mf_0(x)\le C|x|^{-N}\int_{B(0,1)}f(y)dy\le C|x|^{-N}$$

for $x\in\mathbf{R}^N\setminus B(0,r)$. By Lemmas 6 and 4

$$r^N\||\cdot|^{-N}\|_{L^{\Phi}(\mathbf{R}^N\setminus B(0,r))}\le C\|M\chi_{B(0,r/2)}\|_{L^{\Phi}(\mathbf{R}^N\setminus B(0,r))}$$
$$\le C\|\chi_{B(0,r/2)}\|_{L^{\Phi}(\mathbf{R}^N)}\le C\{\Phi_\infty^{-1}(r^{-N})\}^{-1}. \tag{5}$$

Hence

$$\|Mf_0\|_{L^{\Phi}(A(0,r))}\le Cr^{-N}\{\Phi_\infty^{-1}(r^{-N})\}^{-1}.$$

Since $r^{\varepsilon_2 - N}\omega(r)\{\Phi_\infty^{-1}(r^{-N})\}^{-1}$ is almost decreasing in $[1, \infty)$ by $(\Phi_\infty\omega 2; -N)$, it follows that

$$\int_2^\infty (\omega(r)\|Mf_0\|_{L^\Phi(A(0,r))})^q \frac{dr}{r} \le C\int_2^\infty (r^{-N}\omega(r)\{\Phi_\infty^{-1}(r^{-N})\}^{-1})^q \frac{dr}{r}$$
$$\le C. \tag{6}$$

For $f_{1,r}$, we find for $x \in \mathbf{R}^N \setminus B(0,r)$

$$Mf_{1,r}(x) \le C|x|^{-N}\int_{B(0,r/2)\setminus B(0,1)} f(y)dy \le C(|x|/r)^{-N} H_{-N}^0 f(r/2)$$
$$\le C|x|^{-N} r^{N-\varepsilon_2'}\omega(r)^{-1}\Phi_\infty^{-1}(r^{-N})\left(\int_{1/2}^r (t^{\varepsilon_2'}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t}\right)^{1/q} \tag{7}$$

for $0 < \varepsilon_2' < \varepsilon_2$ by Lemma 8. Hence, using (5), we have

$$\int_2^\infty (\omega(r)\|Mf_{1,r}\|_{L^\Phi(A(0,r))})^q \frac{dr}{r}$$
$$\le C\int_2^\infty r^{-\varepsilon_2' q}\left\{\int_{1/2}^r (t^{\varepsilon_2'}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t}\right\}\frac{dr}{r}$$
$$\le C\int_{1/2}^\infty (t^{\varepsilon_2'}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \left(\int_t^\infty r^{-\varepsilon_2' q}\frac{dr}{r}\right)\frac{dt}{t}$$
$$\le C\int_{1/2}^\infty (\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t} \le C. \tag{8}$$

For $f_{2,r}$, by Lemma 6

$$\|Mf_{2,r}\|_{L^\Phi(A(0,r))} \le C\|f_{2,r}\|_{L^\Phi(\mathbf{R}^N)} = C\|f\|_{L^\Phi(B(0,4r)\setminus B(0,r/2))},$$

which implies

$$\int_2^\infty (\omega(r)\|Mf_{2,r}\|_{L^\Phi(A(0,r))})^q \frac{dr}{r} \le C. \tag{9}$$

For $f_{3,r}$, we find for $x \in B(0,2r)$

$$Mf_{3,r}(x) \le C\int_{\mathbf{R}^N\setminus B(0,4r)} f(y)|y|^{-N}dy \le CH_0^\infty f(2r)$$
$$\le Cr^{\varepsilon_1'}\omega(r)^{-1}\Phi_\infty^{-1}(r^{-N})\left(\int_r^\infty (t^{-\varepsilon_1'}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t}\right)^{1/q} \tag{10}$$

for $0 < \varepsilon_1' < \varepsilon_1$ by Lemma 7. Hence, by Lemma 4

$$\begin{aligned}\|Mf_{3,r}\|_{L^\Phi(A(0,r))} &\le \|Mf_{3,r}\|_{L^\Phi(B(0,2r))} \\ &\le Cr^{\varepsilon_1'}\omega(r)^{-1}\left(\int_r^\infty (t^{-\varepsilon_1'}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t}\right)^{1/q},\end{aligned}$$

so that

$$\begin{aligned}&\int_2^\infty (\omega(r)\|Mf_{3,r}\|_{L^\Phi(A(0,r))})^q \frac{dr}{r} \\ &\quad\le C\int_2^\infty r^{\varepsilon_1' q}\left(\int_r^\infty (t^{-\varepsilon_1'}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t}\right)\frac{dr}{r} \\ &\quad\le C\int_2^\infty (t^{-\varepsilon_1'}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q\left(\int_2^t r^{\varepsilon_1' q}\frac{dr}{r}\right)\frac{dt}{t} \\ &\quad\le C\int_2^\infty (\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t} \le C. \qquad (11)\end{aligned}$$

Combining (6), (8), (9) and (11), we obtain (4).

Finally we show

$$\|Mf\|_{L^\Phi(B(0,4))} \le C$$

since

$$\|Mf\|_{L^\Phi(B(0,2))} + \left(\int_1^2 (\omega(r)\|Mf\|_{L^\Phi(A(0,r))})^q \frac{dr}{r}\right)^{1/q} \le C\|Mf\|_{L^\Phi(B(0,4))}.$$

Set

$$f = f\chi_{B(0,8)} + f\chi_{\mathbf{R}^N\setminus B(0,8)} = f_4 + f_5.$$

By Lemmas 6 and 2,

$$\|Mf_4\|_{L^\Phi(B(0,4))} \le C\|f\|_{L^\Phi(B(0,8))} \le C$$

and

$$\begin{aligned}\|Mf_5\|_{L^\Phi(B(0,4))} &= \|Mf_{3,2}\|_{L^\Phi(B(0,4))} \\ &\le C\left(\int_2^\infty (\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t}\right)^{1/q} \le C\end{aligned}$$

by (10). □

## 4. Boundedness of the Hardy operator

For a locally integrable function $f$ on $\mathbf{R}^N$ and $\beta \in \mathbf{R}$, the Hardy functions $\hat{H}_\beta^\infty f$ and $\hat{H}_\beta^0 f$ are defined by

$$\hat{H}_\beta^\infty f(x) = |x|^\beta \int_{\mathbf{R}^N \setminus (B(0,|x|) \cup B(0,1))} |y|^{-N-\beta} |f(y)| dy$$

and

$$\hat{H}_\beta^0 f(x) = |x|^\beta \int_{B(0,|x|) \setminus B(0,1)} |y|^{-N-\beta} |f(y)| dy,$$

respectively.

Lemma 9. *For a real number $\beta$, suppose that $\Phi(x,t)$ satisfies $(\Phi 3; \infty; q)$ and $(\Phi 5; \eta)$ for $q \geq 1$ and $\eta > 0$ satisfying $\eta \leq q/N$. Assume that*
$(\Phi\omega; \beta)$ $\quad t^{-\varepsilon_3 + \beta} \{\bar{\Phi}^{-1}(0, t^{-N})\}^{-1}$ *is almost increasing in $(0,1]$ for some $\varepsilon_3 > 0$.*
*Then there exists a constant $C > 0$ such that*

$$\| |\cdot|^\beta \|_{L^\Phi(B(0,r))} \leq C r^\beta \{\bar{\Phi}^{-1}(0, r^{-N})\}^{-1}$$

*for all $0 < r \leq 1$.*

Proof. Let $0 < r \leq 1$. First note from (1), $(\Phi 2)$ and $(\Phi 3; \infty; q)$ that

$$r^{-N} < |x|^{-N} \leq 2 A_1 A_{2,\infty,q} A_3 \bar{\Phi}(0, |x|^{-N/q})$$

for $x \in B(0,r)$, so that we have by Lemma 3 (2) and (4)

$$|x| \leq C \{\bar{\Phi}^{-1}(0, r^{-N})\}^{-q/N}$$

for $x \in B(0,r)$ and

$$\bar{\Phi}^{-1}(0, r^{-N}) \geq \bar{\Phi}^{-1}(0,1) > 0.$$

Therefore we find by $(\Phi 5; \eta)$ and $\eta \leq q/N$

$$\int_{B(0,r)} \bar{\Phi}(x, \bar{\Phi}^{-1}(0, r^{-N})) dx \leq C \int_{B(0,r)} \bar{\Phi}(0, \bar{\Phi}^{-1}(0, r^{-N})) dx \leq C,$$

so that

$$\|\chi_{B(0,r)}\|_{L^\Phi(\mathbf{R}^N)} \leq C \{\bar{\Phi}^{-1}(0, r^{-N})\}^{-1}.$$

Hence we have by $(\Phi\omega; \beta)$

$$\begin{aligned}\| |\cdot|^{\beta} \|_{L^{\Phi}(B(0,r))} &\le \sum_{j=1}^{\infty} \| |\cdot|^{\beta} \|_{L^{\Phi}(A(0,2^{-j}r))} \\ &\le C \sum_{j=1}^{\infty} (2^{-j}r)^{\beta} \{\bar{\Phi}^{-1}(0,(2^{-j}r)^{-N})\}^{-1} \\ &\le C r^{-\varepsilon_3+\beta} \{\bar{\Phi}^{-1}(0,r^{-N})\}^{-1} \sum_{j=1}^{\infty} (2^{-j}r)^{\varepsilon_3} \\ &\le C r^{\beta} \{\bar{\Phi}^{-1}(0,r^{-N})\}^{-1},\end{aligned}$$

as required. □

THEOREM 2. *For a real number $\beta$, suppose that $\Phi(x,t)$ satisfies $(\Phi 3;\infty;q)$ and $(\Phi 5;\eta)$ for $q \ge 1$ and $\eta > 0$ satisfying $\eta \le q/N$. Assume that $\Phi(x,t)$ satisfies $(\Phi\omega;\beta)$ and $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty \omega 1;\beta)$. Then there exists a constant $C > 0$ such that*

$$\|\hat{H}_\beta^\infty f\|_{\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)} \le C \|f\|_{\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)}$$

