Aspects of invariant submanifolds of a f_λ-Hsu manifold with complemented frames

Praveen Kumar Mathur[1] and Pravin Kumar Srivastava[2]

[1]Department of Mathematics, Surya School of Planning and Engineering Management, Lucknow, India
[2]Department of Applied Science and Humanities, Bundelkhand Institute of Engineering and Technology, Jhansi, India

E-mail: drpraveenmathur8@gmail.com[1], drpravinsrivastava@gmail.com[2]

Abstract

In this paper, we have discussed the aspects of invariant sub manifolds of a f_λ–Hsu manifold with complemented frames. The relation between the integrability of Hsu-structure F and that of manifold having f_λ–Hsu structure with complemented frames has been established. Some other results concerning the normality of f_λ–Hsu structure with complemented frames have also been obtained.

2010 Mathematics Subject Classification. **53C15**. 53C50
Keywords. invariant submanifold, f_λ- Hsu manifold, complemented frame.

1 Introduction

In an m-dimensional differentiable manifold M of class C^∞, if a non-null tensor field of type (1, 1) satisfies $f^3 - \lambda^r f$=0 , where λ is a non zero complex number and is of constant rank p at each point of M then f is called f_λ - Hsu structure of rank p and M with f_λ- Hsu structure an f_λ-Hsu manifold. The integrability conditions of a f_λ-structure has been discussed by Upadhyay and Gupta [3]. If we put

$$\ell = f^2/\lambda^r \text{ and } m = I - f^2/\lambda^r \tag{1.1}$$

where I denotes the unit tensor field, then it is easy to see that

$$\ell^2 = \ell,\ \ m^2 = m,\ \ \ell + m = I,\ \ \ell m = m\ell = 0. \tag{1.2}$$

This implies that the tensor fields f^2/λ^r and I- f^2/λ^r are complementary projection operators. Let L, M be distribution corresponding to the projection operators f^2/λ^r and I- f^2/λ^r respectively. The distributions corresponding to f^2/λ^r and I- f^2/λ^r , are p and (m - p) dimensional. Let there exist (m - p) vector fields U_α (α =1,2,3,...m-p) spanning the distribution corresponding to I- f^2/λ^r and (m - p) I -form u^α satisfying

$$f^2/\lambda^r = I - \sum_{\alpha=1}^{m-p}[u^\alpha \bigotimes U_\alpha] \tag{1.3}$$

and

$$fU_\alpha = 0, u^\alpha \circ f = 0, u^\alpha(U_\beta) = \delta^\alpha_\beta, \alpha, \beta = 1, 2, ..., (m-p), \tag{1.4}$$

where, δ^α_β is the Kronecker delta. Then we call the set $\{f_\lambda, U_\alpha, u^\alpha\}$ an f_λ -Hsu structure with complemented frames and the manifold M an f_λ - manifold with complemented frames.

Tbilisi Mathematical Journal 12(1) (2019), pp. 45–54.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 10 August 2016.
Accepted for publication: 10 November 2018.

Invariant submanifold

Suppose that an n -dimensional differentiable manifold $\tilde{m}$ is immersed in a manifold m by the immersion $I : \tilde{m} \to m$. If the tangent space of $(\tilde{m})$ is invariant by the action of f , then $I(\tilde{m})$ is called an invariant sub-manifold of m.

In the present chapter, we consider a f_λ- structure with complement frames such that $r= m$ - 3.

2 f_λ-Hsu structure with complemented frames

Let M be an m -dimensional differentiable manifold of class C^∞ and let there be given a tensor field f of type **(1, 1)** and of rank (m - 2), two vector fields U, V and two 1-forms u, v . If the set $\{f_\lambda, U, V, W, u, v, w\}$ satisfies

$$f^2/\lambda^r = I - u \bigotimes U - v \bigotimes V - w \bigotimes W \tag{2.1}$$

$$fU = 0, fV = 0, fW = 0, u \circ f = 0, v \circ f = 0, w \circ f = 0....(a)$$

$$v(U) = 0, u(V) = 0, u(W) = 0, w(U) = 0, v(W) = 0, w(V) = 0.....(b) \tag{2.2}$$

where, λ is any complex number not equal to zero, then we call $(f_\lambda, U, V, W, u, v, w)$ an f_λ - Hsu structure with complemented frames and M and f_λ-manifold with complemented frame on an f_λ-Hsu manifold with complemented frame M.

