Simulation functions and Boyd-Wong type results

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Abstract

In this paper we give new and much shorter proofs of Boyd-Wong and Meir-Keeler type results. Also, we define a new form of Boyd-Wong type contraction mappings using simulation functions and obtain some sufficient conditions for the existence and uniqueness of a fixed point for such class of mappings in the setting of metric spaces. Also, we present some results regarding the property P for these well-known types of self-mappings.

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1 Introduction and preliminaries

The Banach contraction mapping principle is an important tool (or method) in nonlinear analysis and other mathematical fields for solving existence problems. It has been generalized in many directions. In some resent papers, the concept of a metric space was generalized, and rectangular metric spaces, b-metric spaces, cone metric spaces, cone rectangular metric spaces and cone b-metric spaces were introduced. Also, weak contractive conditions have been studied by several authors. A mapping $T : X \to X$ on a metric space (X, d) is said to be a $\varphi-$contraction if there exists a function $\varphi : [0, \infty) \to [0, \infty)$ such that $d(Tx, Ty) \leq \varphi(d(x, y))$ for all $x, y \in X$. In [3] it was proved that a self-mapping T of a complete metric space (X, d) has a unique fixed point x^* and $\lim_{n\to\infty} T^n x_0 = x^*$ for any given $x_0 \in X$, if φ is non-decreasing and right continuous function. After that, a weaker sufficient condition with a right upper semi-continuous function φ was obtained in [2]. For further information, we refer the interested readers to ([5], [14]).

Let $\mathcal{B}$ be the class of all mappings $\beta : [0, \infty) \to [0, 1)$ satisfying the condition: $\beta(t_n) \to 1$ whenever $t_n \to 0$. Since $\beta(t) = \frac{1}{1+t^2}$, $t \in [0, \infty)$, belongs to $\mathcal{B}$, the set $\mathcal{B}$ is nonempty.

In [6], Geraghty introduced the notation of Geraghty-contraction and proved the next fixed point theorem.

Theorem 1.1 ([6]). Let (X, d) be a complete metric space and T a self mapping on X. If there exists $\beta \in \mathcal{B}$ such that

$$d(Tx, Ty) \leq \beta(d(x, y))\, d(x, y) \text{ for all } x, y \in X,$$

then T has a unique fixed point and, for any given $x_0 \in X$, the iterative sequence $\{T^n x_0\}$ converges to the fixed point.

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Alternatively, in [15] the following generalized contraction mapping principle was proved.

Theorem 1.2 ([15])**.** Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that

$$\psi\left(d\left(Tx, Ty\right)\right) \leq \varphi\left(d\left(x, y\right)\right) \quad \text{for all } x, y \in X,$$

where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are two functions satisfying the conditions:

(a) $\psi(a) \leq \varphi(b)$ implies $a \leq b$;

(b) if $a_n \to \varepsilon$, $b_n \to \varepsilon$ and $\psi(a_n) \leq \varphi(b_n)$, then $\varepsilon = 0$.

Then T has a unique fixed point and, for any given $x_0 \in X$, the iterative sequence $T^n x_0$ converges to the fixed point.

Now, we give some basic definitions and introduce notations and some basic concepts used in the sequel.

Definition 1.3 ([8])**.** A mapping $\zeta : [0, \infty)^2 \to \mathbb{R}$ is called simulation function if it satisfies the following:

(ζ_1) $\zeta(0, 0) = 0$;

(ζ_2) $\zeta(t, s) < s - t$ for all $t, s > 0$;

(ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0, +\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

We will denote by $\mathcal{Z}$ the set of all simulation functions. Next, we provide some examples of the simulation functions.

Example 1.4. The following functions are simulation functions.

1) $\zeta(t, s) = \frac{s}{1+s} - t$ for all $t, s \in [0, \infty)$;

2) $\zeta(t, s) = \lambda s - t$ for all $t, s \in [0, \infty)$, where $\lambda \in [0, 1)$;

3) $\zeta(t, s) = s\varphi(s) - t$ for all $t, s \in [0, \infty)$, where $\varphi : [0, \infty) \to [0, 1)$ is such that $\lim_{t\to r^+} \varphi(t) < 1$ for all $r > 0$.

