Perturbed fourth-order Kirchhoff-type problems

Shapour Heidarkhani[1], Shahin Moradi[1], Giuseppe Caristi[2] and Bin Ge[3]

[1]Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran
[2]Department of Economics, University of Messina, via dei Verdi, 75, Messina, Italy
[3]Department of Applied Mathematics, Harbin Engineering University, Harbin, 150001, P.R. China

E-mail: sh.heidarkhani@yahoo.com, shahin.moradi86@yahoo.com, gcaristi@unime.it, gebin04523080261@163.com

Abstract

We establish the existence of at least three distinct weak solutions for a perturbed nonlocal fourth-order Kirchhoff-type problem with Navier boundary conditions under appropriate hypotheses on nonlinear terms. Our main tools are based on variational methods and some critical points theorems. We give some examples to illustrate the obtained results.

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1 Introduction

The purpose of this paper is to establish the existence of at least three distinct weak solutions for the following perturbed nonlocal fourth-order problem of Kirchhoff-type under Navier boundary condition

$$\begin{cases} T(u) = \lambda f(x,u) + \mu g(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega \end{cases} \tag{$P^{f,g}_{\lambda,\mu}$}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded smooth domain,

$$T(u) = \Delta\left(|\Delta u|^{p-2}\Delta u\right) - \left[M\left(\int_\Omega |\nabla u|^p dx\right)\right]^{p-1} \Delta_p u + \rho|u|^{p-2}u$$

in which $p > \max\left\{1, \frac{N}{2}\right\}$, $\rho > 0$ and $M : [0, +\infty[\to \mathbb{R}$ is a continuous function such that there are two positive constants m_0 and m_1 with $m_0 \leq M(t) \leq m_1$ for all $t \geq 0$, and $\lambda > 0$, $\mu \geq 0$ and $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions.

The problem $(P^{f,g}_{\lambda,\mu})$ is related to the stationary analogue of the equation

$$u_{tt} - M\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(x,u) \tag{1.1}$$

proposed by Kirchhoff [30] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Similar nonlocal problems also model several physical and biological systems where u describes a process which depends on the average of itself, for example the population density. There are a number of papers concerned with Kirchhoff-type boundary value problems, for instance see [9, 12, 14, 24, 36, 38, 39, 41, 42]. For example, in [41] Perera and Zhang employing the Yang index and critical groups, obtained nontrivial solutions of

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a class of nonlocal quasilinear elliptic boundary value problems. More precisely, they proved the existence of at least one nontrivial weak solution, and under additional assumptions, the existence of infinitely many weak solutions. Ricceri in an interesting paper [42] established the existence of at least three weak solutions to a class of Kirchhoff-type doubly eigenvalue boundary value problems. In [24], based on variational methods, the existence of infinitely many solutions for a class of nonlocal elliptic systems of $(p_1,\dots,p_n)$-Kirchhoff type was studied. Also in [12] employing a three critical point theorem due to Ricceri, the existence of at least three weak solutions for the following Kirchhoff-type problem involving two parameters

$$\begin{cases} -K\left(\int_\Omega |\nabla u(x)|^2dx\right)\Delta u(x) = \lambda f(x,u) + \mu g(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega \end{cases}$$

was discussed. The existence and multiplicity of stationary problems of Kirchhoff type were also studied in some recent papers, via variational methods like the symmetric mountain pass theorem in [10] and via a three critical point theorem in [4]. Moreover, in [2, 3] some evolutionary higher order Kirchhoff problems, mainly focusing on the qualitative properties of the solutions were treated.

The fourth-order equation of nonlinearity furnishes a model to study travelling waves in suspension bridges; therefore this becomes very significant in Physics. Many authors consider this type of equation, we refer to [8, 13, 15, 20, 28, 32, 33, 34, 35, 40] and the references therein. For example, Li and Tang in [33] by using a three critical points theorem due to Ricceri, established the existence of at least three weak solutions for the following p-biharmonic Navier boundary value problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(x,u) + \mu g(x,u), & x\in\Omega,\\ u=\Delta u = 0, & x\in\partial\Omega \end{cases}$$

where $\lambda,\mu\in[0,+\infty)$ and $f:\bar\Omega\times\mathbb{R}\to\mathbb{R}$ is a continuous function, and $g:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function, while in [32] the authors based on a three critical points theorem due to Ricceri, studied the existence of at least three solutions to a Navier boundary problem involving the (p,q)-biharmonic systems. Also, in [13, 28] the existence of multiple solutions for $(p_1,\dots,p_n)$-biharmonic systems was discussed based on variational methods and critical point theory. By using variational methods, Molica Bisci and Repovš in [40] investigated the existence of multiple weak solutions for a class of elliptic Navier boundary problems involving the p-biharmonic operator, and presented a concrete example of an application. The problem $(P^{f,g}_{\lambda,\mu})$ models the bending equilibrium of simply supported extensible beams on nonlinear foundations. The function f represents the force that the foundation exerts on the beam and $M\left(\int_\Omega|\nabla u|^pdx\right)$ models the effects of the small changes in the length of the beam. Recently many authors looked for the existence and multiplicity of solutions to fourth-order Kirchhoff-type problems, for an overview on this subject, we cite the papers [1, 23, 25, 27, 37, 43, 44, 45]. In [43], using the mountain pass theorem, Wang and An established the existence and multiplicity of solutions for the following fourth-order nonlocal elliptic problem

$$\begin{cases} \Delta^2 u - M\left(\int_\Omega|\nabla u|^2dx\right)\Delta u = \lambda f(x,u) & \text{in } \Omega\\ u=\Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Wang et al. in [44] studied the existence of positive solutions to a class of fourth order elliptic equations of Kirchhoff type on $\mathbb{R}^N$ by using variational methods and the truncation method. In particular, Massar et al. in [37] by employing Ricceri's variational principle, studied the existence of infinitely many solutions for the problem $(P^{f,g}_{\lambda,\mu})$, in the case $\mu = 0$. In [25] employing two three critical points theorems the existence of three distinct weak solutions for the problem $(P^{f,g}_{\lambda,\mu})$ was ensured. Also in [1] based on a variational method, the existence of one weak solution for the problem $(P^{f,g}_{\lambda,\mu})$, in the case $\mu = 0$ was discussed. Moreover, in [27] by using a variational method and some critical points theorems due to Ricceri, the existence of multiple solutions for the problem $(P^{f,g}_{\lambda,\mu})$ was studied.

