Measure of noncompactness and semilinear differential equations in Fréchet spaces

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Abstract

This paper deals with the existence of mild and integral solutions for a class of functional differential equations. The technique used is a generalization of the classical Darbo fixed point theorem for Fréchet spaces associated with the concept of measure of noncompactness.

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1 Introduction

Differential equations on infinite intervals frequently occur in mathematical modelling of various applied problems see [2, 21, 22]. For example, in the study of unsteady flow of a gas through a semi-infinite porous medium Agarwal and O'Regan [3], Kidder [17], analysis of the mass transfer on a rotating disk in a non-Newtonian fluid Agarwal and O'Regan [4], heat transfer in the radial flow between parallel circular disks Na [20], investigation of the temperature distribution in the problem of phase change of solids with temperature dependent thermal conductivity Na [20], as well as numerous problems arising in the study of circular membranes Agarwal and O'Regan [2], Dickey [9, 10], plasma physics Agarwal and O'Regan [4], nonlinear mechanics, and non-Newtonian fluid flows Agarwal and O'Regan [2].

Measures of noncompactness are very useful tools in functional analysis, for instance in metric fixed pint theory and in the theory of operator equations in Banach spaces. They are also used in the studies of functional equations, ordinary and partial differential equations, fractional partial differential equations, integral and integro-differential equations, optimal control theory, and in the characterizations of compact operators between Banach spaces. The first measure of noncompactness, denoted by α, was defined and studied by Kuratowski [18] in 1930. In 1955, Darbo [8] used the function α to prove his fixed point theorem. Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem, and includes the existence part of Banach's fixed point theorem.

In the present paper, we consider the following class of semilinear differential equations:

$$y'(t) = Ay(t) + f(t, y(t)), \quad t \in [0, \infty), \tag{1}$$

with the initial condition

$$y(0) = y_0 \in E, \tag{2}$$

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where $f : \mathbb{R}_+ \times E \to E$ is given function, $(E, \|\cdot\|)$ is a (real or complex) Banach space. The theory of semilinear differential equations has emerged as an important branch of nonlinear analysis, existence and uniqueness of mild, integral, strong and classical solutions of semilinear differential equations has been studied extensively by many authors using the semigroup theory, fixed point argument, degree theory and measures of noncompactness. We mention, for instance, the books of Abbas and Benchohra [1], Ahmed [5], Engel and Nagel [13], Kamenski *et al.* [15], Pazy [23] and Wu [24].

This paper initiates the existence of solutions for differential equations with an application of a generalization of the classical Darbo fixed point theorem, and the concept of measure of noncompactness in Fréchet spaces.

The paper is organized as follows. In Section 2 some preliminary results are introduced. The main results is presented in section 3, we discuss the existence of mild solutions of the problem (1)-(2), in the case where A is densely defined operator generating a C_0-semigroup $(T(t))_{t\geq 0}$ on E, while in Section 4, we discuss the existence of integral solutions of the problem (1)-(2), in the case where A is a Hille–Yosida operator and nondensely defined on E generating an integrated semigroup $(S(t))_{t\geq 0}$ on E. Section 5 is devoted to illustrative examples. A conclusion is presented in Section 6.

2 Preliminaries

Let $I =: [0, T]; \quad T > 0$. A measure function $y : I \to E$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable. For properties of the Bochner integral, see for instance, Yosida[25].
By $B(E)$ we denote the Banach space of all bounded linear operators from E into E, with the norm

$$\|N\|_{B(E)} = \sup_{\|y\|=1} \|N(y)\|.$$

As usual, $L^1(I, E)$ denotes the Banach space of measurable functions $y : I \to E$ which are Bochner integrable and normed by

$$\|y\|_{L^1} = \int_0^T \|y(t)\|dt.$$

As usual, $C := C(I)$ we denote the Banach space of all continuous functions from I into E with the norm $\|\cdot\|_\infty$ defined by

$$\|y\|_\infty = \sup_{t\in I} \|y(t)\|.$$

It is well known that the operator A generates a semigroup if A satisfies:

(ii) $\overline{D(A)} = E$.