The next result will prove extremely useful in the sequel.

Lemma 1.5 ([12])**.** Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$, as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence in X, then there exist $\varepsilon > 0$ and two sequences $\{n(k)\}$ and $\{m(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the sequences

$$\left\{d\left(x_{m(k)}, x_{n(k)}\right)\right\}, \left\{d\left(x_{m(k)}, x_{n(k)+1}\right)\right\}, \left\{d\left(x_{m(k)-1}, x_{n(k)}\right)\right\},$$

$$\left\{d\left(x_{m(k)-1}, x_{n(k)+1}\right)\right\}, \left\{d\left(x_{m(k)+1}, x_{n(k)+1}\right)\right\}$$

tend to ε^+, as $k \to +\infty$.

Remark 1.6. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d). If for all $n \in \mathbb{N}$ holds $d(x_{n+1}, x_n) < d(x_n, x_{n-1})$, then $n \neq m$ implies $x_n \neq x_m$ whenever $n, m \in \mathbb{N}$.

2 Main results

Using Lemma 1.5, we prove the following two enough known results in literature: Boyd-Wong and Meir-Keeler theorems but with much shorter proofs.

Theorem 2.1. (Boyd and Wong [2]) Let (X, d) be a complete metric space and $T : X \to X$ satisfies

$$d(Tx, Ty) \leq \varphi(d(x, y)) \quad \text{for all } x, y \in X, \tag{2.1}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a real function, upper semi-continuous from the right, satisfying

$$\varphi(t) < t \text{ for } t > 0. \tag{2.2}$$

Then T has a unique fixed point $u \in X$, and $T^n x \to u$, as $n \to \infty$, for each $x \in X$.

Remark 2.2. Upper semi-continuity from the right of φ means that

$$\limsup_{\alpha_n \to \alpha^+} \varphi(\alpha_n) \leq \varphi(\alpha) = \varphi\left(\lim_{n\to\infty} \alpha_n\right). \tag{2.3}$$

Proof. Let us choose $x_0 \in X$ arbitrary and define the corresponding Picard sequence $\{x_n\}$ as follows $x_n = Tx_{n-1}$. If there exists $k \in \mathbb{N}$ such that $x_{k-1} = x_k$, then x_{k-1} is a (unique) fixed point of T. Therefore, let $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. Assuming (2.1) to hold for all $n \in \mathbb{N}$, we have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \varphi(d(x_n, x_{n-1})) < d(x_n, x_{n-1}).$$

Suppose that $d(x_{n+1}, x_n) \to \alpha^+$ as $n \to \infty$, for some $\alpha > 0$. From the inequality

$$d(x_{n+1}, x_n) \leq \varphi(d(x_n, x_{n-1})),$$

and the property of φ, it follows

$$\begin{aligned} \alpha &= \lim_{n\to+\infty} d(x_{n+1}, x_n) = \limsup_{n\to+\infty} d(x_{n+1}, x_n) \\ &\leq \limsup_{n\to+\infty} \varphi(d(x_n, x_{n-1})) \leq \limsup_{d(x_n, x_{n-1})\to\alpha^+} \varphi(d(x_n, x_{n-1})) \leq \varphi(\alpha) < \alpha, \end{aligned}$$

which is a contradiction. Hence, $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$.

Now, we prove that $\{x_n\}$ is a Cauchy sequence. On the contrary, suppose that $\{x_n\}$ is not a Cauchy sequence. By Lemma 1.5, there exist $\varepsilon > 0$ and two sequences $\{n(k)\}$ and $\{m(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the sequences $\{d(x_{m(k)}, x_{n(k)})\}$ and $\{d(x_{m(k)+1}, x_{n(k)+1})\}$ tend to $\varepsilon^+ > 0$ as $k \to \infty$. Substituting $x = x_{m(k)}$, $y = x_{n(k)}$ into (2.1), we obtain

$$d\left(x_{m(k)+1}, x_{n(k)+1}\right) \leq \varphi\left(d\left(x_{m(k)}, x_{n(k)}\right)\right).$$