In the present paper, motivated by the above results, using two kinds of three critical points theorems obtained in [5, 6] which we recall in the next section (Theorems 2.1 and 2.2), we ensure the existence of at least three weak solutions for the problem $(P^{f,g}_{\lambda,\mu})$ for appropriate values of the parameters λ and μ belonging to real intervals (see Theorems 3.1 and 3.7). In fact, in Theorem 3.1 we prove the existence of an interval of positive real parameters λ and an interval of positive real parameters μ for which the problem $(P^{f,g}_{\lambda,\mu})$ possesses at least three weak bounded solutions, while in Theorem 3.7 we establish the existence of two intervals of positive real parameters λ and an interval of positive real parameters μ for which the problem $(P^{f,g}_{\lambda,\mu})$ possesses three weak solutions, whose norms are uniformly bounded in respect to λ belonging to one of the two intervals. In Theorem 3.1, no asymptotic conditions on f and g are needed and only algebraic conditions on f are imposed to guarantee the existence of the weak solutions, while in Theorem 3.7 in addition we need an asymptotic condition on g. Theorem 3.3 is a consequence of Theorem 3.1. Theorem 3.4 is a simple consequence of Theorem 3.3, in which the function f has separated variables. Theorem 3.5 is a consequence of Theorem 3.3, in the case f does not depend upon x. Examples 3.6 and 3.9 are presented in order to illustrate Theorems 3.5 and 3.7, respectively. Theorem 3.13 is a particular case of Theorem 3.7, in which the function f has separated variables. In Theorem 3.14, we study the autonomous version of Theorem 3.7. Theorem 3.15 is a special case of Theorem 3.7. In Theorem 3.16 we present an application of Theorem 3.7. Finally, we give Example 3.17 to illustrate Theorem 3.16.

The rest of this paper is organized as follows. Section 2 contains some preliminary lemmas and Section 3 contains our main results and their proofs.

2 Preliminaries

Our main tools are the following three critical points theorems. In the first one a suitable sign hypothesis is assumed. In the second one the coercivity of the functional $\Phi - \lambda\Psi$ is required.

Let X be a nonempty set and $\Phi, \Psi : X \to \mathbb{R}$ be two functions. For all $r,\ r_1,\ r_2 > \inf_X \Phi,\ r_2 > r_1,\ r_3 > 0$, we define

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)) - \Psi(u)}{r - \Phi(u)},$$

$$\beta(r_1, r_2) := \inf_{u\in\Phi^{-1}(-\infty,r_1)} \sup_{v\in\Phi^{-1}[r_1,r_2)} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)},$$

$$\gamma(r_2, r_3) := \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)} \Psi(u)}{r_3},$$

$$\alpha(r_1, r_2, r_3) := \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\}.$$

Theorem 2.1. [6, Theorem 3.3] Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that

(a_1) $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;

(a_2) for every $u_1, u_2 \in X$ such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are three positive constants r_1, r_2, r_3 with $r_1 < r_2$, such that

(a_3) $\varphi(r_1) < \beta(r_1, r_2)$;

(a_4) $\varphi(r_2) < \beta(r_1, r_2)$;

(a_5) $\gamma(r_2, r_3) < \beta(r_1, r_2)$.

Then, for each $\lambda \in]\frac{1}{\beta(r_1,r_2)}, \frac{1}{\alpha(r_1,r_2,r_3)}[$ the functional $\Phi - \lambda\Psi$ admits three distinct critical points u_1, u_2, u_3 such that $u_1 \in \Phi^{-1}(-\infty, r_1)$, $u_2 \in \Phi^{-1}[r_1, r_2)$ and $u_3 \in \Phi^{-1}(-\infty, r_2 + r_3)$.

Theorem 2.2. [5, Theorem 3.1] Let X be a separable and reflexive real Banach space; $\Phi : X \longrightarrow \mathbb{R}$ a nonnegative continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \longrightarrow \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exists $u_0 \in X$ such that $\Phi(u_0) = \Psi(u_0) = 0$ and that

(b_1) $\lim_{\|u\|\to+\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty$ for all $\lambda \in [0, +\infty[$.

Further, assume that there are $r > 0$, $u_1 \in X$ such that:

(b_2) $r < \Phi(u_1)$;

(b_3) $\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^w} \Psi(u) < \frac{r}{r+\Phi(u_1)}\Psi(u_1)$.

Then, for each

$$\lambda \in \Lambda_1 =]\frac{\Phi(u_1)}{\Psi(u_1) - \sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^w} \Psi(u)}, \frac{r}{\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^w} \Psi(u)}[,$$

the equation

$$\Phi'(u) - \lambda\Psi'(u) = 0 \tag{2.1}$$

has at least three solutions in X and, moreover, for each $h > 1$, there exist an open interval

$$\Lambda_2 \subseteq [0, \frac{hr}{r\frac{\Psi(u_1)}{\Phi(u_1)} - \sup_{u\in\overline{\Phi^{-1}(-\infty,r[)}^w} \Psi(u)}]$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2$, the equation (2.1) has at least three solutions in X whose norms are less than σ.

We refer the interested reader to the papers [7, 18, 19, 21, 22, 26, 31] and the papers [16, 17, 29] in which Theorems 2.1 and 2.2 have been successfully employed to the existence of at least three solutions for boundary value problems, respectively.

Here and in the sequel, X will denote the space $W^{2,p}(\Omega)\bigcap W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| := \left(\int_\Omega (|\Delta u(x)|^p + |\nabla u(x)|^p + |u(x)|^p)dx \right)^{\frac{1}{p}}.$$

Put

$$k = \sup_{u\in X\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}} |u(x)|}{\|u\|}. \tag{2.2}$$

For $p > \max\{1, \frac{N}{2}\}$, since the embedding $X \hookrightarrow C^0(\overline{\Omega})$ is compact, one has $k < +\infty$.

Set

$$F(x,t) = \int_0^t f(x,\xi)d\xi, \quad \text{for all } (x,t) \in \Omega\times\mathbb{R},$$

$$G(x,t) = \int_0^t g(x,\xi)d\xi, \quad \text{for all } (x,t) \in \Omega\times\mathbb{R},$$

$$\widetilde{M}(t) = \int_0^t [M(s)]^{p-1}\, ds \text{ for all } t \geq 0,$$

$$M^- := \min\left\{1, m_0^{p-1}, \rho\right\}$$

and

$$M^+ := \max\left\{1, m_1^{p-1}, \rho\right\}.$$

We say that a function $u \in X$ is a (weak) solution of the problem $(P^{f,g}_{\lambda,\mu})$ if

$$\int_\Omega |\Delta u|^{p-2}\Delta u\Delta v dx + \left[M\left(\int_\Omega |\nabla u|^p dx\right)\right]^{p-1} \int_\Omega |\nabla u|^{p-2}\nabla u\nabla v dx + \rho\int_\Omega |u|^{p-2}uv dx$$
$$-\lambda\int_\Omega f(x,u)v dx - \mu\int_\Omega g(x,u)v dx = 0$$

for every $v \in X$.

3 Main results

In this section, we formulate our main results on the existence of at least three weak solutions for the problem $(P^{f,g}_{\lambda,\mu})$.

Fix $x^0 \in \Omega$ and pick $s > 0$ such that $B(x^0, s) \subset \Omega$ where $B(x^0, s)$ denotes the ball with center at x^0 and radius of s. Put

$$c_1 := \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \int_{\frac{s}{2}}^s \left| \frac{12(N+1)}{s^3} r - \frac{24N}{s^2} + \frac{9(N-1)}{s}\frac{1}{r} \right|^p r^{N-1} dr,$$

$$c_2 := \int_{B(x^0,s)\setminus B(x^0,\frac{s}{2})} \left[\Sigma_{i=1}^{N} \left(\frac{12\ell(x_i - x_i^0)}{s^3} - \frac{24(x_i - x_i^0)}{s^2} + \frac{9(x_i - x_i^0)}{s\ell} \right)^2 \right]^{\frac{p}{2}} dx,$$

$$c_3 := \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \left[\frac{(\frac{s}{2})^N}{N} + \int_{\frac{s}{2}}^{s} \left| \frac{4}{s^3} r^3 - \frac{12}{s^2} r^2 + \frac{9}{s} r - 1 \right|^p r^{N-1} dr \right]$$

where Γ denotes the Gamma function, and

$$L := c_1 + c_2 + c_3.$$

In the sequel $\text{meas}(\Omega)$ denotes the Lebesgue measure of the set Ω.

