Structural properties for (m, n)-quasi-hyperideals in ordered semihypergroups

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Abstract

In this paper, we first introduce the notion of an (m, n)-quasi-hyperideal in an ordered semihypergroup and, then, study some properties of (m, n)-quasi-hyperideals for any positive integers m and n. Thereafter, we characterize the minimality of an (m, n)-quasi-hyperideal in terms of $(m, 0)$-hyperideals and $(0, n)$-hyperideals respectively. The relation $\mathcal{Q}_m^n$ on an ordered semihypergroup is, then, introduced for any positive integers m and n and proved that the relation $\mathcal{Q}_m^n$ is contained in the relation $\mathcal{Q} = \mathcal{Q}_1^1$. We also show that, in an (m, n)-regular ordered semihypergroup, the relation $\mathcal{Q}_m^n$ coincides with the relation $\mathcal{Q}$. Finally, the notion of an (m, n)-quasi-hypersimple ordered semihypergroup is introduced and some properties of (m, n)-quasi-hypersimple ordered semihypergroups are studied. We further show that, on any (m, n)-quasi-hypersimple ordered semihypergroup, the relations $\mathcal{Q}_m^n$ and $\mathcal{Q}$ are equal and are universal relations.

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1 Introduction and preliminaries

In 1934, Marty [13] introduced the concept of a hyperstructure and defined a hypergroup which had been also studied by several authors. In classical algebraic structures, the multiplication of two elements is an element while in an algebraic hyperstructure the multiplication of two elements is a set. In [8], Heidari and Davvaz introduced the notion of an ordered semihypergroup as a generalization of the notion of an ordered semigroup. Many classical notion such as of an ideal, a quasi-ideal and a bi-ideal defined for an ordered semigroup as well as for a regular ordered semigroup had been generalized to an ordered semihypergroup.

It is worth noting that the notion of a quasi-ideal is a generalization of the notion of a one sided ideal. In 1953, Steinfeld introduced quasi-ideals for rings [19] and, in 1956, for semigroups [20]. These notions have been widely studied by several authors in different algebraic structures. The notion of an (m, n)-ideal for a semigroup was introduced by Lajos [9] (see [10, 11, 12] for related notions and results on (m, n)-ideals of semigroups). In [7], the notion of an (m, n)-quasi-hyperideal have been introduced and different characterizations and properties of an (m, n)-quasi-hyperideal and a minimal (m, n)-quasi-hyperideal have been obtained.

A hyperoperation on a non-empty set H is a map $\circ : H \times H \to \mathcal{P}^*(H)$ where $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\varnothing\}$ (the set of all non-empty subsets of H). In such a case, H is called a hypergroupoid. Let H be a hypergroupoid and $A,\ B$ be any non-empty subsets of H. Then $A \circ B$ is defined as follows:

$$A \circ B = \bigcup_{a \in A,\ b \in B} a \circ b.$$

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We shall write, in whatever follows, $A \circ x$ instead of $A \circ \{x\}$ and $x \circ A$ instead of $\{x\} \circ A$ for any $x \in H$. Also, for simplicity, throughout the paper, we shall write A^n for $A \circ A \circ \cdots \circ A$ ($n-$copies of A) for any $n \in \mathbb{Z}^+$. Moreover, the hypergroupoid H is called a semihypergroup if, for all $x, y, z \in H$,

$$(x \circ y) \circ z = x \circ (y \circ z)$$

i.e.,

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

A non-empty subset T of semihypergroup H is called a subsemihypergroup of H if $T \circ T \subseteq T$.

Let H be a non-empty set. Then the triplet $(H, \circ, \leq)$ is called an ordered semihypergroup if $(H, \circ)$ is a semihypergroup and $(H, \leq)$ is a partially ordered set such that

$$x \leq y \Rightarrow x \circ z \leq y \circ z \text{ and } z \circ x \leq z \circ y$$

for all $x, y, z \in H$. Here, if A and B are non-empty subsets of H, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. For more details of ordered semihypergroups, the reader is referred to [1, 2, 3, 4, 5, 6, 7, 8, 14, 15, 17, 18].

Let H be an ordered semihypergroup and A be a non-empty subset of H. Then A is called a left(right)-hyperideal [2] of H if

(1) $H \circ A \subseteq A(A \circ H \subseteq A)$; and

(2) $(A] \subseteq A$.

A is called a hyperideal of H if A is both a left-hyperideal and a right-hyperideal of H.

Let H be an ordered semihypergroup and let $(A] = \{x \in H \mid x \leq a$ for some $a \in A\}$ for any non-empty subset A of H. Then H is called a regular (a left-regular, a right-regular) [2] if for each $x \in H$, $x \in (x \circ H \circ x]$ $(x \in (H \circ x \circ x], x \in (x \circ x \circ H])$.

Lemma 1.1. [2] Let H be an ordered semihypergroup and A, B be any non-empty subsets of H. Then

(1) $A \subseteq (A]$;

(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $(A] \circ (B] \subseteq (A \circ B]$;

(4) $((A] \circ (B]] = (A \circ B]$;

(5) $(A] \cup (B] = (A \cup B]$.

2 (m, n)-quasi-hyperideals

Definition 2.1. Let H be an ordered semihypergroup and Q be a non-empty subset of H. Then Q is called a quasi-hyperideal of H if

(1) $(Q \circ H] \cap (H \circ Q] \subseteq Q$; and

(2) $(Q] \subseteq Q$.

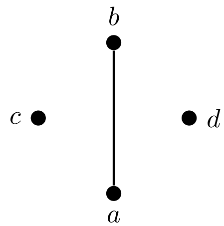
Example 2.2. Let $H = \{a,b,c,d\}$. Define hyperoperation $\circ$ and order $\leq$ on H as follows:

$\circ$	a	b	c	d
a	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
b	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
c	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{b\}$
d	$\{a,b\}$	$\{a,b\}$	$\{b\}$	$\{c\}$

$$\leq = \{(a,a),(b,b),(c,c),(d,d),(a,b)\}.$$

The covering relation $\prec$ and the figure of H are as follows;

$$\prec = \{(a,b)\}$$



Then H is an ordered semihypergroup. It is easy to verify that $Q = \{a,b\}$ is a quasi-hyperideal of H.

Definition 2.3. Let H be an ordered semihypergroup and m,n be any positive integers. Then a subsemihypergroup Q of H is called an (m,n)-quasi-hyperideal of H if

(1) $(Q^m \circ H] \cap (H \circ Q^n] \subseteq Q$; and

(2) $(Q] \subseteq Q$.

Clearly, every quasi-hyperideal of an ordered semihypergroup H is a $(1,1)$-quasi-hyperideal of H. It may also be easily verified that every quasi-hyperideal of an ordered semihypergroup H is an (m,n)-quasi-hyperideal of H for each positive integers m and n provided $(Q^m \circ H] \cap (H \circ Q^n]$ is non-empty, but the converse is not true in general as illustrated by the following examples.

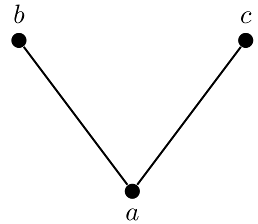
Example 2.4. Let $H = \{a,b,c,d\}$. Define hyperoperation $\circ$ and order $\leq$ on H as follows:

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a,b\}$
d	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a\}$

$$\leq = \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c)\}.$$

The covering relation $\prec$ and the figure of H are as follows;

$$\prec = \{(a,b),(a,c)\}$$



Then H is an ordered semihypergroup. It is easy to verify that the subset $A = \{a, d\}$ of H is an (m,n)-quasi-hyperideal of H for all integers $m, n \geq 2$, but not a quasi-hyperideal of H.