*for all $f \in \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$.*

PROOF. Let $f$ be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)} \le 1$. Let $r \ge 2$. Then we have by Lemmas 4 and 7

$$\begin{aligned}\|\hat{H}_\beta^\infty f\|_{L^{\Phi}(A(0,r))} &\le C H_\beta^\infty f(r/2) \|1\|_{L^{\Phi}(A(0,r))} \\ &\le C r^{\varepsilon_1'} \omega(r)^{-1} \left( \int_{r/2}^{\infty} (t^{-\varepsilon_1'} \omega(t) \|f\|_{L^{\Phi}(A(0,t))})^q \frac{dt}{t} \right)^{1/q}\end{aligned}$$

for $0 < \varepsilon_1' < \varepsilon_1$. Therefore, as in the proof of Theorem 1, we obtain

$$\int_2^{\infty} (\omega(r) \|\hat{H}_\beta^\infty f\|_{L^{\Phi}(A(0,r))})^q \frac{dr}{r} \le C.$$

Finally we show

$$\|\hat{H}_\beta^\infty f\|_{L^{\Phi}(B(0,4))} \le C.$$

Note from Lemmas 7 and 2 that

$$\begin{aligned}\hat{H}_\beta^\infty f(x) &\le 2^{\beta} |x|^{\beta} H_\beta^\infty f(1/2) \\ &\le C|x|^{\beta} \left\{ \left( \int_1^{\infty} (\omega(t) \|f\|_{L^{\Phi}(A(0,t))})^q \frac{dt}{t} \right)^{1/q} + \int_{B(0,2)\setminus B(0,1)} f(y)dy \right\} \\ &\le C|x|^{\beta}.\end{aligned}$$

Hence we obtain by Lemma 9

$$\|\hat{H}_{\beta}^{\infty} f\|_{L^{\Phi}(B(0,4))} \le C\| \, |\cdot|^{\beta}\|_{L^{\Phi}(B(0,4))} \le C,$$

as required. □

In the same manner, using Lemma 8 instead of Lemma 7, we can prove the following result.

Theorem 3. *For a real number $\beta$, suppose that $\Phi(x,t)$ satisfies $(\Phi3;\infty;q)$ and $(\Phi5;\eta)$ for $q \ge 1$ and $\eta > 0$ satisfying $\eta \le q/N$. Assume that $\Phi(x,t)$ satisfies $(\Phi\omega;\beta)$ and $\Phi_{\infty}(t)$ satisfies $(\Phi_{\infty}2)$ and $(\Phi_{\infty}\omega2;\beta)$. Then there exists a constant $C > 0$ such that*

$$\|\hat{H}_{\beta}^{0} f\|_{\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)} \le C\|f\|_{\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)}$$

*for all $f \in \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$.*

In fact, note that

$$\hat{H}_{\beta}^{0} f(x) \le C|x|^{\beta} \int_{B(0,4)\setminus B(0,1)} |f(y)|dy \le C|x|^{\beta}$$

for $x \in B(0,4)$ and $f \in \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$ with $\|f\|_{\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)} \le 1$.

## 5. Sobolev's inequality

For $0 < \alpha < N$, the Riesz potential $I_{\alpha}f$ is defined by

$$I_{\alpha}f(x) = \int_{\mathbf{R}^N} |x-y|^{\alpha-N} f(y)dy$$

for a locally integrable function $f$ on $\mathbf{R}^N$.

Lemma 10. *Assume that $\Phi_{\infty}(t)$ satisfies $(\Phi_{\infty}2)$ and $(\Phi_{\infty}\omega1;-\alpha)$ for $\varepsilon_1 > 0$. Then, for $0 < \varepsilon < \varepsilon_1$, there exists a constant $C > 0$ such that, for all $x \in B(0,2r)$ with $r \ge 1$ and nonnegative functions $f \in L^1_{loc}(\mathbf{R}^N)$,*

$$|I_{\alpha}(f\chi_{\mathbf{R}^N \setminus B(0,4r)})(x)| \\ \le Cr^{\varepsilon+\alpha}\omega(r)^{-1}\Phi_{\infty}^{-1}(r^{-N})\left(\int_r^{\infty} (t^{-\varepsilon}\omega(t)\|f\|_{L^{\Phi}(A(0,t))})^q \frac{dt}{t}\right)^{1/q}.$$

Proof. Let $f \in L^1_{loc}(\mathbf{R}^N)$ be a nonnegative function on $\mathbf{R}^N$. Let $r \ge 1$, $x \in B(0,2r)$ and $0 < \varepsilon < \varepsilon_1$. Note from Lemma 7 with $\beta = -\alpha$ that

$$
\begin{aligned}
&|I_\alpha(f\chi_{\mathbf{R}^N\setminus B(0,4r)})(x)| \\
&\quad \le C\int_{\mathbf{R}^N\setminus B(0,4r)} |y|^{\alpha-N} f(y)dy \\
&\quad \le Cr^{\varepsilon+\alpha}\omega(r)^{-1}\Phi_\infty^{-1}(r^{-N})\left(\int_r^\infty (t^{-\varepsilon}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t}\right)^{1/q},
\end{aligned}
$$

as required. □

LEMMA 11. *Assume that $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty\omega 2; -N)$ for $\varepsilon_2 > 0$. Then, for $0 < \varepsilon < \varepsilon_2$, there exists a constant $C > 0$ such that for all $x \in \mathbf{R}^N\setminus B(0,r)$ with $r \ge 1$ and nonnegative functions $f \in L^1_{loc}(\mathbf{R}^N)$,*

$$
\begin{aligned}
|I_\alpha(f\chi_{B(0,r/2)\setminus B(0,1)})(x)| &\le C(|x|/r)^{\alpha-N} r^{-\varepsilon+\alpha}\omega(r)^{-1}\Phi_\infty^{-1}(r^{-N}) \\
&\quad \times \left(\int_{1/2}^r (t^{\varepsilon}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t}\right)^{1/q}.
\end{aligned}
$$

PROOF. Let $f \in L^1_{loc}(\mathbf{R}^N)$ be a nonnegative function on $\mathbf{R}^N$. Let $r \ge 1$, $x \in \mathbf{R}^N\setminus B(0,r)$ and $0 < \varepsilon < \varepsilon_2$. Note that

$$
\begin{aligned}
|I_\alpha(f\chi_{B(0,r/2)\setminus B(0,1)})(x)| &\le C|x|^{\alpha-N}\int_{B(0,r/2)\setminus B(0,1)} f(y)dy \\
&= C(|x|/r)^{\alpha-N} r^\alpha H^0_{-N} f(r/2),
\end{aligned}
$$

so that Lemma 8 with $\beta = -N$ gives the required result. □

We consider a function

$$\Psi(x,t) = t\psi(x,t) : \mathbf{R}^N \times [0,\infty) \to [0,\infty)$$

satisfying the conditions $(\Phi 1)$–$(\Phi 4)$ with $\phi$ replaced by $\psi$.

Now we consider the following conditions:

$(\Phi_\infty 2')$ there exists a constant $Q \ge 1$ such that

$$\Phi_\infty(g^*(x)) \le Q(1+|x|)^{-N}$$

for all $x \in \mathbf{R}^N$, where $g^*(x) = \max\{g(x), Mg(x)\}$;

$(\Phi\alpha)$ $r \mapsto r^{\varepsilon+\alpha}\bar{\Phi}^{-1}(x, r^{-N})$ is uniformly almost decreasing on $(0,\infty)$ for some $\varepsilon > 0$;

$(\Psi\Phi\alpha)$ there exists a constant $Q \ge 1$ such that

$$\Psi(x, t\Phi(x,t)^{-\alpha/N}) \le Q\Phi(x,t)$$

for all $x \in \mathbf{R}^N$ and $t > 0$.

Remark 10. Let $\Phi(x,t)$ be as in Example 1 and let $\Phi_\infty(t)$ be as in Remark 8 with $\tau = 1$. Assume

$$\inf_{x\in\mathbf{R}^N}(N-\alpha p_1(x)) > 0 \qquad \text{and} \qquad \inf_{x\in\mathbf{R}^N}(N-\alpha p_2(x)) > 0.$$

Then $\Phi(x,t)$ satisfies $(\Phi\alpha)$ and $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2')$.

Set

$$\Psi(x,t) = \begin{cases} t^{p_1^*(x)}\prod_{j=1}^{k_1}(L_{e-1}^{(j)}(1/t))^{-q_{1,j}(x)p_1^*(x)/p_1(x)} & \text{if } 0 \le t \le 1; \\ t^{p_2^*(x)}\prod_{j=1}^{k_2}(L_{e-1}^{(j)}(t))^{q_{2,j}(x)p_2^*(x)/p_2(x)} & \text{if } t \ge 1, \end{cases}$$

where

$$\frac{1}{p_i^*(x)} = \frac{1}{p_i(x)} - \frac{\alpha}{N}$$

for $i = 1, 2$. Then $\Psi(x,t)$ satisfies $(\Psi\Phi\alpha)$.

