An alternative proof of the generalized Littlewood Tauberian theorem for Cesàro summable double sequences

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Abstract

In this paper, we first examine the relationships between a double sequence and its arithmetic means in different senses (i. e. $(C,1,0)$, $(C,0,1)$ and $(C,1,1)$ means) in terms of slow oscillation in certain senses and investigate some properties of oscillatory behaviors of the difference sequence between the double sequence and its arithmetic means in different senses. Next, we give an alternative proof of the generalized Littlewood Tauberian theorem for Cesàro summability method as an application of the results obtained in the first part.

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1 Introduction

Imposing some conditions on the difference sequence between a single sequence of real numbers and its arithmetic means, such a sequence is called a generator sequence, is based on the second theorem of Tauber [13]. Tauber [13] proved that the generator sequence's convergence to zero is a Tauberian condition for Abel summability method. Later, Szász [12] proved that one-sided boundedness of the generator sequence is sufficient to recover Cesàro summability of a sequence from its Abel summability.

The concept of slowly oscillating sequence is introduced by Schmidt [10]. It is well-known that every convergent sequence is slowly oscillating. Nevertheless, the converse is not true in general. Schmidt [10] proved that every sequence which is Abel summable and slowly oscillating is convergent. Dik [4] and Móricz [8] proved that sequence of arithmetic means of a slowly oscillating sequence is slowly oscillating. Moreover, Dik [4] showed that a sequence is slowly oscillating if and only if the generator sequence is bounded and slowly oscillating.

Schmidt [10] obtained convergence of (u_n) from its Cesàro summability by introducing the concept of a slowly oscillating sequence. Schmidt's Tauberian theorem is called the generalized Littlewood Tauberian theorem in the literature. The generalized Littlewood Tauberian theorem is one of the well-known classical Tauberian theorems. The generalized Littlewood Tauberian theorem for a single sequence is proved by several authors such as Landau [6] and Vijayaraghavan [16].

Stanojević [11] introduced an equivalent definition for a slowly oscillating sequence. Using Stanojević's definition, Dik [4] proved that the slow oscillation of the generator sequence is a Tauberian condition for Abel summability method. Çanak [3] obtained an alternative proof of the generalized Littlewood Tauberian theorem stating that (u_n) is Cesàro summable to s and slowly oscillating, then (u_n) is convergent to s.

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The concept of a regularly generated sequence is introduced by Stanojević [11]. Çanak [2] proved a new Tauberian theorem to retrieve the slow oscillation of (u_n) out of the Cesàro summability of the generator sequence and some additional condition(s).

On the other hand, Móricz [7] has given the Tauberian theorems of Landau and Hardy type for double sequences and deduced an analogue of the generalized Littlewood Tauberian theorem for double sequences. Totur [14] has proved some classical type Tauberian theorems, which are called Landau's theorem and generalized Littlewood Tauberian theorem for Cesàro summability of a double sequence. Moreover, Totur [14] has given sufficient conditions for a $(C,1,1)$ summable sequence to be P-convergent by imposing conditions on the sequence or one of its generators. Recently, Totur [15] has shown that if a sequence of arithmetic means of a double sequence has both the limit inferior and the limit superior, then the double sequence itself has the limit inferior and the limit superior under suitable Tauberian conditions.

The main results in this paper are given in two parts. In the first part, we examine the relationships between a double sequence and its arithmetic means in different senses (i.e. $(C,1,0)$, $(C,0,1)$ and $(C,1,1)$ means) in terms of slow oscillation in different senses, and investigate the oscillatory behaviors of the generator sequence. In the second part, we give an alternative proof of the generalized Littlewood Tauberian theorem as an application of the results obtained in the first part.

2 Preliminaries

We now give some necessary definitions and notations for double sequences.

A double sequence $u=(u_{mn})$ is called Pringsheim convergent (in short P-convergent) to s [9], if for a given $\varepsilon>0$ there exists a positive integer N_0 such that $|u_{mn}-s|<\varepsilon$ for all nonnegative integers $m,n\geq N_0$. Note that in this paper we use convergence in Pringsheim's sense for double sequences.

A double sequence (u_{mn}) is said to be slowly oscillating in sense $(1,1)$ if

$$\lim_{\lambda\to 1^+}\limsup_{m,n\to\infty}\max_{\substack{m<j\leq\lambda_m\\ n<k\leq\lambda_n}}\left|\sum_{r=m+1}^{j}\sum_{s=n+1}^{k}\Delta_{r,s}u_{rs}\right|=0,$$

where λ_m and λ_n denote the integer parts of λm and λn, respectively. Equivalently, (u_{mn}) is slowly oscillating in sense $(1,1)$ if for each $\varepsilon>0$ there exist $n_1>0$ and $\lambda>1$ such that $|u_{jk}-u_{mk}-u_{jn}+u_{mn}|<\varepsilon$ whenever $n_1<m<j\leq\lambda_m$ and $n_1<n<k\leq\lambda_n$. A double sequence (u_{mn}) is said to be slowly oscillating in sense $(1,0)$ if

$$\lim_{\lambda\to 1^+}\limsup_{m,n\to\infty}\max_{m<j\leq\lambda_m}\left|\sum_{r=m+1}^{j}\Delta_{r}u_{rn}\right|=0$$

Equivalently, (u_{mn}) is slowly oscillating in sense $(1,0)$ if for each $\varepsilon>0$ there exist $n_1>0$ and $\lambda>1$ such that $|u_{jn}-u_{mn}|<\varepsilon$ whenever $n_1<m<j\leq\lambda_m$ and $n_1<n$.
A double sequence (u_{mn}) is said to be slowly oscillating in sense $(0,1)$ if

$$\lim_{\lambda\to 1^+}\limsup_{m,n\to\infty}\max_{n<k\leq\lambda_n}\left|\sum_{s=n+1}^{k}\Delta_{s}u_{ms}\right|=0$$

Equivalently, (u_{mn}) is slowly oscillating in sense $(0,1)$ if for each $\varepsilon > 0$ there exist $n_1 > 0$ and $\lambda > 1$ such that $|u_{mk} - u_{mn}| < \varepsilon$ whenever $n_1 < n < k \leq \lambda_n$ and $n_1 < m$.

Notice that every P-convergent sequence is slowly oscillating in senses $(1,1)$, $(1,0)$ and $(0,1)$. However, the converses may not be true. For instance, the double sequence $(u_{mn}) = (\log m \log n)$ is slowly oscillating in sense $(1,1)$, but not P-convergent.

The $(C,1,1)$ means of (u_{mn}) are defined by

$$\sigma^{(11)}_{mn}(u) := \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} u_{ij}$$

for all nonnegative integers m and n. The $(C,1,0)$ and $(C,0,1)$ means of (u_{mn}) are defined respectively by

$$\sigma^{(10)}_{mn}(u) := \frac{1}{m+1} \sum_{i=0}^{m} u_{in} \quad \text{and} \quad \sigma^{(01)}_{mn}(u) := \frac{1}{n+1} \sum_{j=0}^{n} u_{mj}$$

for all nonnegative integers m and n.