(ii) The Hille-Yosida condition, that is, there exists $M \geq 0$ and $\tau \in \mathbb{R}$ such that $(\tau, \infty) \subset \rho(A), \sup\{(\lambda I - \tau)^n |(\lambda I - A)^{-n}| : \lambda > \tau, n \in \mathbb{N}\} \leq M$,

where $\rho(A)$ is the resolvent operator set of A and I is the identity operator. Existence and uniqueness, among other things, are derived. See, for example, the books of Heikkila and Lakshmikantham [14], Pazy [23].

Let $C(\mathbb{R}_+)$ be the Fréchet space of all continuous functions v from $\mathbb{R}_+$ into E, equipped with the family of seminorms

$$\|v\|_n = \sup_{t\in[0,n]} \|v(t)\|; \ n \in \mathbb{N},$$

and the distance

$$d(u,v) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|u-v\|_n}{1+\|u-v\|_n};\ u,v \in C(\mathbb{R}_+).$$

We recall the following definition of the notion of a sequence of measures of noncompactness [11, 12].

Definition 2.1. Let $\mathcal{M}_X$ be the family of all nonempty and bounded subsets of Fréchet space X. A family of functions $\{\mu_n\}_{n\in\mathbb{N}}$ where $\mu_n : \mathcal{M}_X \to [0,\infty)$ is said to be a family of measures of noncompactness in the Fréchet space X if it satisfies the following conditions for all $B, B_1, B_2 \in \mathcal{M}_X$:

(a) $\{\mu_n\}_{n\in\mathbb{N}}$ is full, that is: $\mu_n(B) = 0$ for $n \in \mathbb{N}$ if and only if B is precompact,

(b) $\mu_n(B_1) \le \mu_n(B_2)$ for $B_1 \subset B_2$ and $n \in \mathbb{N}$,

(c) $\mu_n(ConvB) = \mu_n(B)$ for $n \in \mathbb{N}$,

(d) If $\{B_i\}_{i=1,...}$ is sequence of closed sets from $\mathcal{M}_X$ such that $B_{i+1} \subset B_i$; $i = 1,\ldots$ and if $\lim_{i\to\infty} \mu(B_i) = 0$, for each $n \in \mathbb{N}$, then the intersection set $B_\infty := \cap_{i=1}^{\infty} B_i$ is nonempty.

Some Properties:

(e) We say that the family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$ is homogeneous if $\mu_n(\lambda B) = |\lambda|\mu_n(B)$, for $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$.

(f) If the family $\{\mu_n\}_{\mathbb{N}}$ satisfies the condition $\mu_n(B_1 + B_2) \le \mu_n(B_1) + \mu_n(B_2)$, for $n \in \mathbb{N}$, it is called subadditive.

(g) It is sublinear if both conditions (e) and (f) hold.

(h) We say that the family of measures $\{\mu_n\}_{n\in\mathbb{N}}$ has the maximum property if

$$\mu_n(B_1 \cup B_2) = \max(\mu_n(B_1), \mu_n(B_2)),$$

(i) The family of measure of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$ is said to be regular if and only if the conditions (a), (g) and (h) hold; (full sublinear and has maximum property).

Definition 2.2. A nonempty subset $B \subset X$ is said to be bounded if

$$\sup_{v\in X} \|v\|_n < \infty \quad \text{for} \quad n \in \mathbb{N}.$$

Lemma 2.3. [6] If Y is a bounded subset of Fréchet space X, then for each $\varepsilon > 0$, there is a sequence $\{y_k\}_{k=1}^{\infty} \subset Y$ such that

$$\mu_n(Y) \le 2\mu(\{y_k\}_{k=1}^{\infty}) + \varepsilon; \quad \text{for} \quad n \in \mathbb{N}.$$

Lemma 2.4. [19] If $\{u_k\}_{k=1}^{\infty} \subset L^1(I)$ is uniformly integrable, then $\mu_n(\{u_k\}_{k=1}^{\infty})$ is measurable for $n \in \mathbb{N}$, and

$$\mu_n\left(\left\{\int_0^t u_k(s)ds\right\}_{k=1}^{\infty}\right) \leq 2\int_0^t \mu_n(\{u_k(s)\}_{k=1}^{\infty})ds,$$

for each $t \in [0, n]$.