$$u(U) = 1, v(V) = 1, w(W) = 1. \tag{2.3}$$

Let us define a tensor field S^* of type (1, 2, 3) as

$$S^*(X,Y) = N(X,Y) - \lambda^r(du)(X,Y)U - \lambda^r(dv)(X,Y)V - \lambda^r(dw)(X,Y)W \tag{2.4}$$

where du, dv, dw are 3-forms and N is the Nijenhuis tensor formed with f defined by [1] as

$$N(X,Y) = [fX, fY] - f[fX,Y] - f[X, fY] + f^2[X,Y]. \tag{2.5}$$

Definition 2.1. If the tensor field S^* vanishes identically then the structure is said to be normal.

In view of Eqs. (2.2), (2.4) and (2.5), we have

$$S^*(X,U) = -f[X,U] + f^2[X,U] - \lambda^r(du)(X,U)U - \lambda^r(dv)(X,U)V - \lambda^r(dw)(X,U)W \tag{2.6}$$

Let $\pounds_U$ be called the Lie derivative with respect to a field U. Then we have

$$-f[fX,U] + f^2[X,U] = f(f[X,U]) - [fX,U] = f(\pounds_U f)X$$

and

$$du(X,U) = X(u(U)) - U(u(X)) - u([X,U]) = -[u([X,U] - [u(X),U])] = -(\pounds_U u)(X).$$

Similarly,

$$dv(X,V) = -(\pounds_U v)(X) \ and \ dw(X,W) = -(\pounds_U w)(X)$$

Therefore from Eq. (2.6), we obtain

$$S^*(X,U) = f(\pounds_U f)X + \lambda^r(\pounds_U u)(X)U + \lambda^r(\pounds_U v)(X)V + \lambda^r(\pounds_U w)(X)W \tag{2.7}$$

We can also prove that

$$S^*(X,V) = f(\pounds_V f)X + \lambda^r(\pounds_V u)(X)U + \lambda^r(\pounds_V v)(X)V + \lambda^r(\pounds_V w)(X)W \tag{2.8}$$

and

$$S^*(X,W) = f(\pounds_W f)X + \lambda^r(\pounds_W u)(X)U + \lambda^r(\pounds_W v)(X)V + \lambda^r(\pounds_W w)(X)W \tag{2.9}$$

Also, as a consequence of Eqs.(2.2), (2.4) and (2.5), we have

$$u\Big(S^*(X,Y)\Big) = U([fX,Y]) - \lambda^r(du)(X,Y) \tag{2.10}$$

But we have

$$du(fX,fY) = (fX)u(fY) - (fY)u(fX) - u(fX,fY),$$

which in view of Eq.(2.2) implies that

$$u([fX,fY]) = -(du)(fX,fY).$$

Thus from Eq. (2.10) we obtain

$$u(S^*(X,Y)) = -\lambda^r(du)(X,Y) - \lambda^r(du)(fX,fY) \tag{2.11}$$

Replacing X by fX in Eq. (2.11) and using Eq. (2.1), we get

$$u(S^*(fX,W)) = -\lambda^r(du)(fX,Y) - du(\lambda^r X - \lambda^r u(X)U - \lambda^r v(X)V, fV - \lambda^r W, fW)$$

$$= -\lambda^r\{(du)(fX,Y) - du(X,fY) - u(X)(du)(U,fY) - v(X)(dv)(V,fY) - w(X)(dw)(W,fY)\} \tag{2.12}$$

But we have

$$(du)(U,fU) = Uu(fY) - (fY)uU - u([U,fY])$$
$$= u\{u([fY,U]) - [u(fY),U]\} = (\pounds_U u)(fY)$$

Similarly,

$$(dv)(V,fY) = (\pounds_V u)(fY) \ and \ (dw)(W,fY) = (\pounds_W u)(fY)$$

Hence, from Eq.(2.12), we have

$$u(S^*(fX,Y)) = -\lambda^r[(du)(fX,Y) + (du)(X,fY) - u(X)(\pounds_U u)(fY) -v(X)(\pounds_V u)(fY) - w(X)(\pounds_W u)(fY)] \tag{2.13}$$