Therefore, we have

$$\begin{aligned} \varepsilon &= \lim_{k\to+\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) = \limsup_{k\to+\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) \\ &\leq \limsup_{k\to+\infty} \varphi\left(dx_{m(k)}, x_{n(k)})\right) \leq \limsup_{d(x_{m(k)}, x_{n(k)})\to\varepsilon^+} \varphi\left(d(x_{m(k)}, x_{n(k)})\right) \\ &\leq \varphi(\varepsilon) < \varepsilon, \end{aligned}$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence.

Completeness of X implies existence of $u \in X$ such that $x_n \to u$, as $n \to \infty$. We conclude from (2.1) that T is continuous, hence $\lim_{n\to\infty} Tx_n = Tu$. By the uniqueness of limits, we obtain that u is a fixed point of T and its uniqueness is obvious.

Q.E.D.

Theorem 2.3 ([9])**.** Let (X, d) be a complete metric space and f be a self-mapping of X satisfying the following property: given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(f(x), f(y)) < \varepsilon. \tag{2.4}$$

Then f has a unique fixed point $u \in X$ and for each $x \in X$, $\lim_{n\to\infty} f^n(x) = u$.

Remark 2.4. The mappings which satisfy the condition (2.4) are called weakly uniformly strict contractions. It is obvious that if f satisfies the contractive condition of Meir-Keeler, then f is contractive, that is $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$, $x \neq y$.

Proof. Let us choose $x \in X$ arbitrary. Consider the sequence of iterates of x generated by the recurrence relation $x_n = f^n(x)$, $n \in \mathbb{N} \cup \{0\}$. If

$$d\left(f^k(x), f^{k+1}(x)\right) = d\left(f^k(x), f\left(f^k(x)\right)\right) = 0$$

for some $k \in \mathbb{N} \cup \{0\}$, then $x_k = f^k(x)$ is a fixed point of f. Let we assume that $d\left(f^n(x), f^{n+1}(x)\right) > 0$ for all $n \in \mathbb{N}$. Since f is contractive, the sequence $\left\{d\left(f^n(x), f^{n+1}(x)\right)\right\}_{\mathbb{N}\cup\{0\}}$ is strictly decreasing. Therefore, there exists $\varepsilon \geq 0$ such that $\lim_{n\to\infty} d\left(f^n(x), f^{n+1}(x)\right) = \varepsilon$ and $d\left(f^n(x), f^{n+1}(x)\right) > \varepsilon$, for all $n \in \mathbb{N}$. Suppose that $\varepsilon > 0$. Then, by hypothesis, there exists a suitable $\delta = \delta(\varepsilon) > 0$ such that (2.4) holds. By the definition of limit, there exists $n \in \mathbb{N}$ such that

$$\varepsilon < d\left(f^n(x), f^{n+1}(x)\right) < \varepsilon + \delta. \tag{2.5}$$

Combining (2.5) with (2.4), we conclude that

$$d\left(f^{n+1}(x), f^{n+2}(x)\right) = d\left(f(f^n(x)), f\left(f^{n+1}(x)\right)\right) < \varepsilon,$$

which contradicts the fact that $d\left(f^n(x), f^{n+1}(x)\right) > \varepsilon$ for all $n \in \mathbb{N}$. Therefore, $\lim_{n\to\infty} d\left(f^n(x), f^{n+1}(x)\right) = 0$.

Now, we show that $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence. On the contrary, suppose that it is not a Cauchy sequence. Then, by Lemma 1.5, there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and

$$\lim_{k\to\infty} d\left(f^{m(k)}(x), f^{n(k)}(x)\right) = \lim_{k\to\infty} d\left(f^{m(k)+1}(x), f^{n(k)+1}(x)\right) = \varepsilon^+. \tag{2.6}$$

By hypothesis, there exists a suitable $\delta = \delta(\varepsilon) > 0$ such that (2.4) is satisfied. Further, substituting $x = x_{m(k)}$ and $y = x_{n(k)}$ in (2.4), we get

$$\varepsilon \leq d\left(x_{m(k)}, x_{n(k)}\right) < \varepsilon + \delta(\varepsilon) \Rightarrow d\left(x_{m(k)+1}, x_{n(k)+1}\right) < \varepsilon,$$

contrary to (2.6).