Definition 2.5. Let Ω be a nonempty subset of a Fréchet space X, and let $A : \Omega \to X$ be a continuous operator which transforms bounded subsets of onto bounded ones. One says that A satisfies the Darbo condition with constants $(k_n)_{n\in\mathbb{N}}$ with respect to a family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$, if

$$\mu_n(A(B)) \leq k_n\mu_n(B)$$

for each bounded set $B \subset \Omega$ and $n \in \mathbb{N}$.
If $k_n < 1;\ n \in \mathbb{N}$, then A is called a contraction with respect to $\{\mu_n\}_{n\in\mathbb{N}}$.

In the sequel we will make use of the following generalization of the classical Darbo fixed point theorem for Fréchet spaces.

Theorem 2.6. [11, 12] Let Ω be a nonempty, bounded, closed, and convex subset of a Fréchet space X and let $V : \Omega \to \Omega$ be a continuous mapping. Suppose that V is a contraction with respect to a family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$. Then V has at least one fixed point in the set Ω.

3 Existence of mild solutions

In this section, we present the main results for the global existence of solutions for our problem. Let us introduce the definition of the mild solution of the problem (1)-(2).

Definition 3.1. We say that a continuous function $y(\cdot) : \mathbb{R}_+ \to E$ is mild solution of the problem (1)-(2), if y satisfies the following integral equation

$$y(t) = T(t)y_0 + \int_0^t T(t-s)f(s, y(s))ds, \quad t \in [0, +\infty). \tag{3}$$

We will consider the hypotheses (1)-(2) and we will need to introduce the following one which is assumed hereafter:

(H_1) The operator A is the infinitesimal generator of a C_0-semigroup $T(t),\ t \in \mathbb{R}_+$ in E and there exists a positive constant $M \geq 1$ such that

$$\|T(t)\|_{B(E)} \leq M,\ t \geq 0.$$

(H_2) The function $t \to f(t, y)$ is measurable on $\mathbb{R}_+$ for each $y \in E$, and the function $y \mapsto f(t, y)$ is continuous on E for a.e. $t \in \mathbb{R}_+$.

(H_3) There exists a continuous function $p : \mathbb{R}_+ \to (0, \infty)$ such that

$$\|f(t, y)\| \leq p(t)(1 + \|y\|) \quad \text{for a.e. } t \in \mathbb{R}_+ \quad \text{and each} \quad y \in E.$$

(H_4) For each bounded and measurable set $B \subset E$ and for each $t \in \mathbb{R}_+$, we have

$$\mu(f(t,B)) \leq p(t)\mu(B),$$

where μ is a measure of noncompactness on the Banach space E.
For $n \in \mathbb{N}$, let

$$p_n^* = \sup_{t\in[0,n]} p(t),$$

and define on $C(\mathbb{R}_+)$ the family of noncompactness by

$$\mu_n(D) = \sup_{t\in[0,n]} e^{-4Mp_n^*\tau t}\mu_n(D(t))$$

where $D(t) = \{v(t) \in E : v \in D\};\ t \in [0,n]$ and $\tau > 1$.

Theorem 3.2. Assume that the hypotheses (H_1)- (H_6) are satisfied, and

$$l_n := nMp_n^* < 1; \quad \text{for each} \quad n \in \mathbb{N}.$$

Then the problem (1)-(2) has at least one mild solution.

Proof. Consider the operator $N : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by:

$$(Ny)(t) = T(t)y_0 + \int_0^t T(t-s)f(s,y(s))ds, \quad t \in [0,+\infty). \tag{4}$$

For any $n \in \mathbb{N}$, let R_n be a positive real number with

$$R_n \geq \frac{M\|y_0\| + nMp_n^*}{1 - nMp_n^*},$$

and we consider the ball

$$B_{R_n} := B(0,R_n) = \{y \in C(R_+) : \|y\|_n \leq R_n\}.$$

For any $n \in \mathbb{N}$, and each $y \in B_{R_n}$ and $t \in [0,n]$ we have

$$\begin{aligned}
\|(Ny)(t)\| &\leq \|T(t)\|_{B(E)}\|y_0\| + \int_0^t \|T(t-s)\|_{B(E)}\|f(s,y(s))\|ds \\
&\leq M\|y_0\| + M\left(\int_0^t p(s)(1+\|y(s)\|)ds\right) \\
&\leq M\|y_0\| + M(1+\|y\|_n)\int_0^t p(s)ds \\
&\leq M\|y_0\| + Mnp_n^*(1+R_n) \\
&\leq R_n.
\end{aligned}$$