We can also see that

$$v(S^*(fX,Y)) = -\lambda^r\{(dv)(fX,Y) + (dv)(X,fY) - v(X)(\pounds_U v)(fY) - v(X)(\pounds_V v)(fY)\} \tag{2.14}$$

$$w(S^*(fX,Y)) = -\lambda^r\{(dw)(fX,Y) + (dw)(X,fY) - w(X)(\pounds_U v)(fY) - v(X)(\pounds_V v)(fY)\} \tag{2.15}$$

Theorem 2.1. If an f_λ -Hsu structure with complemented $\{f_\lambda , U , V , W , u , v , w\}$ is normal, then

$$\pounds_U f = 0, \pounds_U u = 0, \pounds_U v = 0, \pounds_U w = 0, \tag{2.16}$$

$$\pounds_V f = 0, \pounds_V u = 0, \pounds_V V = 0, \pounds_V w = 0, \tag{2.17}$$

$$\pounds_W f = 0, \pounds_W u = 0, \pounds_W v = 0, \pounds_W w = 0, \tag{2.18}$$

$$du\pi f = 0, dv\pi f = 0, dw\pi f = 0, [U, V, W] = 0 \tag{2.19}$$

Proof. Let us suppose that f_λ -Hsu structures with complemented frames $\{f_\lambda, U , V , W , u , v , w \}$ is normal. Then from Eq. (2.7), we have

$$f(\pounds_U f)X + \lambda^r(\pounds_U u)(X)U + \lambda^r(\pounds_U v)(X)V + \lambda^r(\pounds_U w)(X)W = 0,$$

which in view of Eq.(2.2) and Eq.(2.3) implies that

$$\pounds_U u = 0,\ \pounds_U v = 0,\ \pounds_U w = 0,\ f(\pounds_U f) = 0 \tag{2.20}$$

Applying f to the last equation of Eq. (2.20) and using Eq. (2.1), we obtain
$\lambda^r[(\pounds_U$ f) -u $\circ$ $(\pounds_U$ f) $\bigotimes$ U-v $\circ$ $(\pounds_U$ f) $\bigotimes$ V-w $\circ$ $(\pounds_U$ f) $\bigotimes$ W]=0
or
$\lambda^r[(\pounds_U$ f) +[$(\pounds_U$ u) $\circ$ f] $\bigotimes$ U-[$(\pounds_U$ v) $\circ$ f] $\bigotimes$ V-[$(\pounds_U$ w) $\circ$ f] $\bigotimes$ W]=0.
$\lambda^r[(\pounds_U$ f) +[$(\pounds_U$ u) $\circ$ f] $\bigotimes$ U+[$(\pounds_U$ v) $\circ$ f] $\bigotimes$ V+[$(\pounds_U$ w) $\circ$ f] $\bigotimes$ W]=0.
Hence, in view of Eq. (2.20), we have

$$\pounds_U f = 0, \quad since \ \ \lambda \neq 0 \tag{2.21}$$

Similarly, from Eq. (2.8), we can prove that

$$\pounds_U u = 0,\ \pounds_V v = 0,\ \pounds_W w = 0,\ \pounds_V f = 0 \tag{2.22}$$

Let us put

$$(Z\pi f)(X, Y) = Z(fX, Y) + Z(X, fY),$$

for a 2-form Z .Then by virtue of Eqs. (2.13), (2.14), (2.20), (2.21) we have $(du)\ \pi\ f = 0$ and $(dv)\ \pi\ f$=0. Now computing $\pounds_U(fV)$=0,we find
$\pounds_V\ v$=0.
Applying f to Eq. (2.24) and using Eq. (2.1), we get

$$\lambda^r[\pounds_U V - u(\pounds_U V)U - v(\pounds_U V)V - w(\pounds_U V)W] = 0.$$

and $\pounds_U\ (V)$=0, which implies $[U,\ V,\ W] = 0$. Q.E.D.

3 Hsu-structure F

Let us define a tensor field F of type (1, 1) as follows

$$FX = fX + \lambda^{\frac{r}{2}} v(X)U + \lambda^{\frac{r}{2}} u(X)V + \lambda^{\frac{r}{2}} u(X)W + \lambda^{\frac{r}{2}} w(X)U + \lambda^{\frac{r}{2}} v(X)V + \lambda^{\frac{r}{2}} v(X)W \tag{3.1}$$

for an arbitrary vector field X.