Every ordered semigroup can be regarded as an ordered semihypergroup [17]. In fact, if $(S, \cdot, \leq)$ is an ordered semigroup and we define $x \circ y = \{xy\}$ for all $a, b \in S$, then obviously $(S, \cdot, \leq)$ is an ordered semihypergroup.

Example 2.5. Let H be the set of all strictly upper triangular matrices over the set of all non-negative real numbers. Define operation $\circ$ and order relation $\leq$ on H by the usual matrix multiplication and

$$(a_{ij})_{4\times 4} \leq (b_{ij})_{4\times 4} \Leftrightarrow a_{ij} \leq b_{ij} \text{ for all } i, j,$$

with $(a_{ij})_{4\times 4}, (b_{ij})_{4\times 4} \in H$. Clearly, H is an ordered semihypergroup (ordered semigroup) with operation $\circ$ and order relation $\leq$. Let subset

$$Q = \left\{ \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{R}^+ \cup \{0\} \right\}.$$

Then Q is a non-empty subset of H. Clearly $(Q] \subseteq Q$ and

$$Q \circ Q = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\} \subseteq Q$$

$$(Q^m \circ H] \cap (H \circ Q^n] = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\} \subseteq Q \qquad (m, n \geq 2),$$

but

$$(Q \circ H] \cap (H \circ Q] = \left\{ \begin{bmatrix} 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid c \in \mathbb{R}^+ \cup \{0\} \right\} \nsubseteq Q.$$

Thus Q is an (m,n)-quasi-hyperideal of H for each positive integers $m, n \geq 2$, but is not a quasi-hyperideal of H.

Theorem 2.6. Let H be an ordered semihypergroup and $\{Q_i, i \in I\}$ be a family of (m,n)-quasi-hyperideals of H. If $\bigcap_{i\in I} Q_i \neq \varnothing$, then $\bigcap_{i\in I} Q_i$ is an (m,n)-quasi-hyperideal of H.

Proof. Assume that $\bigcap_{i\in I} Q_i \neq \varnothing$. Let $x, y \in \bigcap_{i\in I} Q_i$. Then, $x, y \in Q_i$ for each $i \in I$. As, for each $i \in I$, Q_i is (m,n)-quasi-hyperideal, $x \circ y \subseteq Q_i$. Therefore $x \circ y \subseteq \bigcap_{i\in I} Q_i$. Thus $\bigcap_{i\in I} Q_i$ is a subsemihypergroup of H. Next we show that $((\bigcap_{i\in I} Q_i)^m \circ H] \cap (H \circ (\bigcap_{i\in I} Q_i)^n] \subseteq \bigcap_{i\in I} Q_i$. Now

$$\begin{aligned}
&\big((\bigcap_{i\in I} Q_i\big)^m \circ H\big] \cap \big(H \circ \big(\bigcap_{i\in I} Q_i\big)^n\big] \\
\subseteq&((Q_i)^m \circ H] \cap (H \circ (Q_i)^n] \quad (\text{as } \bigcap_{i\in I} Q_i \subseteq Q_i, \forall i \in I) \\
\subseteq&Q_i \quad (\text{as } Q_i\text{'s are } (m,n)\text{-quasi-hyperideals}).
\end{aligned}$$

Thus, $\big(\big(\bigcap_{i\in I} Q_i\big)^m \circ H\big] \cap \big(H \circ \big(\bigcap_{i\in I} Q_i\big)^n\big] \subseteq \bigcap_{i\in I} Q_i$. Finally to show that $(\bigcap_{i\in I} Q_i] \subseteq \bigcap_{i\in I} Q_i$, take any $h \in (\bigcap_{i\in I} Q_i]$. Then $h \leq x$ for some $x \in \bigcap_{i\in I} Q_i$. As $x \in Q_i$ for each $i \in I$ and Q_i's are (m,n)-quasi-hyperideals, $h \in Q_i$ for each $i \in I$. Therefore $h \in \bigcap_{i\in I} Q_i$. Thus $(\bigcap_{i\in I} Q_i] \subseteq \bigcap_{i\in I} Q_i$, as required. Q.E.D.

Example 2.7. Let $H = [0,1]$. Then H, with hyperoperation defined by $x \circ y = xy$ and under the usual order relation, is an ordered semihypergroup (is also an ordered semigroup). Let $Q_n = [0, 1/n]$, where $n \in \mathbb{N}$. Clearly Q_n is an (m,n)-quasi-hyperideal of H for each $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} Q_n \neq \varnothing$. Also it is easy to check that $\bigcap_{n\in\mathbb{N}} Q_n$ is an (m,n)-quasi-hyperideal.

In the following example, the statement of the Theorem 2.6 has been verified for an ordered semihypergroup which is not an ordered semigroup.

Example 2.8. Let H be an ordered semihypergroup of Example 2.2. It is easy to check that all $Q_1 = \{a,b\}, Q_2 = \{a,b,c\}, Q_3 = \{a,b,d\}$ and $Q_4 = \{a,b,c,d\}$ are (m,n)-quasi-hyperideals of H for each positive integers m and n. Let $\mathcal{F} = \{Q_i \mid i \in \{1,2,3,4\}\}$. Then $\mathcal{F}$ is a family of (m,n)-quasi-hyperideals of H. Clearly $\bigcap_{n\in\{1,2,3,4\}} Q_i \neq \varnothing$. Also it is easy to check that $\bigcap_{n\in\{1,2,3,4\}} Q_i$ is an (m,n)-quasi-hyperideal of H.

Let H be an ordered semihypergroup and Q be any non-empty subset of H. Let $\mathcal{P} = \{J | J \text{ is an } (m,n)\text{-quasi-hyperideal of } H \text{ containing } Q\}$. As $H \in \mathcal{P}$, $\mathcal{P} \neq \varnothing$. Let $[Q]_{q(m,n)} = \bigcap_{J\in\mathcal{P}} J$. As $Q \subseteq J$ for each $J \in \mathcal{P}$, $\bigcap_{J\in\mathcal{P}} J (= [Q]_{q(m,n)}) \neq \varnothing$. By Theorem 2.6, $[Q]_{q(m,n)} (= \bigcap_{J\in\mathcal{P}} J)$ is an (m,n)-quasi-hyperideal of H containing Q. This (m,n)-quasi-hyperideal $[Q]_{q(m,n)}$ of H is called the (m,n)-quasi-hyperideal of H generated by Q.