As Sobolev's inequality for Riesz potentials of functions in Musielak-Orlicz spaces $L^\Phi(\mathbf{R}^N)$, we give the following lemma ([30, Corollary 6.5]). Here we shall state our result without assumptions

$(\Phi_\infty 3)$ $r \mapsto r^\gamma \Phi_\infty^{-1}(r^{-N})$ is almost increasing on $[1,\infty)$ for some $0 < \gamma < N$ and

$(\Phi_\infty \alpha)$ $r \mapsto r^{\varepsilon+\alpha}\Phi_\infty^{-1}(r^{-N})$ is almost decreasing on $[1,\infty)$ for some $\varepsilon > 0$,

which are assumed in [30] (see Remark 11 below).

Lemma 12. *Suppose $\Phi(x,t)$ satisfies $(\Phi 3;0;p)$, $(\Phi 3;\infty;q)$, $(\Phi 5;\eta)$, $(\Phi 6;\tau)$ and $(\Phi\alpha)$ for $p > 1$, $q > 1$, $\eta > 0$ and $\tau > 0$ satisfying $\eta \le q/N$ and $\tau \le p$. For the function $\Phi_\infty(t)$, assume $(\Phi_\infty 2')$ holds. Further, assume that $\Psi(x,t)$ satisfies $(\Psi\Phi\alpha)$. Then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{L^\Psi(\mathbf{R}^N)} \le C\|f\|_{L^\Phi(\mathbf{R}^N)}$$

*for all $f \in L^\Phi(\mathbf{R}^N)$.*

Remark 11. Assumptions $(\Phi 3;0;p)$ and $(\Phi 3;\infty;q)$ imply $(\Phi_\infty 3)$ and $(\Phi\alpha)$ implies $(\Phi_\infty\alpha)$.

In fact, we see from $(\Phi 3;0;p)$ and $(\Phi 3;\infty;q)$ that there exist constants $c > 0$ and $\varepsilon > 0$ such that

$$t^{-(1+\varepsilon)}\Phi(x,t) \le cs^{-(1+\varepsilon)}\Phi(x,s)$$

for all $0 < t < s$, so that for $0 < t < s \le \max\{1, \Phi_\infty^{-1}(1)\}$, there exists a point $x_0 \in \mathbf{R}^N$ such that $g(x_0) \le t < s \le \max\{1, \Phi_\infty^{-1}(1)\}$ and

$$t^{-(1+\varepsilon)}\Phi_\infty(t) \le Qt^{-(1+\varepsilon)}\Phi(x_0,t) \le cQs^{-(1+\varepsilon)}\Phi(x_0,s) \le cQ^2 s^{-(1+\varepsilon)}\Phi_\infty(s)$$

by $(\Phi_\infty 1)$ and $g \in L^\tau(\mathbf{R}^N)$, where $Q$ is the constant appearing in $(\Phi_\infty 1)$. Since $\Phi_\infty^{-1}(r^{-N}) \le \Phi_\infty^{-1}(1)$ for all $r \ge 1$, we see that $(\Phi_\infty 3)$ holds with $\gamma = N/(1+\varepsilon)$. Similarly we can show that $(\Phi\alpha)$ implies $(\Phi_\infty\alpha)$.

Further, as the definition of $\Phi_\infty(t)$, we consider a convex function $\Psi_\infty(t) = t\psi_\infty(t) : [0,\infty) \to [0,\infty)$ such that $\psi_\infty(t) > 0$ for $t > 0$, $\psi_\infty(t)$ is increasing on $[0,\infty)$ and satisfies the doubling condition and

$(\Psi_\infty 1)$ there exists a constant $Q \ge 1$ such that

$$Q^{-1}\Psi(x,t) \le \Psi_\infty(t) \le Q\Psi(x,t) \qquad \text{whenever } g(x) \le t \le 1,$$

where $g$ is the function appearing in $(\Phi 6;\tau)$.

Now we show the Sobolev type inequality for Riesz potentials of functions in $\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$.

THEOREM 4. *Suppose $\Phi(x,t)$ satisfies $(\Phi 3;0;p)$, $(\Phi 3;\infty;q)$, $(\Phi 5;\eta)$, $(\Phi 6;\tau)$ and $(\Phi\alpha)$ for $p>1$, $q>1$, $\eta>0$ and $\tau>0$ satisfying $\eta \le q/N$ and $\tau \le p$. Assume that $\Psi(x,t)$ satisfies $(\Psi\Phi\alpha)$. For the function $\Phi_\infty(t)$, assume $(\Phi_\infty 2')$, $(\Phi_\infty\omega 1;-\alpha)$ and $(\Phi_\infty\omega 2;-N)$ hold. Then there exists a constant $C>0$ such that*

$$\|I_\alpha f\|_{\mathscr{H}^{\Psi,q,\omega}(\mathbf{R}^N)} \le C\|f\|_{\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)}$$

*for all $f \in \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$.*

REMARK 12. Suppose $\Phi(x,t)$ satisfies $(\Phi\alpha)$ and $\Psi(x,t)$ satisfies $(\Psi\Phi\alpha)$. For the function $\Phi_\infty(t)$, assume $(\Phi_\infty 2')$ holds.

To show Theorem 4, we need to verify that the following conditions hold:

$(\Psi_\infty 2)$ there exists a constant $Q \ge 1$ such that

$$\Psi_\infty(g(x)) \le Q(1+|x|)^{-N}$$

for all $x \in \mathbf{R}^N$;

$(\Psi_\infty\Phi_\infty\alpha)$ there exists a constant $Q>0$ such that

$$\sup_{t\ge 1} t^\alpha \Phi_\infty^{-1}(t^{-N})\{\Psi_\infty^{-1}(t^{-N})\}^{-1} \le Q.$$

First we show that $(\Psi_\infty 2)$ holds. Since we have by $(\Psi\Phi\alpha)$ and $(\Psi_\infty 1)$

$$\begin{aligned}\Phi(x,g(x)) &\ge C\Psi(x, g(x)\Phi(x,g(x))^{-\alpha/N}) \\ &\ge C\Psi(x, g(x)\Phi(x,1)^{-\alpha/N}) \ge C\Psi(x,g(x)) \ge C\Psi_\infty(g(x)),\end{aligned}$$

we find by $(\Phi_\infty 1)$ and $(\Phi_\infty 2)$

$$\Psi_\infty(g(x)) \le C\Phi(x,g(x)) \le C\Phi_\infty(g(x)) \le C(1+|x|)^{-N}.$$

Next we show that $(\Psi_\infty \Phi_\infty \alpha)$ holds. For $0 < t \le \max\{1, \Phi_\infty^{-1}(1)\}$, there exists a point $x_0 \in \mathbf{R}^N$ such that $g(x_0) \le t \le \max\{1, \Phi_\infty^{-1}(1)\}$ by $g \in L^\tau(\mathbf{R}^N)$. Then note that

$$t\Phi(x_0, t)^{-\alpha/N} \ge Ct \ge Cg(x_0)$$

and $t\Phi(x_0, t)^{-\alpha/N} \le C$ by $(\Phi\alpha)$, so that we find by $(\Phi_\infty 1)$ and $(\Psi_\infty 1)$

$$\Psi_\infty(t\Phi_\infty(t)^{-\alpha/N}) \le C\Phi_\infty(t).$$

Since $\Phi_\infty^{-1}(r^{-N}) \le \Phi_\infty^{-1}(1)$ for all $r \ge 1$, we obtain that $(\Psi_\infty \Phi_\infty \alpha)$ holds.

Proof (Proof of Theorem 4). Let $f$ be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)} \le 1$. For $r \ge 2$ set

$$\begin{aligned} f &= f\chi_{B(0,1)} + f\chi_{B(0,r/2)\setminus B(0,1)} + f\chi_{B(0,4r)\setminus B(0,r/2)} + f\chi_{\mathbf{R}^N\setminus B(0,4r)} \\ &= f_0 + f_{1,r} + f_{2,r} + f_{3,r}. \end{aligned}$$

For $f_0$, we see that

$$I_\alpha f_0(x) \le C|x|^{\alpha-N} \int_{B(0,1)} f(y)dy \le C|x|^{\alpha-N}$$

for $x \in \mathbf{R}^N \setminus B(0,r)$ by Lemma 2. By Lemmas 12 and 4

$$\begin{aligned} r^N & \| |\cdot|^{\alpha-N} \|_{L^\Psi(\mathbf{R}^N\setminus B(0,r))} \\ &\le C\|I_\alpha \chi_{B(0,r/2)}\|_{L^\Psi(\mathbf{R}^N\setminus B(0,r))} \\ &\le C\|\chi_{B(0,r/2)}\|_{L^\Phi(\mathbf{R}^N)} \le C\{\Phi_\infty^{-1}(r^{-N})\}^{-1}. \end{aligned} \tag{12}$$

Hence

$$\|I_\alpha f_0\|_{L^\Psi(A(0,r))} \le Cr^{-N}\{\Phi_\infty^{-1}(r^{-N})\}^{-1},$$

and using $(\Phi_\infty \omega 2; -N)$, we have (cf. (6))

$$\int_2^\infty (\omega(r)\|I_\alpha f_0\|_{L^\Psi(A(0,r))})^q \frac{dr}{r} \le C. \tag{13}$$

