On the second radical elements of lattice modules

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Abstract

Let L be a C-lattice and M be a lattice module over L. For a non-zero element $N \in M$, join of all second elements X of M with $X \leq N$ is called the second radical of N, and it is denoted by $\sqrt[s]{N}$. In this paper, we study some properties of second radical of elements of M and obtain some related results.

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1 Introduction

A lattice L is said to be complete, if for any subset S of L, we have $\vee S, \wedge S \in L$. A complete lattice L with least element 0_L and greatest element 1_L is said to be a *multiplicative lattice*, if there is defined a binary operation " $\cdot$ " called multiplication on L satisfying the following conditions:

1. $a \cdot b = b \cdot a$, for all $a, b \in L$,
2. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, for all $a, b, c \in L$,
3. $a \cdot (\vee_\alpha b_\alpha) = \vee_\alpha (a \cdot b_\alpha)$, for all $a, b_\alpha \in L$,
4. $a \cdot 1_L = a$, for all $a \in L$.

Henceforth, $a \cdot b$ will be simply denoted by ab.

An element a in L is called *compact*, if $a \leq \bigvee_{\alpha \in I} b_\alpha$ (I is an indexed set) implies $a \leq b_{\alpha_1} \vee b_{\alpha_2} \vee \cdots \vee b_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of I. By a *C-lattice*, we mean a multiplicative lattice L with greatest element 1_L which is compact as well as multiplicative identity, that is generated under joins by a multiplicatively closed subset C of compact elements of L.

An element $m \in L$ said to be proper, if $m < 1_L$. A proper element m of L is said to be *maximal*, if for every $x \in L$ with $m < x \leq 1_L$ implies $x = 1_L$.

In [3], Alarcon et. al., defined the concept of the *radical* of an element $a \in L$ as, $\sqrt{a} = \vee\{x \in L : x^n \leq a$ for some natural number n$\}$. If $\sqrt{a} = a$, then an element a is called *radical* or *semiprime*. A proper element p of L is said to be *prime*, if $ab \leq p$ implies $a \leq p$ or $b \leq p$.

Thakare et.al.([12], [13]), studied the properties of radical of an element of multiplicative lattices and proved that, for $a \in L$, $\sqrt{a} = \wedge\{p \in L : p$ is prime and $a \leq p\}$.

A complete lattice M with smallest element 0_M and greatest element 1_M is said to be a *lattice module* over the multiplicative lattice L or L-module if there is a multiplication between elements of M and L, denoted by aN for $a \in L$ and $N \in M$, which satisfies the following properties:

1. $(ab)N = a(bN)$;

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2. $(\vee_\alpha a_\alpha)(\vee_\beta N_\beta) = \vee_{\alpha,\beta}(a_\alpha N_\beta)$;

3. $1_L N = N$;

4. $0_L N = 0_M$; for $a, b, a_\alpha \in L$ and for $N, N_\beta \in M$.

Let M be a lattice module over a multiplicative lattice L. For $N \in M$ and $b \in L$, denote $(N : b) = \vee\{X \in M : aX \leq N\}$. If $a, b \in L$, we write $(a : b) = \vee\{x \in L : bx \leq a\}$. If $A, B \in M$, then $(A : B) = \vee\{x \in L : xB \leq A\}$.

An element $N \in M$ is said to be *meet principal* (respectively *join principal*) if it satisfies the identity $A \wedge aN = (a \wedge (A : N))N$ (respectively $((aN \vee A) : N) = (a \vee (A : N))$ for all $a \in L$ and for all $A \in M$. Also, N is said to be *principal* if it is both join as well as meet principal. If each element of M is a join of principal elements of M, then M is called *principally generated*.

An element $N < 1_M$ of M is said to be *maximal* element if $N \leq B$ implies either $N = B$ or $B = 1_M$, $B \in M$.

