Fekete-Szegö results for a class of non-Bazilevič functions defined by convolution

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Abstract

In the present paper using the principal of subordination we obtain sharp bounds for a class of non-Bazilevič functions with complex order defined by convolution.

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1 Introduction

Denote by $\mathbb{A}$ the class of univalent analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z\in\mathbb{U}= \{z\in \mathbb{C} : |z| < 1\}). \tag{1.1}$$

For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, the function $f(z)$ is subordinate to $g(z)$ $(f(z) \prec g(z))$ in $\mathbb{U}$, if there exists a function $\omega(z)$, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, $f(z) = g(\omega(z))$ $(z \in \mathbb{U})$ and if $g(z)$ is univalent in $\mathbb{U}$, then (see for details [1], [4] and also [9]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

The Hadamard product of $f(z)$ and $g(z)$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \ , \tag{1.2}$$

is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

Let $\phi(z)$ be an analytic function with positive real part on $\mathbb{U}$ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps $\mathbb{U}$ onto a region starlike with respect to 1 and is symmetric with respect to the real axis. For $b \in \mathbb{C}^* = \mathbb{C}\backslash\{\mathbf{0}\}$, Ravichandran et al. [13] defined the classes $\mathcal{S}_b^*(\phi)$ and $\mathcal{C}_b(\phi)$ as follow:

$$\mathcal{S}_b^*(\phi) = \left\{ f \in \mathbb{A} : 1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z) \quad (z\in\mathbb{U}) \right\} \tag{1.3}$$

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and

$$\mathcal{C}_b(\phi) = \left\{ f \in \mathbb{A} : 1 + \frac{1}{b}\frac{zf^{''}(z)}{f^{'}(z)} \prec \phi(z) \quad (z\in\mathbb{U}) \right\}. \tag{1.4}$$

We note that:
(i) $\mathcal{S}_b^*(\frac{1+z}{1-z}) = \mathcal{S}^*(b)$ (see [11]);
(ii) $\mathcal{C}_b(\frac{1+z}{1-z}) = \mathcal{C}(b)$ (see [19] and [10]);
(iii) $\mathcal{S}_b^*(\frac{1+(1-2\alpha)z}{1-z}) = \mathcal{S}_\alpha^*(b)$ (see [7]) $(0 \leq \alpha < 1)$;
(iv) $\mathcal{C}_b(\frac{1+(1-2\alpha)z}{1-z}) = \mathcal{C}_\alpha(b)$ (see [7]) $(0 \leq \alpha < 1)$;
(v) $\mathcal{S}_1^*(\phi) = \mathcal{S}^*(\phi)$ and $\mathcal{C}_1(\phi) = \mathcal{C}(\phi)$ (see [8]).

For $-1 \leq B < A \leq 1, 0 < \alpha < 1, \gamma \in \mathbb{C}$, Wang et al. [18] (see also [2]) introduced and studied the class $N(\gamma, \alpha; A, B)$ of $f(z) \in \mathbb{A}$ satisfying

$$(1+\gamma)\left(\frac{z}{f(z)}\right)^\alpha - \gamma\frac{zf^{'}(z)}{f(z)}\left(\frac{z}{f(z)}\right)^\alpha \prec \frac{1+Az}{1+Bz}. \tag{1.5}$$

By making use of the convolution, $b \in \mathbb{C}^*$ and the principle of subordination between analytic functions, we now introduce the following class of non-Bazilevič functions.

Definition 1. Let $\phi(z)$ be an univalent starlike function with respect to 1 which maps $\mathbb{U}$ onto a region in the right half plane which is symmetric with respect to the real axis, $\phi(0) = 1$, $\phi^{'}(0) > 0, \gamma \in \mathbb{C}$ and $0 < \alpha < 1$. A function $f(z) \in \mathbb{A}$ is said to be in the class $R_g^{\alpha,\gamma}(b, \phi)$ if it satisfies the following subordination condition:

$$1 + \tfrac{1}{b}\left\{(1+\gamma)\left(\tfrac{z}{(f*g)(z)}\right)^\alpha - \gamma\frac{z\left((f*g)(z)\right)^{'}}{(f*g)(z)}\left(\tfrac{z}{(f*g)(z)}\right)^\alpha - 1\right\} \prec \phi(z). \tag{1.6}$$