Theorem 2.9. Let H be an ordered semihypergroup and Q be a non-empty subset of H. Then for each positive integers m and n

$$[Q]_{q(m,n)} = \big(\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H] \cap (H \circ Q^n]\big).$$

Proof. Clearly $(\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H] \cap (H \circ Q^n]) \neq \varnothing$. Also

$$\begin{aligned}
&\Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H] \cap (H \circ Q^n]))\Big) \circ \Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H] \\
&\cap (H \circ Q^n])\Big) \\
=&(\bigcup_{i=1}^{\max\{m,n\}} Q^i] \circ (\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (\bigcup_{i=1}^{\max\{m,n\}} Q^i] \circ ((Q^m \circ H] \cap (H \circ Q^n]) \\
\cup&((Q^m \circ H] \cap (H \circ Q^n]) \circ (\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H] \cap (H \circ Q^n]) \circ ((Q^m \circ H] \\
&\cap (H \circ Q^n]) \\
\subseteq&(\bigcup_{i=1}^{\max\{m,n\}} Q^i] \circ (\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (\bigcup_{i=1}^{\max\{m,n\}} Q^i] \circ (Q^m \circ H] \cup (H \circ Q^n] \\
\circ&(\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H] \circ (H \circ Q^n] \\
\subseteq&(\bigcup_{i=1}^{\max\{m,n\}} Q^i] \circ (\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H] \cap (H \circ Q^n]) \\
\subseteq&((\bigcup_{i=1}^{\max\{m,n\}} Q^i) \circ (\bigcup_{i=1}^{\max\{m,n\}} Q^i)] \cup ((Q^m \circ H] \cap (H \circ Q^n]). \qquad (1)
\end{aligned}$$

Let $x \in (\bigcup_{i=1}^{\max\{m,n\}} Q^i) \circ (\bigcup_{i=1}^{\max\{m,n\}} Q^i)$. Then, $x \in z_1 \circ z_2$ for some $z_1, z_2 \in \bigcup_{i=1}^{\max\{m,n\}} Q^i$. As $z_1, z_2 \in \bigcup_{i=1}^{\max\{m,n\}} Q^i$, $z_1 \in Q^p, z_2 \in Q^q$ for some $1 \leq p, q \leq \max\{m,n\}$. If $p + q \leq \max\{m,n\}$, then $z_1 \circ z_2 \subseteq \bigcup_{i=1}^{\max\{m,n\}} Q^i$; otherwise $z_1 \circ z_2 \subseteq Q^m \circ H$ and $z_1 \circ z_2 \subseteq H \circ Q^n$. Thus $z_1 \circ z_2 \subseteq \bigcup_{i=1}^{\max\{m,n\}} Q^i \cup (Q^m \circ H \cap H \circ Q^n)$. As $x \in z_1 \circ z_2$, $x \in \bigcup_{i=1}^{\max\{m,n\}} Q^i \cup (Q^m \circ H \cap H \circ Q^n)$. So, $(\bigcup_{i=1}^{\max\{m,n\}} Q^i) \circ (\bigcup_{i=1}^{\max\{m,n\}} Q^i) \subseteq \bigcup_{i=1}^{\max\{m,n\}} Q^i \cup (Q^m \circ H \cap H \circ Q^n)$. Therefore $((\bigcup_{i=1}^{\max\{m,n\}} Q^i) \circ (\bigcup_{i=1}^{\max\{m,n\}} Q^i)] \subseteq (\bigcup_{i=1}^{\max\{m,n\}} Q^i \cup (Q^m \circ H \cap H \circ Q^n)] = (\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H \cap H \circ Q^n)] \subseteq$

$((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H] \cap (H \circ Q^n])$. Now, from (1), $(\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H] \cap (H \circ Q^n]) \circ (\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H] \cap (H \circ Q^n]) \subseteq (\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H] \cap (H \circ Q^n])$. Hence $(\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H] \cap (H \circ Q^n])$ is a subsemihypergroup of H containing Q. Now

$$\begin{aligned}
&\Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H] \cap (H \circ Q^n])\Big)^m \circ H \\
\subseteq&\Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H]\Big)^m \circ H \\
=&\Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H]\Big)^{m-1} \circ \Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H]\Big) \circ H \\
\subseteq&\Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H]\Big)^{m-1} \circ \Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i \circ H] \cup (Q^m \circ H \circ H]\Big) \\
\subseteq&\Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H]\Big)^{m-1} \circ (Q \circ H] \\
=&\Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H]\Big)^{m-2} \circ \Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H]\Big) \circ (Q \circ H] \\
\subseteq&\Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H]\Big)^{m-2} \circ \Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i \circ Q \circ H] \cup (Q^m \circ H \circ Q \circ H]\Big) \\
\subseteq&\Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H]\Big)^{m-2} \circ (Q^2 \circ H] \\
&\vdots \\
=&(Q^m \circ H].
\end{aligned}$$

Similarly $H \circ \Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H] \cap (H \circ Q^n])\Big)^n \subseteq (H \circ Q^n]$. Therefore

$$(P^m \circ H] \cap (H \circ P^n] \subseteq (Q^m \circ H] \cap (H \circ Q^n] \subseteq P,$$

where $P = (\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H] \cap (H \circ Q^n])$. Also

$$\begin{aligned}
&\Big(\big(\bigcup_{i=1}^{\max\{m,n\}} Q^i\big] \cup \big((Q^m \circ H] \cap (H \circ Q^n]\big)\Big] \\
=&\Big(\big(\bigcup_{i=1}^{\max\{m,n\}} Q^i\big]\Big] \cup \Big(\big((Q^m \circ H] \cap (H \circ Q^n]\big)\Big] \\
\subseteq&\big(\bigcup_{i=1}^{\max\{m,n\}} Q^i\big] \cup \big(((Q^m \circ H]] \cap ((H \circ Q^n]]\big) \\
=&\big(\bigcup_{i=1}^{\max\{m,n\}} Q^i\big] \cup \big((Q^m \circ H] \cap (H \circ Q^n]\big).
\end{aligned}$$

Therefore, $(\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H] \cap (H \circ Q^n])$ is an (m,n)-quasi-hyperideal of H containing Q. So $[Q]_{q(m,n)} \subseteq (\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup \big((Q^m \circ H] \cap (H \circ Q^n]\big)$. For the reverse inclusion, take any $x \in (\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup \big((Q^m \circ H] \cap (H \circ Q^n]\big)$ i.e., $x \in (\bigcup_{i=1}^{\max\{m,n\}} Q^i]$ or $x \in \big((Q^m \circ H] \cap (H \circ Q^n]\big)$. If $x \in (\bigcup_{i=1}^{\max\{m,n\}} Q^i]$, then there exists $z_1 \in \bigcup_{i=1}^{\max\{m,n\}} Q^i$ such that $x \leq z_1$. As $z_1 \in \bigcup_{i=1}^{\max\{m,n\}} Q^i$, $z_1 \in Q^p$ for some $1 \leq p \leq \max\{m,n\}$. Therefore, $x \in [Q]_{q(m,n)}$. In the other case when $x \in \big((Q^m \circ H] \cap (H \circ Q^n]\big)$ i.e., $x \in (Q^m \circ H]$ and $x \in (H \circ Q^n]$. Thus there exist $z_2 \in Q^m \circ H$ and $z_3 \in H \circ Q^n$ such that $x \leq z_2$ and $x \leq z_3$. Now

$$Q^m \circ H \subseteq ([Q]_{m,n})^m \circ H \subseteq [Q]_{m,n}$$

and

$$H \circ Q^n \subseteq H \circ ([Q]_{m,n})^n \subseteq [Q]_{m,n}.$$

Therefore $z_2 \in [Q]_{q(m,n)}$ and $z_3 \in [Q]_{q(m,n)}$. Thus $x \in [Q]_{q(m,n)}$. Hence, $[Q]_{q(m,n)} = (\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H] \cap (H \circ Q^n])$, as required. Q.E.D.

Theorem 2.10. Let H be an ordered semihypergroup and Q be a non-empty subset of H. Then

(1) $(([Q]_{q(m,n)})^m \circ H] = (Q^m \circ H]$ for any positive integer m.

(2) $(H \circ ([Q]_{q(m,n)})^n] = (H \circ Q^n]$ for any positive integer n.