In [2], Eaman A. Al-Khouja, defined the concept of *Jacobson radical* of a lattice module M as the intersection of the maximal elements of a lattice module M and denoted it by $J(M)$. An element $N < 1_M$ of M is said to be *prime* if $aX \leq N$ implies $X \leq N$ or $a1_M \leq N$, i.e., $a \leq (N : 1_M)$ for every $a \in L$ and $X \in M$.

Ballal and Kharat [5], unified various generalizations of prime and primary elements in multiplicative lattices and lattice modules as φ-absorbing elements and φ-absorbing primary elements.

Phadatare et. al. [10], introduced the concept of *second* elements of a lattice module as a generalization of second submodules of a module (see [14]). A non-zero element N of a lattice module M is said to be *second*, if for $a \in L$ either $aN = N$ or $aN = 0_M$.

In [1], Ansari-Toroghy and Farshadifar studied the dual notion of the concept of the prime radical of a submodule of a module and obtain some related results.

In this paper, we introduce second radical of an element of a lattice module M and study some properties of it as a generalization of the dual notion of the prime radical of a submodule.

Further all these concepts and for more information on multiplicative lattices and lattice modules, the reader may refer ([2], [4]-[13]).

2 The second radical

We begin this section with the definition of second element for a lattice module M over a C-lattice L due to Phadatare et. al.[10].

Definition 2.1. Let M be a lattice module over a C-lattice L. A non-zero element $N \in M$ is said to be *second*, if for $a \in L$ either $aN = N$ or $aN = 0_M$.

Lemma 2.2. *[10] Let M be a lattice module over a C-lattice L and $N \in M$. If N is second then $(0_M : N)$ is a prime element of L.*

If a non-zero element $S \in M$ is second and $(0_M : S) = p$ is a prime element of L then S is said to be $p-$*second*(see [10]).

Following is the result due to Johnson [9] which has been used in sequel.

Lemma 2.3. *[9] Let M be a lattice module over a C-lattice L. Then for $x \in L$ and $A, B, C \in M$, following holds:*

1. $x \leq (0_M : (0_M : x))$.
2. $A \leq (0_M : (0_M : A))$.
3. If $A \leq B$ then $(C : B) \leq (C : A)$.
4. $(0_M : A) = (0_M : (0_M : (0_M : A)))$.
5. $(A : B \vee C) = (A : B) \wedge (A : C)$.
6. $(A : B)B \leq A$.

Lemma 2.4. *Let M be a lattice module over a C-lattice L and S be a p-second element of M. If for $N, K \in M$, $S \leq N \vee K$ and $(0_M : N) \nleq p$, then $S \leq K$.*

Proof. Suppose that for $N, K \in M$, $S \leq N \vee K$, where S is a p-second element of M and $(0_M : N) \nleq p$. Then by Lemma 2.3(3), $(0_M : N \vee K) \leq (0_M : S)$. Therefore by Lemma 2.3(5), $(0_M : N)(0_M : K) \leq (0_M : N) \wedge (0_M : K) \leq (0_M : N \vee K) \leq (0_M : S) = p$. Since S is p-second, $(0_M : S) = p$ is prime, this implies $(0_M : N) \leq p$ or $(0_M : K) \leq p$. Note that, $(0_M : N) \nleq p$ therefore $(0_M : K) \leq p = (0_M : S)$ and so $S \leq K$ by Lemma 2.3(3). Q.E.D.