The following identities are satisfied for a double sequence (u_{mn}):

$$\sigma^{(10)}_{mn}(\sigma^{(01)}(u)) = \sigma^{(01)}_{mn}(\sigma^{(10)}(u)) = \sigma^{(11)}_{mn}(u), \tag{1}$$
$$\sigma^{(10)}_{mn}(\sigma^{(11)}(u)) = \sigma^{(11)}_{mn}(\sigma^{(10)}(u)), \tag{2}$$
$$\sigma^{(01)}_{mn}(\sigma^{(11)}(u)) = \sigma^{(11)}_{mn}(\sigma^{(01)}(u)) \tag{3}$$

for all nonnegative integers m and n.

A double sequence (u_{mn}) is said to be bounded if there exists a real number $C > 0$ such that $|u_{mn}| \leq C$ for all nonnegative m and n. A sequence (u_{mn}) is called (C,α,β) bounded if the sequence $(\sigma^{(\alpha\beta)}_{mn}(u))$ is bounded, where $(\alpha,\beta) = (1,1)$, $(1,0)$ and $(0,1)$.

Throughout this paper, the symbols $u_{mn} = o(1)$ and $u_{mn} = O(1)$ mean respectively that (u_{mn}) is P-convergent to zero as $m,n \to \infty$, and (u_{mn}) is bounded. For a double sequence (u_{mn}), the backward differences in sense $(1,0)$, $(0,1)$ and $(1,1)$ are defined respectively by

$$\begin{aligned}
\Delta_{10} u_{mn} &:= u_{mn} - u_{m-1,n}, \\
\Delta_{01} u_{mn} &:= u_{mn} - u_{m,n-1}, \\
\Delta_{11} u_{mn} &:= u_{mn} - u_{m,n-1} - u_{m-1,n} + u_{m-1,n-1}
\end{aligned}$$

for all nonnegative integers $m,n \geq 1$.

Let (u_{mn}) be a double sequence. The $(C,1,1)$ means of the sequence $(mn\Delta_{11}u_{mn})$ is defined by

$$V^{(11)}_{mn}(\Delta_{11}u) := \frac{1}{(m+1)(n+1)} \sum_{i=1}^{m} \sum_{j=1}^{n} ij\Delta_{11}u_{ij}$$

for all nonnegative integers m and n ([5]). The $(C,1,0)$ means of the sequence $(m\Delta_{10}u_{mn})$ is defined by

$$V^{(10)}_{mn}(\Delta_{10}u) := \frac{1}{m+1} \sum_{i=1}^{m} i\Delta_{10}u_{in}$$

for all nonnegative integers m and n. The $(C, 0, 1)$ means of sequence $(n\Delta_{01}u_{mn})$ is defined by

$$V_{mn}^{(01)}(\Delta_{01}u) := \frac{1}{n+1}\sum_{j=1}^{n} j\Delta_{01}u_{mj}$$

for all nonnegative integers m and n.

We define the Kronecker identities for a double sequence (u_{mn}) as follows:

$$u_{mn} - \sigma_{mn}^{(10)}(u) - \sigma_{mn}^{(01)}(u) + \sigma_{mn}^{(11)}(u) = V_{mn}^{(11)}(\Delta_{11}u), \tag{4}$$

$$u_{mn} - \sigma_{mn}^{(10)}(u) = V_{mn}^{(10)}(\Delta_{10}u), \tag{5}$$

$$u_{mn} - \sigma_{mn}^{(01)}(u) = V_{mn}^{(01)}(\Delta_{01}u) \tag{6}$$

for all nonnegative integers m and n ([5]).

For a double sequence (u_{mn}), we have the following identities:

$$mn\Delta_{11}\sigma_{mn}^{(11)}(u) = V_{mn}^{(11)}(\Delta_{11}u),$$
$$m\Delta_{10}\sigma_{mn}^{(10)}(u) = V_{mn}^{(10)}(\Delta_{10}u),$$
$$n\Delta_{01}\sigma_{mn}^{(01)}(u) = V_{mn}^{(01)}(\Delta_{01}u) \tag{7}$$

for all nonnegative integer m and n. Indeed, we can easily obtain the identity (7) from the calculations

$$\begin{aligned} n\Delta_{01}\sigma_{mn}^{(01)}(u) &= n\left(\frac{1}{n+1}\sum_{j=0}^{n} u_{mj} - \frac{1}{n}\sum_{j=0}^{n-1} u_{mj}\right) \\ &= \frac{1}{n+1}\left(n\sum_{j=0}^{n} u_{mj} - n\sum_{j=0}^{n-1} u_{mj} - \sum_{j=0}^{n-1} u_{mj}\right) \\ &= \frac{1}{n+1}\left(nu_{mn} - \sum_{j=0}^{n-1} u_{mj}\right) \\ &= \frac{1}{n+1}\sum_{j=0}^{n} j\Delta_{01}u_{mj}. \end{aligned}$$

The other identities can be obtained similarly.

For a double sequence (u_{mn}), we obtain

$$\sigma_{mn}^{(11)}(u) = \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{V_{ij}^{(11)}(\Delta_{11}u)}{ij} + \sigma_{0,n}^{(11)}(u) + \sigma_{m,0}^{(11)}(u) - \sigma_{0,0}^{(11)}(u),$$

$$\sigma_{mn}^{(10)}(u) = \sum_{i=1}^{m}\frac{V_{in}^{(10)}(\Delta_{10}u)}{i} + u_{0n},$$

$$\sigma_{mn}^{(01)}(u)=\sum_{j=1}^{n}\frac{V_{mj}^{(01)}(\Delta_{01}u)}{j}+u_{m0}$$

by using the identities above.

The Kronecker identity (4) can be rewritten as

$$V_{mn}^{(10)}(\Delta_{10}u)+V_{mn}^{(01)}(\Delta_{01}u)-V_{mn}^{(11)}(\Delta_{11}u)=u_{mn}-\sigma_{mn}^{(11)}(u) \tag{8}$$

by (5) and (6).

A double sequence (u_{mn}) can be expressed by (8) as

$$u_{mn}=V_{mn}^{(10)}(\Delta_{10}u)+V_{mn}^{(01)}(\Delta_{01}u)+\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{V_{ij}^{(11)}(\Delta_{11}u)}{ij}$$
$$+\sigma_{0,n}^{(11)}(u)+\sigma_{m,0}^{(11)}(u)-\sigma_{0,0}^{(11)}(u)-V_{mn}^{(11)}(\Delta_{11}u)$$

in terms of the sequences $\left(V_{mn}^{(10)}(\Delta_{10}u)\right)$, $\left(V_{mn}^{(01)}(\Delta_{01}u)\right)$ and $\left(V_{mn}^{(11)}(\Delta_{11}u)\right)$ which generate (u_{mn}).

A double sequence (u_{mn}) can be also represented as

$$u_{mn}=V_{mn}^{(10)}(\Delta_{10}u)+\sum_{i=1}^{m}\frac{V_{in}^{(10)}(\Delta_{10}u)}{i}+u_{0,n}$$

or

$$u_{mn}=V_{mn}^{(01)}(\Delta_{01}u)+\sum_{j=1}^{n}\frac{V_{mj}^{(01)}(\Delta_{01}u)}{j}+u_{m,0}$$

The sequences $\left(V_{mn}^{(10)}(\Delta_{10}u)\right)$ and $\left(V_{mn}^{(01)}(\Delta_{01}u)\right)$ are said to be generators of (u_{mn}).

3 Lemmas

In this part of the paper, we give the required lemmas to prove our main theorem.