For our convenience, set

$$G^{\theta} := \int_{\Omega} \max_{|\xi| \leq \theta} G(x,\xi) dx \quad \text{for all } \theta > 0$$

and

$$G_{\eta} := \text{meas}(\Omega) \inf_{\overline{\Omega} \times [0,\eta]} G(x,\xi) \quad \text{for all } \eta > 0.$$

If g is sign-changing, then clearly $G^{\theta} \geq 0$ and $G_{\eta} \leq 0$.

We fix four positive constants θ_1, θ_2, θ_3 and η, put

$$\begin{aligned} \delta_{\lambda,g} := \min \Bigg\{ & \frac{1}{pk^p} \min \Bigg\{ \frac{M^- \theta_1^p - \lambda p k^p \int_{\Omega} F(x,\theta_1) dx}{G^{\theta_1}}, \frac{M^- \theta_2^p - \lambda p k^p \int_{\Omega} F(x,\theta_2) dx}{G^{\theta_2}}, \\ & \frac{M^-(\theta_3^p - \theta_2^p) - \lambda p k^p \int_{\Omega} F(x,\theta_3) dx}{G^{\theta_3}} \Bigg\}, \\ & \frac{\frac{\eta^p}{p} M^+ L - \lambda \left(\int_{B(x^0,\frac{s}{2})} F(x,\eta) dx - \int_{\Omega} F(x,\theta_1) dx \right)}{G_{\eta} - G^{\theta_1}} \Bigg\}. \end{aligned} \tag{3.1}$$

We present our first existence result as follows.

Theorem 3.1. Assume that there exist four positive constants θ_1, θ_2, θ_3 and η with $\theta_1 < k\sqrt[p]{L}\eta$, $\max\{\eta, k\sqrt[p]{\frac{M^+L}{M^-}}\eta\} < \theta_2$ and $\theta_2 < \theta_3$ such that

(A_1) $f(x,t) \geq 0$ for each $(x,t) \in \Omega \times [-\theta_3, \theta_3]$;

(A_2)

$$\max \left\{ \frac{\int_{\Omega} F(x,\theta_1) dx}{\theta_1^p}, \frac{\int_{\Omega} F(x,\theta_2) dx}{\theta_2^p}, \frac{\int_{\Omega} F(x,\theta_3) dx}{\theta_3^p - \theta_2^p} \right\} < \frac{M^-}{k^p M^+ L} \frac{\int_{B(x^0,\frac{s}{2})} F(x,\eta) dx - \int_{\Omega} F(x,\theta_1) dx}{\eta^p}.$$

Then, for every

$$\lambda \in \Big(\frac{\frac{\eta^p}{p} M^+ L}{\int_{B(x^0, \frac{s}{2})} F(x,\eta)dx - \int_\Omega F(x,\theta_1)dx},$$
$$\frac{M^-}{pk^p} \min\Big\{ \frac{\theta_1^p}{\int_\Omega F(x,\theta_1)dx}, \frac{\theta_2^p}{\int_\Omega F(x,\theta_2)dx}, \frac{\theta_3^p - \theta_2^p}{\int_\Omega F(x,\theta_3)dx} \Big\} \Big)$$

for every non-negative L^1-Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ there exists $\delta_{\lambda,g} > 0$ given by (3.1) such that, for each $\mu \in [0, \delta_{\lambda,g})$, the problem $(P^{f,g}_{\lambda,\mu})$ possesses at least three weak solutions u_1, u_2, and u_3 such that

$$\max_{x\in\Omega} |u_1(x)| < \theta_1, \ \max_{x\in\Omega} |u_2(x)| < \theta_2 \text{ and } \max_{x\in\Omega} |u_3(x)| < \theta_3.$$

Proof. Our aim is to apply Theorem 2.1 to our problem. We consider the auxiliary problem

$$\begin{cases} T(u) = \lambda \hat{f}(x,u) + \mu g(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega \end{cases} \qquad (P^{\hat{f},g}_{\lambda,\mu})$$

where $\hat{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, defined as follows

$$\hat{f}(x,\xi) = \begin{cases} f(x,0), & \text{if } \xi < -\theta_3, \\ f(x,\xi), & \text{if } -\theta_3 \le \xi \le \theta_3, \\ f(x,\theta_3), & \text{if } \xi > \theta_3. \end{cases}$$

If any solution of the problem $(P^{f,g}_{\lambda,\mu})$ satisfies the condition $-\theta_3 \le u(x) \le \theta_3$ for every $x \in \Omega$, then, any weak solution of the problem $(P^{\hat{f},g}_{\lambda,\mu})$ clearly turns to be also a weak solution of $(P^{f,g}_{\lambda,\mu})$. Therefore, for our goal, it is enough to show that our conclusion holds for $(P^{f,g}_{\lambda,\mu})$. We introduce the functionals Φ, Ψ for $u \in X$, as follows

$$\Phi(u) = \frac{1}{p}\int_\Omega |\Delta u(x)|^p dx + \frac{1}{p}\widetilde{M}\left[\int_\Omega |\nabla u(x)|^p dx\right] + \frac{\rho}{p}\int_\Omega |u(x)|^p dx \tag{3.2}$$

and

$$\Psi(u) = \int_\Omega F(x,u(x))dx + \frac{\mu}{\lambda}\int_\Omega G(x,u(x))dx, \tag{3.3}$$

and we put

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$$

for $u \in X$. Now we show that the functionals Φ and Ψ satisfy the required conditions in Theorem 2.1. We easily observe that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)(v) = \int_\Omega f(x,u(x))v(x)dx + \frac{\mu}{\lambda}\int_\Omega g(x,u(x))v(x)dx$$

for every $v \in X$, and Ψ is sequentially weakly upper semicontinuous. Moreover, since $m_0 \le M(s) \le m_1$ for all $s \in [0, +\infty[$, from (3.2), we have

$$\frac{M^-}{p}\|u\|^p \le \Phi(u) \le \frac{M^+}{p}\|u\|^p \tag{3.4}$$

for all $u \in X$ and it follows $\lim_{\|u\| \to +\infty} \Phi(u) = +\infty$, namely Φ is coercive. Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ is $\Phi'(u) \in X^*$, given by

$$\begin{aligned}\Phi'(u)(v) &= \int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)dx \\ &+ \left[M\left(\int_\Omega |\nabla u(x)|^p dx\right)\right]^{p-1} \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx \\ &+\rho \int_\Omega |u(x)|^{p-2}u(x)v(x)dx\end{aligned}$$

for every $v \in X$. Furthermore, the functional Φ defined by (3.2) is sequentially weakly lower semicontinuous, while [11, Proposition 2.3] gives that its Gâteaux derivative admits a continuous inverse on X^*. Put $r_1 := \frac{M^-}{p}(\frac{\theta_1}{k})^p$, $r_2 := \frac{M^-}{p}(\frac{\theta_2}{k})^p$, $r_3 := \frac{M^-}{p}(\frac{\theta_3^p - \theta_2^p}{k^p})$ and