Theorem 2.5. *Let M be a lattice module over a C-lattice L and $S \in M$. If S is a p-second element of M with $S \leq (0_M : a) \vee N$, then $S \leq (0_M : a)$ or $S \leq N$, where $a \in L$ and $N \in M$.*

Proof. Suppose that for $a \in L$ and $N \in M$, $S \leq (0_M : a) \vee N$, where S is a p-second element of M. If $(0_M : (0_M : a)) \nleq p$, then $S \leq N$ by Lemma 2.4. Now, if $(0_M : (0_M : a)) \leq p$, then by Lemma 2.3(1), $a \leq (0_M : (0_M : a)) \leq p$ therefore $(0_M : p) \leq (0_M : a)$ by Lemma 2.3(3). Since S is $p-$second, $(0_M : S) = p$ therefore by Lemma 2.3(2), $S \leq (0_M : (0_M : S)) = (0_M : p)$ and so $S \leq (0_M : p) \leq (0_M : a)$, consequently, $S \leq (0_M : a)$. Q.E.D.

Callialp et. al.[8] introduced the concept of *comultiplication* lattice modules and also, investigated some properties of comultiplication lattice modules.

Definition 2.6. [8] Let M be a lattice module over a C-lattice L. Then M is said to be a *comultiplication* lattice module, if for each $N \in M$ there exists an element $a \in L$ such that $N = (0_M : a)$.

Lemma 2.7. *[8] Let M be a lattice module over a C-lattice L. Then M is a comultiplication lattice module if and only if $N = (0_M : (0_M : N))$ for each $N \in M$.*

Converse of Lemma 2.2 is true for comultiplication lattice module.

Lemma 2.8. *[8] Let M be a comultiplication lattice module over a C-lattice L and $N \in M$. Then N is second if and only if $(0_M : N)$ is a prime element of L.*

Lemma 2.9. *[8] Let M be a comultiplication lattice module over a C-lattice L. Then for $a \in L$ and $N \in M$, $(N : a) = ((0_M : a) : (0_M : N))$.*

Theorem 2.10. *Let M be a comultiplication lattice module over a C-lattice L and p be a prime element of L with $(0_M : 1_M) \leq p$, then $(0_M : p)$ is a second element of M.*

Proof. Suppose that M is a comultiplication lattice module over a C-lattice L and p is a prime element of L with $(0_M : 1_M) \leq p$. By Lemma 2.3(1), we have $p \leq (0_M : (0_M : p))$.
Now, suppose that $r \leq (0_M : (0_M : p))$, where $r \in L$. Then $(0_M : p) \leq (0_M : r)$ therefore $((0_M : p) : (0_M : p1_M)) \leq ((0_M : r) : (0_M : p1_M))$ and so $(p1_M : p) \leq (p1_M : r)$ by Lemma 2.9. Since $(p1_M : p) = 1_M$, we have $1_M = (p1_M : r)$ therefore $r1_M \leq p1_M$ and hence $r \leq p$, consequently, $(0_M : (0_M : p)) = p$. But p is prime, therefore by Lemma 2.8, $(0_M : p)$ is second. Q.E.D.

Denote the set of all second elements of M by $Spec^s(M)$. For $N \in M$, the *second radical* of N is denoted by $\sqrt[s]{N}$ and defined as, $\sqrt[s]{N} = \vee\{K \in Spec^s(M) | K \leq N\}$. If N does not contain any second element of M, then $\sqrt[s]{N} = 0_M$ and also, if $\sqrt[s]{N} = N$ then N is said to be *second radical element* of M.

Lemma 2.11. *Let M be a lattice module over a C-lattice L and $N, K \in M$. Then the following statements hold:*

1. $\sqrt[s]{N} \leq N$.
2. *If* $N \leq K$ *then* $\sqrt[s]{N} \leq \sqrt[s]{K}$.
3. $\sqrt[s]{\sqrt[s]{N}} \leq \sqrt[s]{N}$.
4. $\sqrt[s]{N} \vee \sqrt[s]{K} \leq \sqrt[s]{N \vee K}$.
5. $\sqrt[s]{N \wedge K} = \sqrt[s]{\sqrt[s]{N} \wedge \sqrt[s]{K}}$.
6. $\sqrt[s]{(0_M : a)} = \sqrt[s]{(0_M : \sqrt{a})}$ *for* $a \in L$.
7. $\sqrt[s]{N} \leq (0_M : \sqrt{(0_M : N)})$.
8. *If* $N \vee K = \sqrt[s]{N} \vee \sqrt[s]{K}$, *then* $\sqrt[s]{N \vee K} = N \vee K$.