We need the following lemma for the proof of the next lemma. The following two Lemmas are double sequence analogues of lemmas given by Badiozzaman [1] for single sequences.

Lemma 3.1. Let (u_{mn}) be a double sequence. If (γ_n) is an increasing sequence of positive integers, then

$$\left|\sum_{i=p}^{q}\sum_{j=r}^{s}\gamma_i\gamma_j u_{ij}\right|\le\gamma_q\gamma_s\max_{\substack{p\le x\le q\\ r\le y\le s}}\left|\sum_{i=x}^{q}\sum_{j=y}^{s}u_{ij}\right|.$$

Proof. Let $t_{mn} = \sum_{i=m}^{q}\sum_{j=n}^{s} u_{ij}$. Then using summation by parts, we have

$$\begin{aligned}
\sum_{i=p}^{q}\sum_{j=r}^{s}\gamma_i\gamma_j u_{ij} &= \sum_{i=p}^{q}\sum_{j=r}^{s}\gamma_i\gamma_j\left(t_{ij}-t_{i+1,j}-t_{i,j+1}+t_{i+1,j+1}\right)\\
&= \sum_{i=p}^{q}\sum_{j=r}^{s}\gamma_i\gamma_j t_{ij} - \sum_{i=p+1}^{q+1}\sum_{j=r}^{s}\gamma_{i-1}\gamma_j t_{ij}\\
&\quad - \sum_{i=p}^{q}\sum_{j=r+1}^{s+1}\gamma_i\gamma_{j-1} t_{ij} + \sum_{i=p+1}^{q+1}\sum_{j=r+1}^{s+1}\gamma_{i-1}\gamma_{j-1} t_{ij}\\
&= \sum_{i=p+1}^{q}\sum_{j=r+1}^{s}\gamma_i\gamma_j t_{ij} + \gamma_r\sum_{i=p}^{q}\gamma_i t_{ir} + \gamma_p\sum_{j=r+1}^{s}\gamma_j t_{pj}\\
&\quad - \sum_{i=p+1}^{q}\sum_{j=r+1}^{s}\gamma_{i-1}\gamma_j t_{ij} - \gamma_r\sum_{i=p+1}^{q}\gamma_{i-1} t_{ir} - \gamma_q\sum_{j=r}^{s}\gamma_j t_{q+1,j}\\
&\quad - \sum_{i=p+1}^{q}\sum_{j=r+1}^{s}\gamma_i\gamma_{j-1} t_{ij} - \gamma_p\sum_{j=r+1}^{s}\gamma_{j-1} t_{pj} - \gamma_s\sum_{i=p}^{q}\gamma_i t_{i,s+1}\\
&\quad + \sum_{i=p+1}^{q}\sum_{j=r+1}^{s}\gamma_{i-1}\gamma_{j-1} t_{ij} + \gamma_q\sum_{j=r+1}^{s}\gamma_{j-1} t_{q+1,j}\\
&\quad + \gamma_s\sum_{i=p+1}^{q}\gamma_{i-1} t_{i,s+1} + \gamma_s\gamma_q t_{q+1,s+1}\\
&= \sum_{i=p+1}^{q}\sum_{j=r+1}^{s} t_{ij}\gamma_i\gamma_j\left(\gamma_i\gamma_j-\gamma_{i-1}\gamma_j-\gamma_i\gamma_{j-1}+\gamma_{i-1}\gamma_{j-1}\right)\\
&\quad + \gamma_r\sum_{i=p+1}^{q} t_{ir}\left(\gamma_i-\gamma_{i-1}\right) + \gamma_p\sum_{j=r+1}^{s} t_{pj}\left(\gamma_j-\gamma_{j-1}\right) + \gamma_r\gamma_p t_{pr}\\
&\le \sum_{i=p+1}^{q}\sum_{j=r+1}^{s} t_{pr}(\gamma_i\gamma_j-\gamma_{i-1}\gamma_j-\gamma_i\gamma_{j-1}+\gamma_{i-1}\gamma_{j-1}\\
&\qquad + \gamma_r\left(\gamma_i-\gamma_{i-1}\right)+\gamma_p\left(\gamma_j-\gamma_{j-1}\right)+\gamma_r\gamma_p).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\left|\sum_{i=p}^{q}\sum_{j=r}^{s}\gamma_i\gamma_j u_{ij}\right| &\le \max_{\substack{p\le x\le q\\ r\le y\le s}}\left|\sum_{i=x}^{q}\sum_{j=y}^{s}u_{ij}\right|\sum_{i=p+1}^{q}\sum_{j=r+1}^{s}\big|\big(\gamma_i\gamma_j-\gamma_{i-1}\gamma_j-\gamma_i\gamma_{j-1}\\ &\quad+\gamma_{i-1}\gamma_{j-1}+\gamma_r\left(\gamma_i-\gamma_{i-1}\right)+\gamma_p\left(\gamma_j-\gamma_{j-1}\right)+\gamma_r\gamma_p\big)\big|\\ &= \max_{\substack{p\le x\le q\\ r\le y\le s}}\left|\sum_{i=x}^{q}\sum_{j=y}^{s}u_{ij}\right|\big|\gamma_p\gamma_r-\gamma_p\gamma_s-\gamma_r\gamma_q+\gamma_q\gamma_s\\ &\quad+\gamma_r\left(\gamma_q-\gamma_p\right)+\gamma_p\left(\gamma_s-\gamma_r\right)+\gamma_r\gamma_p\big|\\ &= \gamma_q\gamma_s\max_{\substack{p\le x\le q\\ r\le y\le s}}\left|\sum_{i=x}^{q}\sum_{j=y}^{s}u_{ij}\right|.\end{aligned}$$

Q.E.D.

The following lemma is required for the proof of Theorem 4.1.

Lemma 3.2. Let (u_{mn}) be a double sequence. If $\lambda > 1$ can be choosen such that for $m, n \ge 0$

$$\max_{\substack{m\le p\le \lambda m\\ n\le s\le \lambda n}}\left|\sum_{i=m}^{p}\sum_{j=n}^{s}u_{ij}\right|\le\beta$$

then

$$\left|\sum_{i=1}^{m}\sum_{j=1}^{n}iju_{ij}\right| = O(mn).$$

Proof. If $m < x < m+1$ and $n < y < n+1$, by Lemma 3.1, we have

$$\begin{aligned}\left|\sum_{i=1}^{m}\sum_{j=1}^{n}iju_{ij}\right| &\le \sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\left|\sum_{x\lambda^{-q-1}<i<x\lambda^{-q}}\ \sum_{y\lambda^{-r-1}<j<\lambda^{-r}}iju_{ij}\right|\\ &\le \sum_{q=0}^{\infty}\sum_{r=0}^{\infty}xy\lambda^{-q-r}\max_{\substack{x\lambda^{-q-1}\le t\le t'<x\lambda^{-q}\\ y\lambda^{-r-1}\le v\le v'<y\lambda^{-r}}}\left|\sum_{i=t}^{t'}\sum_{j=v}^{v'}u_{ij}\right|\\ &\le \beta xy\left(\frac{\lambda}{\lambda-1}\right)^2\end{aligned}$$

and the result follows on letting $x \to m^+$ and $y \to n^+$. Q.E.D.

The following identities are important because of showing the relationships between the generators of a double sequence (u_{mn}).