$$w(x) := \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus B(x^0, s) \\ \eta\left(\frac{4}{s^3}\ell^3 - \frac{12}{s^2}\ell^2 + \frac{9}{s}\ell - 1\right) & \text{if } x \in B(x^0, s) \setminus B(x^0, \frac{s}{2}) \\ \eta & \text{if } x \in B(x^0, \frac{s}{2}) \end{cases} \tag{3.5}$$

with $\ell = dist(x, x^0) = \sqrt{\Sigma_{i=1}^N (x_i - x_i^0)^2}$. We see that $w \in X$ and

$$\frac{\partial w(x)}{\partial x_i} =$$

$$\begin{cases} 0 & \text{if } \quad x \in \overline{\Omega} \setminus B(x^0, s) \cup B(x^0, \frac{s}{2}) \\ \eta\left(\frac{12\ell(x_i - x_i^0)}{s^3} - \frac{24(x_i - x_i^0)}{s^2} + \frac{9}{s}\frac{(x_i - x_i^0)}{\ell}\right) & \text{if } \quad x \in B(x^0, s) \setminus B(x^0, \frac{s}{2}) \end{cases}$$

and

$$\frac{\partial^2 w(x)}{\partial x_i^2} =$$

$$\begin{cases} 0 & \text{if } \quad x \in \overline{\Omega} \setminus B(x^0, s) \cup B(x^0, \frac{s}{2}) \\ \eta\left(\frac{12}{s^3}\frac{(x_i - x_i^0)^2 + \ell^2}{\ell} - \frac{24}{s^2} + \frac{9}{s}\frac{\ell^2 - (x_i - x_i^0)^2}{\ell^3}\right) & \text{if } \quad x \in B(x^0, s) \setminus B(x^0, \frac{s}{2}) \end{cases}$$

and so that

$$\sum_{i=1}^N \frac{\partial^2 w(x)}{\partial x_i^2} = \begin{cases} 0 & \text{if } \quad x \in \overline{\Omega} \setminus B(x^0, s) \cup B(x^0, \frac{s}{2}) \\ \eta\left(\frac{12l(N+1)}{s^3} - \frac{24N}{s^2} + \frac{9}{s}\frac{N-1}{\ell}\right) & \text{if } \quad x \in B(x^0, s) \setminus B(x^0, \frac{s}{2}). \end{cases}$$

Since

$$\int_\Omega |\Delta w(x)|^p dx = \eta^p \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \int_{\frac{s}{2}}^s \left|\frac{12(N+1)}{s^3}r - \frac{24N}{s^2} + \frac{9(N-1)}{s}\frac{1}{r}\right|^p r^{N-1}dr,$$

$$\int_\Omega |\nabla w(x)|^p dx =$$

$$\int_{B(x^0,s)\setminus B(x^0,\frac{s}{2})} \left[\sum_{i=1}^N \eta^2 \left(\frac{12l(x_i-x_i^0)}{s^3} - \frac{24(x_i-x_i^0)}{s^2} + \frac{9}{s}\frac{(x_i-x_i^0)}{l}\right)^2\right]^{\frac{p}{2}} dx$$

$$= \eta^p \int_{B(x^0,s)\setminus B(x^0,\frac{s}{2})} \left[\sum_{i=1}^N \left(\frac{12l(x_i-x_i^0)}{s^3} - \frac{24(x_i-x_i^0)}{s^2} + \frac{9}{s}\frac{(x_i-x_i^0)}{l}\right)^2\right]^{\frac{p}{2}} dx$$

and

$$\int_\Omega |w(x)|^p dx = \eta^p \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\left(\frac{(\frac{s}{2})^N}{N} + \int_{\frac{s}{2}}^s \left|\frac{4}{s^3}r^3 - \frac{12}{s^2}r^2 + \frac{9}{s}r - 1\right|^p r^{N-1}dr\right).$$

In particular, one has

$$\frac{\eta^p}{p}M^-L \le \frac{1}{p}(c_1\eta^p + m_0^{p-1}c_2\eta^p + \rho c_3\eta^p) \le \Phi(w) = \frac{1}{p}(c_1\eta^p + \widehat{M}(c_2\eta^p) + \rho c_3\eta^p)$$

$$\le \frac{1}{p}(c_1\eta^p + m_1^{p-1}c_2\eta^p + \rho c_3\eta^p) \le \frac{\eta^p}{p}M^+L.$$

From the conditions $\theta_3 > \theta_2$, $\theta_1 < k\sqrt[p]{L}\eta$ and $k\sqrt[p]{\frac{M^+L}{M^-}}\eta < \theta_2$, we get $r_3 > 0$ and $r_1 < \Phi(w) < r_2$. Moreover, for all $u \in X$ with $\Phi(u) < r_1$, from the definition of Φ and taking (3.4) into account, one has

$$\begin{aligned}\Phi^{-1}(-\infty, r_1) \quad &\subseteq \quad \left\{u \in X;\ \frac{M^-}{p}\|u\|^p \le r_1\right\} \\ &\subseteq \quad \{u \in X;\ |u(x)| \le \theta_1 \text{ for each } x \in \Omega\}.\end{aligned}$$

Hence, by using the assumption (A_1), one has

$$\sup_{u\in\Phi^{-1}(-\infty,r_1)} \int_\Omega F(x,u(x))dx \le \int_\Omega \max_{|t|\le\theta_1} F(x,t)dx \le \int_\Omega F(x,\theta_1)dx.$$

In a similar way, we have

$$\sup_{u\in\Phi^{-1}(-\infty,r_2)} \int_\Omega F(x,u(x))dx \le \int_\Omega F(x,\theta_2)dx$$

and

$$\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)} \int_\Omega F(x,u(x))dx \le \int_\Omega F(x,\theta_3)dx.$$

Therefore, since $0 \in \Phi^{-1}(-\infty, r_1)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\begin{aligned}
\varphi(r_1) &= \inf_{u \in \Phi^{-1}(-\infty, r_1)} \frac{(\sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u)) - \Psi(u)}{r_1 - \Phi(u)} \\
&\leq \frac{\sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u)}{r_1} \\
&= \frac{\sup\limits_{u \in \Phi^{-1}(-\infty, r_1)} \displaystyle\int_\Omega [F(x, u(x)) + \frac{\mu}{\lambda} G(x, u(x))] dx}{r_1} \\
&\leq \frac{pk^p}{M^-} \frac{\displaystyle\int_\Omega F(x, \theta_1) dx + \frac{\mu}{\lambda} G^{\theta_1}}{\theta_1^p},
\end{aligned}$$

$$\begin{aligned}
\varphi(r_2) \leq \frac{\sup\limits_{u \in \Phi^{-1}(-\infty, r_2)} \Psi(u)}{r_2} &= \frac{\sup\limits_{u \in \Phi^{-1}(-\infty, r_2)} \displaystyle\int_\Omega [F(x, u(x)) + \frac{\mu}{\lambda} G(x, u(x))] dx}{r_2} \\
&\leq \frac{pk^p}{M^-} \frac{\displaystyle\int_\Omega F(x, \theta_2) dx + \frac{\mu}{\lambda} G^{\theta_2}}{\theta_2^p}
\end{aligned}$$

and

$$\begin{aligned}
\gamma(r_2, r_3) &\leq \frac{\sup\limits_{u \in \Phi^{-1}(-\infty, r_2 + r_3)} \Psi(u)}{r_3} \\
&= \frac{\sup\limits_{u \in \Phi^{-1}(-\infty, r_2 + r_3)} \displaystyle\int_\Omega [F(x, u(x)) + \frac{\mu}{\lambda} G(x, u(x))] dx}{r_3} \\
&\leq \frac{pk^p}{M^-} \frac{\displaystyle\int_\Omega F(x, \theta_3) dx + \frac{\mu}{\lambda} G^{\theta_3}}{\theta_3^p - \theta_2^p}.
\end{aligned}$$