Proof. 1) By definition, $\sqrt[s]{N} = \vee\{K \in Spec^s(M) | K \leq N\} \leq N$.
2) Follows from (1).
3) By definition $\sqrt[s]{\sqrt[s]{N}} = \sqrt[s]{(\vee\{K \in Spec^s(M) | K \leq N\})} =$
$\vee \{P \in Spec^s(M) | P \leq \vee\{K \in Spec^s(M) | K \leq N\}\} \leq \sqrt[s]{N}$.
4) Note that $\sqrt[s]{N}, \sqrt[s]{K} \leq \vee\{X \in Spec^s(M) | X \leq N \vee K\} = \sqrt[s]{N \vee K}$. Therefore $\sqrt[s]{N} \vee \sqrt[s]{K} \leq \sqrt[s]{N \vee K}$.
5) $\sqrt[s]{N \wedge K} = \vee\{X \in Spec^s(M) | X \leq N \wedge K\} = \vee\{X \in Spec^s(M) | (X \leq N) \wedge (X \leq K)\}$. Since X is second, $X = \sqrt[s]{X}$ therefore by (2), $\sqrt[s]{N \wedge K} = \vee\{X \in Spec^s(M) | (X \leq N) \wedge (X \leq K)\} = \vee\{X \in Spec^s(M) | (X \leq \sqrt[s]{N}) \wedge (X \leq \sqrt[s]{K})\} = \vee\{X \in Spec^s(M) | X \leq \sqrt[s]{N} \wedge \sqrt[s]{K}\} = \sqrt[s]{\sqrt[s]{N} \wedge \sqrt[s]{K}}$.
6) Suppose that $S \leq (0_M : a)$ for $S \in Spec^s(M)$. Then $a \leq (0_M : S)$. Since $(0_M : S)$ is prime, we have $\sqrt{a} \leq (0_M : S)$ therefore $S \leq \sqrt[s]{(0_M : \sqrt{a})}$ and so $\sqrt[s]{(0_M : a)} \leq \sqrt[s]{(0_M : \sqrt{a})}$.
Conversely, suppose that $P \leq (0_M : \sqrt{a})$ for $P \in Spec^s(M)$.
Then $a \leq \sqrt{a} \leq (0_M : P)$ therefore $P \leq (0_M : a)$. This implies that $\sqrt[s]{(0_M : \sqrt{a})} \leq \sqrt[s]{(0_M : a)}$, consequently, $\sqrt[s]{(0_M : a)} = \sqrt[s]{(0_M : \sqrt{a})}$.
7) By Lemma 2.3(2), we have $N \leq (0_M : (0_M : N))$ therefore by (1), $\sqrt[s]{N} \leq \sqrt[s]{(0_M : (0_M : N))}$ and so by (6), $\sqrt[s]{N} \leq \sqrt[s]{(0_M : \sqrt{(0_M : N)})}$. Again by (2), $\sqrt[s]{N} \leq \sqrt[s]{(0_M : \sqrt{(0_M : N)})} \leq (0_M :$

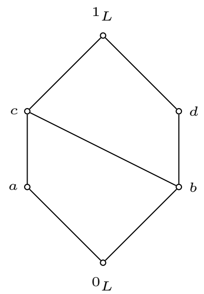
$\sqrt{(0_M : N)})$, consequently, $\sqrt[s]{N} \leq (0_M : \sqrt{(0_M : N)})$.