For $f_{1,r}$, by Lemma 11 and (12), we have

$$\|I_\alpha f_{1,r}\|_{L^\Psi(A(0,r))} \le Cr^{-\varepsilon_2'}\omega(r)^{-1}\left(\int_{1/2}^r (t^{\varepsilon_2'}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t}\right)^{1/q}$$

for $0 < \varepsilon_2' < \varepsilon_2$, which implies (cf. (8))

$$\int_2^\infty (\omega(r)\|I_\alpha f_{1,r}\|_{L^\Psi(A(0,r))})^q \frac{dr}{r} \le C. \tag{14}$$

For $f_{2,r}$, we use Lemma 12 and have

$$\|I_\alpha f_{2,r}\|_{L^{\Psi}(A(0,r))} \le C\|f\|_{L^{\Phi}(B(0,4r)\setminus B(0,r/2))}.$$

Hence

$$\int_2^\infty (\omega(r)\|I_\alpha f_{2,r}\|_{L^{\Psi}(A(0,r))})^q \frac{dr}{r} \le C. \tag{15}$$

To treat $f_{3,r}$, we remark that Lemma 4 holds for $\Psi$ by $(\Psi_\infty 2)$ and hence

$$\|1\|_{L^{\Psi}(B(0,r))} \le C\{\Psi_\infty^{-1}(r^{-N})\}^{-1} \le Cr^{-\alpha}\{\Phi_\infty^{-1}(r^{-N})\}^{-1}$$

by $(\Psi_\infty\Phi_\infty\alpha)$. Thus, we find by Lemma 10

$$\begin{aligned}
&\|I_\alpha f_{3,r}\|_{L^{\Psi}(A(0,r))} \\
&\quad \le \|I_\alpha f_{3,r}\|_{L^{\Psi}(B(0,r))} \\
&\quad \le Cr^{\varepsilon_1'}\omega(r)^{-1}\left(\int_r^\infty (t^{-\varepsilon_1'}\omega(t)\|f\|_{L^{\Phi}(A(0,t))})^q \frac{dt}{t}\right)^{1/q}
\end{aligned} \tag{16}$$

for $0 < \varepsilon_1' < \varepsilon_1$. It then follows that

$$\int_2^\infty (\omega(r)\|I_\alpha f_{3,r}\|_{L^{\Psi}(A(0,r))})^q \frac{dr}{r} \le C. \tag{17}$$

Combining (13), (14), (15) and (17), we obtain

$$\int_2^\infty (\omega(r)\|I_\alpha f\|_{L^{\Psi}(A(0,r))})^q \frac{dr}{r} \le C.$$

Finally, Lemma 12 and (16) with $r = 2$ yield

$$\begin{aligned}
&\|I_\alpha f\|_{L^{\Psi}(B(0,2))} + \left(\int_1^2 (\omega(r)\|I_\alpha f\|_{L^{\Psi}(A(0,r))})^q \frac{dr}{r}\right)^{1/q} \\
&\quad \le C\|I_\alpha f\|_{L^{\Psi}(B(0,4))} \\
&\quad \le C\left(\|f\|_{L^{\Phi}(B(0,8))} + \int_1^\infty (\omega(t)\|f\|_{L^{\Phi}(A(0,t))})^q \frac{dt}{t}\right) \le C,
\end{aligned}$$

which proves the theorem. □

REMARK 13. Let $\Phi$ be as in Example 1 and let $\Phi_\infty(t)$ be as in Remark 8. If $\omega(r) = r^\nu$, then $(\Phi_\infty\omega 1; -\alpha)$ and $(\Phi_\infty\omega 2; -N)$ hold when

$$\alpha - N/p_1(\infty) < \nu < N(1 - 1/p_1(\infty)).$$

REMARK 14. Let $\Phi(x,t)$, $\Phi_\infty(t)$ and $\Psi(x,t)$ be as in Example 1, Remark 8 and Remark 10. Assume

$$\inf_{x \in \mathbf{R}^N} (N - \alpha p_1(x)) > 0.$$

Set $\bar{\psi}_\infty(t) = \sup_{0 \le s \le t} \{ s^{p_1^*(\infty)-1} \prod_{j=1}^{k_1} (L_{e-1}^{(j)}(1/s))^{-q_{1,j}(\infty)p_1^*(\infty)/p_1(\infty)} \}$ and

$$\Psi_\infty(t) = \int_0^t \bar{\psi}_\infty(r) \frac{dr}{r},$$

where

$$\frac{1}{p_1^*(\infty)} = \frac{1}{p_1(\infty)} - \frac{\alpha}{N}.$$

Then $\Psi_\infty(t)$ satisfies $(\Psi_\infty 1)$.

## 6. Sobolev's inequality for the generalized Riesz potential

To obtain general results, for $0 < \alpha < N$ and an integer $k \ge 1$, we define the generalized Riesz potential $I_{\alpha,k} f$ of order $\alpha$ of a locally integrable function $f$ on $\mathbf{R}^N$ by

$$I_{\alpha,k} f(x) = \int_{B(0,1)} I_\alpha(x-y) f(y) dy + \int_{\mathbf{R}^N \setminus B(0,1)} \left\{ I_\alpha(x-y) - \sum_{\{\mu : |\mu| \le k-1\}} \frac{x^\mu}{\mu!} (D^\mu I_\alpha)(-y) \right\} f(y) dy,$$

where $I_\alpha(x) = |x|^{\alpha - N}$ (see [32, 33]).

Set

$$\tilde{I}_{\alpha,k}(x,y) = I_\alpha(x-y) - \sum_{\{\mu : |\mu| \le k-1\}} \frac{x^\mu}{\mu!} (D^\mu I_\alpha)(-y)$$

and

$$\tilde{I}_{\alpha,k} f(x) = \int_{\mathbf{R}^N \setminus B(0,1)} \tilde{I}_{\alpha,k}(x,y) f(y) dy$$

for a locally integrable function $f$ on $\mathbf{R}^N$.

The following estimates are fundamental (see [32], [33] and [49]).

LEMMA 13. (1) *If* $2|x| < |y|$, *then* $|\tilde{I}_{\alpha,k}(x,y)| \le C|x|^k |y|^{\alpha - N - k}$.
(2) *If* $|x|/2 \le |y| \le 2|x|$, *then* $|\tilde{I}_{\alpha,k}(x,y)| \le C|x-y|^{\alpha - N}$.
(3) *If* $1 \le |y| \le |x|/2$, *then* $|\tilde{I}_{\alpha,k}(x,y)| \le C|x|^{k-1} |y|^{\alpha - N - (k-1)}$.

LEMMA 14. *Assume that $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty\omega 1; k-\alpha)$ for $\varepsilon_1 > 0$. Then, for $0 < \varepsilon < \varepsilon_1$, there exists a constant $C > 0$ such that, for all $x \in B(0,2r)$ with $r \ge 1$ and nonnegative functions $f \in L^1_{loc}(\mathbf{R}^N)$,*

$$|\tilde{I}_{\alpha,k}(f\chi_{\mathbf{R}^N\setminus B(0,4r)})(x)|$$
$$\le Cr^{\varepsilon+\alpha}\omega(r)^{-1}\Phi_\infty^{-1}(r^{-N})\left(\int_r^\infty (t^{-\varepsilon}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t}\right)^{1/q}.$$

PROOF. Let $f \in L^1_{loc}(\mathbf{R}^N)$ be nonnegative, $r \ge 1$ and $x \in B(0,2r)$. By Lemma 13 (1),

$$|\tilde{I}_{\alpha,k}(f\chi_{\mathbf{R}^N\setminus B(0,4r)})(x)| \le C|x|^k \int_{\mathbf{R}^N\setminus B(0,4r)} |y|^{\alpha-N-k} f(y)dy$$
$$\le Cr^\alpha H^\infty_{k-\alpha} f(2r),$$

so that we obtain the required inequality by Lemma 7. □

LEMMA 15. *Assume $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty\omega 2; k-1-\alpha)$ for $\varepsilon_2 > 0$. Then, for $0 < \varepsilon < \varepsilon_2$, there exists a constant $C > 0$ such that for all $x \in B(0,2r)$ with $r \ge 1$ and nonnegative functions $f \in L^1_{loc}(\mathbf{R}^N)$,*

$$|\tilde{I}_{\alpha,k}(f\chi_{B(0,|x|/2)})(x)|$$
$$\le Cr^{-\varepsilon+\alpha}\omega(r)^{-1}\Phi_\infty^{-1}(r^{-N})\left(\int_{1/2}^r (t^{\varepsilon}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t}\right)^{1/q}.$$

PROOF. Let $f \in L^1_{loc}(\mathbf{R}^N)$ be nonnegative, $r \ge 1$ and $x \in B(0,2r)$. By Lemma 13 (3),

$$|\tilde{I}_{\alpha,k}(f\chi_{B(0,|x|/2)})(x)| \le C|x|^{k-1} \int_{B(0,|x|/2)\setminus B(0,1)} |y|^{\alpha-N-(k-1)} f(y)dy$$
$$\le Cr^\alpha H^0_{k-1-\alpha} f(r),$$

so that we obtain the required inequality by Lemma 8. □

Now we give the Sobolev type inequality for generalized Riesz potentials of functions in $\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$.

THEOREM 5. *Suppose $\Phi(x,t)$ satisfies $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \eta)$, $(\Phi 6; \tau)$ and $(\Phi\alpha)$ for $p > 1$, $q > 1$, $\eta > 0$ and $\tau > 0$ satisfying $\eta \le q/N$ and $\tau \le p$. Assume that $\Psi(x,t)$ satisfies $(\Psi\Phi\alpha)$. Assume $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2')$, $(\Phi_\infty\omega 1;$*

$k-\alpha)$ *and* $(\Phi_\infty\omega 2; k-1-\alpha)$. *Then there exists a constant* $C>0$ *such that*

$$\|I_{\alpha,k}f\|_{\mathscr{H}^{\Psi,q,\omega}(\mathbf{R}^N)} \le C\|f\|_{\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)}$$

*for all* $f \in \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$.