We note that:
(i) $R^{\alpha,\gamma}_{\frac{z}{1-z}}(1, \phi) = R^{\alpha,\gamma}(\phi)$ (see [16]);
(ii) $R^{\alpha,\gamma}_{\frac{z}{1-z}}\left(1, \frac{1+Az}{1+Bz}\right) = R^{\alpha,\gamma}(A, B)$ $(-1 \leq B < A \leq 1)$ (see [18]);
(iii) $R^{\alpha,-1}_{\frac{z}{1-z}}\left(1, \frac{1+(1-2\rho)z)}{1-z}\right) = R^\alpha(\rho)$ $(0 \leq \rho < 1)$ (see [17]);
(iv) $R^{\alpha,-1}_{\frac{z}{1-z}}\left(1, \frac{1+z}{1-z}\right) = R^\alpha$ (see [12]).

Also, we can have following new subclasses for different forms of $g(z)$:
(i) For

$$g(z) = z + \sum_{k=2}^{\infty}\left[\frac{l+1+\lambda(k-1)}{l+1}\right]^m z^k \quad (\lambda > 0, l \geq 0, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\}),$$

the class $R_g^{\alpha,\gamma}(b, \phi)$ reduces to the class $R^{\alpha,\gamma}_{m,\lambda,l}(b, \phi)$ which satisfies:

$$1 + \frac{1}{b}\left\{(1+\gamma)\left(\tfrac{z}{D^m_{\lambda,l}f(z)}\right)^\alpha - \gamma\frac{z\left(D^m_{\lambda,l}f(z)\right)^{'}}{D^m_{\lambda,l}f(z)}\left(\tfrac{z}{D^m_{\lambda,l}f(z)}\right)^\alpha - 1\right\} \prec \phi(z),$$

where $D_{\lambda,l}^{m}f(z)$ generalized multiplier operator introduced by Catas et al. (see [5]);

(ii) For

$$g(z)=z+\sum_{k=2}^{\infty}k^{m}z^{k}\,(m\in\mathbb{N}_0),$$

the class $R_g^{\alpha,\gamma}(b,\phi)$ reduces to the class $R_m^{\alpha,\gamma}(b,\phi)$ which satisfies:

$$1+\frac{1}{b}\left\{(1+\gamma)\left(\tfrac{z}{D^m f(z)}\right)^{\alpha}-\gamma\frac{z\left(D^m f(z)\right)'}{D^m f(z)}\left(\tfrac{z}{D^m f(z)}\right)^{\alpha}-1\right\}\prec\phi(z),$$

where $D^m f(z)$ Salagean operator (see [14] and also [3]);

(iii) For

$$\begin{aligned} g(z) &= z+\sum_{k=2}^{\infty}\frac{(\alpha_1)_{k-1}\ldots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\ldots(\beta_s)_{k-1}(1)_{k-1}}z^k \\ &= z+\sum_{k=2}^{\infty}\Gamma_k(\alpha_1)z^k \quad \left(\begin{array}{l} q \text{ and } s\in\mathbb{N}_0, q<s+1, \alpha_1\in\mathbb{C} \text{ and} \\ \beta_j\notin Z_0=\{0,-1,-2,\ldots\}; j=1,\ldots,s \end{array}\right), \end{aligned}$$

the class $R_g^{\alpha,\gamma}(b,\phi)$ reduces to the class $R_{q,s,\alpha_1}^{\alpha,\gamma}(b,\phi)$ which satisfies:

$$1+\frac{1}{b}\left\{(1+\gamma)\left(\tfrac{z}{H_{q,s}(\alpha_1)f(z)}\right)^{\alpha}-\gamma\frac{z\left(H_{q,s}(\alpha_1)f(z)\right)'}{H_{q,s}(\alpha_1)f(z)}\left(\tfrac{z}{H_{q,s}(\alpha_1)f(z)}\right)^{\alpha}-1\right\}\prec\phi(z),$$

where $H_{q,s}(\alpha_1)f(z)$ Dizok Srivastava operator (see [6]);