Since (X,d) is complete, the iterative sequence has a limit in the set X, i.e. $\lim_{n\to\infty} f^n(x) = u$, $u \in X$. By continuity of f, u is the fixed point of f. The uniqueness is a consequence of contractivity of f. The known result of Meir-Keeler is completely proved. Q.E.D.

Boyd and Wong type simulation functions

In this section, we establish some results on the existence and uniqueness of a fixed point for Boyd-Wong type contraction mappings using the simulation functions in metric spaces.

Definition 2.5. Let (X,d) be a metric space and $T : X \to X$ be a self-mapping. The mapping T is called a $\mathcal{Z}_{BW}-$contraction if there exists $\zeta \in \mathcal{Z}$ such that

$$\zeta\left(d\left(Tx,Ty\right),\varphi\left(d\left(x,y\right)\right)\right) \geq 0 \text{ for all } x,y \in X, \tag{2.7}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a real function, upper semi-continuous from the right, satisfying

$$\varphi\left(t\right) < t \text{ for } t > 0. \tag{2.8}$$

We will prove that the inequality (2.7) implies the inequality (2.1).

Theorem 2.6. Let (X,d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}_{BW}-$contraction. Then T has a unique fixed point $u \in X$ and $T^n x \to u$, as $n \to \infty$, for each $x \in X$.

Proof. Let $\varphi : [0,\infty) \to [0,\infty)$ be a real function, upper semi-continuous from the right, satisfying $\varphi(t) < t$ for $t > 0$. Let $\zeta \in \mathcal{Z}$ be a simulation function which satisfies inequality

$$\zeta\left(d\left(Tx,Ty\right),\varphi\left(d\left(x,y\right)\right)\right) \geq 0,\ \forall\ x,y \in X.$$

If $x = y$, then (2.1) holds. Let $x \neq y$ and $Tx = Ty$. Then (2.1) again holds. Suppose that $Tx \neq Ty$. Then $d\left(Tx,Ty\right)$ and $\varphi\left(d\left(x,y\right)\right)$ are both positive numbers. According to the property (ζ_2) of ζ, we have

$$0 \leq \zeta\left(d\left(Tx,Ty\right),\varphi\left(d\left(x,y\right)\right)\right) < \varphi\left(d\left(x,y\right)\right) - d\left(Tx,Ty\right),$$

and consequently (2.1) holds. From Theorem 2.1, it follows that T has a (unique) fixed point. Since ζ is arbitrary chosen simulation function, the proof is complete. Q.E.D.

Theorem 2.7. Let (X,d) be a complete metric space. Assume that $T : X \to X$ is a self-mapping such that

$$\zeta\left(\psi\left(d\left(Tx,Ty\right)\right),\varphi\left(d\left(x,y\right)\right)\right) \geq 0 \text{ for all } x,y \in X, \tag{2.9}$$

where $\zeta \in \mathcal{Z}$ and $\psi,\varphi : [0,\infty) \to [0,\infty)$ are two functions satisfying the conditions:

(*a*) $\psi\left(a\right) \leq \varphi\left(b\right)$ implies $a \leq b$;

(*b*) if $a_n \to \varepsilon$, $b_n \to \varepsilon$ and $\psi(a_n) \leq \varphi(b_n)$, then $\varepsilon = 0$.

Then T has a unique fixed point and, for any given $x_0 \in X$, the iterative sequence $T^n x_0$ converges to the fixed point.