Theorem 3.1. In order that a manifold M may admit an f_λ -Hsu structure with complemented frames [f_λ , U , V , W , u , v , w], it is necessary and sufficient that the manifold admits a Hsu-structure F , a vector field U and an 1-form u such that

$$u(U) = 1, u(FU) = 0, v(V) = 1, v(FV) = 0, w(W) = 1, w(FW) = 0.$$

Proof. In view of Eqs. (2.1), (2.2), (2.3) and (3.1), we have

$$\begin{aligned} F^2X = F(FX) = {} & f(FX + \lambda^{\frac{r}{2}} v(X)U + \lambda^{\frac{r}{2}} u(X)V + \lambda^{\frac{r}{2}} w(X)U + \lambda^{\frac{r}{2}} w(X)V \\ & + \lambda^{\frac{r}{2}} u(X)W + \lambda^{\frac{r}{2}} v(X)W) + \lambda^{\frac{r}{2}} v(fX + \lambda^{\frac{r}{2}} v(X)U + \lambda^{\frac{r}{2}} u(X)V \\ & + \lambda^{\frac{r}{2}} w(X)U + \lambda^{\frac{r}{2}} w(X)V + \lambda^{\frac{r}{2}} u(X)W + \lambda^{\frac{r}{2}} v(X)W)U \\ & + \lambda^{\frac{r}{2}} u(fX + \lambda^{\frac{r}{2}} v(X)U + \lambda^{\frac{r}{2}} u(X)V + \lambda^{\frac{r}{2}} w(X)U + \lambda^{\frac{r}{2}} w(X)V \\ & + \lambda^{\frac{r}{2}} u(X)W + \lambda^{\frac{r}{2}} v(X)W)V + \lambda^{\frac{r}{2}} w(fX + \lambda^{\frac{r}{2}} v(X)U + \lambda^{\frac{r}{2}} u(X)V \\ & + \lambda^{\frac{r}{2}} w(X)U + \lambda^{\frac{r}{2}} w(X)V + \lambda^{\frac{r}{2}} u(X)W + \lambda^{\frac{r}{2}} v(X)W)U \\ & + \lambda^{\frac{r}{2}} w(fX + \lambda^{\frac{r}{2}} v(X)U + \lambda^{\frac{r}{2}} u(X)V + \lambda^{\frac{r}{2}} w(X)U + \lambda^{\frac{r}{2}} w(X)V \\ & + \lambda^{\frac{r}{2}} u(X)W + \lambda^{\frac{r}{2}} v(X)W)V + \lambda^{\frac{r}{2}} u(fX + \lambda^{\frac{r}{2}} v(X)U + \lambda^{\frac{r}{2}} u(X)V \\ & + \lambda^{\frac{r}{2}} w(X)U + \lambda^{\frac{r}{2}} w(X)V + \lambda^{\frac{r}{2}} u(X)W + \lambda^{\frac{r}{2}} v(X)W)W \\ & + \lambda^{\frac{r}{2}} v(fX + \lambda^{\frac{r}{2}} v(X)U + \lambda^{\frac{r}{2}} u(X)V + \lambda^{\frac{r}{2}} w(X)U \\ & + \lambda^{\frac{r}{2}} w(X)V + \lambda^{\frac{r}{2}} u(X)W + \lambda^{\frac{r}{2}} v(X)W)U \\ = {} & f^2X + \lambda^r u(X)U + \lambda^r v(X)V + \lambda^r w(X)W = \lambda^r X. \end{aligned}$$

Therefore, $F^2 = \lambda^r I$. Thus, F defines a Hsu-structure.
Also, by virtue of Eqs. (2.2), (2.3) and (3.1), we can easily verify that

$$\begin{gathered} FU = \lambda^{\frac{r}{2}} V, FV = \lambda^{\frac{r}{2}} U, FW = \lambda^{\frac{r}{2}} U \\ FW = \lambda^{\frac{r}{2}} V, FU = \lambda^{\frac{r}{2}} W, FV = \lambda^{\frac{r}{2}} W \end{gathered} \tag{3.2}$$

$$\begin{gathered} u \circ F = \lambda^{\frac{r}{2}} v, v \circ F = \lambda^{\frac{r}{2}} u, w \circ F = \lambda^{\frac{r}{2}} u, \\ u \circ F = \lambda^{\frac{r}{2}} w, v \circ F = \lambda^{\frac{r}{2}} w, w \circ F = \lambda^{\frac{r}{2}} v, \end{gathered} \tag{3.3}$$