8) Suppose that for $N, K \in M$, $N \vee K = \sqrt[s]{N} \vee \sqrt[s]{K}$. Since $\sqrt[s]{N \vee K} \leq N \vee K$ by (1) and $\sqrt[s]{N} \vee \sqrt[s]{K} \leq \sqrt[s]{N \vee K}$ by (4), we have $\sqrt[s]{N \vee K} \leq N \vee K = \sqrt[s]{N} \vee \sqrt[s]{K} \leq \sqrt[s]{N \vee K}$, consequently, $\sqrt[s]{N \vee K} = N \vee K$. Q.E.D.

Definition 2.12. Let M be a lattice module over a $C-$lattice L. A non-zero element $K \neq 1_M$ of M is said to be *minimal*, whenever $0_M \leq N < K$ implies $N = 0_M$, $N \in M$.

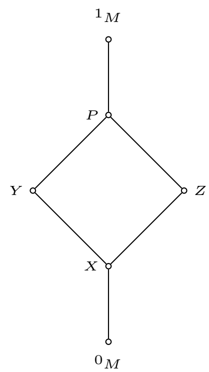
Note that, every minimal element of M is second. But the converse is not true in general.

Example 2.13. The lattice depicted in Fig.(a) is a multiplicative lattice L and the lattice depicted in Fig.(b) is a lattice module M over L. Note that, X is minimal and hence second but Y, Z and P are second but not minimal.



.	0_L	a	b	c	d	1_L
0_L	0_L	0_L	0_L	0_L	0_L	0_L
a	0_L	a	0_L	a	0_L	a
b	0_L	0_L	0_L	0_L	b	b
c	0_L	a	0_L	a	b	c
d	0_L	0_L	b	b	d	d
1_L	0_L	a	b	c	d	1_L

Fig.(a) Multiplicative lattice L



.	0_M	X	Y	Z	P	1_M
0_L	0_M	0_M	0_M	0_M	0_M	0_M
a	0_M	X	Y	Z	P	1_M
b	0_M	0_M	0_M	0_M	0_M	0_M
c	0_M	X	Y	Z	P	1_M
d	0_M	0_M	0_M	0_M	0_M	0_M
1_L	0_M	X	Y	Z	P	1_M

Fig.(b) Lattice module M over L

Theorem 2.14. *Let M be a lattice module over a C-lattice L with each non-zero element of M contains a minimal element. Then following statements hold.*

1. $\sqrt[s]{1_M} \neq 0_M$,*i.e., for $N \in M$, $\sqrt[s]{N} = 0_M$ if and only if $N = 0_M$.*
2. *For $N, K \in M$, $\sqrt[s]{N} \wedge \sqrt[s]{K} = 0_M$ if and only if $N \wedge K = 0_M$.*

Proof. 1) Since every minimal element is second and each non-zero element of M contains a minimal element, we have $\sqrt[s]{1_M} \neq 0_M$.
2) Suppose that for $N, K \in M$, $\sqrt[s]{N} \wedge \sqrt[s]{K} = 0_M$. Then by Lemma 2.11(5), we have $\sqrt[s]{N \wedge K} = \sqrt[s]{\sqrt[s]{N} \wedge \sqrt[s]{K}} = \sqrt[s]{0_M} = 0_M$, consequently, $N \wedge K = 0_M$ by (1). Conversely, suppose that $N \wedge K = 0_M$ for $N, K \in M$. Then by (1), $0_M = \sqrt[s]{N \wedge K}$. Therefore by Lemma 2.11(5), $\sqrt[s]{N \wedge K} = \sqrt[s]{\sqrt[s]{N} \wedge \sqrt[s]{K}} = 0_M$ and so $\sqrt[s]{N} \wedge \sqrt[s]{K} = 0_M$ by (1). Q.E.D.