Proof. Suppose that the contractive condition (2.9) holds for an arbitrary simulation function ζ. We shall established the theorem if we prove that the inequality (2.9) implies the following

$$\psi\left(d\left(Tx, Ty\right)\right) \leq \varphi\left(d\left(x, y\right)\right) \quad \text{for all } x, y \in X. \tag{2.10}$$

It is easily seen that if $x = y$, then (2.10) holds. Also, in the case $x \neq y$ and $Tx = Ty$, the inequality (2.10) is satisfied. Now, if $x \neq y$ and $Tx \neq Ty$, then by the property (ζ_2) of ζ we have

$$0 \leq \zeta\left(\psi\left(d\left(Tx, Ty\right)\right), \varphi\left(d\left(x, y\right)\right)\right) < \varphi\left(d\left(x, y\right)\right) - \psi\left(d\left(Tx, Ty\right)\right),$$

i.e., (2.10) holds.

By the above, (2.9) holds for an arbitrary simulation function ζ and consequently the assertion follows directly from Theorem 1.2. Q.E.D.

Simulation functions and cyclic (α, β)-admissible mappings

In this section, we prove that the proofs of the results obtained in [11] could be much simpler and shorter using Lemma 1.5.

Alizadeh et al. [1] introduced the notion of cyclic (α, β)-admissible mapping which is defined as follows:

Definition 2.8. Let X be a nonempty set, S be a self-mapping on X and $\alpha, \beta : X \to [0, \infty)$ be two mappings. We say that S is a cyclic (α, β)-admissible mapping if $x \in X$ with $\alpha(x) \geq 1$ implies $\beta(Sx) \geq 1$ and $\beta(x) \geq 1$ implies $\alpha(Sx) \geq 1$.

Definition 2.9 ([11])**.** Let (X, d) be a complete metric space, $S : X \to X$ be a mapping and $\alpha, \beta : \mathbb{R} \to [0, \infty)$ be two functions. Then S is said to be a generalized (α, β, Z) contraction mapping if S satisfies the following conditions:
(1) S is cyclic (α, β)-admissible;
(2) there exits a simulation function $\zeta \in \mathcal{Z}$ such that

$$\alpha(x)\beta(y) \geq 1 \implies \zeta(d(Sx, Sy), M(x, y)) \geq 0, \tag{2.11}$$

holds for all $x, y \in X$, where $M(x, y) = \max\{d(x, y), d(x, Sx), d(y, Sy)\}$.

Theorem 2.10. [11, Theorems 2.4-2.5] Let (X, d) be a complete metric space, $S : X \to X$ be a mapping and $\alpha, \beta : X \times X \to [0, 1)$ be two functions. Suppose that the following conditions hold.

(1) S is a generalized (α, β, Z) contraction mapping;

(2) There exists an element $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$;

(3) S is continuous;

or

If sequence $\{x_n\}$ in X converges to $x \in X$ with the property $\alpha(x_n) \geq 1$ (or $\beta(x_n) \geq 1$) for all $n \in N$, then $\alpha(x) \geq 1$ (or $\beta(x) \geq 1$).

Then S has a fixed point $u \in X$ such that $Su = u$.

Proof. Analysis similar to that in the proof of [11, Theorem 2.4] shows that $d(x_n, x_{n+1}) \to 0$, as $n \to \infty$.

Now, we prove that $\{x_n\}$ is a Cauchy sequence. On the contrary, suppose that $\{x_n\}$ is not a Cauchy sequence. By Lemma 1.5, there exist $\varepsilon > 0$ and two sequences $\{n(k)\}$ and $\{m(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the sequences $\{d(x_{m(k)}, x_{n(k)})\}$ and $\{d(x_{m(k)+1}, x_{n(k)+1})\}$ tend to $\varepsilon^+ > 0$ as $k \to \infty$. Substituting $x = x_{m(k)}$ and $y = x_{n(k)}$ into the inequality (2.11), we obtain

$$\alpha\left(x_{m(k)}\right)\beta\left(x_{n(k)}\right) \geq 1 \Rightarrow \zeta\left(d\left(Sx_{m(k)}, Sx_{n(k)}\right), M\left(x_{m(k)}, x_{n(k)}\right)\right) \geq 0, \tag{2.12}$$

where

$$M\left(x_{m(k)}, x_{n(k)}\right) = \max\left\{d\left(x_{m(k)}, x_{n(k)}\right), d\left(x_{m(k)}, x_{m(k)+1}\right), d\left(x_{n(k)}, x_{n(k)+1}\right)\right\}.$$