Proof. (1). We have

$$\begin{aligned}
&([Q]_{q(m,n)})^m \circ H \\
=&\Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H] \cap (H \circ Q^n]) \Big)^m \circ H \\
\subseteq&\Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H] \Big)^m \circ H \\
=&\Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H] \Big)^{m-1} \circ \Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H] \Big) \circ H \\
\subseteq&\Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H] \Big)^{m-1} \circ \Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i \circ H] \cup (Q^m \circ H \circ H] \Big) \\
\subseteq&\Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H] \Big)^{m-1} \circ (Q \circ H] \\
=&\Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H] \Big)^{m-2} \circ \Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H] \Big) \circ (Q \circ H] \\
\subseteq&\Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H] \Big)^{m-2} \circ \Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i \circ Q \circ H] \cup (Q^m \circ H \circ Q \circ H] \Big) \\
\subseteq&\Big((\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup (Q^m \circ H] \Big)^{m-2} \circ (Q^2 \circ H] \\
&\vdots \\
=&(Q^m \circ H].
\end{aligned}$$

Therefore $(([Q]_{q(m,n)})^m \circ H] \subseteq (Q^m \circ H]$. Reverse inclusion is obvious. Hence $(([Q]_{q(m,n)})^m \circ H] = (Q^m \circ H]$.

On the lines similar to the proof of (1), we may prove (2). Q.E.D.

Lemma 2.11. Let H be an ordered semihypergroup and m, n be positive integers. Then

$$\big[[a]_{q(m,n)}\big]_{q(m,n)} = [a]_{q(m,n)}$$

for each $a \in H$.

Proof. For any $a \in H$, we have

$$\big[[a]_{q(m,n)}\big]_{q(m,n)} = \big(\bigcup_{i=1}^{\max\{m,n\}} [a]^i_{q(m,n)}\big] \cup \big((([a]_{q(m,n)})^m \circ H] \cap (H \circ ([a]_{q(m,n)})^n]\big).$$

Now, by Theorem 2.10, $(([a]_{q(m,n)})^m \circ H] = (a^m \circ H]$ and $(H \circ ([a]_{q(m,n)})^n] = (H \circ a^n]$. So $\big((([a]_{q(m,n)})^m \circ H] \cap (H \circ ([a]_{q(m,n)})^n]\big) = (a^m \circ H] \cap (H \circ a^n]$. Therefore

$$\begin{aligned}\big[[a]_{q(m,n)}\big]_{q(m,n)} &= \Big(\bigcup_{i=1}^{\max\{m,n\}} [a]^i_{q(m,n)}\Big] \cup \big((([a]_{q(m,n)})^m \circ H] \cap (H \circ ([a]_{q(m,n)})^n]\big)\\ &= \Big(\bigcup_{i=1}^{\max\{m,n\}} [a]^i_{q(m,n)}\Big] \cup \big((a^m \circ H] \cap (H \circ a^n]\big). \qquad (2)\end{aligned}$$

As $[a]_{q(m,n)}$ is an (m,n)-quasi-hyperideal, $[a]_{q(m,n)} \circ [a]_{q(m,n)} \subseteq [a]_{q(m,n)}$. Therefore $([a]_{q(m,n)})^i \subseteq [a]_{q(m,n)}$ for each $i \in \{1, 2, ..., \max\{m,n\}\}$. Thus $\bigcup_{i=1}^{\max\{m,n\}} [a]^i_{q(m,n)} \subseteq [a]_{q(m,n)}$. Now, from (2), we have

$$\begin{aligned}\big[[a]_{q(m,n)}\big]_{q(m,n)} &= \Big(\bigcup_{i=1}^{\max\{m,n\}} [a]^i_{q(m,n)}\Big] \cup \big((a^m \circ H] \cap (H \circ a^n]\big)\\ &= \big([a]_{q(m,n)}\big] \cup \big((a^m \circ H] \cap (H \circ a^n]\big)\\ &= [a]_{q(m,n)} \cup \big((a^m \circ H] \cap (H \circ a^n]\big)\\ &= \Big(\bigcup_{i=1}^{\max\{m,n\}} a^i\Big] \cup \big((a^m \circ H] \cap (H \circ a^n]\big) \cup \big((a^m \circ H] \cap (H \circ a^n]\big)\\ &= \Big(\bigcup_{i=1}^{\max\{m,n\}} a^i\Big] \cup \big((a^m \circ H] \cap (H \circ a^n]\big)\\ &= [a]_{q(m,n)}.\end{aligned}$$

Hence $\big[[a]_{q(m,n)}\big]_{q(m,n)} = [a]_{q(m,n)}$, as required. Q.E.D.

Let H be an ordered semihypergroup and L be any subsemihypergroup of H. Then L is called an $(m,0)$-hyperideal of H if $L^m \circ H \subseteq L$ and $(L] \subseteq L$ for any positive integer m. Dually, if $H \circ R^n \subseteq R$ and $(R] \subseteq R$, then R is called a $(0,n)$-hyperideal of H, where n is any positive integer.

Theorem 2.12. Let H be an ordered semihypergroup. Then following conditions hold:

(1) Let $\{L_i, i \in I\}$ be a set of $(m,0)$-hyperideals of H. If $\bigcap_{i\in I} L_i \neq \varnothing$, then $\bigcap_{i\in I} L_i$ is an $(m,0)$-hyperideal of H.

(2) Let $\{R_i, i \in I\}$ be a set of $(0,n)$-hyperideals of H. If $\bigcap_{i\in I} R_i \neq \varnothing$, then $\bigcap_{i\in I} R_i$ is a $(0,n)$-hyperideal of H.

Proof. (1). Assume that $\bigcap_{i\in I} L_i \neq \varnothing$. Let $x, y \in \bigcap_{i\in I} L_i$. Then, $x, y \in L_i$ for each $i \in I$. As, each L_i $(i \in I)$ is an $(m,0)$-hyperideal, $x \circ y \subseteq L_i$ for each $i \in I$. Therefore $x \circ y \subseteq \bigcap_{i\in I} L_i$. Thus $\bigcap_{i\in I} L_i$ is a subsemihypergroup of H. Also

$$\begin{aligned}&\big(\bigcap_{i\in I} L_i\big)^m \circ H\\ \subseteq&(L_i)^m \circ H \quad (\text{as } \bigcap_{i\in I} L_i \subseteq L_i, \forall i \in I)\\ \subseteq&L_i \quad (\text{as } L_i\text{'s are } (m,0)\text{-hyperideals}).\end{aligned}$$

Finally to show that $(\bigcap_{i\in I} L_i] \subseteq \bigcap_{i\in I} L_i$, take any $h \in (\bigcap_{i\in I} L_i]$. Then $h \leq x$ for some $x \in \bigcap_{i\in I} L_i$. As $x \in L_i$ and each L_i is an $(m,0)$-hyperideal of H for each $i \in I$, $h \in L_i$ for each $i \in I$. Therefore $h \in \bigcap_{i\in I} L_i$. Thus $(\bigcap_{i\in I} L_i] \subseteq \bigcap_{i\in I} L_i$, as required.
(2). It may be proved similar to the proof of part (1). Q.E.D.

Let H be an ordered semihypergroup, A be any non-empty subset of H and let $\mathcal{P} = \{L | L \text{ is an } (m,0)\text{-hyperideal of } H \text{ containing } A\}$. As $H \in \mathcal{P}$, $\mathcal{P} \neq \varnothing$. Let $[A]_{(m,0)} = \bigcap_{J\in\mathcal{P}} J$. As $A \subseteq J$ for each $L \in \mathcal{P}$, $\bigcap_{L\in\mathcal{P}} L(= [A]_{m,0}) \neq \varnothing$. By Theorem 2.12, $[A]_{(m,0)}(= \bigcap_{L\in\mathcal{P}} L)$ is an $(m,0)$-hyperideal of H containing A.