Conversely, suppose that a manifold M admits a Hsu-structure F, a vector field U and a 1-form u such that

$$u(U) = 1, u(FU) = 0 \tag{3.4}$$

Let us define a vector field V, a 1-form v and a tensor field, f, as

$$\lambda^{\frac{r}{2}}V = FU, \lambda^{\frac{r}{2}}V = FW \tag{3.5}$$

$$\lambda^{\frac{r}{2}}v = u \circ F, \lambda^{\frac{r}{2}}v = u \circ F \tag{3.6}$$

$$f = F - \lambda^{\frac{r}{2}}v \bigotimes U - \lambda^{\frac{r}{2}}u \bigotimes V - \lambda^{\frac{r}{2}}w \bigotimes U - \lambda^{\frac{r}{2}}w \bigotimes V - \lambda^{\frac{r}{2}}v \bigotimes W - \lambda^{\frac{r}{2}}u \bigotimes W. \tag{3.7}$$

Now as a consequence of Eqs.(3.4), (3.5) and (3.6), we have

$$u(V) = 0, v(U) = 0, u(W) = 0, v(W) = 0, w(U) = 0,$$

$$w(V) = 0, u(U) = 1, v(V) = 1, w(W) = 1. \tag{3.8}$$

Also, in view of Eqs. (3.4), (3.6), (3.7) and (3.8), we have

$$fU = 0, fV = 0, fW = 0,$$

$$u \circ f = 0, v \circ f = 0, w \circ f = 0. \tag{3.9}$$

Further, by virtue of Eqs. (3.6), (3.7) and (3.8), we have

$$f^2X = f(fX) = f(fX - \lambda^{\frac{r}{2}}v(X)U - \lambda^{\frac{r}{2}}u(X)V - \lambda^{\frac{r}{2}}w(X)U - \lambda^{\frac{r}{2}}w(X)V - \lambda^{\frac{r}{2}}v(X)W$$

$$-\lambda^{\frac{r}{2}}u(X)W)$$

$$= F(FX) - \lambda^{\frac{r}{2}}v(X)U - \lambda^{\frac{r}{2}}u(X)V - \lambda^{\frac{r}{2}}w(X)U - \lambda^{\frac{r}{2}}w(X)V$$

$$-\lambda^{\frac{r}{2}}v(X)W - \lambda^{\frac{r}{2}}u(X)W$$

$$= F^2X - \lambda^{\frac{r}{2}}(v \circ F)(X)U - \lambda^{\frac{r}{2}}(u \circ F)(X)V - \lambda^{\frac{r}{2}}(w \circ F)(X)U$$

$$-\lambda^{\frac{r}{2}}(u \circ F)(X)W - \lambda^{\frac{r}{2}}(v \circ F)(X)W - \lambda^{\frac{r}{2}}(w \circ F)(X)V$$

$$= F^2X - \lambda^r u(X)U - \lambda^r v(X)V - \lambda^r w(X)W$$

$$= \lambda^r X - \lambda^r u(X)U - \lambda^r v(X)V - \lambda^r w(X)W$$

$$f^2/\lambda^r = I - u \bigotimes U - v \bigotimes V - w \bigotimes W \tag{3.10}$$

Equations (3.8), (3.9) and (3.10) show that M admits an f_λ -Hsu structure with complemented frames [f_λ , U , V , W , u , v , w] . Q.E.D.

4 Integrability conditions

In this section, we shall obtain the relation between the integrability of Hsu-structure F and that of the manifold having f_λ -Hsu structure with complemented frames.