(iv) For $b=(1-\beta)\cos\theta\; e^{-i\theta}\left(|\theta|<\frac{\pi}{2}, 0\le\beta<1\right)$, the class $R_{m,\lambda,l}^{\alpha,\gamma}(b,\phi)$ reduces to the class

$$R_g^{\alpha,\gamma}(\beta,\theta,\phi)=\left\{f:\frac{e^{-i\theta}\Psi f(z)-\beta\cos\theta-i\sin\theta}{(1-\beta)\cos\theta}\prec\phi(z)\right\}, \tag{1.7}$$

where

$$\Psi f(z)=(1+\gamma)\left(\frac{z}{(f*g)(z)}\right)^{\alpha}-\gamma\frac{z\left((f*g)(z)\right)'}{(f*g)(z)}\left(\frac{z}{(f*g)(z)}\right)^{\alpha}$$

and also, putting $\gamma=-1$ in (1.7), we have the following subclass:

$$R_g^{\alpha,-1}(\beta,\theta,\phi)=\left\{f:\frac{e^{-i\theta}\frac{z\left((f*g)(z)\right)'}{(f*g)(z)}\left(\frac{z}{(f*g)(z)}\right)^{\alpha}-\beta\cos\theta-i\sin\theta}{(1-\beta)\cos\theta}\prec\phi(z)\right\}.$$

In order to prove our results, we need the following lemmas.

Lemma 1 [8]. If $p(z)=1+c_1z+c_2z^2+\ldots$ is a function with positive real part in $\mathbb{U}$ and μ is a complex number, then

$$\left|c_2-\mu c_1^2\right|\le 2\max\{1;|2\mu-1|\}.$$

The result is sharp for the function

$$p(z)=\frac{1+z^2}{1-z^2} \text{ and } p(z)=\frac{1+z}{1-z}.$$

Lemma 2 [8]. If $p(z) = 1 + c_1 z + c_2 z^2 + ...$ is an analytic function with a positive real part in $\mathbb{U}$, then

$$\left|c_2 - v c_1^2\right| \leq \begin{cases} -4v + 2 & \text{if } v \leq 0, \\ 2 & \text{if } 0 \leq v \leq 1, \\ 4v - 2 & \text{if } v \geq 1, \end{cases}$$

when $v < 0$ or $v > 1$, the equality holds if and only if $p(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < v < 1$, then the equality holds if and only if $p(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$p(z) = \left(\frac{1+\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\lambda}{2}\right)\frac{1-z}{1+z} \quad (0 \leq \lambda \leq 1),$$

or one of its rotations. If $v = 1$, the equality holds if and only if p is the reciprocal of one of the functions such that equality holds in the case of $v = 0$. Also the above upper bound is sharp, and it can be improved as follows when $0 < v < 1$.

$$\left|c_2 - v c_1^2\right| + v\left|c_1\right|^2 \leq 2 \quad \left(0 \leq v \leq \frac{1}{2}\right)$$

and

$$\left|c_2 - v c_1^2\right| + (1 - v)\left|c_1\right|^2 \leq 2 \quad \left(\frac{1}{2} \leq v \leq 1\right).$$

2 Main results

Unless otherwise mentioned, we assume throughout this paper that $\phi(0) = 1, \phi'(0) > 0, \gamma \in \mathbb{C}$, $0 < \alpha < 1$ and $g(z)$ is given by (1.2) with $b_2, b_3 > 0$.

Theorem 1. Let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ with $B_1 > 0$. If $f(z)$ given by (1.1) belongs to the class $R_g^{\alpha,\gamma}(b, \phi)$ with $\alpha + \gamma \neq 0$ and $\alpha + 2\gamma \neq 0$, then

$$\left|a_3 - \mu a_2^2\right| \leq \tfrac{|b| B_1}{|\alpha+2\gamma| b_3} \max\left\{1; \left|\tfrac{B_2}{B_1} - \tfrac{(\alpha+1)(\alpha+2\gamma) b B_1}{2(\alpha+\gamma)^2} + \tfrac{\mu b (\alpha+2\gamma) b_3 B_1}{(\alpha+\gamma)^2 b_2^2}\right|\right\}. \tag{2.1}$$

The result is sharp.