Thus

$$\|N(y)\|_n \leq R_n. \tag{5}$$

This proves that N transforms the ball B_{R_n} into itself. We shall show that the operator $N : B_{R_n} \to B_{R_n}$ satisfies all the assumptions of Theorem 3.2. The proof will be given several steps.
Step 1. $N : B_{R_n} \to B_{R_n}$ is continuous.
Let $\{y_k\}_{k\in\mathbb{N}}$ be a sequence such that $y_k \to y$ in B_{R_n}. Then, for each $t \in [0, n]$, we have

$$\begin{aligned}\|(Ny_k)(t) - (Ny)(t)\| &\leq \int_0^t \|T(t-s)\|_{B(E)}\|f(s, y_k(s)) - f(s, y(s))\|ds \\ &\leq M\int_0^t \|f(s, y_k(s)) - f(s, y(s))\|ds.\end{aligned}$$

Since $y_k \to y$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\|Ny_k - Ny\|_n \to 0 \quad \text{as} \quad k \to \infty.$$

Step 2. $N(B_{R_n})$ is bounded.
Since $N(B_{R_n}) \subset B_{R_n}$ is bounded, then $N(B_{R_n})$ is bounded.

Step 3. For each equicontinuous subset D of B_{R_n}, $\mu_n(ND) \leq l_n\mu_n(D)$.
From lemmas 2.3 and 2.4 , for any $D \subset B_{R_n}$ and any $\varepsilon > 0$, there exists a sequence $\{y_k(s)\}_{k=1}^{\infty} \subset D$, such that for all $t \in [0, n]$, we have

$$\begin{aligned}\mu(ND)(t)) &= \mu\left(\left\{T(t)y_0 + \int_0^t T(t-s)f(s, y(s))ds; y \in D\right\}\right) \\ &\leq 2\mu(\{\int_0^t T(t-s)f(s, y_k(s))ds\}_{k=1}^{\infty}) + \varepsilon \\ &\leq 2\mu\left(\left\{\int_0^t T(t-s)f(s, y_k(s))ds\right\}_{k=1}^{\infty}\right) + \varepsilon \\ &\leq 4M\int_0^t \mu(\{f(s, y_k(s))\}_{k=1}^{\infty})ds + \varepsilon \\ &\leq 4M\int_0^t p(s)\mu(\{y_k(s)\}_{k=1}^{\infty})ds + \varepsilon \\ &\leq 4Mp_n^*\int_0^t \mu(\{y_k(s)\}_{k=1}^{\infty})ds + \varepsilon \\ &\leq 4Mp_n^*\int_0^t e^{4Mp_n^*\tau s}e^{-4Mp_n^*\tau s}\mu(\{y_k(s)\}_{k=1}^{\infty})ds + \varepsilon \\ &\leq 4Mp_n^*\mu_n(D)\int_0^t e^{4Mp_n^*\tau s}ds + \varepsilon \\ &\leq \frac{e^{4Mp_n^*\tau t}}{\tau}\mu_n(D) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then

$$\mu(ND)(t) \leq \frac{e^{4Mp_n^*\tau t}}{\tau}\mu_n(D).$$

Thus

$$\mu_n(ND) \leq \frac{1}{\tau}\mu_n(D).$$

As a consequence of steps to 1 to 3 together with Theorem 3.2, we can conclude that N has a least one fixed in B_{R_n} which is a mild solution of problem (1).

4 Existence of integral solutions

In this section we give our main existence result for problem (1)-(2) in the case when the operator A is nondensely defined. Before that, we present some examples of nondensely defined linear operators. We refer to [7] for more details.