Let $N^*(X,Y)$ be the Nijenhuis tensor formed with the help of F . Then we have

$$N^*(X,Y) = [FX, FY] - F[FX, Y] - [X, FY] + F^2[X, Y] \tag{4.1}$$

Now from Eqs.(2.1), (3.1) and (4.1), we obtain

$$\begin{aligned} N^*(X,Y) = {}& [fX + \lambda^{\frac{r}{2}}v(X)U + \lambda^{\frac{r}{2}}u(X)V + \lambda^{\frac{r}{2}}w(X)U + \lambda^{\frac{r}{2}}w(X)V + \lambda^{\frac{r}{2}}v(X)W + \lambda^{\frac{r}{2}}u(X)W, fY \\ &+\lambda^{\frac{r}{2}}v(Y)U + \lambda^{\frac{r}{2}}u(Y)V + \lambda^{\frac{r}{2}}w(Y)U + \lambda^{\frac{r}{2}}w(Y)V + \lambda^{\frac{r}{2}}v(Y)W + \lambda^{\frac{r}{2}}u(Y)W] - F[fX + \lambda^{\frac{r}{2}}v(X)U \\ &+\lambda^{\frac{r}{2}}u(x)V + \lambda^{\frac{r}{2}}w(X)U + \lambda^{\frac{r}{2}}w(X)V + \lambda^{\frac{r}{2}}v(X)W + \lambda^{\frac{r}{2}}u(X)W, Y] - F[X, fY + \lambda^{\frac{r}{2}}v(X)U \\ &+\lambda^{\frac{r}{2}}u(X)V + \lambda^{\frac{r}{2}}w(X)U + \lambda^{\frac{r}{2}}w(X)V + \lambda^{\frac{r}{2}}v(X)W + \lambda^{\frac{r}{2}}u(X)W] + F^2[X,Y] \end{aligned}$$

The above equation as a consequence of Eqs.(2.3), (2.5) and (2.23) reduces to

$$\begin{aligned} N^*(X,Y) = {}& N(X,Y) - \lambda^r(du)(X,Y)U - \lambda^r(dv)(X,Y)V - \lambda^r(dw)(X,Y)W + \lambda^{\frac{r}{2}}(dv\pi f)(X,Y)U \\ &-\lambda^{\frac{r}{2}}(du\pi f)(X,Y)V + \lambda^{\frac{r}{2}}(dw\pi f)(X,Y)U - \lambda^{\frac{r}{2}}(dw\pi f)(X,Y)V + \lambda^{\frac{r}{2}}(dv\pi f)(X,Y)W \\ &-\lambda^{\frac{r}{2}}(du\pi f)(X,Y)W + \lambda^{\frac{r}{2}}\{v(X)(\pounds_U f)Y - v(Y)(\pounds_U f)X + u(X)(\pounds_V f)Y - u(Y)(\pounds_V f)X \\ &+w(X)(\pounds_U f)Y - u(Y)(\pounds_U f)X + w(X)(\pounds_V f)Y - u(Y)(\pounds_V f)X + v(x)(\pounds_W f)Y - v(Y)(\pounds_W f)X\} \\ &-\lambda^r[v(X)(\pounds_U v)Y - v(Y)(\pounds_U v)X + u(X)(\pounds_V v)Y - u(Y)(\pounds_V v)X \\ &+w(X)(\pounds_V w)Y - w(Y)(\pounds_V w)X]U + \lambda^r[u(X)(\pounds_U u)Y - u(Y)(\pounds_U u)X \\ &+v(X)(\pounds_U u)Y - v(Y)(\pounds_U u)X + w(X)(\pounds_U w)Y - w(Y)(\pounds_U w)X]V - \lambda^r[u(X)v(Y) \\ &-u(Y)v(X) - u(X)w(Y) - v(Y)w(X) - v(X)w(Y) - v(Y)w(X)][U,V,W] \end{aligned} \tag{4.2}$$

Theorem 4.1. If an f_λ-Hsu structure with complemented $\{f_\lambda,\ U,\ V,\ W,\ u,\ v,\ w\}$ is normal, then Hsu structure defined by Eq.(3.1) is integrable.

Proof. If an f_λ -Hsu structure with complemented $\{f_\lambda,\ U,\ V,\ W,\ u,\ v,\ w\}$ is normal, then from definition (2.1), S^* is zero.

Thus by virtue of Equations (2.4), (2.14), (2.15), (2.16) and (4.2), we have $N^*(X,\ Y) = 0$.

Hence the Hsu structure F defined by Eq. (3.1) is integrable. Q.E.D.

5 Invariant submanifold

Let $\tilde{M}$ be an n dimensional $(1 < n < m)$ differentiable manifold of class C^∞ and suppose that $\tilde{M}$ is immersed in M by the immersion $I : \tilde{M} \to M$. Let us denote by B the differential d_i of the immersion i. Let us suppose that the vector field U is tangent $i(\tilde{M})$.Then any vector tangent to $i(\tilde{M})$ annihilates the 1-form v, w and the tangent space to $i(\tilde{M})$ is invariant by f. Therefore, we have

$$U = B\tilde{U}. \tag{5.1}$$

For a vector field $\tilde{U}$ of $\tilde{M}$

$$v(B, \tilde{X}) = 0, \tag{5.2}$$

For a vector field $\tilde{X}$ of $\tilde{M}$

$$f(B, \tilde{X}) = B\tilde{f}\tilde{X}, \tag{5.3}$$

for a tensor field $\tilde{f}$ of $\tilde{M}$ and an arbitrary vector field $\tilde{X}$ of $\tilde{M}$. For convenience, we call such a submanifold an invariant submanifold with respect to U and v. Similarly, we can define an invariant submanifold with respect to V and u.