Lemma 3.3. The following identities are satisfied:

$$V^{(01)}_{mn}(\Delta_{01}V^{(10)}(\Delta_{10}u)) = V^{(11)}_{mn}(\Delta_{11}u), \tag{9}$$

and

$$V^{(10)}_{mn}(\Delta_{10}V^{(01)}(\Delta_{01}u)) = V^{(11)}_{mn}(\Delta_{11}u).$$

Proof. First, we prove (9). By definition, we have

$$\begin{aligned} V^{(01)}_{mn}\left(\Delta_{01}V^{(10)}\left(\Delta_{10}u\right)\right) &= \frac{1}{n+1}\sum_{j=1}^{n} j\Delta_{01}\left(\frac{1}{m+1}\sum_{i=1}^{m} i\Delta_{10}u_{ij}\right) \\ &= \frac{1}{(m+1)(n+1)}\sum_{j=1}^{n} j\Delta_{01}\sum_{i=1}^{m} i\Delta_{10}u_{ij} \\ &= \frac{1}{(m+1)(n+1)}\sum_{i=1}^{m}\sum_{j=1}^{n} ij\Delta_{11}u_{ij} \\ &= V^{(11)}_{mn}(\Delta_{11}u). \end{aligned}$$

Proof of the second identity is similar. We omit the proof of it. Q.E.D.

The following two lemmas are due to Móricz [7]. A different proof of Lemma 3.4 is given by Totur [14].

Lemma 3.4. Let (u_{mn}) be a double sequence of real numbers. For sufficiently large m and n,

(i) If $\lambda > 1$,

$$\begin{aligned} u_{mn} - \sigma^{(11)}_{mn}(u) &= \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}\left(\sigma^{(11)}_{\lambda_m,\lambda_n}(u) - \sigma^{(11)}_{\lambda_m,n}(u) - \sigma^{(11)}_{m,\lambda_n}(u) + \sigma^{(11)}_{mn}(u)\right) \\ &+ \frac{\lambda_m+1}{\lambda_m-m}\left(\sigma^{(11)}_{\lambda_m,n}(u) - \sigma^{(11)}_{mn}(u)\right) + \frac{\lambda_n+1}{\lambda_n-n}\left(\sigma^{(11)}_{m,\lambda_n}(u) - \sigma^{(11)}_{mn}(u)\right) \\ &- \frac{1}{(\lambda_m-m)(\lambda_n-n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}(u_{jk}-u_{mn}). \end{aligned}$$

(ii) If $0 < \lambda < 1$,

$$\begin{aligned} u_{mn} - \sigma^{(11)}_{mn}(u) &= \frac{(\lambda_m+1)(\lambda_n+1)}{(m-\lambda_m)(n-\lambda_n)}\left(\sigma^{(11)}_{mn}(u) - \sigma^{(11)}_{\lambda_m,n}(u) - \sigma^{(11)}_{m,\lambda_n}(u) + \sigma^{(11)}_{\lambda_m,\lambda_n}(u)\right) \\ &+ \frac{\lambda_m+1}{m-\lambda_m}\left(\sigma^{(11)}_{mn}(u) - \sigma^{(11)}_{\lambda_m,n}(u)\right) + \frac{\lambda_n+1}{n-\lambda_n}\left(\sigma^{(11)}_{mn}(u) - \sigma^{(11)}_{m,\lambda_n}(u)\right) \\ &+ \frac{1}{(m-\lambda_m)(n-\lambda_n)}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}(u_{mn}-u_{jk}) \end{aligned}$$

where λ_m and λ_n denote the integer parts of λm and λn, respectively.

Lemma 3.5. Let (u_{mn}) be a double sequence of real numbers. For sufficiently large m and n,

(i) If $\lambda > 1$,

$$u_{mn} - \sigma_{mn}^{(10)}(u) = \frac{\lambda_m + 1}{\lambda_m - m}\left(\sigma_{\lambda_m,n}^{(10)}(u) - \sigma_{mn}^{(10)}(u)\right) - \frac{1}{\lambda_m - m}\sum_{j=m+1}^{\lambda_m}(u_{jn} - u_{mn}).$$

(ii) If $0 < \lambda < 1$,

$$u_{mn} - \sigma_{mn}^{(10)}(u) = \frac{\lambda_m + 1}{m - \lambda_m}\left(\sigma_{mn}^{(10)}(u) - \sigma_{\lambda_m,n}^{(10)}(u)\right) + \frac{1}{m - \lambda_m}\sum_{j=\lambda_m+1}^{m}(u_{mn} - u_{jn}).$$

4 Main results

In this section, we present the first part of our main results.

Theorem 4.1. If (u_{mn}) is slowly oscillating in sense $(1,1)$, then $\left(V_{mn}^{(11)}(\Delta_{11}u)\right)$ is bounded.

Proof. Let $m < x < m+1$ and $n < y < n+1$. Since (u_{mn}) is slowly oscillating in sense $(1,1)$, the condition in Lemma 3.2 is satisfied for the sequence $(\Delta_{11}u_{mn})$ sequence. Replacing u_{mn} by $\Delta_{11}u_{mn}$ in Lemma 3.2, then we obtain

$$\left|\sum_{i=1}^{m}\sum_{j=1}^{n} ij\Delta_{11}u_{ij}\right| \leq \beta xy\left(\frac{\lambda}{\lambda - 1}\right)^2.$$

By dividing both sides of the previous inequality by $(m+1)(n+1)$, we have

$$\left|\frac{1}{(m+1)(n+1)}\sum_{i=1}^{m}\sum_{j=1}^{n} ij\Delta_{11}u_{ij}\right| \leq \frac{\beta xy}{(m+1)(n+1)}\left(\frac{\lambda}{\lambda - 1}\right)^2.$$

Because of $m < x < m+1$ and $n < y < n+1$, we get

$$\left|\frac{1}{(m+1)(n+1)}\sum_{i=1}^{m}\sum_{j=1}^{n} ij\Delta_{11}u_{ij}\right| \leq \beta\left(\frac{\lambda}{\lambda - 1}\right)^2.$$

Finally, we conclude that $\left(V_{mn}^{(11)}(\Delta_{11}u_{mn})\right)$ is bounded. Q.E.D.

Dik [4] proved that a single sequence (u_n) is slowly oscillating if and only if the sequence of the difference between u_n and its arithmetic mean $\sigma_n(u)$ is bounded and slowly oscillating. Analogue statements for double sequences are given by the following theorem. Since their proofs can be done similarly as in the proof of single sequence case, then we omit it.

Theorem 4.2. (i) A double sequence (u_{mn}) is slowly oscillating in sense $(1,0)$ if and only if $\left(V_{mn}^{(10)}(\Delta_{10}u)\right)$ is bounded and slowly oscillating in sense $(1,0)$.

(ii) A double sequence (u_{mn}) is slowly oscillating in sense $(0,1)$ if and only if $\left(V_{mn}^{(01)}(\Delta_{01}u)\right)$ is bounded and slowly oscillating in sense $(0,1)$.

It is well-known that the $(C,0,1)$ (or $(C,1,0)$) mean of a slowly oscillating sequence in sense $(0,1)$ (or $(1,0)$) is slowly oscillating in the same sense.

The following theorem shows that the $(C,1,0)$ (or $(C,0,1)$) mean of a slowly oscillating sequence in sense $(0,1)$ (or $(1,0)$) is also slowly oscillating in the same sense under the boundness condition.