Example 1. Let $E = C([0, 1], \mathbb{R})$ and $A : D(A) \to E$ the operator defined by

$$Ay = y',$$

where

$$D(A) = \{y \in C^1([0, 1], \mathbb{R}) : y(0) = 0\}.$$

We have

$$\overline{D(A)} = \{y \in C^1([0, 1], \mathbb{R}) : y(0) = 0\} \neq E.$$

Example 2. Let $E = C([0, 1], \mathbb{R})$ and $A : D(A) \to E$ the operator defined by

$$Ay = y'',$$

where

$$D(A) = \{y \in C^2([0, 1], \mathbb{R}) : y(0) = y(1) = 0\}.$$

We have

$$\overline{D(A)} = \{y \in C^2([0, 1], \mathbb{R}) : y(0) = y(1) = 0\} \neq E.$$

Example 3. Let $\beta \in (0, 1)$,

$$E = C_0^\beta([0, 1], \mathbb{R}) = \{y : [0, 1] \to \mathbb{R};\, y(0) = 0 \text{ and } \sup_{0 \leq t,\, s \leq 1} \frac{|y(t) - y(s)|}{|t - s|^\beta} < +\infty\}$$

and the operator $A : D(A) \to E$ defined by

$$Ay = -y',$$

where

$$D(A) = \{y \in C^{1+\beta}([0, 1], \mathbb{R}) : \, y(0) = y'(0) = 0\}.$$

Then

$$\begin{aligned}\overline{D(A)} &= h_0^\beta([0, 1], \mathbb{R}) \\ &= \{y : [0, 1] \to \mathbb{R};\, \lim_{\delta \to 0} \sup_{0 \leq |t-s| \leq \delta} \frac{|y(t) - y(s)|}{|t - s|^\beta} = 0,\, y(0) = y'(0) = 0\} \neq E.\end{aligned}$$

Here

$$C^{1+\beta}([0, 1], \mathbb{R}) = \{y : [0, 1] \to \mathbb{R};\, |\, y' \in C^1([0, 1], \mathbb{R})\}.$$

The elements of $h^{\beta}([0,1],\mathbb{R})$ are called Holderien functions and the closure of $C^1([0,1],\mathbb{R})$ in $C^{\beta}([0,1],\mathbb{R})$ is $h^{\beta}([0,1],\mathbb{R})$.

Before starting and proving this result, we give the definition of the integral solution.

(P_1) A satisfies the Hille–Yosida condition, namely, there exist $\overline{M} \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega,\infty) \subset \varrho(A)$ and

$$|(\lambda I - A)^{-n}| \leq \frac{\overline{M}}{(\lambda-\omega)^n} \quad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad \lambda > \omega,$$

where $\varrho(A)$ is the resolvent set of A, for more details (see [16]).

Definition 4.1. We say that $y : [0,\infty) \to E$ is an integral solution of (1)-(2) if

(i) $y \in C([0,\infty),E)$,

(ii) $\int_0^t y(s)ds \in D(A)$ for $t \in J$,

(iii) $y(t) = y_0 + A\int_0^t y(s)ds + \int_0^t f(s,y(s))ds, \quad t \in J.$

From the definition it follows that $y(t) \in \overline{D(A)}, \quad t \geq 0$, in particular $y_0 \in \overline{D(A)}$. Moreover, y satisfies the following variation of constants formula:

$$y(t) = S'(t)y_0 + \frac{d}{dt}\int_0^t S(t-s)f(s,y(s))ds, \quad t \in [0,+\infty). \tag{6}$$

We will consider the hypotheses (1)-(2) and we will need to introduce the following one which is assumed hereafter:

(P_2) Let $(S(t))_{t\geq 0}$, be the integrated semigroup generated by A such that

$$\|S'(t)\|_{B(E)} \leq \widetilde{M}, \quad t \geq 0.$$

Theorem 4.2. Assume that the hypotheses (P_1)- (P_2) and (H_2)-(H_4) are satisfied, and

$$l_n := n\widetilde{M}p_n^* < 1; \quad \text{for each} \quad n \in \mathbb{N}.$$

Then the problem (1)-(2) has at least one mild solution.