Theorem 5.1. An invariant submanifold with respect to U and v of a manifold with f_λ-Hsu structure with complemented $\{f_\lambda, U, V, W, u, v, w\}$ admits $\{\tilde{f}_\lambda, \tilde{U}, \tilde{u}\}$ structure.

Proof. Let $\tilde{M}$ be an invariant submanifold with respect to U and v of a manifold M withf_λ -Hsu structure with complemented $\{f_\lambda, U, V, W, u, v, w\}$.

Now applying f to Eq. (5.1) and using Eqs. (2.2) and (5.3), we obtain

$$0 = fU = f(B\tilde{U}) = B\tilde{f}\tilde{U},$$

which gives

$$\tilde{f}\tilde{U} = 0. \tag{5.4}$$

Applying f to Eq. (5.3) and using Eq. (2.1), we get

$$\lambda^r(B\tilde{X}) - u(B\tilde{X})U - v(B\tilde{X})V = B\tilde{f}^2\tilde{X}. \tag{5.5}$$

Let us put

$$\tilde{u}(\tilde{X}) = u(B\tilde{X}), \tag{5.6}$$

then by virtue of Eqs. (5.1) (5.2) and (5.6), Eq. (5.5) yields

$$\tilde{f}^2\tilde{X} = \lambda^r(\tilde{X} - \tilde{u}(\tilde{X})\tilde{U}). \tag{5.7}$$

Also from Eq. (5.3), we have

$$u(f(B\tilde{X})) = u(B\tilde{f}\tilde{X}),$$

which in consequences of Eqs.(2.2) and (5.6) yields

$$\tilde{u}(\tilde{f}\tilde{X}) = 0. \tag{5.8}$$

Further from Eq.(5.1), we have

$$\tilde{u}(U) = \tilde{u}(B\tilde{U}),$$

which in view of Eqs.(2.3) and (5.6) gives

$$\tilde{u}(\tilde{U}) = 1. \tag{5.9}$$

Combining Eqs.(5.4), (5.7) and (5.9) , we have

$$f^2/\lambda^r = I - \tilde{u} \bigotimes \tilde{U},$$

$$\tilde{f}\tilde{U} = 0, \tilde{u} \circ \tilde{f} = 0, \tilde{u}(\tilde{U}) = 1. \tag{5.10}$$

We call a structure satisfying Eq.(5.10), $\{\tilde{f_\lambda},\tilde{U},\tilde{u}\ \}$ – structure. Q.E.D.

Theorem 5.2. An invariant submanifold with respect to V and u of a manifold with f_λ -Hsu structure with complemented [f_λ , U , V , W , u , v , w] admits [$\tilde{f_\lambda}$,$\tilde{V}$,$\tilde{v}$] structure.

Proof. The proof is similar to the proof of theorem 5.1. Q.E.D.

Theorem 5.3. An invariant submanifold with respect to W and u of a manifold with f_λ -Hsu structure with complemented [f_λ , U , V , W , u , v , w] admits [$\tilde{f_\lambda}$,$\tilde{W}$,$\tilde{w}$] structure.

Proof. The proof is similar to the proof of theorem 5.1. Q.E.D.

6 Invariant submanifolds of a normal f_λ -Hsu manifold with complemented frames

In this section we shall compute the expression S^* $(B\tilde{X},\ B\tilde{Y})$ for an invariant submanifold with respect to U, V and W.