Theorem 4.3. (i) If (u_{mn}) is slowly oscillating in sense $(0,1)$ and bounded, then $(\sigma_{mn}^{(10)}(u))$ is slowly oscillating in sense $(0,1)$.

(ii) If (u_{mn}) is slowly oscillating in sense $(1,0)$ and bounded, then $(\sigma_{mn}^{(01)}(u))$ is slowly oscillating in sense $(1,0)$.

Proof. (i) Suppose that (u_{mn}) is slowly oscillating in sense $(0,1)$. Hence given $\varepsilon>0$, there exist $n_1>0$ and $\lambda>1$ such that $|u_{mk}-u_{mn}|<\frac{\varepsilon}{2}$ whenever $n_1<n<k\leq\lambda_n$ and $n_1<m$. Since (u_{mn}) is bounded, there is a constant $K>0$ such that $|u_{mn}|\leq K$ for all nonnegative integers m and n. Then, we have

$$\begin{aligned}
\left|\sigma_{mk}^{(10)}(u)-\sigma_{mn}^{(10)}(u)\right| &= \frac{1}{m+1}\left|\sum_{i=0}^{m}(u_{ik}-u_{in})\right| \\
&\leq \frac{1}{m+1}\sum_{i=0}^{m}|u_{ik}-u_{in}| \\
&= \frac{|u_{0k}-u_{0n}|+|u_{1k}-u_{1n}|+\cdots+|u_{n_1,k}-u_{n_1,n}|}{m+1} \\
&\quad+\frac{|u_{n_1+1,k}-u_{n_1+1,n}+|+\cdots+|u_{mk}-u_{mn}|}{m+1} \\
&\leq \frac{n_1K}{m+1}+\frac{\varepsilon}{2}\frac{m-n_1}{m+1} \\
&< \frac{n_1K}{m+1}+\frac{\varepsilon}{2}.
\end{aligned}$$

Since $\lim\limits_{m\to\infty}\frac{n_1K}{m+1}=0$, there exists n_2 such that $\left|\frac{n_1K}{m+1}\right|<\frac{\varepsilon}{2}$ whenever $m>n_2$. So there exists $n_3=\max(n_1,n_2)$ such that

$$\left|\sigma_{mk}^{(10)}(u)-\sigma_{mn}^{(10)}(u)\right|<\varepsilon$$

whenever $n_3<n<k\leq\lambda_n$ and $n_3<m$. This proves that $\left(\sigma_{mn}^{(10)}\right)$ is slowly oscillating in sense $(0,1)$.

The proof of (ii) is similar to that of (i). Q.E.D.

The following Theorem is presented that the slow oscillation of (u_{mn}) in the certain sense implies the slow oscillation of the sequence $(C,1,0)$ (or $(C,0,1)$) mean of (u_{mn}) in the different sense.

Theorem 4.4. (i) If (u_{mn}) is slowly oscillating in sense $(0,1)$, then $(\sigma_{mn}^{(10)}(u))$ is slowly oscillating in sense $(1,1)$.

(ii) If (u_{mn}) is slowly oscillating in sense $(1,0)$, then $(\sigma_{mn}^{(01)}(u))$ is slowly oscillating in sense $(1,1)$.

Proof. *(i)* Suppose that (u_{mn}) is slowly oscillating in sense $(0,1)$. Hence given $\varepsilon > 0$, there exist $n_1 > 0$ and $\lambda > 1$ such that $|u_{mk} - u_{mn}| < \varepsilon$ whenever $n_1 < n < k \leq \lambda_n$ and $n_1 < m$. Then, we have

$$\left|\sigma_{jk}^{(10)}(u) - \sigma_{jn}^{(10)}(u) + \sigma_{mk}^{(10)}(u) - \sigma_{mn}^{(10)}(u)\right|$$
$$= \left|\frac{1}{j+1}\sum_{i=0}^{j} u_{ik} - \frac{1}{j+1}\sum_{i=0}^{j} u_{in} - \frac{1}{m+1}\sum_{i=0}^{m} u_{ik} + \frac{1}{m+1}\sum_{i=0}^{m} u_{in}\right|$$
$$\leq \frac{1}{j+1}\sum_{i=m+1}^{j} |u_{ik} - u_{in}| < \varepsilon$$

This proves that $\left(\sigma_{mn}^{(10)}\right)$ is slowly oscillating in sense $(0,1)$.

The proof of (ii) is similar to that of (i). Q.E.D.

Theorem 4.5. If (u_{mn}) is slowly oscillating in senses $(1,0)$, $(0,1)$ and $(1,1)$, then $\left(V_{mn}^{(11)}(\Delta_{11}u)\right)$ is slowly oscillating in sense $(1,1)$.

Proof. Suppose that (u_{mn}) is slowly oscillating in senses $(1,0)$, $(0,1)$ and $(1,1)$. Then, $\left(\sigma_{mn}^{(10)}(u)\right)$ and $\left(\sigma_{mn}^{(01)}(u)\right)$ are slowly oscillating in sense $(1,1)$ by Theorem 4.4. So we obtain that $\left(V_{mn}^{(10)}(\Delta_{10}u)\right)$ and $\left(V_{mn}^{(01)}(\Delta_{01}u)\right)$ are slowly oscillating in sense $(1,1)$ by the Kronecker identities (5) and (6), respectively. We conclude that $\left(V_{mn}^{(11)}(\Delta_{11}u)\right)$ is slowly oscillating in sense $(1,1)$ by (8). Q.E.D.

The following theorem allows us to prove the generalized Littlewood Tauberian theorem without imposing a condition on the regularly generated sequence.

Theorem 4.6. If (u_{mn}) is slowly oscillating in senses $(1,0)$ and $(0,1)$, then $\left(V_{mn}^{(11)}(\Delta_{11}u)\right)$ is slowly oscillating in senses $(1,0)$ and $(0,1)$, respectively.

Proof. Replacing u_{mn} by $V_{mn}^{(01)}(\Delta_{01}u)$ in the Kronecker identity (5), we have

$$V_{mn}^{(01)}(\Delta_{01}u) - \sigma_{mn}^{(10)}\left(V^{(01)}(\Delta_{01}u)\right) = V_{mn}^{(11)}(\Delta_{11}u).$$

Since $\left(V_{mn}^{(01)}(\Delta_{01}u)\right)$ is bounded and slowly oscillating in sense $(0,1)$ by Theorem 4.2 (ii), the $(C,1,0)$ means of $\left(V_{mn}^{(01)}(\Delta_{01}u)\right)$ is slowly oscillating in sense (0,1) by Theorem 4.3 (i). Therefore, $\left(V_{mn}^{(11)}(\Delta_{11}u)\right)$ is slowly oscillating in sense $(0,1)$. Similarly, we obtain that $\left(V_{mn}^{(11)}(\Delta_{11}u)\right)$ is slowly oscillating in sense $(1,0)$. Q.E.D.

5 Applications to Tauberian theorems

A sequence (u_{mn}) is said to be (C,α,β) summable to s if $\lim_{m,n\to\infty}\sigma_{mn}^{(\alpha\beta)}(u)=s$, where $(\alpha,\beta)=(1,1)$, $(1,0)$ and $(0,1)$. In this case, we write $u_{mn}\to s\ (C,\alpha,\beta)$.