Proof. Let $f$ be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)} \le 1$. For $r \ge 2$ and fixed $x \in A(0,r)$, set

$$\begin{aligned} f &= f\chi_{B(0,1)} + f\chi_{B(0,|x|/2)\setminus B(0,1)} + f\chi_{B(0,4r)\setminus B(0,|x|/2)} + f\chi_{\mathbf{R}^N\setminus B(0,4r)} \\ &= f_0 + f_{1,x} + f_{2,r,x} + f_{3,r}. \end{aligned}$$

For $f_0$, we note

$$I_{\alpha,k}f_0(x) = I_\alpha f_0(x) \le C|x|^{\alpha-N}.$$

For $f_{2,r,x}$, by Lemma 13 (1), (2), we see that

$$|I_{\alpha,k}f_{2,r,x}(x)| = |\tilde{I}_{\alpha,k}f_{2,r,x}(x)| \le CI_\alpha(f\chi_{B(0,4r)\setminus B(0,r/2)})(x).$$

Since $I_{\alpha,k}f_{1,x} = \tilde{I}_{\alpha,k}(f\chi_{B(0,|x|/2)})$ and $I_{\alpha,k}f_{3,r} = \tilde{I}_{\alpha,k}(f\chi_{\mathbf{R}^N\setminus B(0,4r)})$, it follows from Lemmas 15 and 14 that

$$\begin{aligned} I_{\alpha,k}f(x) \le C\Bigg\{ & |x|^{\alpha-N} + I_\alpha(f\chi_{B(0,4r)\setminus B(0,r/2)})(x) \\ & + r^{-\varepsilon_2'+\alpha}\omega(r)^{-1}\Phi_\infty^{-1}(r^{-N})\left(\int_{1/2}^{r}(t^{\varepsilon_2'}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q\frac{dt}{t}\right)^{1/q} \\ & + r^{\varepsilon_1'+\alpha}\omega(r)^{-1}\Phi_\infty^{-1}(r^{-N})\left(\int_{r}^{\infty}(t^{-\varepsilon_1'}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q\frac{dt}{t}\right)^{1/q}\Bigg\} \end{aligned} \tag{18}$$

for $x \in A(0,r)$, with $0<\varepsilon_1'<\varepsilon_1$ and $0<\varepsilon_2'<\varepsilon_2$.

Then, we obtain

$$\int_2^\infty (\omega(r)\|I_{\alpha,k}f\|_{L^\Psi(A(0,r))})^q\frac{dr}{r} \le C$$

by the same arguments as in the proof of Theorem 4.

By Lemma 13, we see that $|\tilde{I}_{\alpha,k}(x,y)| \le CI_\alpha(x-y)$ for $|x|\le 4$ and $|y|\ge 1$. Hence, as in the proof of Theorem 4 we can show that $\|I_{\alpha,k}f\|_{L^\Psi(B(0,4))} \le C$, which implies

$$\|I_{\alpha,k}f\|_{L^\Psi(B(0,2))} + \int_1^2(\omega(r)\|I_{\alpha,k}f\|_{L^\Psi(A(0,r))})^q\frac{dr}{r} \le C. \qquad \square$$

Remark 15. Let $\Phi$ be as in Example 1 and let $\Phi_\infty(t)$ be as in Remark 8. If $\omega(r) = r^\nu$, then $(\Phi_\infty\omega1; k-\alpha)$ and $(\Phi_\infty\omega2; k-1-\alpha)$ hold when

$$\alpha - N/p_1(\infty) - \nu < k < \alpha - N/p_1(\infty) - \nu + 1.$$

## 7. $\underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$ and $\overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$

We further consider the space $\underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$ of locally integrable functions $f$ on $\mathbf{R}^N$ satisfying

$$\|f\|_{\underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)} = \left(\int_1^\infty (\omega(r)\|f\|_{L^\Phi(B(0,r))})^q \frac{dr}{r}\right)^{1/q} < \infty$$

and the space $\overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$ consisting of all measurable functions $f$ on $\mathbf{R}^N$ satisfying

$$\|f\|_{\overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)} = \|f\|_{L^\Phi(B(0,2))} + \left(\int_1^\infty (\omega(r)\|f\|_{L^\Phi(\mathbf{R}^N\setminus B(0,r))})^q \frac{dr}{r}\right)^{1/q} < \infty.$$

If $\omega(r)$ satisfies

($\omega3$) $\displaystyle\int_1^\infty \omega(r)^q \frac{dr}{r} < \infty,$

then

$$L^\Phi(\mathbf{R}^N) = \overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N) \hookrightarrow \underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N) \hookrightarrow \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N) \tag{19}$$

and if $\omega$ satisfies

($\omega4$) $\displaystyle\int_1^\infty \omega(r)^q \frac{dr}{r} = \infty,$

then

$$\{0\} = \underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N) \subset \overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N) \hookrightarrow \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N) \cap L^\Phi(\mathbf{R}^N).$$

Therefore, it is natural to assume ($\omega3$) when we treat the space $\underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$; and we assume ($\omega4$) when we treat the space $\overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$.

Proposition 1. (1) *Suppose* $\omega(r)$ *satisfies*

($\omega5a$) $r \mapsto r^a\omega(r)$ *is almost decreasing on* $[1,\infty)$ *for some* $a > 0$.

*Then,* $\underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N) = \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$.

(2) *Suppose* $\omega(r)$ *satisfies*

($\omega5b$) $r \mapsto r^{-b}\omega(r)$ *is almost increasing on* $[1,\infty)$ *for some* $b > 0$.

*Then,* $\overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N) = \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$.

Proof. (1) Assume ($\omega5a$). Let $X = \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$ and $Y = \underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$. Since $Y \hookrightarrow X$ in general, we have to show $X \hookrightarrow Y$.

Let $f \in X$ and let $K$ be a compact set in $\mathbf{R}^N$. Then note from Lemma 2 and $(\omega 5a)$ ($(\omega 5a)$ implies $(\omega 3)$) that $\|f\chi_K\|_Y < \infty$. Set $F(r) = \|f\chi_K\|_{L^{\Phi}(A(0,r))}$ and $G(r) = \|f\chi_K\|_{L^{\Phi}(B(0,r))}$. Then

$$G(2r) \le F(r) + G(r). \tag{20}$$

Set $\omega_*(r) = r^{-a} \inf_{1 \le s \le r} s^a \omega(s)$ for $r \ge 1$. Then

$$\omega_*(r) \le \omega(r) \le C_0 \omega_*(r) \tag{21}$$

by $(\omega 5a)$ with a constant $C_0 \ge 1$.

Set

$$A = \left( \int_1^{\infty} (\omega_*(r) F(r))^q \frac{dr}{r} \right)^{1/q},$$

$$B_1 = \left( \int_1^{2} (\omega_*(r) G(r))^q \frac{dr}{r} \right)^{1/q} \quad \text{and} \quad B_2 = \left( \int_2^{\infty} (\omega_*(r) G(r))^q \frac{dr}{r} \right)^{1/q}.$$

All of these are finite values.

By (20)

$$\left( \int_1^{\infty} (\omega_*(r) G(2r))^q \frac{dr}{r} \right)^{1/q} \le \begin{cases} A + (B_1^q + B_2^q)^{1/q} & \text{if } q \ge 1, \\ (A^q + B_1^q + B_2^q)^{1/q} & \text{if } 0 < q < 1. \end{cases}$$

Since $r^a \omega_*(r)$ is decreasing, $\omega_*(r/2) \ge 2^a \omega_*(r)$ for $r \ge 2$, so that

$$\int_1^{\infty} (\omega_*(r) G(2r))^q \frac{dr}{r} = \int_2^{\infty} (\omega_*(r/2) G(r))^q \frac{dr}{r} \ge 2^{aq} B_2^q.$$

Hence

$$2^a B_2 \le \begin{cases} A + (B_1^q + B_2^q)^{1/q} & \text{if } q \ge 1, \\ (A^q + B_1^q + B_2^q)^{1/q} & \text{if } 0 < q < 1, \end{cases}$$

which implies

$$(B_1^q + B_2^q)^{1/q} \le C(A + 2^a B_1) \tag{22}$$

with $C > 0$ depending only on $a$ and $q$. Note that $B_1 \le C_1 \|f\chi_K\|_{L^{\Phi}(B(0,2))}$ with $C_1 = \omega(1)(\log 2)^{1/q}$. By (21) $\|f\chi_K\|_Y \le C_0 (B_1^q + B_2^q)^{1/q}$ and $\|f\chi_K\|_X \ge A + \|f\chi_K\|_{L^{\Phi}(B(0,2))}$. Hence (22) implies

$$\|f\chi_K\|_Y \le C \|f\chi_K\|_X$$

with a constant $C > 0$ independent of $K$. By the monotone convergence theorem, we obtain the required result.

(2) Assume ($\omega 5b$). Let $X$ be as above and $Z = \overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$. Since $Z \hookrightarrow X$ in general, we shall show $X \hookrightarrow Z$.