The above $(m,0)$-hyperideal $[A]_{(m,0)}$ of H will be called, in the sequel, as the $(m,0)$-hyperideal of H generated by A. Analogously the $(0,n)$-hyperideal $[A]_{(0,n)}$ of H generated by the subset A of H may be defined.

Theorem 2.13. Let H be an ordered semihypergroup and let A be any non-empty subset of H. Then following conditions hold:

(1) $[A]_{m,0} = \big(\bigcup_{i=1}^{m} A^i \cup A^m \circ H]$;

(2) $[A]_{0,n} = \big(\bigcup_{i=1}^{n} A^i \cup H \circ A^n]$.

Proof. The proof follows on the lines similar to the proof of Theorem 2.9. Q.E.D.

Theorem 2.14. Let H be an ordered semihypergroup and A be a non-empty subset of H. Then

(1) $(([A]_{m,0})^m \circ H] = (A^m \circ H]$ for any positive integer m.

(2) $(H \circ ([A]_{(0,n)})^n] = (H \circ A^n]$ for any positive integer n.

Proof. The proof follows on the lines similar to the proof of Theorem 2.10. Q.E.D.

Lemma 2.15. Let H be an ordered semihypergroup and let R, L be an $(m,0)$-hyperideal and a $(0,n)$-hyperideal of H respectively. Then $L \cap R$ is an (m,n)-quasi-hyperideal of H.

Proof. Let R, L be an $(m,0)$-hyperideal and a $(0,n)$-hyperideal of H respectively. Then $L \cap R \neq \varnothing$ because $(R^m \circ L^n] \subseteq (R^m \circ H] \cap (H \circ L^n] \subseteq (R] \cap (L] = R \cap L$. Now we have

$$(L \cap R) \circ (L \cap R) \subseteq L^2 \cap R^2 \subseteq L \cap R$$

and

$$((L\cap R)^m\circ H]\cap(H\circ(L\cap R)^n]\subseteq(R^m\circ H]\cap(H\circ L^n]\subseteq R\cap L.$$

Hence $R\cap L$ is an (m,n)-quasi-hyperideal. Q.E.D.

The following lemma easily follows.

Lemma 2.16. Let H be an ordered semihypergroup, $a\in H$ and m,n be positive integers. Then the following conditions hold:

(1) $(a^m\circ H]$ is an $(m,0)$-hyperideal of H;

(2) $(H\circ a^n]$ is a $(0,n)$-hyperideal of H;

(3) $(a^m\circ H]\cap(H\circ a^n]$ is an (m,n)-quasi-hyperideal of H.

Theorem 2.17. Let H be an ordered semihypergroup and Q be an (m,n)-quasi-hyperideal of H. Then

(1) For each $a\in H$, $[a]_{q(m,n)}=[a]_{m,0}\cap[a]_{0,n}$; and

(2) $Q=[Q]_{m,0}\cap[Q]_{0,n}$.

Proof. (1). Let $a\in H$. As $a\in[a]_{m,0}\cap[a]_{0,n}$ and, by Lemma 2.15, $[a]_{m,0}\cap[a]_{0,n}$ is an (m,n)-quasi-hyperideal of H, $[a]_{q(m,n)}\subseteq[a]_{m,0}\cap[a]_{0,n}$. On the other hand, we have

$$\begin{aligned}[a]_{m,0}\cap[a]_{0,n}&=(\bigcup_{i=1}^{m}a^i\cup a^m\circ H]\cap(\bigcup_{i=1}^{n}a^i\cup H\circ a^n]\\&\subseteq(\bigcup_{i=1}^{m}([a]_{q(m,n)})^i\cup a^m\circ H]\cap(\bigcup_{i=1}^{n}([a]_{q(m,n)})^i\cup H\circ a^n]\\&\subseteq([a]_{q(m,n)}\cup a^m\circ H]\cap([a]_{q(m,n)}\cup H\circ a^n]\\&=([a]_{q(m,n)}]\cup((a^m\circ H]\cap(H\circ a^n])\\&=[a]_{q(m,n)}\cup((a^m\circ H]\cap(H\circ a^n])\\&=[a]_{q(m,n)}.\end{aligned}$$

Therefore $[a]_{q(m,n)}=[a]_{m,0}\cap[a]_{0,n}$, as required.
(2). As $Q\subseteq(\bigcup_{i=1}^{m}Q^i\cup Q^m\circ H]=[Q]_{m,0}$ and $Q\subseteq(\bigcup_{i=1}^{n}Q^i\cup H\circ Q^n]=[Q]_{0,n}$, $Q\subseteq[Q]_{m,0}\cap[Q]_{0,n}$. To prove the reverse inclusion, take any $x\in[Q]_{m,0}\cap[Q]_{0,n}\subseteq(\bigcup_{i=1}^{m}Q^i\cup Q^m\circ H]\cap(\bigcup_{i=1}^{n}Q^i\cup H\circ Q^n]$. As Q is an (m,n)-quasi-hyperideal of H, $Q^2\subseteq Q$ which implies that $Q^k\subseteq Q$ for each positive integers k. Therefore $\bigcup_{i=1}^{m}Q^i\subseteq Q$ and $\bigcup_{i=1}^{n}Q^i\subseteq Q$. Thus $x\in(Q\cup Q^m\circ H]\cap(Q\cup H\circ Q^n]=(Q]\cup((Q^m\circ H]\cap(H\circ Q^n])=Q$. Hence $Q=[Q]_{m,0}\cap[Q]_{0,n}$, as required. Q.E.D.

Remark 2.18. By the above Theorem, every (m,n)-quasi-hyperideal of H can be expressed as the intersection of an $(m,0)$-hyperideal and a $(0,n)$-hyperideal of H.

3 Minimality of (m,n)-quasi-hyperideals

Let H be an ordered semihypergroup, m,n be any positive integers and Q be an (m,n)-quasi-hyperideal of H. Then, Q is said to be a minimal (m,n)-quasi-hyperideal of H if for every (m,n)-quasi-hyperideal Q' of H, $Q' \subseteq Q$ implies $Q' = Q$.

Similarly a minimal $(m,0)$-hyperideal and a minimal $(0,n)$-hyperideal of H are defined.

Theorem 3.1. Let H be an ordered semihypergroup. Then

(1) An $(m,0)$-hyperideal R is minimal if and only if $(a^m \circ H] = R$ $(\forall a \in R)$, where m is any positive integer;

(2) An $(0,n)$-hyperideal L is minimal if and only if $(H \circ a^n] = L$ $(\forall a \in L)$, where n is any positive integer;

(3) An (m,n)-quasi-hyperideal Q is minimal if and only if $(a^m \circ H] \cap (H \circ a^n] = Q$ $(\forall a \in Q)$, where m,n are any positive integers.

Proof. Let R be a minimal $(m,0)$-hyperideal of H and $a \in R$. Then $(a^m \circ H] \subseteq (R^m \circ H] \subseteq (R] = R$. As $(a^m \circ H]$ is an $(m,0)$-hyperideal of H, so, by minimality of R, $(a^m \circ H] = R$.

Conversely assume that $(a^m \circ H] = R$ for each element $a \in R$. Let R' be any $(m,0)$-hyperideal of H such that $R' \subseteq R$. Take any $x \in R'$. Then $x \in R$. By hypothesis, $R = (x^m \circ H] \subseteq (R' \circ H] \subseteq (R'] = R'$. Thus $R' = R$. Hence R is a minimal $(m,0)$-hyperideal.