Note that P-convergent double sequences need not be bounded (see [14]).

By the regularity of the $(C,1,1)$ summability method under the boundedness condition, we mean that if a double sequence is P-convergent to s and bounded, then it is $(C,1,1)$ summable to s. However, the converse is not necessarily true in general. Namely, a double sequence which is bounded and $(C,1,1)$ summable may not be P-convergent. We can recover P-convergence of a double sequence from its $(C,1,1)$ summability under some suitable condition. Such a condition is called a Tauberian condition and the resulting theorem is called a Tauberian theorem.

Our goal in this part is to give an alternative proof of the generalized Littlewood Tauberian theorem for Cesàro summability method for double sequences by using the concept of the regularly generated sequence. Our main theorem adapts the proof given by Çanak [3] for single sequences to double sequences.

Theorem 5.1. Let (u_{mn}) be $(C,1,1)$ bounded. If (u_{mn}) is $(C,1,1)$ summable to s and slowly oscillating in senses $(1,1)$, $(1,0)$ and $(0,1)$, then (u_{mn}) is P-convergent to s.

Corollary 5.2. Let (u_{mn}) be $(C,1,1)$ bounded. If (u_{mn}) is $(C,1,1)$ summable to s and the conditions

$$m\Delta_{10}u_{mn}=O(1),\quad n\Delta_{01}u_{mn}=O(1)\quad\text{and}\quad mn\Delta_{11}u_{mn}=O(1)$$

are satisfied, then (u_{mn}) is P-convergent to s.

Proof. Since $mn\Delta_{11}u_{mn}=O(1)$, there exists a constant $C>0$ such that $|mn\Delta_{11}u_{mn}|\le C$. Then, we have

$$|u_{jk}-u_{mk}-u_{jn}+u_{mn}|\le\sum_{i=m+1}^{j}\sum_{s=n+1}^{k}|\Delta_{11}u_{is}|\le\sum_{i=m+1}^{j}\sum_{s=n+1}^{k}\frac{C}{is}\le C\log\left(\frac{j}{m}\right)\log\left(\frac{k}{n}\right).$$

For $\lambda>1$, we obtain

$$\max_{\substack{m<j\le\lambda_m\\ n<k\le\lambda_n}}|u_{jk}-u_{mk}-u_{jn}+u_{mn}|\le C\log\left(\frac{\lambda_m}{m}\right)\log\left(\frac{\lambda_n}{n}\right).$$

By the definition of slowly oscillating sequence in sense $(1,1)$, we obtain that (u_{mn}) is slowly oscillating in sense $(1,1)$.

Similarly, we can easily obtain the slow oscillation of (u_{mn}) in senses $(1,0)$ and $(0,1)$. Q.E.D.

In the following theorem, we present the generalized Littlewood Tauberian theorem for $(C,1,0)$ and $(C,0,1)$ summability methods.

Theorem 5.3. (i) Let (u_{mn}) be $(C,1,0)$ bounded. If (u_{mn}) is $(C,1,0)$ summable to s and slowly oscillating in sense $(1,0)$, then (u_{mn}) is P-convergent to s.

(ii) Let (u_{mn}) be $(C,0,1)$ bounded. If (u_{mn}) is $(C,0,1)$ summable to s and slowly oscillating in sense $(0,1)$, then (u_{mn}) is P-convergent to s.

The following result can be given as a corollary for Theorem 5.3.

Corollary 5.4. (i) Let (u_{mn}) be $(C,1,0)$ bounded. If (u_{mn}) is $(C,1,0)$ summable to s and the condition $m\Delta_{10}u_{mn} = O(1)$ is satisfied, then (u_{mn}) is P-convergent to s.

(ii) Let (u_{mn}) be $(C,0,1)$ bounded. If (u_{mn}) is $(C,0,1)$ summable to s and the condition $n\Delta_{01}u_{mn} = O(1)$ is satisfied, then (u_{mn}) is P-convergent to s.

6 The proof of Theorem 5.1

Now, we prove our main theorem in this section.

Proof. Taking the $(C,1,1)$ means of both sides of the Kronecker identity (4), we get

$$\sigma^{(11)}_{mn}(u) - \sigma^{(11)}_{mn}\left(\sigma^{(10)}(u)\right) - \sigma^{(11)}_{mn}\left(\sigma^{(01)}(u)\right) + \sigma^{(11)}_{mn}\left(\sigma^{(11)}(u)\right) = \sigma^{(11)}_{mn}\left(V^{(11)}\left(\Delta_{11}u\right)\right).$$

Then, we obtain

$$\sigma^{(11)}_{mn}(u) - \sigma^{(10)}_{mn}\left(\sigma^{(11)}(u)\right) - \sigma^{(01)}_{mn}\left(\sigma^{(11)}(u)\right) + \sigma^{(11)}_{mn}\left(\sigma^{(11)}(u)\right) = \sigma^{(11)}_{mn}\left(V^{(11)}\left(\Delta_{11}u\right)\right)$$

by the identities (2) and (3).

Taking the limit of both sides of the last identity as $m,n \to \infty$, then we have

$$\lim_{m,n\to\infty}\left(\sigma^{(11)}_{mn}(u) - \sigma^{(10)}_{mn}\left(\sigma^{(11)}(u)\right) - \sigma^{(01)}_{mn}\left(\sigma^{(11)}(u)\right) + \sigma^{(11)}_{mn}\left(\sigma^{(11)}(u)\right)\right) = \lim_{m,n\to\infty}\sigma^{(11)}_{mn}\left(V^{(11)}\left(\Delta_{11}u\right)\right)$$

Since $(C,0,1)$, $(C,1,0)$ and $(C,1,1)$ summability methods are regular under the boundedness condition, and (u_{mn}) is $(C,1,1)$ bounded and $(C,1,1)$ summable to s, it follows that

$$\lim_{m,n\to\infty}\sigma^{(11)}_{mn}\left(V^{(11)}\left(\Delta_{11}u\right)\right) = 0.$$

Therefore, $\left(V^{(11)}_{mn}\left(\Delta_{11}u\right)\right)$ is $(C,1,1)$ summable to 0. Replacing u_{mn} by $V^{(11)}_{mn}(\Delta_{11}u)$ in Lemma 3.4 (i), we obtain

$$\begin{aligned}
V^{(11)}_{mn}(\Delta_{11}u) - \sigma^{(11)}_{mn}(V^{(11)}(\Delta_{11}u)) \\
&= \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}\Big(\sigma^{(11)}_{\lambda_m,\lambda_n}(V^{(11)}(\Delta_{11}u)) - \sigma^{(11)}_{\lambda_m,n}(V^{(11)}(\Delta_{11}u)) \\
&\qquad - \sigma^{(11)}_{m,\lambda_n}(V^{(11)}(\Delta_{11}u)) + \sigma^{(11)}_{mn}(V^{(11)}(\Delta_{11}u))\Big) \\
&\quad + \frac{\lambda_m+1}{\lambda_m-m}\Big(\sigma^{(11)}_{\lambda_m,n}(V^{(11)}(\Delta_{11}u)) - \sigma^{(11)}_{mn}(V^{(11)}(\Delta_{11}u))\Big) \\
&\quad + \frac{\lambda_n+1}{\lambda_n-n}\Big(\sigma^{(11)}_{m,\lambda_m}(V^{(11)}(\Delta_{11}u)) - \sigma^{(11)}_{mn}(V^{(11)}(\Delta_{11}u))\Big) \\
&\qquad - \frac{1}{(\lambda_m-m)(\lambda_n-n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}(V^{(11)}_{jk}(\Delta_{11}u) - V^{(11)}_{mn}(\Delta_{11}u))
\end{aligned} \tag{10}$$