Let $f \in X$ and let $K$ be a compact set in $\mathbf{R}^N$. Then note from Lemma 2 that $\|f\chi_K\|_Z < \infty$. Set $F(r) = \|f\chi_K\|_{L^\Phi(A(0,r))}$ and $H(r) = \|f\chi_K\|_{L^\Phi(\mathbf{R}^N \setminus B(0,r))}$. Then

$$H(r) \le F(r) + H(2r). \tag{23}$$

Set $\omega^*(r) = r^b \sup_{1 \le s \le r} s^{-b}\omega(s)$ for $r \ge 1$. Then

$$\omega(r) \le \omega^*(r) \le C_2\omega(r) \tag{24}$$

by ($\omega 5b$) with a constant $C_2 \ge 1$. Since $r^{-b}\omega^*(r)$ is increasing, $\omega^*(r/2) \le 2^{-b}\omega^*(r)$ for $r \ge 2$, so that

$$\int_1^\infty (\omega^*(r)H(2r))^q \frac{dr}{r} \le 2^{-bq} \int_2^\infty (\omega^*(r)H(r))^q \frac{dr}{r}.$$

Hence, by (23), we have

$$\int_1^\infty (\omega^*(r)H(r))^q \frac{dr}{r} \le C \int_1^\infty (\omega^*(r)F(r))^q \frac{dr}{r},$$

which implies $\|f\chi_K\|_Z \le C\|f\chi_K\|_X$ in view of (24) with a constant $C > 0$ independent of $K$. Hence, by the monotone convergence theorem, we obtain the required result. □

The following example shows that there are $\omega(r)$ satisfying ($\omega 3$) for which $L^\Phi(\mathbf{R}^N) \ne \underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N) \ne \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$; and also there are $\omega(r)$ satisfying ($\omega 4$) for which $\{0\} \ne \overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N) \ne \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$.

EXAMPLE 3. Let $\Phi(x,t) = t^p$, $p \ge 1$ and

$$\omega(r) \sim \log(e+r)^\nu, \qquad \nu \in \mathbf{R}.$$

(1) If $\nu < -1/q$, then $\omega(r)$ satisfies ($\omega 3$) and

$$L^p(\mathbf{R}^N) \ne \underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N) \ne \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N).$$

(2) If $\nu \ge -1/q$, then $\omega(r)$ satisfies ($\omega 4$) and

$$\{0\} \ne \overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N) \ne \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N).$$

In fact, consider the function

$$f_a(x) = |x|^{-N/p}(\log(e+|x|))^{-a}\chi_{\mathbf{R}^N \setminus B(0,2)}(x)$$

for $a \in \mathbf{R}$. Then, $\|f_a\|_{L^p(A(0,r))} \sim (\log(e+r))^{-a}$ for $r \ge 2$, so that

$$f_a \in \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N) \qquad \text{if (and only if) } a > \nu + 1/q.$$

On the other hand, for $r \ge 3$, $\|f_a\|_{L^p(B(0,r))} \sim (\log(e+r))^{-a+1/p}$ in case $a < 1/p$, $\sim (\log(\log(e+r)))^{1/p}$ in case $a = 1/p$ and $\sim C$ in case $a > 1/p$, so that

$$f_a \in \underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N) \qquad \text{if and only if } a > v + 1/p + 1/q$$

when $v < -1/q$. Thus, $f_a \in \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N) \setminus \underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$ if $v + 1/q < a \le v + 1/p + 1/q$ when $v < -1/q$.

Since $f_a \in L^p(\mathbf{R}^N)$ if and only if $ap > 1$, $f_a \in \underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N) \setminus L^p(\mathbf{R}^N)$ if $v + 1/p + 1/q < a \le 1/p$. Such $a$ exists when $v < -1/q$.

Next, for $r \ge 2$, $\|f_a\|_{L^p(\mathbf{R}^N \setminus B(0,r))} \sim (\log(e+r))^{-a+1/p}$ in case $a > 1/p$ and $= \infty$ in case $a \le 1/p$. Hence, in case $v \ge -1/q$, $f_a \in \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N) \setminus \overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$ if $v + 1/q < a \le v + 1/p + 1/q$. Since $\chi_{B(0,1)} \in \overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$, $\overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N) \neq \{0\}$.

Remark 16. Since $\underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N) \hookrightarrow \mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$, the second inequality (3) also holds with $\|f\|_{\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)}$ replaced by $\|f\|_{\underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)}$.

Analogous inequality is trivial for $\|f\|_{\overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)}$, since $\overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N) \hookrightarrow L^{\Phi}(\mathbf{R}^N)$.

For the boundedness of the maximal operator, we have the following results (cf. [9]).

Theorem 6. *Suppose that $\Phi(x,t)$ satisfies $(\Phi 3;0;p)$, $(\Phi 3;\infty;q)$, $(\Phi 5;\eta)$ and $(\Phi 6;\tau)$ for $p > 1$, $q > 1$, $\eta > 0$ and $\tau > 0$ satisfying $\eta \le q/N$ and $\tau \le p$. Assume that $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty \omega 1;0)$. Then the maximal operator $M$ is bounded from $\underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$ to itself.*

Proof. Let $f$ be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{\underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)} \le 1$. For $r \ge 1$, set

$$f = f\chi_{B(0,2r)} + f\chi_{\mathbf{R}^N \setminus B(0,2r)} = g_{1,r} + g_{2,r}.$$

By Lemma 6,

$$\|Mg_{1,r}\|_{L^{\Phi}(B(0,r))} \le C\|g_{1,r}\|_{L^{\Phi}(\mathbf{R}^N)} = C\|f\|_{L^{\Phi}(B(0,2r))},$$

so that

$$\int_1^\infty (\omega(r)\|Mg_{1,r}\|_{L^{\Phi}(B(0,r))})^q \frac{dr}{r} \le C\int_1^\infty (\omega(r)\|f\|_{L^{\Phi}(B(0,2r))})^q \frac{dr}{r} \le C.$$

For $g_{2,r}$, we argue as for $f_{3,r}$ in the proof of Theorem 1 to obtain

$$Mg_{2,r}(x) \le Cr^{\varepsilon_1'}\omega(r)^{-1}\Phi_\infty^{-1}(r^{-N})\left(\int_r^\infty (t^{-\varepsilon_1'}\omega(t)\|f\|_{L^{\Phi}(A(0,t))})^q \frac{dt}{t}\right)^{1/q}$$

for $x \in B(0, r)$ with $0 < \varepsilon_1' < \varepsilon_1$, which implies

$$\int_1^{\infty} (\omega(r)\|Mg_{2,r}\|_{L^{\Phi}(B(0,r))})^q \frac{dr}{r} \le C \int_1^{\infty} (\omega(r)\|f\|_{L^{\Phi}(B(0,2r))})^q \frac{dr}{r} \le C. \qquad \square$$

THEOREM 7. *Suppose that $\Phi(x,t)$ satisfies $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \eta)$ and $(\Phi 6; \tau)$ for $p > 1$, $q > 1$, $\eta > 0$ and $\tau > 0$ satisfying $\eta \le q/N$ and $\tau \le p$. Assume that $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty \omega 2; -N)$. Then the maximal operator $M$ is bounded from $\overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$ to itself.*

PROOF. Let $f$ be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{\overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)} \le 1$. Then $\|f\|_{L^{\Phi}(\mathbf{R}^N)} \le C$.

For $r \ge 2$, set

$$f = f\chi_{B(0,1)} + f\chi_{B(0,r/2)\setminus B(0,1)} + f\chi_{\mathbf{R}^N \setminus B(0,r/2)} = f_0 + f_{1,r} + h_{2,r}.$$

By (5), we see that

$$\|Mf_0\|_{L^{\Phi}(\mathbf{R}^N \setminus B(0,r))} \le Cr^{-N}\{\Phi_\infty^{-1}(r^{-N})\}^{-1}$$

and using $(\Phi_\infty \omega 2; -N)$, we have

$$\int_2^{\infty} (\omega(r)\|Mf_0\|_{L^{\Phi}(\mathbf{R}^N \setminus B(0,r))})^q \frac{dr}{r} \le C$$

by the same arguments as to obtain (6) in the proof of Theorem 1.

In view of (7) in the proof of Theorem 1, we see

$$\begin{aligned}
&\int_2^{\infty} (\omega(r)\|Mf_{1,r}\|_{L^{\Phi}(\mathbf{R}^N \setminus B(0,r))})^q \frac{dr}{r} \\
&\le C \int_{1/2}^{\infty} (\omega(r)\|f\|_{L^{\Phi}(A(0,r))})^q \frac{dr}{r} \\
&\le C\left\{\|f\|_{L^{\Phi}(B(0,2))} + \int_1^{\infty} (\omega(r)\|f\|_{L^{\Phi}(\mathbf{R}^N \setminus B(0,r))})^q \frac{dr}{r}\right\} \\
&\le C.
\end{aligned}$$

By Lemma 6,

$$\|Mh_{2,r}\|_{L^{\Phi}(\mathbf{R}^N \setminus B(0,r))} \le C\|h_{2,r}\|_{L^{\Phi}(\mathbf{R}^N)} = C\|f\|_{L^{\Phi}(\mathbf{R}^N \setminus B(0,r))},$$

so that

$$\begin{aligned}
\int_2^{\infty} (\omega(r)\|Mh_{2,r}\|_{L^{\Phi}(\mathbf{R}^N \setminus B(0,r))})^q \frac{dr}{r} &\le C \int_2^{\infty} (\omega(r)\|f\|_{L^{\Phi}(\mathbf{R}^N \setminus B(0,r))})^q \frac{dr}{r} \\
&\le C.
\end{aligned}$$

Thus,

$$\int_2^\infty (\omega(r)\|Mf\|_{L^\Phi(\mathbf{R}^N\setminus B(0,r))})^q \frac{dr}{r} \le C.$$

Finally, since $\|Mf\|_{L^\Phi(\mathbf{R}^N)} \le C\|f\|_{L^\Phi(\mathbf{R}^N)} \le C$ by Lemma 6,

$$\|Mf\|_{L^\Phi(B(0,2))} + \int_1^2 (\omega(r)\|Mf\|_{L^\Phi(\mathbf{R}^N\setminus B(0,r))})^q \frac{dr}{r} \le C. \qquad \square$$

As to Sobolev's inequalities, we have the following results (see also [10]).