Similarly we may prove (2) and (3). Q.E.D.

Theorem 3.2. Let H be an ordered semihypergroup and Q be an (m,n)-quasi-hyperideal of H. Then Q is a minimal (m,n)-quasi-hyperideal of H if and only if Q is the intersection of a minimal $(m,0)$-hyperideal and a minimal $(0,n)$-hyperideal of H.

Proof. Let Q be any minimal (m,n)-quasi-hyperideal of H and $x \in Q$. As $(x^m \circ H]$ and $(H \circ x^n]$ are $(m,0)$-hyperideal and $(0,n)$-hyperideal of H respectively, by Lemma 2.16, $(x^m \circ H] \cap (H \circ x^n]$ is an (m,n)-quasi-hyperideal of H. Since $(x^m \circ H] \cap (H \circ x^n] \subseteq (Q^m \circ H] \cap (H \circ Q^n] \subseteq Q$, so, by minimality of Q, $(x^m \circ H] \cap (H \circ x^n] = Q$. Now to complete the proof, it is sufficient to show that $(x^m \circ H]$ and $(H \circ x^n]$ are minimal $(m,0)$-hyperideal and minimal $(0,n)$-hyperideal of H respectively. To show that $(x^m \circ H]$ is a minimal $(m,0)$-hyperideal of H, take any $(m,0)$-hyperideal R of H such that $R \subseteq (x^m \circ H]$. Then $R \cap (H \circ x^n] \subseteq (x^m \circ H] \cap (H \circ x^n] = Q$. Since $R \cap (H \circ x^n]$ is an (m,n)-quasi-hyperideal of H and Q is a minimal $(m,0)$-quasi-hyperideal of H, $R \cap (H \circ x^n] = Q$. Therefore $Q \subseteq R$. Now $(x^m \circ H] \subseteq (Q^m \circ H] \subseteq (R^m \circ H] \subseteq R$. Thus $(x^m \circ H] = R$; i.e., $(x^m \circ H]$ is a minimal $(m,0)$-hyperideal of H. Similarly we may show that $(H \circ x^n]$ is a minimal $(0,n)$-hyperideal of H.

Conversely assume that $Q = L \cap R$ for some minimal $(0,n)$-hyperideal L and minimal $(m,0)$-hyperideal R of H. Therefore $Q \subseteq L$ and $Q \subseteq R$. Let Q' be any (m,n)-quasi-ideal of H such that $Q' \subseteq Q$. Then $(Q'^m \circ H] \subseteq (Q^m \circ H] \subseteq (R^m \circ H] \subseteq R$ and $(H \circ Q'^n] \subseteq (H \circ Q^n] \subseteq (H \circ L^n] \subseteq L$. As $(Q'^m \circ H]$ and $(H \circ Q'^n]$ are $(m,0)$-hyperideal and $(0,n)$-hyperideal of H respectively, by minimality of L and R, $(Q'^m \circ H] = R$ and $(H \circ Q'^n] = L$. Therefore $Q = R \cap L = (Q'^m \circ H] \cap (H \circ Q'^n] \subseteq Q'$. So $Q' = Q$. Hence Q is a minimal (m,n)-quasi-hyperideal of H. Q.E.D.

Corollary 3.3. Let H be an ordered semihypergroup. Then H has at least one minimal (m,n)-quasi-hyperideal if and only if H has at least one minimal $(m,0)$-hyperideal and at least one minimal $(0,n)$-hyperideal of H.

Lemma 3.4. Let H be an ordered semihypergroup and Q be an (m,n)-quasi-hyperideal of H. Then Q is a minimal (m,n)-quasi-hyperideal of H if and only if $[x]_{q(m,n)} = [y]_{q(m,n)}$ for each $x, y \in Q$.

Proof. Let Q be a minimal (m,n)-quasi-hyperideal of H and $x, y \in Q$. Now $[x]_{q(m,n)} = (\bigcup_{i=1}^{\max\{m,n\}} x^i] \cup ((x^m \circ H] \cap (H \circ x^n]) \subseteq (\bigcup_{i=1}^{\max\{m,n\}} Q^i] \cup ((Q^m \circ H] \cap (H \circ Q^n]) = Q$. As Q is minimal (m,n)-quasi-hyperideal and $[x]_{q(m,n)}$ is an (m,n)-quasi-hyperideal of H, $[x]_{q(m,n)} = Q$. Similarly $[y]_{q(m,n)} = Q$. Hence $[x]_{q(m,n)} = [y]_{q(m,n)}$.

Conversely assume that $[x]_{q(m,n)} = [y]_{q(m,n)}$ for each $x, y \in Q$. Let Q' be any (m,n)-quasi-hyperideal of H such that $Q' \subseteq Q$. Let $x \in Q'$. Then for each $y \in Q$, $[x]_{q(m,n)} = [y]_{q(m,n)}$. As $y \in [y]_{q(m,n)}$ and $[x]_{q(m,n)} \subseteq Q'$, $y \in Q'$. Hence Q is a minimal (m,n)-quasi-hyperideal of H. Q.E.D.

Let H be an ordered semihypergroup and m, n be positive integers. Define a relation $\mathcal{Q}_m^n$ on H as follows:

$$\mathcal{Q}_m^n = \{(x,y) \in H \times H \mid [x]_{q(m,n)} = [y]_{q(m,n)}\}.$$

In particular when $m = 1 = n$,

$$\mathcal{Q}_1^1 = \{(x,y) \in H \times H \mid Q(x) = Q(y)\}.$$

Clearly $\mathcal{Q}_m^n$ and $\mathcal{Q} = \mathcal{Q}_1^1$ are equivalence relations on H.

Theorem 3.5. Let H be an ordered semihypergroup and Q be an (m,n)-quasi-hyperideal of H. Then Q is a minimal (m,n)-quasi-hyperideal of H if and only if it is a $\mathcal{Q}_m^n$-class.

Proof. Let Q be a minimal (m,n)-quasi-hyperideal of H and $x, y \in Q$. Then, by Lemma 3.4, $[x]_{q(m,n)} = [y]_{q(m,n)}$. Thus $(x,y) \in \mathcal{Q}_m^n$. Therefore Q is a $\mathcal{Q}_m^n$-class.

Conversely assume that Q is a $\mathcal{Q}_m^n$-class. Let Q' be (m,n)-quasi-hyperideal of H such that $Q' \subseteq Q$. Let $x \in Q$ and $y \in Q'$. Then, by hypothesis, $[x]_{q(m,n)} = [y]_{q(m,n)}$. As $x \in [x]_{q(m,n)}$ and $[y]_{q(m,n)} \subseteq Q'$, $x \in Q'$. Thus $Q' = Q$, as required. Q.E.D.

Lemma 3.6. Let H be an order semihypergroup and $a, b \in H$. If a and b are $\mathcal{Q}_m^n$-related, then, $(a^m \circ H] = (b^m \circ H]$, $(H \circ a^n] = (H \circ b^n]$ and $(a^m \circ H \circ a^n] = (b^m \circ H \circ b^n]$.