Since $0 \leq w(x) \leq \eta$ for each $x \in \Omega$, the assumption (A_1) ensures that

$$\Psi(w) \geq \int_{B(x^0, \frac{s}{2})} F(x, \eta) dx + \frac{\mu}{\lambda} \int_\Omega G(x, w(x)) dx \geq \int_{B(x^0, \frac{s}{2})} F(t, \eta) dt + \frac{\mu}{\lambda} G_\eta.$$

On the other hand, for each $u \in \Phi^{-1}(-\infty, r_1)$ one has

$$\begin{aligned}
\beta(r_1, r_2) &\geq \frac{\int_{B(x^0, \frac{s}{2})} F(x, \eta) dx - \int_\Omega F(x, \theta_1) dx + \frac{\mu}{\lambda}(G_\eta - G^{\theta_1})}{\Phi(w) - \Phi(u)} \\
&\geq \frac{\int_{B(x^0, \frac{s}{2})} F(x, \eta) dx - \int_\Omega F(x, \theta_1) dx + \frac{\mu}{\lambda}(G_\eta - G^{\theta_1})}{\frac{\eta^p}{p} M^+ L}.
\end{aligned}$$

Due to (A_2) we get

$$\alpha(r_1, r_2, r_3) < \beta(r_1, r_2).$$

Now, we show that the functional I_λ satisfies the assumption (a_2) of Theorem 2.1. Let u_1 and u_2 be two local minima for I_λ. Then u_1 and u_2 are critical points for I_λ, and so, they are weak solutions for the problem $(P^{f,g}_{\lambda,\mu})$. Since we could assume that f is non-negative, and g is non-negative, for fixed $\lambda > 0$ and $\mu \geq 0$ we have $(\lambda f + \mu g)(x, su_1 + (1-s)u_2) \geq 0$ for all $s \in [0,1]$, and consequently, $\Psi(su_1 + (1-s)u_2) \geq 0$, for every $s \in [0,1]$. Hence, Theorem 2.1 implies that for every

$$\lambda \in \Big(\frac{\frac{\eta^p}{p} M^+ L}{\int_{B(x^0, \frac{s}{2})} F(x,\eta)dx - \int_\Omega F(x,\theta_1)dx},$$

$$\frac{M^-}{pk^p} \min\Big\{ \frac{\theta_1^p}{\int_\Omega F(x,\theta_1)dx}, \frac{\theta_2^p}{\int_\Omega F(x,\theta_2)dx}, \frac{\theta_3^p - \theta_2^p}{\int_\Omega F(x,\theta_3)dx} \Big\}\Big)$$

and $\mu \in [0, \delta_{\lambda,g})$, the functional I_λ has three critical points $u_i,\ i = 1,2,3$, in X such that $\Phi(u_1) < r_1$, $\Phi(u_2) < r_2$ and $\Phi(u_3) < r_2 + r_3$, that is,

$$\max_{x\in\Omega} |u_1(x)| < \theta_1,\ \max_{x\in\Omega} |u_2(x)| < \theta_2 \text{ and } \max_{x\in\Omega} |u_3(x)| < \theta_3.$$

Then, taking into account the fact that the weak solutions of the problem $(P^{f,g}_{\lambda,\mu})$ are exactly critical points of the functional I_λ we have the desired conclusion. Q.E.D.

Remark 3.2. We observe that, in Theorem 3.1, no asymptotic conditions on f and g are needed and only algebraic conditions on f are imposed to guarantee the existence of the weak solutions.

For positive constants $\theta_1,\ \theta_4$ and η, set

$$\delta'_{\lambda,g} := \min\Bigg\{ \frac{1}{pk^p} \min\Big\{ \frac{M^-\theta_1^p - \lambda pk^p \int_\Omega F(x,\theta_1)dx}{G^{\theta_1}}, \tag{3.6}$$

$$\frac{M^-\theta_4^p - 2\lambda pk^p \int_\Omega F(x, \frac{1}{\sqrt[p]{2}}\theta_4)dx}{2G^{\frac{1}{\sqrt[p]{2}}\theta_4}}, \frac{M^-\theta_4^p - 2\lambda pk^p \int_\Omega F(x,\theta_4)dx}{2G^{\theta_4}} \Big\},$$

$$\frac{\frac{\eta^p}{p} M^+ L - \lambda\big(\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega F(x,\theta_1)dx \big)}{G_\eta - G^{\theta_1}} \Bigg\}.$$

Now, we deduce the following straightforward consequence of Theorem 3.1.

Theorem 3.3. Assume that there exist three positive constants θ_1, θ_4 and η with $\theta_1 < \min\{\eta, k\sqrt[p]{L}\eta\}$ and $\max\{\eta, k\sqrt[p]{\frac{2M^+L}{M^-}}\eta\} < \theta_4$ such that

(A$_3$) $f(x,t) \geq 0$ for each $(x,t) \in \Omega \times [-\theta_4, \theta_4]$;

(A$_4$)
$$\max\Big\{\frac{\int_\Omega F(x,\theta_1)dx}{\theta_1^p},\ \frac{2\int_\Omega F(x,\theta_4)dx}{\theta_4^p}\Big\}<\frac{M^-}{M^-+k^pM^+L}\frac{\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx}{\eta^p}.$$

Then, for every
$$\lambda\in\Big(\frac{\frac{M^-+k^pM^+L}{pk^p}\eta^p}{\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx},\frac{M^-}{pk^p}\min\Big\{\frac{\theta_1^p}{\int_\Omega F(x,\theta_1)dx},\frac{\theta_4^p}{2\int_\Omega F(x,\theta_4)dx}\Big\}\Big)$$

and for every non-negative L^1-Carathéodory function $g:\Omega\times\mathbb{R}\to\mathbb{R}$, there exists $\delta'_{\lambda,g}>0$ given by (3.6) such that, for each $\mu\in[0,\delta'_{\lambda,g})$, the problem $(P^{f,g}_{\lambda,\mu})$ possesses at least three weak solutions u_1, u_2 and u_3 such that
$$\max_{x\in\Omega}|u_1(x)|<\theta_1,\ \max_{x\in\Omega}|u_2(x)|<\frac{1}{\sqrt[p]{2}}\theta_4 \text{ and } \max_{x\in\Omega}|u_3(x)|<\theta_4.$$