Proof. If $f(z) \in R_g^{\alpha,\gamma}(b, \phi)$, then there is a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$ such that

$$1 + \tfrac{1}{b}\left\{(1+\gamma)\left(\tfrac{z}{(f*g)(z)}\right)^{\alpha} - \gamma \frac{z\left((f*g)(z)\right)'}{(f*g)(z)}\left(\tfrac{z}{(f*g)(z)}\right)^{\alpha} - 1\right\} = \phi(\omega(z)). \tag{2.2}$$

Define the function $p(z)$ by

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + c_1 z + c_2 z^2 + ... \tag{2.3}$$

Since $\omega(z)$ is a function, we see that $\operatorname{Re}\{p(z)\}>0$ and $p(0)=1$. Therefore,

$$\begin{aligned}
\phi(\omega(z)) &= \phi\left(\frac{p(z)-1}{p(z)+1}\right) \\
&= \phi\left\{\frac{1}{2}\left[c_1 z+\left(c_2-\frac{c_1^2}{2}\right)z^2+\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)z^3+\ldots\right]\right\} \\
&= 1+\frac{1}{2}c_1B_1z+\left[\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}c_1^2B_2\right]z^2+\ldots\ . \qquad (2.4)
\end{aligned}$$

Now by substituting (2.4) in (2.2), we have

$$\begin{aligned}
&1+\frac{1}{b}\left\{(1+\gamma)\left(\frac{z}{(f*g)(z)}\right)^{\alpha}-\gamma\frac{z\left((f*g)(z)\right)'}{(f*g)(z)}\left(\frac{z}{(f*g)(z)}\right)^{\alpha}-1\right\} \\
=\ &1+\frac{1}{2}c_1B_1z+\left[\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}c_1^2B_2\right]z^2+\ldots\ .
\end{aligned}$$

So, we obtain

$$-(\alpha+\gamma)\,b_2a_2=\frac{1}{2}bc_1B_1,$$

$$\begin{aligned}
&-(\alpha+2\gamma)\left[b_3a_3-\frac{1}{2}(\alpha+1)\,b_2^2a_2^2\right] \\
=\ &\frac{1}{2}bB_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}bB_2c_1^2,
\end{aligned}$$

or, equivalenty,

$$a_2=\frac{-bc_1B_1}{2(\alpha+\gamma)\,b_2},$$

$$a_3=\frac{-bB_1}{2(\alpha+2\gamma)\,b_3}\left\{c_2-\frac{1}{2}\left[1-\frac{B_2}{B_1}+\frac{(\alpha+1)(\alpha+2\gamma)bB_1}{2(\alpha+\gamma)^2}\right]c_1^2\right\}.$$

Therefore,

$$a_3-\mu a_2^2=\frac{-bB_1}{2(\alpha+2\gamma)\,b_3}\left[c_2-vc_1^2\right], \qquad (2.5)$$

where

$$v=\frac{1}{2}\left[1-\frac{B_2}{B_1}+\frac{(\alpha+1)(\alpha+2\gamma)bB_1}{2(\alpha+\gamma)^2}-\frac{\mu b(\alpha+2\gamma)b_3B_1}{(\alpha+\gamma)^2b_2^2}\right]. \qquad (2.6)$$

Our result now follows by using Lemma 1. The result is sharp for the functions

$$1+\frac{1}{b}\left\{(1+\gamma)\left(\frac{z}{(f*g)(z)}\right)^{\alpha}-\gamma\frac{z\left((f*g)(z)\right)'}{(f*g)(z)}\left(\frac{z}{(f*g)(z)}\right)^{\alpha}-1\right\}=\phi(z^2)$$

and

$$1+\frac{1}{b}\left\{(1+\gamma)\left(\frac{z}{(f*g)(z)}\right)^{\alpha}-\gamma\frac{z\left((f*g)(z)\right)'}{(f*g)(z)}\left(\frac{z}{(f*g)(z)}\right)^{\alpha}-1\right\}=\phi(z).$$

This completes the proof of Theorem 1. ∎

Putting $g(z)=\frac{z}{1-z}$ in Theorem 1, we obtain the following result.