Proof. Suppose that $(a,b) \in \mathcal{Q}_m^n$. Then, by definition, $[a]_{q(m,n)} = [b]_{q(m,n)}$. Therefore $\{a\} \subseteq [b]_{q(m,n)}$ and $\{b\} \subseteq [b]_{q(m,n)}$. Thus, by Theorem 2.10, $(a^m \circ H] \subseteq (([b]_{q(m,n)})^m \circ H] = (b^m \circ H]$. Similarly, as $\{b\} \subseteq [a]_{q(m,n)}$, we have $(b^m \circ H] \subseteq (a^m \circ H]$. Hence, $(a^m \circ H] = (b^m \circ H]$. Similarly, $(H \circ a^n] = (H \circ b^n]$. So $(a^m \circ H \circ a^n] = ((a^m \circ H] \circ a^n] = ((b^m \circ H] \circ a^n] = (b^m \circ H \circ a^n] = (b^m \circ (H \circ a^n]] = (b^m \circ (H \circ b^n]] = (b^m \circ H \circ b^n]$, as required. Q.E.D.

Corollary 3.7. Let H be an order semihypergroup and $a, b \in H$ are $\mathcal{Q}$-related. Then $(a \circ H] = (b \circ H]$, $(H \circ a] = (H \circ b]$ and $(a \circ H \circ a] = (b \circ H \circ b]$.

Definition 3.8. Let H be an ordered semihypergroup and m, n be non-negative integers. An element $a \in H$ is said to be an (m,n)-regular element if $a \in (a^m \circ H \circ a^n]$. The ordered semihypergroup H is said to be (m,n)-regular if each element of H is (m,n)-regular or equivalently for each subset A of H, $A \subseteq (A^m \circ H \circ A^n]$. Here, $A^0 \circ H = H \circ A^0 = H$.

Remark 3.9. Let H be an (m,n)-regular ordered semihypergroup, m,n be positive integers and A be any subset of H. Then

(1) $[A]_{m,0} = (A^m \circ H]$.

(2) $[A]_{0,n} = (H \circ A^n]$.

(3) $[A]_{q(m,n)} = (A^m \circ H] \cap (H \circ A^n]$.

Theorem 3.10. Let H be an ordered semihypergroup and m,n be positive integers. Then

(1) $\mathcal{Q}_m^n \subseteq \mathcal{Q}$.

(2) If H is (m,n)-regular, then $\mathcal{Q}_m^n = \mathcal{Q}$.

Proof. (1). Let $(x,y) \in \mathcal{Q}_m^n$. Then $[x]_{q(m,n)} = [y]_{q(m,n)}$. So $\{x\} \subseteq [y]_{q(m,n)}$ and $\{y\} \subseteq [x]_{q(m,n)}$. Also $\{x\} \subseteq [y]_{m,n} = (\bigcup_{i=1}^{\max\{m,n\}} y^i] \cup ((y^m \circ H] \cap (H \circ y^n]) \subseteq \{y\} \cup ((y \circ H] \cap (H \circ y])$ and $\{y\} \subseteq [x]_{m,n} = (\bigcup_{i=1}^{\max\{m,n\}} x^i] \cup ((x^m \circ H] \cap (H \circ x^n]) \subseteq \{x\} \cup ((x \circ H] \cap (H \circ x])$. Thus

$$\begin{aligned}(x \circ H] &\subseteq ((y \cup ((y \circ H] \cap (H \circ y])) \circ H] \\ &= (y \circ H \cup \big(((y \circ H] \cap (H \circ y]) \circ H\big)] \\ &\subseteq (y \circ H].\end{aligned}$$

Similarly $(y \circ H] \subseteq (x \circ H]$. Therefore $(x \circ H] = (y \circ H]$. Similarly $(H \circ x] = (H \circ y]$. Now

$$\begin{aligned}Q(x) &= (x] \cup ((x \circ H] \cap (H \circ x]) \\ &\subseteq (y \cup ((y \circ H] \cap (H \circ y])] \cup ((x \circ H] \cap (H \circ x]) \\ &\subseteq (y] \cup (((y \circ H] \cap (H \circ y])] \cup ((x \circ H] \cap (H \circ x]) \\ &\subseteq (y] \cup ((y \circ H] \cap (H \circ y]) \qquad (\because (x \circ H] = (y \circ H], (H \circ x] = (H \circ y]) \\ &= Q(y).\end{aligned}$$

Similarly we may show that $Q(y) \subseteq Q(x)$. Thus $Q(x) = Q(y)$. Hence $(x,y) \in \mathcal{Q}$.

(2). Let H be an (m,n)-regular ordered semihypergroup and $(x,y) \in \mathcal{Q}$. Then $Q(x) = Q(y)$. So $x \in (y] \cup ((y \circ H] \cap (H \circ y])$ and $y \in (x] \cup ((x \circ H] \cap (H \circ x])$. As H is (m,n)-regular, $(y \circ H] \cap (H \circ y] \subseteq ((y^m \circ H \circ y^n] \circ H] \cap (H \circ (y^m \circ H \circ y^n]] \subseteq (y^m \circ H] \cap (H \circ y^n] \subseteq (y \circ H] \cap (H \circ y]$ and $(x \circ H] \cap (H \circ x] \subseteq ((x^m \circ H \circ x^n] \circ H] \cap (H \circ (x^m \circ H \circ x^n]] \subseteq (x^m \circ H] \cap (H \circ x^n] \subseteq (x \circ H] \cap (H \circ x]$. So $(y \circ H] \cap (H \circ y] = (y^m \circ H] \cap (H \circ y^n]$ and $(x \circ H] \cap (H \circ x] = (x^m \circ H] \cap (H \circ x^n]$. As H is (m,n)-regular, by Remark 3.9, $[x]_{q(m,n)} = (x^m \circ H] \cap (H \circ x^n]$ and $[y]_{q(m,n)} = (y^m \circ H] \cap (H \circ y^n]$. Thus, by Corollary 3.7, $(x \circ H] = (y \circ H], (H \circ x] = (H \circ y]$. Therefore $(x \circ H] \cap (x \circ H] = (y \circ H] \cap (y \circ H]$. This implies that $[x]_{q(m,n)} = [y]_{q(m,n)}$. So $(x,y) \in \mathcal{Q}_m^n$. Therefore $\mathcal{Q} \subseteq \mathcal{Q}_m^n$, and, hence, by Part (1), $\mathcal{Q}_m^n = \mathcal{Q}$, as required. Q.E.D.

Lemma 3.11. Let H be an ordered semihypergroup. If the sets of all $(m,0)$-hyperideals, $(0,n)$-hyperideals and (m,n)-hyperideals are denoted by $I_{(m,0)}, I_{(0,n)}$ and $I_{(m,n)}$ respectively, then

(1) H is $(m,0)$-regular if and only if $R=(R^m \circ H]$ $(\forall R \in I_{(m,0)})$, where m is any positive integer;

(2) H is $(0,n)$-regular if and only if $L=(H \circ L^n]$ $(\forall L \in I_{(0,n)})$, where n is any positive integer.

Proof. (1) Let H is $(m,0)$-regular ordered semihypergroup and R be any $(m,0)$-hyperideal of H. Then, by definition of $(m,0)$-regularity, we have $R \subseteq (R^m \circ H]$ and by definition of $(m,0)$-hyperideal, we have $(R^m \circ H] \subseteq (R] = R$. Hence $R=(R^m \circ H]$.

For the converse, assume that $R=(R^m \circ H]$ for each $R \in I_{(m,0)}$. Take any $a \in H$, so $[a]_{m,0} \in I_{(m,0)}$. From Theorem 2.14 and by assumption, $[a]_{m,0} = (([a]_{m,0})^m \circ H] = (a^m \circ H]$. As $\{a\} \subseteq [a]_{m,0}$, $a \in (a^m \circ H]$. Hence H is $(m,0)$-regular.