Theorem 2.15. *Let M be a lattice module over a C-lattice L with each non-zero element of M contains a minimal element. If m is a maximal element of L and $\sqrt{(0_M : Q)} = m$ for non-zero $Q \in M$, then $\sqrt[s]{Q}$ is m-second.*

Proof. Suppose that for $0_M \neq Q \in M$, $\sqrt{(0_M : Q)} = m$, where m is a maximal element of L. By Lemma 2.11(7), we have $\sqrt[s]{Q} \leq (0_M : \sqrt{(0_M : Q)})$,
therefore $m = \sqrt{(0_M : Q)} \leq (0_M : \sqrt[s]{Q})$. Since m is maximal, either $(0_M : \sqrt[s]{Q}) = m$ or $(0_M : \sqrt[s]{Q}) = 1_L$. If $(0_M : \sqrt[s]{Q}) = 1_L$, then $\sqrt[s]{Q} = 0_M$ and so by Theorem 2.14(1), $Q = 0_M$, a contradiction, consequently, $(0_M : \sqrt[s]{Q}) = m$. Since m is maximal, $\sqrt[s]{Q}$ is minimal, indeed if $\sqrt[s]{Q}$ is not minimal, then there exists a minimal element K such that $K \leq \sqrt[s]{Q}$ and so by Lemma 2.3(3),

$m = (0_M : \sqrt[s]{Q}) \leq (0_M : K)$, a contradiction to maximality of m, consequently, $\sqrt[s]{Q}$ is minimal and hence is a second element of M. Q.E.D.

Lemma 2.16. *Let M be a comultiplication lattice module over a C-lattice L. Then for $N, K \in M$, $\sqrt[s]{N \vee K} = \sqrt[s]{N} \vee \sqrt[s]{K}$.*

Proof. By Lemma 2.11(4), $\sqrt[s]{N} \vee \sqrt[s]{K} \leq \sqrt[s]{N \vee K}$. Now, suppose that S is a second element of M with $S \leq N \vee K$, where $N, K \in M$. Since M is comultiplication, by Lemma 2.7 we have, $N = (0_M : (0_M : N))$, therefore $S \leq N \vee K = (0_M : (0_M : N)) \vee K$ and so by Theorem 2.5, either $S \leq (0_M : (0_M : N)) = N$ or $S \leq K$, consequently, $\sqrt[s]{N \vee K} \leq \sqrt[s]{N} \vee \sqrt[s]{K}$. Q.E.D.

Definition 2.17. Let M be a lattice module over a C-lattice L. Then the map $\psi^s : Spec^s(M) \to Spec(L/(0_M : 1_M))$ defined by $\psi^s(N) = \overline{(0_M : N)}$ is called the *natural map* of $Spec^s(M)$.

The following remark is immediate from Theorem 2.10.

Remark 2.18. Let M be a comultiplication lattice module over a C-lattice L. Then the natural map ψ^s is surjective.

Lemma 2.19. *Let M be a lattice module over a C-lattice L and the natural map ψ^s be surjective. Then $(0_M : (0_M : \sqrt{a})) = \sqrt{a}$, for $a \in L$ with $(0_M : 1_M) \leq a$.*

Proof. Suppose that the natural map ψ^s of $Spec^s(M)$ is surjective and $(0_M : 1_M) \leq a$ for $a \in L$. Then $(0_M : 1_M) \leq a \leq \sqrt{a} = \wedge p$, where p is prime element of L with $a \leq p$. Since ψ^s is surjective and $(0_M : 1_M) \leq p$, $p = (0_M : S)$ for $S \in Spec^s(M)$. Therefore $\sqrt{a} \leq (0_M : (0_M : \sqrt{a})) \leq (0_M : (0_M : \wedge p)) \leq \wedge(0_M : (0_M : p)) = (0_M : (0_M : (0_M : S)))$ by Lemma 2.3(1) and Lemma 2.3(3). But by Lemma 2.3(4), $(0_M : (0_M : (0_M : S))) = (0_M : S)$, therefore $\sqrt{a} \leq (0_M : (0_M : \sqrt{a})) \leq (0_M : (0_M : \wedge a)) \leq \wedge(0_M : (0_M : a)) = (0_M : (0_M : (0_M : S))) = \wedge(0_M : S) = \wedge a = \sqrt{a}$. Consequently, $(0_M : (0_M : \sqrt{a})) = \sqrt{a}$. Q.E.D.