From the identity (10), we get

$$
\begin{aligned}
&|V_{mn}^{(11)}(\Delta_{11}u) - \sigma_{mn}^{(11)}(V^{(11)}(\Delta_{11}u))| \\
&\leq \left| \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)} \Big(\sigma_{\lambda_m,\lambda_n}^{(11)}(V^{(11)}(\Delta_{11}u)) - \sigma_{\lambda_m,n}^{(11)}(V^{(11)}(\Delta_{11}u)) \right. \\
&\qquad \left. - \sigma_{m,\lambda_n}^{(11)}(V^{(11)}(\Delta_{11}u)) + \sigma_{mn}^{(11)}(V^{(11)}(\Delta_{11}u)) \Big) \right| \\
&\quad + \left| \frac{\lambda_m+1}{\lambda_m-m} \Big(\sigma_{\lambda_m,n}^{(11)}(V^{(11)}(\Delta_{11}u)) - \sigma_{mn}^{(11)}(V^{(11)}(\Delta_{11}u)) \Big) \right| \\
&\quad + \left| \frac{\lambda_n+1}{\lambda_n-n} \Big(\sigma_{m,\lambda_m}^{(11)}(V^{(11)}(\Delta_{11}u)) - \sigma_{mn}^{(11)}(V^{(11)}(\Delta_{11}u)) \Big) \right| \\
&\quad + \left| -\frac{1}{(\lambda_m-m)(\lambda_n-n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} (V_{jk}^{(11)}(\Delta_{11}u) - V_{mn}^{(11)}(\Delta_{11}u)) \right| \qquad (11)
\end{aligned}
$$

For the last term on the right-hand side of the inequality (11), we have

$$
\begin{aligned}
&\left| -\frac{1}{(\lambda_m-m)(\lambda_n-n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} (V_{jk}^{(11)}(\Delta_{11}u) - V_{mn}^{(11)}(\Delta_{11}u)) \right| \\
&\leq \frac{1}{(\lambda_m-m)(\lambda_n-n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \left| \sum_{r=m+1}^{j} \sum_{s=n+1}^{k} \Delta_{11} V_{rs}^{(11)}(\Delta_{11}u) \right| \\
&\quad + \frac{1}{(\lambda_m-m)(\lambda_n-n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \left| \sum_{r=m+1}^{j} \Delta_{10} V_{rn}^{(11)}(\Delta_{11}u) \right| \\
&\quad + \frac{1}{(\lambda_m-m)(\lambda_n-n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \left| \sum_{s=n+1}^{k} \Delta_{01} V_{ms}^{(11)}(\Delta_{11}u) \right|.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&\left| -\frac{1}{(\lambda_m-m)(\lambda_n-n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} (V_{jk}^{(11)}(\Delta_{11}u) - V_{mn}^{(11)}(\Delta_{11}u)) \right| \\
&\leq \max_{\substack{m+1\leq j\leq \lambda_m \\ n+1\leq k\leq \lambda_n}} \left| \sum_{r=m+1}^{j} \sum_{s=n+1}^{k} \Delta_{11} V_{rs}^{(11)}(\Delta_{11}u) \right| + \max_{m+1\leq j\leq \lambda_m} \left| \sum_{r=m+1}^{j} \Delta_{10} V_{rn}^{(11)}(\Delta_{11}u) \right| \\
&\quad + \max_{n+1\leq k\leq \lambda_n} \left| \sum_{s=n+1}^{k} \Delta_{01} V_{ms}^{(11)}(\Delta_{11}u) \right|.
\end{aligned}
$$

Taking the lim sup of both sides of the inequality (11) as $m, n \to \infty$, we have

$$\limsup_{m,n\to\infty} \left| V_{mn}^{(11)}(\Delta_{11}u) - \sigma_{mn}^{(11)}(V^{(11)}(\Delta_{11}u)) \right| \leq \frac{\lambda^2}{(\lambda-1)^2} \limsup_{m,n\to\infty} |\sigma_{\lambda_m,\lambda_n}^{(11)}(V^{(11)}(\Delta_{11}u))|$$
$$+ \left(\frac{\lambda^2}{(\lambda-1)^2} + \frac{\lambda}{\lambda-1} \right) \limsup_{m,n\to\infty} |\sigma_{\lambda_m,n}^{(11)}(V^{(11)}(\Delta_{11}u))|$$
$$+ \left(\frac{\lambda^2}{(\lambda-1)^2} + \frac{\lambda}{\lambda-1} \right) \limsup_{m,n\to\infty} |\sigma_{m,\lambda_n}^{(11)}(V^{(11)}(\Delta_{11}u))|$$
$$+ \left(\frac{\lambda^2}{(\lambda-1)^2} + \frac{2\lambda}{\lambda-1} \right) \limsup_{m,n\to\infty} |\sigma_{m,n}^{(11)}(V^{(11)}(\Delta_{11}u))|$$
$$+ \limsup_{m,n\to\infty} \max_{\substack{m+1\leq j\leq\lambda_m \\ n+1\leq k\leq\lambda_n}} \left| \sum_{r=m+1}^{j} \sum_{s=n+1}^{k} \Delta_{11} V_{rs}^{(11)}(\Delta_{11}u) \right|$$
$$+ \limsup_{m,n\to\infty} \max_{m<j\leq\lambda_m} \left| \sum_{r=m+1}^{j} \Delta_{10} V_{rn}^{(11)}(\Delta_{11}u) \right|$$
$$+ \limsup_{m,n\to\infty} \max_{n<k\leq\lambda_n} \left| \sum_{s=n+1}^{k} \Delta_{01} V_{ms}^{(11)}(\Delta_{11}u) \right|$$

Since $\left(V_{mn}^{(11)}(\Delta_{11}u)\right)$ is $(C,1,1)$ summable to 0, the first four terms on the right-hand side of the last inequality vanish. Taking the limit of both sides as $\lambda \to 1^+$, we obtain

$$\limsup_{m,n\to\infty} \left| V_{mn}^{(11)}(\Delta_{11}u) - \sigma_{mn}^{(11)}(V^{(11)}(\Delta_{11}u)) \right|$$
$$\leq \lim_{\lambda\to1^+} \limsup_{m,n\to\infty} \max_{\substack{m+1\leq j\leq\lambda_m \\ n+1\leq k\leq\lambda_n}} \left| \sum_{r=m+1}^{j} \sum_{s=n+1}^{k} \Delta_{11} V_{rs}^{(11)}(\Delta_{11}u) \right|$$
$$+ \lim_{\lambda\to1^+} \limsup_{m,n\to\infty} \max_{m<j\leq\lambda_m} \left| \sum_{r=m+1}^{j} \Delta_{10} V_{rn}^{(11)}(\Delta_{11}u) \right| + \lim_{\lambda\to1^+} \limsup_{m,n\to\infty} \max_{n<k\leq\lambda_n} \left| \sum_{s=n+1}^{k} \Delta_{01} V_{ms}^{(11)}(\Delta_{11}u) \right|$$