Proof. Choose $\theta_2=\frac{1}{\sqrt[p]{2}}\theta_4$ and $\theta_3=\theta_4$. So, from (A_4) one has
$$\begin{aligned}\frac{\int_\Omega F(x,\theta_2)dx}{\theta_2^p}&=\frac{2\int_\Omega F(x,\frac{1}{\sqrt[p]{2}}\theta_4)dx}{\theta_4^p}\le\frac{2\int_\Omega F(x,\theta_4)dx}{\theta_4^p}\\&<\frac{M^-}{M^-+k^pM^+L}\frac{\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx}{\eta^p}\end{aligned}\tag{3.7}$$

and
$$\frac{\int_\Omega F(x,\theta_3)dx}{\theta_3^p-\theta_2^p}=\frac{2\int_\Omega F(x,\theta_4)dx}{\theta_4^p}<\frac{M^-}{M^-+k^pM^+L}\frac{\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx}{\eta^p}.\tag{3.8}$$

Moreover, taking into account that $\theta_1<\eta$, by using (A_4) we have
$$\begin{aligned}&\frac{M^-}{k^pM^+L}\frac{\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx-\int_\Omega F(x,\theta_1)dx}{\eta^p}\\&>\frac{M^-}{k^pM^+L}\frac{\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx}{\eta^p}-\frac{M^-}{k^pM^+L}\frac{\int_\Omega F(x,\theta_1)dx}{\theta_1^p}\\&>\frac{M^-}{k^pM^+L}\Big(\frac{\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx}{\eta^p}-\frac{M^-}{M^-+k^pM^+L}\frac{\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx}{\eta^p}\Big)\\&=\frac{M^-}{M^-+k^pM^+L}\frac{\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx}{\eta^p}.\end{aligned}$$

Hence, from (A_4), (3.7) and (3.8), it is easy to see that the assumption (A_2) of Theorem 3.1 is satisfied, and it follows the conclusion. Q.E.D.

We want to point out a simple consequence of Theorem 3.3, in which the function f has separated variables.

Theorem 3.4. Let $f_1 \in L^1(\Omega)$ and $f_2 \in C(\mathbb{R})$ be two functions. Put $\tilde{F}(t) = \int_0^t f_2(\xi)d\xi$ for all $t \in \mathbb{R}$ and assume that there exist three positive constants θ_1, θ_4 and η with $\theta_1 < \min\{\eta, k\sqrt[p]{L}\eta\}$ and $\max\{\eta, k\sqrt[p]{\frac{2M^+L}{M^-}}\eta\} < \theta_4$ such that

(A_5) $f_1(x) \geq 0$ for each $x \in \Omega$ and $f_2(t) \geq 0$ for each $t \in [-\theta_4, \theta_4]$;

(A_6)

$$\int_\Omega f_1(x)dx \max\left\{\frac{\max_{|t|\leq\theta_1} \tilde{F}(t)}{\theta_1^p}, \frac{2\max_{|t|\leq\theta_4} \tilde{F}(t)}{\theta_4^p}\right\} < \frac{M^-}{M^- + k^pM^+L}\frac{\tilde{F}(\eta)}{\eta^p}\int_{B(x^0,\frac{s}{2})} f_1(x)dx.$$

Then, for every

$$\lambda \in \left(\frac{\frac{M^-+k^pM^+L}{pk^p}\eta^p}{\tilde{F}(\eta)\int_{B(x^0,\frac{s}{2})} f_1(x)dx}, \frac{M^-}{pk^p\int_\Omega f_1(x)dx}\min\left\{\frac{\theta_1^p}{\max_{|t|\leq\theta_1}\tilde{F}(t)}, \frac{\theta_4^p}{2\max_{|t|\leq\theta_4}\tilde{F}(t)}\right\}\right)$$

and for every non-negative L^1-Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, for each

$$\mu \in \left[0, \min\left\{\frac{1}{pk^p}\min\left\{\frac{M^-\theta_1^p - \lambda pk^p \max\limits_{|t|\leq\theta_1}\tilde{F}(t)\int_\Omega f_1(x)dx}{G^{\theta_1}}, \frac{M^-\theta_4^p - 2\lambda pk^p \max\limits_{|t|\leq\frac{1}{\sqrt[p]{2}}\theta_4}\tilde{F}(t)\int_\Omega f_1(x)dx}{2G^{\frac{1}{\sqrt[p]{2}}\theta_4}}, \frac{M^-\theta_4^p - 2\lambda pk^p \max\limits_{|t|\leq\theta_4}\tilde{F}(t)\int_\Omega f_1(x)dx}{2G^{\theta_4}}\right\}, \frac{\frac{\eta^p}{p}M^+L - \lambda\left(\tilde{F}(\eta)\int_{B(x^0,\frac{s}{2})} f_1(x)dx - \max\limits_{|t|\leq\theta_1}\tilde{F}(t)\int_\Omega f_1(x)dx\right)}{G_\eta - G^{\theta_1}}\right\}\right),$$

the problem

$$\begin{cases} T(u) = \lambda f_1(x)f_2(u) + \mu g(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega \end{cases} \tag{3.9}$$

possesses at least three weak solutions u_1, u_2 and u_3 such that

$$\max_{x\in\Omega}|u_1(x)| < \theta_1, \ \max_{x\in\Omega}|u_2(x)| < \frac{1}{\sqrt[p]{2}}\theta_4 \text{ and } \max_{x\in\Omega}|u_3(x)| < \theta_4.$$

Proof. Set $f(x,u) = f_1(x)f_2(u)$ for each $(x,u) \in \Omega \times \mathbb{R}$. Since

$$F(x,t) = f_1(x)\tilde{F}(t)$$

from (A_5) and (A_6) we get (A_3) and (A_4) are satisfied, respectively. Hence, the result follows from Theorem 3.3. Q.E.D.

Here, we present a simple consequence of Theorem 3.3, in the case f does not depend upon x.

Theorem 3.5. Assume that there exist three positive constants θ_1, θ_4 and η with $\theta_1 < \min\{\eta, k\sqrt[p]{L}\eta\}$ and $\max\{\eta, k\sqrt[p]{\frac{2M^+L}{M^-}}\eta\} < \theta_4$ such that

(A$_7$) $f(t) \geq 0$ for each $t \in [-\theta_4, \theta_4]$;

(A$_8$)

$$\mathrm{meas}(\Omega)\max\left\{\frac{F(\theta_1)}{\theta_1^p},\ \frac{2F(\theta_4)}{\theta_4^p}\right\} < \frac{M^-}{M^- + k^pM^+L}\frac{\mathrm{meas}(B(x^0,\frac{s}{2}))F(\eta)}{\eta^p}.$$