A lattice module M over a multiplicative lattice L is said to be faithful, if $(0_M : 1_M) = 0_L$ (see [6]).

Theorem 2.20. *Let M be a faithful comultiplication lattice module over a C-lattice L and $a \in L$. Then $\sqrt[s]{(0_M : a)} = (0_M : \sqrt{a})$ if and only if $(0_M : \sqrt[s]{(0_M : a)}) = \sqrt{a}$.*

Proof. Suppose that $\sqrt[s]{(0_M : a)} = (0_M : \sqrt{a})$ where $a \in L$. Since M is faithful, we have $(0_M : 1_M) = 0_L \leq a$. Also, since M is comultiplication, by Remark 2.18, the natural map ψ^s is surjective, therefore by Lemma 2.19, $(0_M : (0_M : \sqrt{a})) = \sqrt{a}$ and hence $(0_M : \sqrt[s]{(0_M : a)}) = \sqrt{a}$. Conversely, suppose that $(0_M : \sqrt[s]{(0_M : a)}) = \sqrt{a}$. Since M is comultiplication, by Lemma 2.7, $\sqrt[s]{(0_M : a)} = (0_M : (0_M : \sqrt[s]{(0_M : a)})$ therefore $\sqrt[s]{(0_M : a)} = (0_M : (0_M : \sqrt[s]{(0_M : a)})) = (0_M : \sqrt{a})$. Q.E.D.

Theorem 2.21. *Let M be a faithful comultiplication lattice module over a C-lattice L. Then the following statements are equivalent.*

1. $\sqrt[s]{(0_M : a)} = (0_M : \sqrt{a})$ *for* $a \in L$.
2. $\sqrt[s]{N} = (0_M : \sqrt{(0_M : N)})$ *for* $N \in M$.
3. $(0_M : \sqrt[s]{N}) = \sqrt{(0_M : N)}$ *for* $N \in M$.

4. $(0_M : \sqrt[s]{(0_M : a)}) = \sqrt{a}$ *for* $a \in L$.

Proof. 1) $\Rightarrow$ 2) Since M is comultiplication, by Lemma 2.7, for $N \in M$, $N = (0_M : (0_M : N))$ therefore $\sqrt[s]{N} = \sqrt[s]{(0_M : (0_M : N))}$ and hence $\sqrt[s]{N} = \sqrt[s]{(0_M : (0_M : N))} = (0_M : \sqrt{(0_M : N)})$ by (1).
2) $\Rightarrow$ 3) Follows from Theorem 2.20.
3) $\Rightarrow$ 4) By Lemma 2.3(6), for $a \in L$, $(0_M : \sqrt[s]{(0_M : a)}) = (0_M : \sqrt[s]{(0_M : \sqrt{a})})$ therefore by (3), $(0_M : \sqrt[s]{(0_M : a)}) = \sqrt{(0_M : (0_M : \sqrt{a}))}$ and hence $(0_M : \sqrt[s]{(0_M : a)}) = \sqrt{(0_M : (0_M : \sqrt{a}))} = \sqrt{\sqrt{a}} = \sqrt{a}$ by Lemma 2.19.
4) $\Rightarrow$ 1) Suppose that $(0_M : \sqrt[s]{(0_M : a)}) = \sqrt{a}$, where $a \in L$. Since M is comultiplication, then by Lemma 2.7,
$\sqrt[s]{(0_M : a)} = (0_M : (0_M : \sqrt[s]{(0_M : a)})$, consequently, by (4) $\sqrt[s]{(0_M : a)} = (0_M : (0_M : \sqrt[s]{(0_M : a)})) = (0_M : \sqrt{a})$. Q.E.D.

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