Corollary 1. Let $\phi(z)=1+B_1z+B_2z^2+...$ with $B_1>0$. If $f(z)$ given by (1.1) belongs to the class $R^{\alpha,\gamma}(b,\phi)$ with $\alpha+\gamma\neq 0$ and $\alpha+2\gamma\neq 0$, then

$$\left|a_3-\mu a_2^2\right|\leq \frac{|b|B_1}{|\alpha+2\gamma|}\max\left\{1;\left|\frac{B_2}{B_1}-\frac{(\alpha+1)(\alpha+2\gamma)bB_1}{2(\alpha+\gamma)^2}+\frac{\mu b(\alpha+2\gamma)B_1}{(\alpha+\gamma)^2}\right|\right\}.$$

The result is sharp.

Putting $\gamma=-1$ and $\alpha=0$ in Corollary 1. We obtain the following result which modifies the result obtained by Ravichandran et al. [13, Theorem 4.1].

Corollary 2. Let $\phi(z)=1+B_1z+B_2z^2+...$ with $B_1>0$. If $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_b^*(\phi)$, then

$$\left|a_3-\mu a_2^2\right|\leq \frac{|b|B_1}{2}\max\left\{1;\left|\frac{B_2}{B_1}+(1-2\mu)\,bB_1\right|\right\}.$$

The result is sharp.

Putting $b=(1-\beta)\cos\theta\ e^{-i\theta}\left(|\theta|<\frac{\pi}{2},0\leq\beta<1\right)$ in Theorem 1. We obtain the following result.

Corollary 3. Let $\phi(z)=1+B_1z+B_2z^2+...$ with $B_1>0$. If $f(z)$ given by (1.1) belongs to the class $R_g^{\alpha,\gamma}(\theta,\beta,\phi)$ with $\alpha+\gamma\neq 0$ and $\alpha+2\gamma\neq 0$, then

$$\begin{aligned}\left|a_3-\mu a_2^2\right| \leq\ & \frac{(1-\beta)\cos\theta B_1}{|\alpha+2\gamma|b_3}\\ &\times\max\left\{1;\left|\begin{array}{c}\frac{B_2}{B_1}e^{i\theta}-\frac{(\alpha+1)(\alpha+2\gamma)(1-\beta)\cos\theta B_1}{2(\alpha+\gamma)^2}\\ +\frac{\mu(1-\beta)\cos\theta(\alpha+2\gamma)b_3B_1}{(\alpha+\gamma)^2b_2^2}\end{array}\right|\right\}.\end{aligned}$$

The result is sharp.

Remark 1. For $\gamma=-1, b=1$ and $\phi(z)=\frac{1+(1-2\rho)z}{1-z}$ $(0\leq\rho<1)$ in Corollary 1, we obtain the result of Tuneski and Darus [17, Theorem 1].

Theorem 2. Let $\phi(z)=1+B_1z+B_2z^2+...$ with $B_1>0$. Let

$$\sigma_1=\frac{b_2^2}{2b_3}\left[(\alpha+1)-\frac{2(\alpha+\gamma)^2(B_2-B_1)}{b(\alpha+2\gamma)B_1^2}\right], \tag{2.7}$$

$$\sigma_2=\frac{b_2^2}{2b_3}\left[(\alpha+1)-\frac{2(\alpha+\gamma)^2(B_2+B_1)}{b(\alpha+2\gamma)B_1^2}\right], \tag{2.8}$$

$$\sigma_3=\frac{b_2^2}{2b_3}\left[(\alpha+1)-\frac{2(\alpha+\gamma)^2B_2}{b(\alpha+2\gamma)B_1^2}\right]. \tag{2.9}$$

If $f(z)$ given by (1.1) belongs to the class $R_g^{\alpha,\gamma}(b,\phi)$ with $\alpha+2\gamma\neq 0$, then