Then, for every

$$\lambda \in \left(\frac{\frac{M^-+k^pM^+L}{pk^p}\eta^p}{\mathrm{meas}(B(x^0,\frac{s}{2}))F(\eta)}, \frac{M^-}{pk^p\mathrm{meas}(\Omega)}\min\left\{\frac{\theta_1^p}{F(\theta_1)}, \frac{\theta_4^p}{2F(\theta_4)}\right\}\right)$$

and for every non-negative L^1-Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, for each

$$\mu \in \Bigg[0, \min\Bigg\{\frac{1}{pk^p}\min\Bigg\{\frac{M^-\theta_1^p - \lambda pk^p\mathrm{meas}(\Omega)F(\theta_1)}{G^{\theta_1}},$$
$$\frac{M^-\theta_4^p - 2\lambda pk^p\mathrm{meas}(\Omega)F(\frac{1}{\sqrt[p]{2}}\theta_4)}{2G^{\frac{1}{\sqrt[p]{2}}\theta_4}}, \frac{M^-\theta_4^p - 2\lambda pk^p\mathrm{meas}(\Omega)F(\theta_4)}{2G^{\theta_4}}\Bigg\},$$
$$\frac{\frac{\eta^p}{p}M^+L - \lambda\big(\mathrm{meas}(B(x^0,\frac{s}{2}))F(\eta) - \mathrm{meas}(\Omega)F(\theta_1)\big)}{G_\eta - G^{\theta_1}}\Bigg\}\Bigg),$$

the problem

$$\begin{cases} T(u) = \lambda f(u) + \mu g(x,u), & \text{in } \Omega,\\ u = \Delta u = 0, & \text{on } \partial\Omega \end{cases} \tag{3.10}$$

possesses at least three weak solutions u_1, u_2 and u_3 such that

$$\max_{x\in\Omega}|u_1(x)| < \theta_1,\ \max_{x\in\Omega}|u_2(x)| < \frac{1}{\sqrt[p]{2}}\theta_4 \text{ and } \max_{x\in\Omega}|u_3(x)| < \theta_4.$$

We now give the following example to illustrate Theorem 3.5.

Example 3.6. Let $p = 3$ and $\rho = 1$. Consider the following problem

$$\begin{cases} \Delta(|\Delta u|\Delta u) - \left[M\left(\int_\Omega |\nabla u|^3 dx\right)\right]^2 \Delta_p u + |u|u = \lambda f(u) + \mu g(u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega \end{cases} \tag{3.11}$$

where $\Omega = \{(x,y);\ x^2 + y^2 < 9\}$, $M(t) = \frac{3}{2} + \frac{\sin(t)}{2}$ for every $t \in [0, +\infty)$ and

$$f(t) = \begin{cases} 6t^5, & \text{if } t \le 1, \\ \frac{6}{t}, & \text{if } t > 1. \end{cases}$$

By the expression of f, we have

$$F(t) = \begin{cases} t^6, & \text{if } t \le 1, \\ 1 + 6\ln(t), & \text{if } t > 1. \end{cases}$$

Direct calculations show $M^- = 1$ and $M^+ = 4$. Choose $x_0 = (0,0)$, $s = 2$, $\theta_1 = 10^{-8}$, $\theta_4 = 10^4$ and $\eta = 1$. Therefore, since $k = \sqrt[3]{\frac{4}{\pi}}$, $L = 58.18309\pi$, we clearly see that all assumptions of Theorem 3.5 are satisfied. Then, for every

$$\lambda \in \Big(\frac{\pi + 16L}{12\pi}, \frac{10^{12}}{216 + 1296\ln(10^4)}\Big)$$

and for every non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\bar{\delta}_{\lambda,g} > 0$ such that, for each $\mu \in [0, \bar{\delta}_{\lambda,g})$, the problem (3.11) possesses at least three weak solutions u_1, u_2 and u_3 such that

$$\max_{x\in\Omega} |u_1(x)| < 10^{-8},\ \max_{x\in\Omega} |u_2(x)| < \frac{10^4}{\sqrt[3]{2}} \text{ and } \max_{x\in\Omega} |u_3(x)| < 10^4.$$

For our second goal, we fix two positive constants θ and η, put

$$\underline{\delta}_{\lambda,g} := \min\left\{ \frac{M^-\theta^p - \lambda p k^p \int_\Omega \max_{|t|\le\theta} F(x,t)dx}{pk^p G^\theta}, \right.$$
$$\left. \frac{\frac{\eta^p}{p} M^+ L - \lambda\Big(\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega \max_{|t|\le\theta} F(x,t)dx\Big)}{G_\eta - G^\theta} \right\}$$

and

$$\bar{\delta}_{\lambda,g} := \min\left\{ \underline{\delta}_{\lambda,g},\ \frac{1}{\max\left\{0, \dfrac{pk^p \text{meas}(\Omega)}{M^-} \limsup\limits_{|t|\to+\infty} \dfrac{\sup_{x\in\Omega} G(x,t)}{t^p}\right\}} \right\} \tag{3.12}$$

where we read $\varepsilon/0 = +\infty$, so that, for instance, $\bar{\delta}_{\lambda,g} = +\infty$ when

$$\limsup_{|t|\to+\infty} \frac{\sup_{x\in\Omega} G(x,t)}{t^p} \le 0,$$

and $G_\eta = G^\theta = 0$.

Theorem 3.7. Assume that there exist two positive constants θ and η with $\theta < k\sqrt[p]{L}\eta$ such that

(B_1) $F(x,t) \geq 0$ for each $(x,t) \in (\Omega \backslash B(x_0, \frac{s}{2})) \times [0,\eta]$;

(B_2) $\int_\Omega \max_{|t|\leq\theta} F(x,t)dx < \frac{1}{2}\frac{M^-\theta^p}{k^pM^+L\eta^p}\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx$;

(B_3) k^p meas$(\Omega)\limsup_{|t|\to\infty}\frac{F(x,t)}{t^p} < \Theta_1$ uniformly with respect to $x \in \Omega$ where

$$\Theta_1 := \max\Big\{\frac{\displaystyle\int_\Omega \max_{|t|\leq\theta} F(x,t)dx}{\frac{M^-}{p}(\frac{\theta}{k})^p}, \frac{\frac{M^-\theta^p}{k^pM^+L\eta^p}\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega \max_{|t|\leq\theta} F(x,t)dx}{\frac{hM^-}{p}(\frac{\theta}{k})^p}\Big\}$$

with $h > 1$.

Then, for every

$$\lambda \in \Lambda_1 := \Big(\frac{\frac{\eta^p}{p}M^+L}{\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega \max_{|t|\leq\theta} F(x,t)dx}, \frac{M^-}{pk^p}\frac{\theta^p}{\displaystyle\int_\Omega \max_{|t|\leq\theta} F(x,t)dx}\Big)$$

and for every L^1-Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the asymptotical condition

$$\limsup_{|t|\to+\infty}\frac{\sup_{x\in\Omega} G(x,t)}{t^p} < +\infty, \tag{3.13}$$

there exists $\overline{\delta}_{\lambda,g} > 0$ given by (3.12) such that, for each $\mu \in [0, \overline{\delta}_{\lambda,g})$, the problem $(P^{f,g}_{\lambda,\mu})$ admits at least three weak solutions in X and, moreover, for each $h > 1$, there exist an open interval

$$\Lambda_2 \subseteq \Big(0, \frac{\frac{hM^-}{p}(\frac{\theta}{k})^p}{\frac{M^-\theta^p}{k^pM^+L\eta^p}\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega \max_{|t|\leq\theta} F(x,t)dx}\Big)$$

and a positive real number σ, for each $\lambda \in \Lambda_2$, and for every L^1-Carathéodory function $g : \Omega\times\mathbb{R} \to \mathbb{R}$ satisfying the condition (3.13), for each

$$\mu \in \left[0, \frac{1}{\max\left\{0, \dfrac{pk^p\text{meas}(\Omega)}{M^-}\displaystyle\limsup_{|t|\to+\infty}\frac{\sup_{x\in\Omega} G(x,t)}{t^p}\right\}}\right),$$

the problem $(P^{f,g}_{\lambda,\mu})$ possesses at least three weak solutions in X whose norms are less than σ.