Since $d\left(x_{m(k)}, x_{m(k)+1}\right) \to 0$ and $d\left(x_{n(k)}, x_{n(k)+1}\right) \to 0$ as $k \to \infty$, then, for sufficiently large k, the inequality

$$M\left(x_{m(k)}, x_{n(k)}\right) = d\left(x_{m(k)}, x_{n(k)}\right) > 0$$

is satisfied. Instead of (2.12), we have

$$\alpha\left(x_{m(k)}\right)\beta\left(x_{n(k)}\right) \geq 1 \Rightarrow \zeta\left(d\left(x_{m(k)+1}, x_{n(k)+1}\right), d\left(x_{m(k)}, x_{n(k)}\right)\right) \geq 0.$$

Using the fact that $\alpha\left(x_{m(k)}\right)\beta\left(x_{n(k)}\right) \geq 1$ holds and that $d\left(x_{m(k)+1}, x_{n(k)+1}\right)$ and $d\left(x_{m(k)}, x_{n(k)}\right)$ are both positive numbers, by the property (ζ_2), we have

$$\begin{aligned} 0 &\leq \zeta\left(d\left(x_{m(k)+1}, x_{n(k)+1}\right), d\left(x_{m(k)}, x_{n(k)}\right)\right) \\ &< d\left(x_{m(k)}, x_{n(k)}\right) - d\left(x_{m(k)+1}, x_{n(k)+1}\right). \end{aligned}$$

Now, we obtain $\limsup_{k\to\infty} \zeta\left(d\left(x_{m(k)+1}, x_{n(k)+1}\right), d\left(x_{m(k)}, x_{n(k)}\right)\right) = 0$, which contradicts the property (ζ_3).

Hence, $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence. The rest of the proof runs as in [11]. We, thus, obtain that S has a fixed point. Q.E.D.

Remark 2.11. Note that, throughout this paper, Lemma 1.5 and the contractive conditions imply that the iterative sequence, i.e. Picard sequence is a Cauchy.

Remark 2.12. Let us provide a new example of the simulation functions.

If $\varphi : [0, \infty) \to [0, \infty)$ is a real function, upper semi-continuous from the right, satisfying $\varphi(t) < t$ for $t > 0$, then

$$\zeta(t, s) = \varphi(s) - t, \text{ for all } s, t \in [0, \infty),$$

is a simulation function.

According to [8, Theorem 2.8], this simulation function ensures a new proof of the Boyd-Wong theorem.

We are interested in finding an example which shows that Theorem 2.8 from [8] is a real generalization of the Boyd-Wong theorem.

Mappings with the property P

Let (X,d) be a metric space and $T:X\to X$. It is said that T has Property P if $F(T)=F(T^n)$ for each $n\in\mathbb{N}$, where $F(T)$ denotes the set of all fixed points of T.

Obviously, if u is a fixed point of the mapping T, then u is also a fixed point of T^n for every $n\in\mathbb{N}$. But, the converse is false, i.e. in general $F(T)\neq F(T^n)$, $n>1$. For example, let $X=[0,1]$ and $T:X\to X$ is defined by $Tx=1-x$. Then T has the unique fixed point $u=\frac{1}{2}$, while all points in the interval $[0,1]$ are fixed points of T^n, $n>1$, since $T^n=I_X$ (identity mapping) for every $n>1$. On the other side, if $X=[0,\pi]$ and $Tx=\cos x$, then T is nonexpansive and every iterate of T has the same fixed point as T. Involutions are also examples where $F(T)\neq F(T^n)$, $n>1$ (see [7] and the references therein). For more details we refer the reader to [7].

We will now present results regarding the property P for some well-known types of self-mappings.