Proof. Consider the operator $Q : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by:

$$(Qy)(t) = S'(t)y_0 + \frac{d}{dt}\int_0^t S(t-s)f(s,y(s))ds, \quad t \in [0,+\infty). \tag{7}$$

For any $n \in \mathbb{N}$, let R_n be a positive real number with

$$R_n \geq \frac{\widetilde{M}\|y_0\| + n\widetilde{M}p_n^*}{1 - n\widetilde{M}p_n^*},$$

and we consider the ball

$$B_{R_n} := B(0, R_n) = \{\omega \in C(R_+) : \|\omega\|_n \leq R_n\}.$$

For any $n \in \mathbb{N}$, and each $y \in B_{R_n}$ and $t \in [0, n]$ we have

$$\begin{aligned}
\|(Qy)(t)\| &\leq \|S'(t)\|_{B(E)}\|y_0\| + \left\|\frac{d}{dt}\int_0^t S(t-s)f(s, y(s))ds\right\| \\
&\leq \widetilde{M}\|y_0\| + \widetilde{M}\left(\int_0^t p(s)(1 + \|y(s)\|)ds\right) \\
&\leq \widetilde{M}\|y_0\| + \widetilde{M}(1 + \|y\|_n)\int_0^t p(s)ds \\
&\leq \widetilde{M}\|y_0\| + \widetilde{M}np_n^*(1 + R_n) \\
&\leq R_n.
\end{aligned}$$

Thus

$$\|Q(y)\|_n \leq R_n. \tag{8}$$

This proves that Q transforms the ball B_{R_n} into itself. We shall show that the operator $Q : B_{R_n} \to B_{R_n}$ satisfies all the assumptions of Theorem 3.2. The proof will be given several steps.

Step 1. $Q : B_{R_n} \to B_{R_n}$ is continuous.

Let $\{y_k\}_{k\in\mathbb{N}}$ be a sequence such that $y_k \to y$ in B_{R_n}. Then, for each $t \in [0, n]$, we have

$$\begin{aligned}
\|(Qy_k)(t) - (Qy)(t)\| &\leq \left\|\frac{d}{dt}\int_0^t S(t-s)(f(s, y_k(s)) - f(s, y(s)))ds\right\| \\
&\leq \widetilde{M}\int_0^t \|f(s, y_k(s)) - f(s, y(s))\|ds.
\end{aligned}$$

Since $y_k \to y$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\|Qy_k - Qy\|_n \to 0 \quad \text{as} \quad k \to \infty.$$

Step 2. $Q(B_{R_n})$ is bounded.

Since $Q(B_{R_n}) \subset B_{R_n}$ is bounded, then $Q(B_{R_n})$ is bounded.

Step 3. For each equicontinuous subset D of B_{R_n}, $\mu_n(QD) \leq l_n\mu_n(D)$.

From lemmas 2.3 and 2.4 , for any $D \subset B_{R_n}$ and any $\varepsilon > 0$, there exists a sequence $\{y_k(s)\}_{k=1}^{\infty} \subset D$, such that for all $t \in [0, n]$, we have

$$\begin{aligned}
\mu(QD)(t)) &= \mu\left(\left\{S'(t)y_0 + \frac{d}{dt}\int_0^t S(t-s)f(s,y(s))ds; y \in D\right\}\right) \\
&\leq 2\mu\left(\{\frac{d}{dt}\int_0^t S(t-s)f(s,y_k(s))ds\}_{k=1}^{\infty}\right) + \varepsilon \\
&\leq 4\widetilde{M}\int_0^t \mu(\{f(s,y_k(s))\}_{k=1}^{\infty})ds + \varepsilon \\
&\leq 4\widetilde{M}\int_0^t p(s)\mu(\{y_k(s)\}_{k=1}^{\infty})ds + \varepsilon \\
&\leq 4\widetilde{M}p_n^*\int_0^t \mu(\{y_k(s)\}_{k=1}^{\infty})ds + \varepsilon \\
&\leq 4\widetilde{M}p_n^*\int_0^t e^{4\widetilde{M}p_n^*\tau s}e^{-4\widetilde{M}p_n^*\tau s}\mu(\{y_k(s)\}_{k=1}^{\infty})ds + \varepsilon \\
&\leq 4\widetilde{M}p_n^*\mu_n(D)\int_0^t e^{4\widetilde{M}p_n^*\tau s}ds + \varepsilon \\
&\leq \frac{e^{4\widetilde{M}p_n^*\tau t}}{\tau}\mu_n(D) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then

$$\mu(QD)(t) \leq \frac{e^{4\widetilde{M}p_n^*\tau t}}{\tau}\mu_n(D).$$

Thus

$$\mu_n(QD) \leq \frac{1}{\tau}\mu_n(D).$$

As a consequence of steps to 1 to 3 together with Theorem 4.2, we can conclude that Q has a least one fixed in B_{R_n} which is an integral solution of problem (1)- (2).