THEOREM 8. *Suppose $\Phi(x,t)$ satisfies $(\Phi3;0;p)$, $(\Phi3;\infty;q)$, $(\Phi5;\eta)$, $(\Phi6;\tau)$ and $(\Phi\alpha)$ for $p>1$, $q>1$, $\eta>0$ and $\tau>0$ satisfying $\eta \le q/N$ and $\tau \le p$. Assume that $\Psi(x,t)$ satisfies $(\Psi\Phi\alpha)$. For the function $\Phi_\infty(t)$, assume $(\Phi_\infty 2')$ and $(\Phi_\infty \omega 1; -\alpha)$ hold. Then there exists a constant $C>0$ such that*

$$\|I_\alpha f\|_{\underline{\mathscr{H}}^{\Psi,q,\omega}(\mathbf{R}^N)} \le C\|f\|_{\underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)}$$

*for all $f \in \underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$.*

PROOF. Let $f$ be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{\underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)} \le 1$. For $r \ge 1$, set

$$f = f\chi_{B(0,2r)} + f\chi_{\mathbf{R}^N\setminus B(0,2r)} = g_{1,r} + g_{2,r}.$$

By Lemma 12,

$$\|I_\alpha g_{1,r}\|_{L^\Psi(B(0,r))} \le C\|g_{1,r}\|_{L^\Phi(\mathbf{R}^N)} = C\|f\|_{L^\Phi(B(0,2r))},$$

so that

$$\int_1^\infty (\omega(r)\|I_\alpha g_{1,r}\|_{L^\Phi(B(0,r))})^q \frac{dr}{r} \le C\int_1^\infty (\omega(r)\|f\|_{L^\Phi(B(0,2r))})^q \frac{dr}{r} \le C.$$

For $g_{2,r}$, we argue as for $f_{3,r}$ in the proof of Theorem 4 to obtain

$$\|I_\alpha g_{2,r}\|_{L^\Psi(B(0,r))} \le Cr^{\varepsilon_1'}\omega(r)^{-1}\left(\int_r^\infty (t^{-\varepsilon_1'}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t}\right)^{1/q}$$

for $0 < \varepsilon_1' < \varepsilon_1$, which implies

$$\int_1^\infty (\omega(r)\|I_\alpha g_{2,r}\|_{L^\Psi(B(0,r))})^q \frac{dr}{r} \le C\int_1^\infty (\omega(r)\|f\|_{L^\Phi(B(0,2r))})^q \frac{dr}{r} \le C. \qquad \square$$

THEOREM 9. *Suppose $\Phi(x,t)$ satisfies $(\Phi3;0;p)$, $(\Phi3;\infty;q)$, $(\Phi5;\eta)$, $(\Phi6;\tau)$ and $(\Phi\alpha)$ for $p>1$, $q>1$, $\eta>0$ and $\tau>0$ satisfying $\eta \le q/N$ and $\tau \le p$. Assume that $\Psi(x,t)$ satisfies $(\Psi\Phi\alpha)$. For the function $\Phi_\infty(t)$, assume $(\Phi_\infty 2')$ and*

$(\Phi_\infty\omega 2; -N)$ *hold. Then there exists a constant* $C > 0$ *such that*

$$\|I_\alpha f\|_{\overline{\mathscr{H}}^{\Psi,q,\omega}(\mathbf{R}^N)} \le C\|f\|_{\overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)}$$

*for all* $f \in \overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$.

PROOF. Let $f$ be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{\overline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)} \le 1$ and for $r \ge 2$, set

$$f = f\chi_{B(0,1)} + f\chi_{B(0,r/2)\setminus B(0,1)} + f\chi_{\mathbf{R}^N\setminus B(0,r/2)} = f_0 + f_{1,r} + h_{2,r}.$$

Since $\int_{B(0,1)} f(y)dy \le C\|f\|_{L^\Phi(B(0,1))} \le C$, we see that

$$\|I_\alpha f_0\|_{L^\Psi(\mathbf{R}^N\setminus B(0,r))} \le Cr^{-N}\{\Phi_\infty^{-1}(r^{-N})\}^{-1},$$

and hence

$$\int_2^\infty (\omega(r)\|I_\alpha f_0\|_{L^\Psi(\mathbf{R}^N\setminus B(0,r))})^q \frac{dr}{r} \le C$$

in the same way as in the proof of Theorem 4.

Also, as in the proof of Theorem 4, we see

$$\int_2^\infty (\omega(r)\|I_\alpha f_{1,r}\|_{L^\Psi(\mathbf{R}^N\setminus B(0,r))})^q \frac{dr}{r} \le C.$$

For $h_{2,r}$, we use Lemma 12 to obtain

$$\|I_\alpha h_{2,r}\|_{L^\Psi(\mathbf{R}^N\setminus B(0,r))} \le C\|h_{2,r}\|_{L^\Phi(\mathbf{R}^N)} = C\|f\|_{L^\Phi(\mathbf{R}^N\setminus B(0,r/2))},$$

which implies

$$\int_2^\infty (\omega(r)\|I_\alpha h_{2,r}\|_{L^\Psi(\mathbf{R}^N\setminus B(0,r))})^q \frac{dr}{r} \le C.$$

Finally, since $\|I_\alpha f\|_{L^\Psi(\mathbf{R}^N)} \le C\|f\|_{L^\Phi(\mathbf{R}^N)} \le C$,

$$\|I_\alpha f\|_{L^\Psi(B(0,2))} + \int_1^2 (\omega(r)\|I_\alpha f\|_{L^\Psi(\mathbf{R}^N\setminus B(0,r))})^q \frac{dr}{r} \le C. \qquad \square$$

THEOREM 10. *Suppose* $\Phi(x,t)$ *satisfies* $(\Phi3; 0; p)$, $(\Phi3; \infty; q)$, $(\Phi5; \eta)$, $(\Phi6; \tau)$ *and* $(\Phi\alpha)$ *for* $p > 1$, $q > 1$, $\eta > 0$ *and* $\tau > 0$ *satisfying* $\eta \le q/N$ *and* $\tau \le p$. *Assume that* $\Psi(x,t)$ *satisfies* $(\Psi\Phi\alpha)$. *For an integer* $k \ge 1$, *assume* $\Phi_\infty(t)$ *satisfies* $(\Phi_\infty 2')$, $(\Phi_\infty\omega 1; k-\alpha)$ *and* $(\Phi_\infty\omega 2; k-1-\alpha)$. *Then there exists a constant* $C > 0$ *such that*

$$\|I_{\alpha,k} f\|_{\underline{\mathscr{H}}^{\Psi,q,\omega}(\mathbf{R}^N)} \le C\|f\|_{\underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)}$$

*for all* $f \in \underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$.

Proof. Let $f$ be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{\underline{\mathscr{H}}^{\Phi,q,\omega}(\mathbf{R}^N)} \le 1$. Noting that

$$|I_{\alpha,k}(f\chi_{B(0,4r)\setminus B(0,|x|/2)\setminus B(0,1)})| \le CI_\alpha(f\chi_{B(0,4r)\setminus B(0,1)})$$

for $r \ge 1$ and $|x| < r$, by the same arguments as to obtain (18) in the proof of Theorem 5, we have

$$I_{\alpha,k}f(x) \le C\Bigg\{ I_\alpha(f\chi_{B(0,4r)})(x) + r^{-\varepsilon_2'+\alpha}\omega(r)^{-1}\Phi_\infty^{-1}(r^{-N})\left(\int_{1/2}^r (t^{\varepsilon_2'}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t}\right)^{1/q} + r^{\varepsilon_1'+\alpha}\omega(r)^{-1}\Phi_\infty^{-1}(r^{-N})\left(\int_r^\infty (t^{-\varepsilon_1'}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t}\right)^{1/q}\Bigg\}$$

for $r \ge 1$ and $x \in B(0,r)$, with $0 < \varepsilon_1' < \varepsilon_1$ and $0 < \varepsilon_2' < \varepsilon_2$.

Now, by Lemma 12

$$\|I_\alpha(f\chi_{B(0,4r)})\|_{L^\Psi(B(0,r))} \le C\|f\|_{L^\Phi(B(0,4r))}.$$

Thus, in the same way as in the proof of Theorem 4 (with $A(0,r)$ replaced by $B(0,r)$), we obtain

$$\int_1^\infty (\omega(r)\|I_{\alpha,k}f\|_{L^\Psi(B(0,r))})^q \frac{dr}{r} \le C. \qquad \square$$

## 8. Variable exponent H-M-M-O spaces

Let $q(r)$ be a measurable function on $[1,\infty)$ satisfying

(Q1) $0 < q^- := \operatorname{ess\,inf}_{r\in[1,\infty)} q(r) \le \operatorname{ess\,sup}_{r\in[1,\infty)} q(r) =: q^+ < \infty$.