(2) On the similar lines to the proof of (1).

Q.E.D.

Theorem 3.12. Let H be an ordered semihypergroup and $m,\ n$ be non-negative integers. Then the following are equivalent:

(1) H is (m,n)-regular;

(2) $Q=(Q^m \circ H \circ Q^n]$ for each (m,n)-quasi-hyperideal Q of H;

(3) $[a]_{q(m,n)} = (([a]_{q(m,n)})^m \circ H \circ ([a]_{q(m,n)})^n]$ $(\forall a \in H)$.

Proof. (1) $\Rightarrow$ (2). If $m=n=0$, then the statement is true because $I_{(0,0)}=\{H\}$. If $m \neq 0$ and $n=0$ or $m=0$ and $n \neq 0$, then the statement follows by Lemma 3.11. So, let $m \neq 0$ and $n \neq 0$ be any Q an (m,n)-quasi-hyperideal of H. Then, by definition of (m,n)-regularity, $Q \subseteq (Q^m \circ H \circ Q^n]$. Also by definition of an (m,n)-quasi-hyperideal, we have $(Q^m \circ H \circ Q^n] \subseteq (Q^m \circ H] \cap (H \circ Q^n] \subseteq Q$. Hence $Q=(Q^m \circ H \circ Q^n]$, as required.

(2) $\Rightarrow$ (3). Obvious as $[a]_{q(m,n)}$ is an (m,n)-quasi-hyperideal.

(3) $\Rightarrow$ (1). Assume that (3) holds. Take any $a \in H$. By hypothesis, $[a]_{q(m,n)} = (([a]_{q(m,n)})^m \circ H \circ ([a]_{q(m,n)})^n]$. Now, by Theorem 2.10, $[a]_{q(m,n)} = (([a]_{q(m,n)})^m \circ H \circ ([a]_{q(m,n)})^n] = (a^m \circ H \circ a^n]$. As $\{a\} \subseteq [a]_{q(m,n)}$, $a \in (a^m \circ H \circ a^n]$. Hence H is (m,n)-regular. Q.E.D.

4 (m,n)-quasi-hypersimple

An ordered semihypergroup H is said to be (m,n)-quasi-hypersimple ($(m,0)$-hypersimple, $(0,n)$-hypersimple) if H does not contain any proper (m,n)-quasi-hyperideal ($(m,0)$-hyperideal, $(0,n)$-hyperideal).

Theorem 4.1. Let H be an ordered semihypergroup. Then

(1) H is $(m,0)$-hypersimple if and only if $(a^m \circ H]=H$ $(\forall a \in H)$, where m is any positive integer.

(2) H is $(0,n)$-hypersimple if and only if $(H \circ a^n]=H$ $(\forall a \in H)$, where n is any positive integer.

(3) H is (m,n)-quasi-hypersimple if and only if $(a^m \circ H] \cap (H \circ a^n]=H$ $(\forall a \in H)$, where m,n are any positive integers.

Proof. (1). Let H be $(m,0)$-simple and $a \in H$. By Lemma 2.16, $(a^m \circ H]$ is $(m,0)$-hyperideal of H. As H is $(m,0)$-simple, $(a^m \circ H] = H$.

Conversely assume that $(a^m \circ H] = H$ for each element $a \in H$. Let R be any $(m,0)$-hyperideal of H. Take an element $a \in R$. Then $(a^m \circ H] \subseteq (R^m \circ H] \subseteq (R] = R$. By assumption $(a^m \circ H] = H$. Thus $H = (a^m \circ H] \subseteq R$. Hence H is $(m,0)$-simple.

The proof of (2) and (3) follows on the lines similar to the proof of (1). Q.E.D.

Lemma 4.2. Let H be an ordered semihypergroup and Q be an (m,n)-quasi-hyperideal of H. If Q is (m,n)-quasi-hypersimple semihypergroup, then Q is a minimal (m,n)-quasi-hyperideal of H.

Proof. Let H be an ordered semihypergroup and Q be an (m,n)-quasi-hyperideal of H. Assume that Q is an (m,n)-quasi-simple semihypergroup. Let Q' be any (m,n)-quasi-hyperideal of H such that $Q' \subseteq Q$. Now $({Q'}^m \circ Q] \cap (Q \circ {Q'}^n] \subseteq ({Q'}^m \circ H] \cap (H \circ {Q'}^n] \subseteq Q'$. Therefore Q' is (m,n)-quasi-hyperideal of Q. As Q is (m,n)-quasi-simple semihypergroup, $Q' = Q$. Hence Q is a minimal (m,n)-quasi-hyperideal of H. Q.E.D.

Theorem 4.3. Let H be an ordered semihypergroup and m,n be positive integers. Then H is (m,n)-quasi-hypersimple if and only if H is both $(m,0)$-hypersimple and $(0,n)$-hypersimple.

Proof. Assume that H is (m,n)-quasi-hypersimple. If R is an $(m,0)$-hyperideal of H, then R is an (m,n)-quasi-hyperideal of H. Hence $R = H$ i.e., H is $(m,0)$-hypersimple. Similarly H is $(0,n)$-hypersimple

Conversely assume that H is both $(m,0)$ and $(0,n)$-hypersimple. Let Q be an (m,n)-quasi-hyperideal of H. Since $(Q^m \circ H]$ and $(H \circ Q^n]$ are $(m,0)$-hyperideal and $(0,n)$-hyperideal of H respectively, by hypothesis, $(Q^m \circ H] = H$ and $(H \circ Q^n] = H$. Therefore $(Q^m \circ H] \cap (H \circ Q^n] = H$. As $(Q^m \circ H] \cap (H \circ Q^n] \subseteq Q$, it follows that $H \subseteq Q$. So $Q = H$. Hence H is (m,n)-quasi-hypersimple. Q.E.D.

Lemma 4.4. Let H be an ordered semihypergroup and m,n be positive integers. If H is (m,n)-quasi-hypersimple, then H is (m,n)-regular.

Proof. Assume that H is (m,n)-quasi-hypersimple. By Theorem 4.3, H is both $(m,0)$ and $(0,n)$-hypersimple. Thus $(a^m \circ H] = H$ and $(H \circ a^n] = H$ for each $a \in H$. Now $(a^m \circ H \circ a^n] = ((a^m \circ H] \circ a^n] = (H \circ a^n] = H$ for each $a \in H$. Therefore, for each $a \in H$, $a \in (a^m \circ H \circ a^n]$. Hence H is (m,n)-regular. Q.E.D.

Theorem 4.5. Let H be an ordered semihypergroup and m,n be positive integers. If H is (m,n)-quasi-hypersimple then $\mathcal{Q}_m^n = \mathcal{Q} = H \times H$.

Proof. Assume that H is (m,n)-quasi-hypersimple and $a, b \in H$. Then $[a]_{q(m,n)}$ and $[b]_{q(m,n)}$ are (m,n)-quasi-hyperideals of H respectively. As H is (m,n)-quasi-hypersimple, $[a]_{q(m,n)} = H$ and $[b]_{q(m,n)} = H$. So $[a]_{q(m,n)} = [b]_{q(m,n)}$. Thus $(a,b) \in \mathcal{Q}_m^n$ and, hence, $\mathcal{Q}_m^n = H \times H$. As, by Lemma 4.4 and Theorem 3.10, $\mathcal{Q}_m^n = \mathcal{Q}$, we have $\mathcal{Q}_m^n = \mathcal{Q} = H \times H$, as required. Q.E.D.

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