Since (u_{mn}) is slowly oscillating in senses $(1,1)$, $(1,0)$ and $(0,1)$, $\left(V_{mn}^{(11)}(\Delta_{11}u)\right)$ is slowly oscillating in sense $(1,1)$ by Theorem 4.5 and $\left(V_{mn}^{(11)}(\Delta_{11}u)\right)$ is slowly oscillating in sense $(1,0)$ and $(0,1)$ by Theorem 4.6. Hence, we get

$$\limsup_{m,n\to\infty} \left| V_{mn}^{(11)}(\Delta_{11}u) - \sigma_{mn}^{(11)}(V^{(11)}(\Delta_{11}u)) \right| \leq 0$$

We then have

$$\lim_{m,n\to\infty} \left(V_{mn}^{(11)}(\Delta_{11}u) - \sigma_{mn}^{(11)}(V^{(11)}(\Delta_{11}u)) \right) = 0$$

Since $\left(V_{mn}^{(11)}(\Delta_{11}u)\right)$ is $(C,1,1)$ summable to 0, we obtain

$$\lim_{m,n\to\infty} V_{mn}^{(11)}(\Delta_{11}u) = 0. \tag{12}$$

On the other hand, we have

$$u_{mn} - \sigma_{mn}^{(10)}(u) - \sigma_{mn}^{(01)}(u) + \sigma_{mn}^{(11)}(u) = V_{mn}^{(11)}(\Delta_{11}u)$$

It follows from the Kronecker identity (5) that

$$V_{mn}^{(10)}(\Delta_{10}u) - \sigma_{mn}^{(01)}(u) + \sigma_{mn}^{(11)}(u) = V_{mn}^{(11)}(\Delta_{11}u)$$

Taking $(C,1,0)$ means of both sides, we get

$$\sigma_{mn}^{(10)}\left(V^{(10)}(\Delta_{10}u)\right) - \sigma_{mn}^{(10)}\left(\sigma^{(01)}(u)\right) + \sigma_{mn}^{(10)}\left(\sigma^{(11)}(u)\right) = \sigma_{mn}^{(10)}\left(V^{(11)}(\Delta_{11}u)\right) \tag{13}$$

Since $\left(V_{mn}^{(11)}(\Delta_{11}u)\right)$ is bounded by Theorem 4.1 and $\left(V_{mn}^{(11)}(\Delta_{11}u)\right)$ is P-convergent to 0, we get

$$\lim_{m,n\to\infty} \sigma_{mn}^{(10)}\left(V^{(11)}(\Delta_{11}u)\right) = 0 \tag{14}$$

by the regularity of the $(C,1,0)$ summability method. Since (u_{mn}) is $(C,1,1)$ bounded and $(C,1,1)$ summable to s, we have

$$\sigma_{mn}^{(10)}\left(\sigma^{(11)}(u)\right) = s \tag{15}$$

It follows by the fact that $\sigma_{mn}^{(10)}\left(\sigma^{(01)}(u)\right) = \sigma_{mn}^{(11)}(u)$ that

$$\lim_{m,n\to\infty} \sigma_{mn}^{(10)}\left(\sigma^{(01)}(u)\right) = \lim_{m,n\to\infty} \sigma_{mn}^{(11)}(u) = s. \tag{16}$$

Taking (14), (15) and (16) into consideration, we see that $\left(V_{mn}^{(10)}(\Delta_{10}u)\right)$ is $(C,1,0)$ summable to 0 by identity (13). Similarly, $\left(V_{mn}^{(01)}(\Delta_{01}u)\right)$ is $(C,0,1)$ summable to 0.

For $\lambda > 1$, replacing u_{mn} by $V_{mn}^{(10)}(\Delta_{10}u)$ in Lemma 3.5 (i), we obtain

$$\begin{aligned}
|V_{mn}^{(10)}(\Delta_{10}u) - \sigma_{mn}^{(10)}(V^{(10)}(\Delta_{10}u))| &\le \left|\frac{\lambda_m+1}{\lambda_m-m}\left(\sigma_{\lambda_m,n}^{(10)}(V^{(10)}(\Delta_{10}u)) - \sigma_{mn}^{(10)}(V^{(10)}(\Delta_{10}u))\right)\right| \\
&\quad + \left|-\frac{1}{\lambda_m-m}\sum_{j=m+1}^{\lambda_m}\left(V_{jn}^{(10)}(\Delta_{10}u) - V_{mn}^{(10)}(\Delta_{10}u)\right)\right| \\
&\le \left|\frac{\lambda_m+1}{\lambda_m-m}\left(\sigma_{\lambda_m,n}^{(10)}(V^{(10)}(\Delta_{10}u)) - \sigma_{mn}^{(10)}(V^{(10)}(\Delta_{10}u))\right)\right| \\
&\quad + \max_{m<j\le\lambda_m}\left|V_{jn}^{(10)}(\Delta_{10}u) - V_{mn}^{(10)}(\Delta_{10}u)\right|
\end{aligned}$$

Taking the lim sup of both sides of the previous inequality as $m, n \to \infty$, we get

$$\limsup_{m,n\to\infty} |V_{mn}^{(10)}(\Delta_{10}u) - \sigma_{mn}^{(10)}(V^{(10)}(\Delta_{10}u))| \leq \frac{\lambda}{\lambda-1} \limsup_{m,n\to\infty} \left|\sigma_{\lambda_m,n}^{(10)}(V^{(10)}(\Delta_{10}u)) - \sigma_{mn}^{(10)}(V^{(10)}(\Delta_{10}u))\right| + \limsup_{m,n\to\infty} \max_{m<j\leq\lambda_m} \left|V_{jn}^{(10)}(\Delta_{10}u) - V_{mn}^{(10)}(\Delta_{10}u)\right|$$

Since $\left(\sigma_{mn}^{(10)}\left(V^{(10)}(\Delta_{10}u)\right)\right)$ is P-convergent to 0, the terms on the right-hand side of the last inequality vanish. Taking the limit of both sides as $\lambda \to 1^+$, we obtain

$$\limsup_{m,n\to\infty} |V_{mn}^{(10)}(\Delta_{10}u) - \sigma_{mn}^{(10)}(V^{(10)}(\Delta_{10}u))| \leq \lim_{\lambda\to 1^+} \limsup_{m,n\to\infty} \max_{m<j\leq\lambda_m} \left|V_{jn}^{(10)}(\Delta_{10}u) - V_{mn}^{(10)}(\Delta_{10}u)\right|$$

Since $\left(V_{mn}^{(10)}(\Delta_{10}u)\right)$ is slowly oscillating in sense $(1,0)$ by Theorem 4.2, we get

$$\limsup_{m,n\to\infty} |V_{mn}^{(10)}(\Delta_{10}u) - \sigma_{mn}^{(10)}(V^{(10)}(\Delta_{10}u))| \leq 0$$

Hence, we obtain

$$\lim_{m,n\to\infty} V_{mn}^{(10)}(\Delta_{10}u) = 0 \tag{17}$$

Similarly, we obtain

$$\lim_{m,n\to\infty} V_{mn}^{(01)}(\Delta_{01}u) = 0 \tag{18}$$

Taking (12), (17) and (18) into consideration and using (8) completes the proof. Q.E.D.

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