Theorem 2.13. If $T:X\to X$ is a mapping of Boyd-Wong type, then T has Property P.

Proof. Suppose that $u\in F(T^n)$, where $n>1$ is a fixed natural number. If $u\neq Tu$, we obtain

$$\begin{aligned} 0 &< d(u,Tu)=d\left(T^n u,T^{n+1}u\right)=d\left(TT^{n-1}u,TT^n u\right)\\ &\leq \varphi\left(d\left(T^{n-1}u,T^n u\right)\right)=\varphi\left(d\left(T^{n-1}u,u\right)\right)<d\left(T^{n-1}u,u\right), \end{aligned}$$

which implies $u\neq T^{n-1}u$.

Since

$$\begin{aligned} 0 &< d(u,Tu)<d\left(u,T^{n-1}u\right)=d\left(TT^{n-1}u,TT^{n-2}u\right)\\ &\leq \varphi\left(d\left(T^{n-1}u,T^{n-2}u\right)\right)<d\left(T^{n-1}u,T^{n-2}u\right), \end{aligned}$$

we have $T^{n-1}u\neq T^{n-2}u$.

We continue in this manner and finally obtain

$$\begin{aligned} 0 &< d(u,Tu)<d\left(T^n u,T^{n-1}u\right)<d\left(T^{n-1}u,T^{n-2}u\right)<\cdots\\ &< d(Tu,u)=d(u,Tu), \end{aligned}$$

a contradiction.

Hence, T has Property P. Q.E.D.

Theorem 2.14. Let $T:X\to X$ be a mapping of Meir-Keeler type. Then T has Property P.

Proof. The mapping $T:X\to X$ is of Meir-Keeler type if for all $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that

$$\varepsilon\leq d(x,y)<\varepsilon+\delta \text{ implies } d(Tx,Ty)<\varepsilon. \tag{2.13}$$

Suppose, contrary to our claim, that $u\in F(T^n)$, $n>1$, and $u\notin F(T)$. Then we have

$$0<d(u,Tu)=d\left(TT^{n-1}u,TT^n u\right)<\varepsilon=d\left(T^{n-1}u,T^n u\right)=d\left(T^{n-1}u,u\right). \tag{2.14}$$

It follows immediately that $u\neq T^{n-1}u$ if and only if $u\neq Tu$. This means that (2.14) holds if in (2.13) we take $\varepsilon=d\left(T^{n-1}u,T^n u\right)=d\left(T^{n-1}u,u\right)$.

Similarly, we get

$$\varepsilon = d\left(T^{n-1}u, T^n u\right) = d\left(TT^{n-2}u, TT^{n-1}u\right) < \varepsilon_1 = d\left(T^{n-2}u, T^{n-1}u\right),$$

where also $\varepsilon_1 > 0$. Indeed, we conclude from $\varepsilon_1 = d\left(T^{n-2}u, T^{n-1}u\right) = 0$ that $T^{n-2}u = T^{n-1}u$, hence that $T^{n-1}u = T^n u = u$, and finally that $T^n u = u = Tu$, which contradicts $u \neq Tu$. Therefore, we obtain

$$\begin{aligned} 0 &< d(u, Tu) = d\left(T^n u, T^{n+1}u\right) < d\left(T^{n-1}u, T^n u\right) < \cdots \\ &< d\left(Tu, T^2 u\right) < d(u, Tu), \end{aligned}$$

a contradiction. Q.E.D.

Theorem 2.15. Let (X, d) be a compact metric space, $T : X \to X$ a contractive mapping, i.e. $d(Tx, Ty) < d(x, y)$, $x \neq y$. Then T has Property P.

Proof. Assume that $u \in F(T^n)$, $n > 1$, as well as that $u \notin F(T)$. We thus get

$$0 < d(u, Tu) = d\left(TT^{n-1}u, TT^n u\right) < d\left(T^{n-1}u, T^n u\right),$$

and, in consequence, $d\left(T^{n-1}u, T^n u\right) = d\left(T^{n-1}u, u\right) > 0$.