5 Examples

Example 1. We consider the following problem

$$\begin{cases}
\dfrac{\partial z}{\partial t}(t,x) = \dfrac{\partial^2 z}{\partial x^2}(t,x) + Q(t,z(t,x)),; & t \in \mathbb{R}_+,\ x \in [0,\pi], \\
z(t,0) = z(t,\pi) = 0; & t \in \mathbb{R}_+, \\
z(0,x) = \Phi(x); & x \in [0,\pi],
\end{cases} \tag{9}$$

where $Q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\Phi : [0,\pi] \to \mathbb{R}$ are continuous functions.

Consider $E = L^2([0,\pi],\mathbb{R})$ and define A by $Aw = w''$ with domain

$$D(A) = \{w \in E : w, w' \text{ are absolutely continuous}, w'' \in E,\ w(0) = w(\pi) = 0\}.$$

Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \quad w \in D(A),$$

where $(\cdot,\cdot)$ is the inner product in L^2 and $w_n(s) = \sqrt{\frac{2}{\pi}} \sin ns, \quad n = 1, 2, \ldots$, is orthogonal set of eigenvectors of A. It is well known (see [23]) that A is the infinitesimal generator of an analytic semigroup $T(t), \quad t \geq 0$, in E and is given by the relation

$$T(t)w = \sum_{n=1}^{\infty} \exp(-n^2 t)(w, w_n) w_n, \quad w \in E,$$

and there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

For $x \in [0, \pi]$, we have

$$y(t)(x) = z(t,x); \quad t \in \mathbb{R}_+,$$
$$f(t, y(t))(x) = Q(t, z(t,x)); \quad t \in \mathbb{R}_+,$$
$$y_0(x) = \Phi(x).$$

Then the system (9) can be represented by the abstract problem (1)- (2), and conditions $(H_1)-(H_4)$ are satisfied. Consequently, Theorem 3.2 implies that the problem (9) has a mild solution.

Example 2. We consider the following problem

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) = \dfrac{\partial z}{\partial x}(t,x) + Q(t, z(t,x)), ; & t \in \mathbb{R}_+, \ x \in [0,\pi], \\ z(t,0) = 0; & t \in \mathbb{R}_+, \\ z(0,x) = \Phi(x); & x \in [0,\pi], \end{cases} \tag{10}$$

where $Q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\Phi : [0,\pi] \to \mathbb{R}$ are continuous functions.

Consider $E = C([0,\pi], \mathbb{R})$ and define A by $Aw = w'$ with domain

$$D(A) = \{w \in C^1([0,\pi], \mathbb{R}) \ w(0) = 0\}.$$

It is well known (see [7]) that the operator A satisfies the Hille–Yosida condition with $(0, +\infty) \subset \varrho(A), \quad \|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}$ for $\lambda > 0$, and

$$\overline{D(A)} = \{w \in E, \ w(0) = 0\} \neq E.$$

It follows that A generates an integrated semigroup $(S(t))_{t \geq 0}$ and $\|S(t)\| \leq 1$ for $t \geq 0$. We can show that problem (1)- (2) is an abstract formulation of problem (10).

For $x \in [0, \pi]$, we have

$$y(t)(x) = z(t,x); \quad t \in \mathbb{R}_+,$$
$$f(t, y(t))(x) = Q(t, z(t,x)); \quad t \in \mathbb{R}_+,$$
$$y_0(x) = \Phi(x).$$

Then the system (10) can be represented by the abstract problem (1)- (2), and conditions (P_1)-(P_2) and (H_2)-(H_4) are satisfied. Consequently, Theorem 4.2 implies that the problem (10) has an integral solution.

6 Conclusion

In this paper we have provided sufficient conditions for the existence of mild and integral solutions for a class of semilinear differential equations on infinite dimensional Banach spaces. We have considered the cases of densely and nondensely defined linear operators. The technique used is a generalization of the classical Darbo fixed point theorem for Fréchet spaces associated with the concept of measure of noncompactness.

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