By virtue of Eqs.(2.4), (2.5) and (5.3), we have

$$S^*(B\tilde{X}, B\tilde{Y}) = [fB\tilde{X}, fB\tilde{Y}] - f[fB\tilde{X}, B\tilde{Y}] - f[B\tilde{X}, fB\tilde{Y}] + f^2[B\tilde{X}, B\tilde{Y}]$$

$$-\lambda^r(du)(B\tilde{X}, B\tilde{Y})U - \lambda^r(dv)(B\tilde{X}, B\tilde{Y})V - \lambda^r(dw)(B\tilde{X}, B\tilde{Y})W$$

$$= [B\tilde{f}\tilde{X}, B\tilde{f}\tilde{Y}] - f[B\tilde{f}\tilde{X}, B\tilde{Y}] - f[B\tilde{X}, B\tilde{f}\tilde{Y}] + f^2[B\tilde{X}, B\tilde{Y}]$$

$$-\lambda^r(du)(B\tilde{X}, B\tilde{Y})U - \lambda^r(dv)(B\tilde{X}, B\tilde{Y})V - \lambda^r(dw)(B\tilde{X}, B\tilde{Y})W$$

But in view of Eqs.(5.1) and (5.6), we have

$$(du)(B\tilde{X}, B\tilde{Y}) = d(\tilde{u})(\tilde{X}, \tilde{Y}), (dv)(B\tilde{X}, B\tilde{Y}) = 0, (dw)(B\tilde{X}, B\tilde{Y}) = 0$$

$$S^*(B\tilde{X}, B\tilde{Y}) = B[\tilde{f}\tilde{X}, \tilde{f}\tilde{Y}] - fB[\tilde{f}\tilde{X}, \tilde{Y}] - fB[\tilde{X}, \tilde{f}\tilde{Y}] + f^2B[\tilde{X}, \tilde{Y}] - \lambda^r d\tilde{u}[\tilde{X}, \tilde{Y}]U. \tag{6.1}$$

Thus, in consequence of Eq.(5.1), Eq. (6.1) yields

$$S^*(B\tilde{X}, B\tilde{Y}) = B[[\tilde{f}\tilde{X}, \tilde{f}\tilde{Y}] - \tilde{f}[\tilde{f}\tilde{X}, \tilde{Y}] - \tilde{f}[\tilde{X}, \tilde{f}\tilde{Y}] + \tilde{f}^2[\tilde{X}, \tilde{Y}] - \lambda^r d\tilde{u}[\tilde{X}, \tilde{Y}]\tilde{U}] \tag{6.2}$$

Theorem 6.1. An invariant submanifold with respect to U, V and W of a manifold having normal f_λ-Hsu structure and complemented frames $\{f_\lambda,\ U,\ V,\ W,\ u,\ v,\ w\}$ admits a normal $\{\tilde{f}_\lambda, \tilde{U}, \tilde{u}\}$–structure.

Proof. If a f_λ -Hsu structure and complemented frames $[f_\lambda\, U, V, W, u, v, w]$ is normal then S^*=0. Therefore, form Eq. (6.2), we have

$$(\tilde{f}\tilde{X}, \tilde{f}\tilde{Y}) - \tilde{f}[\tilde{f}\tilde{X}, \tilde{Y}] - \tilde{f}[\tilde{X}, \tilde{f}\tilde{Y}] + \tilde{f}^2[\tilde{X}, \tilde{Y}] - \lambda^r d\tilde{u}[\tilde{X}, \tilde{Y}]\tilde{U} = 0. \tag{6.3}$$

Hence, in view Eq.(6.3) and using theorem (5.1) we have the result. Q.E.D.

Theorem 6.2. An invariant submanifold with respect to V and u of a manifold having normal f_λ -Hsu structure with complemented frames $[f_\lambda\ ,\ U\ ,\ V\ ,\ W\ ,\ u\ ,\ v\ ,\ w]$ admits $[\tilde{f}_\lambda, \tilde{V}, \tilde{v}\ ]$ structure.

Proof. The proof is similar to that of theorem (6.1). Q.E.D.

References

[1] S. I. Goldberg and K. Yano, *On normal globally framed f- manifolds*, Tohoku Math. Jour., 22 (1970) , 362-370.

[2] S. I. Goldberg and K. Yano, *Globally framed f-manifolds*, Minois Jour. of Math., 15 (1971), 456-474.

[3] M. D. Upadhyay and V. C. Gupta, *Integrability conditions of a structure f_λ satisfying* $f^3 - \lambda^2 f = 0$, Publications Mathematicae, 24(3-4),(1977) 249-255. .

[4] K. Yano, *Invariant submanifolds of an f-manifold with complemented frames*, Kodai Maths. Sem. Report 25,(1973) 163-174.