Further, we have

$$d\left(T^{n-1}u, T^n u\right) = d\left(TT^{n-2}u, TT^{n-1}u\right) < d\left(T^{n-2}u, T^{n-1}u\right),$$

which clearly forces $d\left(T^{n-2}u, T^{n-1}u\right) > 0$. The same reasoning applies, we obtain

$$\begin{aligned} 0 &< d(u, Tu) < d\left(T^{n-1}u, T^n u\right) < d\left(T^{n-2}u, T^{n-1}u\right) < \cdots \\ &< d\left(Tu, T^2 u\right) < d(u, Tu), \end{aligned}$$

which is impossible. This completes the proof. Q.E.D.

Theorem 2.16. If $T : X \to X$ is a $\mathcal{Z}-$contraction, i.e.

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0,$$

for all $x, y \in X$, and some $\zeta \in \mathcal{Z}$, then T has Property P.

Proof. Let $u \in F(T^n)$, $n > 1$. If $u \neq Tu$, then we have

$$\begin{aligned} 0 &\leq \zeta\left(d\left(TT^{n-1}u, TT^n u\right), d\left(T^{n-1}u, T^n u\right)\right) \\ &< d\left(T^{n-1}u, T^n u\right) - d\left(TT^{n-1}u, TT^n u\right) \\ &= d\left(T^{n-1}u, u\right) - d\left(u, T^{n+1}u\right). \end{aligned}$$

It follows that

$$\begin{aligned} d(u, Tu) &= d\left(T^n u, T^{n+1}u\right) < d\left(T^{n-1}u, T^n u\right) \\ &< d\left(T^{n-2}u, T^{n-1}u\right) < \cdots < d(u, Tu), \end{aligned}$$

which is impossible.

We use the fact that $d\left(u, T^{n+1}u\right) = d(u, Tu)$ and $d\left(T^{n-1}u, u\right)$ are both positive numbers. Q.E.D.

Theorem 2.17. If $T : X \to X$ is a mapping such that T^n, $n > 1$, is Boyd-Wong mapping (respectively, Meir-Keeler, contractive) or if X is a compact, and T is $\mathcal{Z}-$contraction, then, in all cases, T has Property P.

Proof. According to the previous results, in all cases, T^n has a unique fixed point (say $u \in X$). Firstly, $F(T) \neq \varnothing$. Indeed, if u is the unique fixed point of T^n, then u is also the fixed point of T. Further, since $F(T) \subseteq F(T^n) = \{u\} \subseteq F(T)$, the proof is complete. Q.E.D.

For the convenience of the reader we now repeat the relevant material from [10], thus making our exposition self-contained.
Let (X, d) be a metric space.
$CB(X) = \{A \mid A \text{ is a nonempty, closed and bounded subset of } X\}$
$D(a, B) = \inf\{d(a, b) \mid b \in B \subset X\}, \quad a \in X$
$H(A, B) = \max\{\sup\{D(a, B) \mid a \in A\}, \sup\{D(b, A) \mid b \in B\}\}$, $A, B \in CB(X)$.

It is easy to see that H is a metric on $CB(X)$, called the Hausdorff-Pompeu metric by d.

A set valued mapping $T : X \to CB(X)$ is said to be multi-valued contraction mapping if there exists a fixed real number λ, $0 < \lambda < 1$, such that

$$H(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in X$. A point $x \in X$ is called a fixed point of T if $x \in Tx$.

In 1969, S.B. Nadler [10] published the following result, which he had announced earlier.

Theorem 2.18 ([10]). Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multi-valued contraction mapping. Then T has a fixed point.

For a treatment of a more general case we state the next assertion.

Claim A. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be multi-valued contraction mapping. Assume that there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta(H(Tx, Ty), d(x, y)) \geq 0 \text{ for all } x, y \in X. \tag{2.15}$$

Then T has a fixed point.

It is worth while discussing here this problem. Indeed, it is clear that **Claim A** generalizes well-known Nadler's theorem (if it holds true!). For $\zeta(t, s) = \lambda s - t$ in (2.15), the result follows.

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