Given $\Phi(x,t)$, $\omega(r)$ and $q(r)$ as above, we denote by $\mathscr{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)$, $\underline{\mathscr{H}}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)$ and $\overline{\mathscr{H}}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)$ the classes of locally integrable functions $f$ on $\mathbf{R}^N$ satisfying

$$\|f\|_{\mathscr{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)} = \|f\|_{L^\Phi(B(0,2))} + \|\omega(\cdot)\|f\|_{L^\Phi(A(0,\cdot))}\|_{L^{q(\cdot)}((1,\infty),dr/r)} < \infty,$$

$$\|f\|_{\underline{\mathscr{H}}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)} = \|\omega(\cdot)\|f\|_{L^\Phi(B(0,\cdot))}\|_{L^{q(\cdot)}((1,\infty),dr/r)} < \infty$$

and

$$\|f\|_{\overline{\mathscr{H}}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)} = \|f\|_{L^\Phi(B(0,2))} + \|\omega(\cdot)\|f\|_{L^\Phi(\mathbf{R}^N\setminus B(0,\cdot))}\|_{L^{q(\cdot)}((1,\infty),dr/r)} < \infty,$$

respectively, where

$$\|g\|_{L^{q(\cdot)}((1,\infty),dr/r)} = \inf\left\{\lambda > 0; \int_1^\infty \left(\frac{|g(r)|}{\lambda}\right)^{q(r)} \frac{dr}{r} \le 1\right\}.$$

PROPOSITION 2. *Suppose $q(r)$ satisfies*

(Q2) *there exists a constant $q(\infty) \in (0,\infty)$ such that*

$$|q(r) - q(\infty)| \le \frac{C_{q,\infty}}{\log(e+r)}$$

*whenever $r \ge 1$ with a constant $C_{q,\infty} \ge 0$.*

*Then*

$$\mathscr{H}^{\Phi,q(\infty),\omega}(\mathbf{R}^N) = \mathscr{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N),$$

$$\underline{\mathscr{H}}^{\Phi,q(\infty),\omega}(\mathbf{R}^N) = \underline{\mathscr{H}}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)$$

*and*

$$\overline{\mathscr{H}}^{\Phi,q(\infty),\omega}(\mathbf{R}^N) = \overline{\mathscr{H}}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N).$$

PROOF. We only prove that $\mathscr{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N) \subset \mathscr{H}^{\Phi,q(\infty),\omega}(\mathbf{R}^N)$, since the remaining assertions can be proved similarly. Let $f$ be a measurable function on $\mathbf{R}^N$ satisfying $\|f\|_{\mathscr{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)} \le 1$. Then note that there exists a constant $c > 0$ such that

$$\int_{1/\sqrt{2}}^\infty (\omega(r)\|f\|_{L^\Phi(A(0,r))})^{q(r)} \frac{dr}{r} \le c.$$

First we show that

$$\omega(r)\|f\|_{L^\Phi(A(0,r))} \le C \qquad \text{for } r \ge 1. \tag{25}$$

Let $J(r) = \omega(r)\|f\|_{L^\Phi(B(0,\sqrt{2}r)\setminus B(0,r))}$. If $r/\sqrt{2} \le t \le r$, then $B(0,\sqrt{2}r)\setminus B(0,r) \subset A(0,t)$, so that

$$J(r) \le c_1 c_2 \omega(t)\|f\|_{L^\Phi(A(0,t))}$$

by ($\omega$2). For $r \ge 1$, if $J(r) \ge c_1 c_2$, then

$$c \ge \int_{r/\sqrt{2}}^r (\omega(t)\|f\|_{L^\Phi(A(0,t))})^{q(t)} \frac{dt}{t} \ge \frac{\log 2}{2}(c_1^{-1}c_2^{-1}J(r))^{q^-},$$

which implies

$$J(r) \le c_1 c_2 (2c/\log 2)^{1/q^-}.$$

Therefore,

$$\omega(r)\|f\|_{L^{\Phi}(A(0,r))} = \omega(r)\|f\|_{L^{\Phi}(B(0,\sqrt{2}r)\setminus B(0,r))} + \omega(r)\|f\|_{L^{\Phi}(B(0,2r)\setminus B(0,\sqrt{2}r))}$$
$$\le J(r) + c_1c_2J(\sqrt{2}r) \le C,$$

which shows (25).

If $r^{-1} < \omega(r)\|f\|_{L^{\Phi}(A(0,r))}$, then we have by (Q2)

$$(\omega(r)\|f\|_{L^{\Phi}(A(0,r))})^{q(\infty)} \le C(\omega(r)\|f\|_{L^{\Phi}(A(0,r))})^{q(r)}$$

for $r \ge 1$, which gives

$$\int_1^{\infty} (\omega(r)\|f\|_{L^{\Phi}(A(0,r))})^{q(\infty)} \frac{dr}{r}$$
$$\le C\int_1^{\infty} (\omega(r)\|f\|_{L^{\Phi}(A(0,r))})^{q(r)} \frac{dr}{r} + \int_1^{\infty} r^{-q(\infty)} \frac{dr}{r} \le C.$$

Thus, we obtain the required result. □

By this proposition, Theorems 1, 2, 3, 4 and 5 are valid with $\mathscr{H}^{\Phi,q,\omega}(\mathbf{R}^N)$ replaced by $\mathscr{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)$, provided that $q(r)$ satisfies (Q2), namely we have the following corollaries.

Corollary 1. *Assume that $q(r)$ satisfies* (Q2). *Suppose that $\Phi(x,t)$ satisfies $(\Phi3;0;p)$, $(\Phi3;\infty;q)$, $(\Phi5;\eta)$ and $(\Phi6;\tau)$ for $p>1$, $q>1$, $\eta>0$ and $\tau>0$ satisfying $\eta \le q/N$ and $\tau \le p$. Assume that $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$, $(\Phi_\infty\omega1;0)$ and $(\Phi_\infty\omega2;-N)$. Then the maximal operator $M$ is bounded from $\mathscr{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)$ to itself.*

Corollary 2. *Assume that $q(r)$ satisfies* (Q2). *For a real number $\beta$, suppose that $\Phi(x,t)$ satisfies $(\Phi3;\infty;q)$ and $(\Phi5;\eta)$ for $q \ge 1$ and $\eta > 0$ satisfying $\eta \le q/N$. Assume that $\Phi(x,t)$ satisfies $(\Phi\omega;\beta)$ and $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty\omega1;\beta)$. Then there exists a constant $C>0$ such that*

$$\|\hat{H}_\beta^\infty f\|_{\mathscr{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)} \le C\|f\|_{\mathscr{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)}$$

*for all $f \in \mathscr{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)$.*

Corollary 3. *Assume that $q(r)$ satisfies* (Q2). *For a real number $\beta$, suppose that $\Phi(x,t)$ satisfies $(\Phi3;\infty;q)$ and $(\Phi5;\eta)$ for $q \ge 1$ and $\eta > 0$ satisfying $\eta \le q/N$. Assume that $\Phi(x,t)$ satisfies $(\Phi\omega;\beta)$ and $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty\omega2;\beta)$. Then there exists a constant $C>0$ such that*

$$\|\hat{H}_\beta^0 f\|_{\mathscr{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)} \le C\|f\|_{\mathscr{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)}$$

*for all $f \in \mathscr{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)$.*

COROLLARY 4. *Assume that* $q(r)$ *satisfies* (Q2). *Suppose* $\Phi(x,t)$ *satisfies* $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \eta)$, $(\Phi 6; \tau)$ *and* $(\Phi \alpha)$ *for* $p > 1$, $q > 1$, $\eta > 0$ *and* $\tau > 0$ *satisfying* $\eta \le q/N$ *and* $\tau \le p$. *Assume that* $\Psi(x,t)$ *satisfies* $(\Psi\Phi\alpha)$. *For the function* $\Phi_\infty(t)$, *assume* $(\Phi_\infty 2')$, $(\Phi_\infty \omega 1; -\alpha)$ *and* $(\Phi_\infty \omega 2; -N)$ *hold. Then there exists a constant* $C > 0$ *such that*

$$\|I_\alpha f\|_{\mathscr{H}^{\Psi, q(\cdot), \omega}(\mathbf{R}^N)} \le C \|f\|_{\mathscr{H}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)}$$

*for all* $f \in \mathscr{H}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)$.

COROLLARY 5. *Assume that* $q(r)$ *satisfies* (Q2). *Suppose* $\Phi(x,t)$ *satisfies* $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \eta)$, $(\Phi 6; \tau)$ *and* $(\Phi \alpha)$ *for* $p > 1$, $q > 1$, $\eta > 0$ *and* $\tau > 0$ *satisfying* $\eta \le q/N$ *and* $\tau \le p$. *Assume that* $\Psi(x,t)$ *satisfies* $(\Psi\Phi\alpha)$. *For an integer* $k \ge 1$, *assume* $\Phi_\infty(t)$ *satisfies* $(\Phi_\infty 2')$, $(\Phi_\infty \omega 1; k - \alpha)$ *and* $(\Phi_\infty \omega 2; k - 1 - \alpha)$. *Then there exists a constant* $C > 0$ *such that*

$$\|I_{\alpha,k} f\|_{\mathscr{H}^{\Psi, q(\cdot), \omega}(\mathbf{R}^N)} \le C \|f\|_{\mathscr{H}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)}$$

*for all* $f \in \mathscr{H}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)$.

Also, Theorems 6, 8 and 10 hold with $\underline{\mathscr{H}}^{\Phi, q, \omega}(\mathbf{R}^N)$ replaced by $\underline{\mathscr{H}}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)$, and Theorems 7 and 9 hold with $\overline{\mathscr{H}}^{\Phi, q, \omega}(\mathbf{R}^N)$ replaced by $\overline{\mathscr{H}}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)$, when $q(r)$ satisfies (Q2).

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