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{|b|B_2}{|\alpha+2\gamma|b_3}-\frac{b|b|(\alpha+1)B_1^2}{2(\alpha+\gamma)^2b_3}+\frac{\mu b|b|B_1^2}{(\alpha+\gamma)^2b_2^2} & \mu\geq\sigma_1,\\ \frac{|b|B_1}{|\alpha+2\gamma|b_3} & \sigma_2\leq\mu\leq\sigma_1,\\ \frac{-|b|B_2}{|\alpha+2\gamma|b_3}+\frac{b|b|(\alpha+1)B_1^2}{2(\alpha+\gamma)^2b_3}-\frac{\mu b|b|B_1^2}{(\alpha+\gamma)^2b_2^2} & \mu\leq\sigma_2.\end{cases}$$

Further, if $\sigma_3\leq\mu\leq\sigma_1$, then

$$\begin{aligned}&\left|a_3-\mu a_2^2\right|+\frac{(\alpha+\gamma)^2b_2^2}{|b|\,|\alpha+2\gamma|\,b_3B_1^2}\\&\times\left[B_1-B_2+\frac{b(\alpha+1)(\alpha+2\gamma)B_1^2}{2(\alpha+\gamma)^2}-\frac{\mu b(\alpha+2\gamma)b_3B_1^2}{(\alpha+\gamma)^2b_2^2}\right]|a_2|^2\\\leq\;&\frac{|b|B_1}{|\alpha+2\gamma|b_3}.\end{aligned}$$

If $\sigma_2\leq\mu\leq\sigma_3$, then

$$\begin{aligned}&\left|a_3-\mu a_2^2\right|+\frac{(\alpha+\gamma)^2b_2^2}{|b|\,|\alpha+2\gamma|\,b_3B_1^2}\\&\times\left[B_1+B_2-\frac{b(\alpha+1)(\alpha+2\gamma)B_1^2}{2(\alpha+\gamma)^2}+\frac{\mu b(\alpha+2\gamma)b_3B_1^2}{(\alpha+\gamma)^2b_2^2}\right]|a_2|^2\\\leq\;&\frac{|b|B_1}{|\alpha+2\gamma|b_3}.\end{aligned}$$

The result is sharp.

Proof. The results of Theorem 2 follows by applying Lemma 2 to (2.5). To show that the bounds are sharp, we define the functions $\chi_{\phi n}\,(n=2,3,4,...)$, F_λ and ξ_λ $(0\leq\lambda\leq 1)$, respectively, by

$$1+\frac{1}{b}\left\{(1+\gamma)\left(\tfrac{z}{(\chi_{\phi n}*g)(z)}\right)^\alpha-\gamma\frac{z\left((\chi_{\phi n}*g)(z)\right)'}{(\chi_{\phi n}*g)(z)}\left(\tfrac{z}{(\chi_{\phi n}*g)(z)}\right)^\alpha-1\right\}=\phi\left(z^{n-1}\right),$$

$$\chi_{\phi n}(0)=0=\chi'_{\phi n}(0)-1,$$

$$1+\frac{1}{b}\left\{(1+\gamma)\left(\tfrac{z}{(F_\lambda*g)(z)}\right)^\alpha-\gamma\frac{z\left((F_\lambda*g)(z)\right)'}{(F_\lambda*g)(z)}\left(\tfrac{z}{(F_\lambda*g)(z)}\right)^\alpha-1\right\}=\phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right),$$

$$F_\lambda(0)=0=F'_\lambda(0)-1$$

and

$$1+\frac{1}{b}\left\{(1+\gamma)\left(\tfrac{z}{(\xi_\lambda*g)(z)}\right)^\alpha-\gamma\frac{z\left((\xi_\lambda*g)(z)\right)'}{(\xi_\lambda*g)(z)}\left(\tfrac{z}{(\xi_\lambda*g)(z)}\right)^\alpha-1\right\}=\phi\left(-\frac{1+\lambda z}{z(z+\lambda)}\right),$$

$$\xi_\lambda(0)=0=\xi'_\lambda(0)-1.$$

Clearly, the functions $\chi_{\phi n}, F_{\lambda}$ and $\xi_{\lambda} \in R_{m,\lambda,l}^{\alpha,\gamma}(b,\phi)$. If $\mu > \sigma_1$ or $\mu < \sigma_2$, then the equality holds if and only if $f(z)$ is $\chi_{\phi 2}$, or one of its rotations. When $\sigma_2 < \mu < \sigma_1$, the equality holds if and only if $f(z)$ is $\chi_{\phi 3}$, or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if $f(z)$ is F_{λ}, or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if $f(z)$ is ξ_{λ}, or one of its rotations. ∎

Taking $g(z) = \frac{z}{1-z}$ in Theorem 2. We obtain the following result for function belonging to the class $R^{\alpha,\gamma}(b,\phi)$.

Corollary 4. Let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ with $B_1 > 0$. Let

$$\sigma_4 = \frac{1}{2}\left[(\alpha+1) - \frac{2(\alpha+\gamma)^2(B_2 - B_1)}{b(\alpha+2\gamma)B_1^2}\right],$$

$$\sigma_5 = \frac{1}{2}\left[(\alpha+1) - \frac{2(\alpha+\gamma)^2(B_2 + B_1)}{b(\alpha+2\gamma)B_1^2}\right],$$

$$\sigma_6 = \frac{1}{2}\left[(\alpha+1) - \frac{2(\alpha+\gamma)^2 B_2}{b(\alpha+2\gamma)B_1^2}\right].$$

If $f(z)$ given by (1.1) belongs to the class $R^{\alpha,\gamma}(b,\phi)$ with $\alpha + 2\gamma \neq 0$, then

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{|b|B_2}{|\alpha+2\gamma|} - \frac{b|b|(\alpha+1)B_1^2}{2(\alpha+\gamma)^2} + \frac{\mu b|b|B_1^2}{(\alpha+\gamma)^2} & \mu \geq \sigma_4, \\ \frac{|b|B_1}{|\alpha+2\gamma|} & \sigma_5 \leq \mu \leq \sigma_4, \\ \frac{-|b|B_2}{|\alpha+2\gamma|} + \frac{b|b|(\alpha+1)B_1^2}{2(\alpha+\gamma)^2} - \frac{\mu b|b|B_1^2}{(\alpha+\gamma)^2} & \mu \leq \sigma_5. \end{cases}$$

Further, if $\sigma_6 \leq \mu \leq \sigma_4$, then

$$\begin{aligned} & \left|a_3 - \mu a_2^2\right| + \frac{(\alpha+\gamma)^2}{|b|\,|\alpha+2\gamma|\,B_1^2} \\ & \times \left[B_1 - B_2 + \frac{b(\alpha+1)(\alpha+2\gamma)B_1^2}{2(\alpha+\gamma)^2} - \frac{\mu b(\alpha+2\gamma)B_1^2}{(\alpha+\gamma)^2}\right]|a_2|^2 \\ \leq \quad & \frac{|b|\,B_1}{|\alpha+2\gamma|}. \end{aligned}$$

If $\sigma_5 \leq \mu \leq \sigma_6$, then

$$\begin{aligned} & \left|a_3 - \mu a_2^2\right| + \frac{(\alpha+\gamma)^2}{|b|\,|\alpha+2\gamma|\,B_1^2} \\ & \times \left[B_1 + B_2 - \frac{b(\alpha+1)(\alpha+2\gamma)B_1^2}{2(\alpha+\gamma)^2} + \frac{\mu b(\alpha+2\gamma)B_1^2}{(\alpha+\gamma)^2}\right]|a_2|^2 \\ \leq \quad & \frac{|b|\,B_1}{|\alpha+2\gamma|}. \end{aligned}$$

The result is sharp.

Remark 2. For $b = 1$ in Corollary 4, we obtain the result of Shanmugam et al. [16, Theorem 1].

Remark 3. For $b = 1$ and $g(z) = z + \sum_{k=2}^{\infty} \frac{(\alpha_1)_{k-1} \ldots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \ldots (\beta_s)_{k-1} (1)_{k-1}} z^k$ $(q \leq s+1; s, q \in \mathbb{N}_0)$ in the above results, we obtain the results of Seoudy [15].

Remark 4. Specializing the parameters γ, α and the function g in the above results, we obtain results corresponding to different classes given in the introduction.

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