Proof. Fix λ, g and μ as in the conclusion and take Φ and Ψ as in the proof of Theorem 3.1. We observe that the regularity assumptions of Theorem 2.2 on Φ and Ψ are satisfied. Then, our aim is to verify (b_1), (b_2) and (b_3). To this end, since

$$\mu < \frac{1}{\max\left\{0, \dfrac{pk^p\text{meas}(\Omega)}{M^-}\displaystyle\limsup_{|t|\to+\infty}\frac{\sup_{x\in\Omega} G(x,t)}{t^p}\right\}},$$

we can fix $l > 0$ such that

$$\limsup_{|t|\to+\infty} \frac{\sup_{x\in\Omega} G(x,t)}{t^p} < l,$$

and $\mu l < \dfrac{M^-}{pk^p \operatorname{meas}(\Omega)}$. Therefore, there exists a function $\varrho \in \mathbb{R}$ such that

$$G(x,t) \le lt^p + \varrho \tag{3.14}$$

for every $(x,t) \in \Omega \times \mathbb{R}$. Fix $\lambda > 0$, from (B_3) there exist two constants $\gamma, \tau \in \mathbb{R}$ with $\gamma < \frac{1}{\lambda\Theta_1}(\frac{M^-}{p} - \mu l k^p \operatorname{meas}(\Omega))$ such that

$$\frac{k^p \ \operatorname{meas}(\Omega)}{\Theta_1} F(x,t) < \gamma t^p + \tau \text{ for all } (x,t) \in \Omega \times \mathbb{R}.$$

Fix $u \in X$. Then

$$F(x,u(x)) < \frac{\Theta_1}{k^p \ \operatorname{meas}(\Omega)}(\gamma|u(x)|^p + \tau) \text{ for all } x \in \Omega. \tag{3.15}$$

Now, to prove the coercivity of the functional $\Phi(u) - \lambda\Psi(u)$, first we assume that $\gamma > 0$. By using (3.15), we have

$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) \ge& \frac{M^-}{p}\|u\|^p - \lambda\int_\Omega [F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))]dx \\
\ge& \frac{M^-}{p}\|u\|^p - \frac{\lambda\Theta_1(\gamma\int_\Omega |u(x)|^p dx + \tau)}{k^p \ \operatorname{meas}(\Omega)} - \mu(l\int_\Omega |u(x)|^p dx + \varrho) \\
\ge& \Big(\frac{M^-}{p} - \lambda\Theta_1\gamma - \mu l k^p \operatorname{meas}(\Omega)\Big)\|u\|^p - \frac{\lambda\Theta_1\tau}{k^p \ \operatorname{meas}(\Omega)} - \mu\varrho
\end{aligned}$$

and so

$$\lim_{\|u\|\to+\infty} \Phi(u) - \lambda\Psi(u) = +\infty.$$

On the other hand, if $\gamma \le 0$, clearly, we obtain $\lim_{\|u\|\to+\infty} \Phi(u) - \lambda\Psi(u) = +\infty$. Both cases lead to the coercivity of functional $\Phi - \lambda\Psi$. Now choose w as given in (3.5), as well as

$$r = \frac{M^-}{p}(\frac{\theta}{k})^p.$$

Thanks to $\theta < k\sqrt[p]{L}\eta$, since $\frac{\eta^p}{p}M^-L \le \Phi(w)$, we have $\Phi(w) > r$. Moreover, by using (2.2), one has

$$\sup_{u\in\Phi^{-1}(-\infty,r]} \int_\Omega F(x,u(x))dx \le \int_\Omega \max_{|t|\le\theta} F(x,t)dx.$$

Since $0 \le w(x) \le \eta$ for each $x \in \Omega$, the assumption (B_1) ensures that

$$\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx + \int_{B(x^0,s)\setminus B(x^0,\frac{s}{2})} F(x,\eta)dx \ge 0.$$

Therefore, owing to our assumptions, we have

$$\sup_{u\in\overline{\Phi^{-1}(-\infty,r)}^{w}} \Psi(u) \le \int_\Omega \max_{|t|\le\theta} F(x,t)dx + \frac{\mu}{\lambda}G^\theta$$

and

$$\begin{aligned}\frac{r}{r+\Phi(w)}\Psi(w) &= \frac{r}{r+\Phi(w)}\Big(\int_\Omega [F(x,w(x)) + \frac{\mu}{\lambda}G(x,w(x))]dx\Big)\\ &\ge \frac{\frac{M^-\theta^p}{pk^p}}{\frac{M^-\theta^p}{pk^p}+\frac{M^+L\eta^p}{p}}\Big(\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx + \frac{\mu}{\lambda}G_\eta\Big)\\ &\ge \frac{1}{2}\frac{\frac{M^-\theta^p}{pk^p}}{\frac{M^+L\eta^p}{p}}\Big(\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx + \frac{\mu}{\lambda}G_\eta\Big).\end{aligned}$$

Now, we can apply Theorem 2.2. Taking into account that

$$\begin{aligned}&\frac{\Phi(w)}{\Psi(w)-\sup_{u\in\overline{\Phi^{-1}(-\infty,r)}^{w}}\Psi(u)}\\ &\le \frac{\frac{\eta^p}{p}M^+L}{\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx - \int_\Omega \max_{|t|\le\theta}F(x,t)dx + \frac{\mu}{\lambda}(G_\eta - G^\theta)}\end{aligned}$$

and

$$\begin{aligned}\frac{r}{\sup_{u\in\overline{\Phi^{-1}(-\infty,r)}^{w}}\Psi(u)} &= \frac{r}{\displaystyle\sup_{u\in\Phi^{-1}(-\infty,r)}\int_\Omega [F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))]dx}\\ &\ge \frac{M^-}{pk^p}\frac{\theta^p}{\displaystyle\int_\Omega \max_{|t|\le\theta}F(x,t)dx + \frac{\mu}{\lambda}G^\theta}.\end{aligned}$$

Since $\mu < \underline{\delta}_{\lambda,g}$, one has

$$\mu < \frac{M^-\theta^p - \lambda pk^p \displaystyle\int_\Omega \max_{|t|\le\theta}F(x,t)dx}{pk^pG^\theta},$$

this means

$$\frac{pk^p}{M^-}\frac{\displaystyle\int_\Omega \max_{|t|\le\theta}F(x,t)dx + \frac{\mu}{\lambda}G^\theta}{\theta^p} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{\frac{\eta^p}{p}M^+L - \lambda\Big(\displaystyle\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx - \int_\Omega \max_{|t|\le\theta}F(x,t)dx\Big)}{G_\eta - G^\theta},$$

this means
$$\frac{\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega \max_{|t|\le\theta} F(x,t)dx + \frac{\mu}{\lambda}(G_\eta - G^\theta)}{\frac{M^-}{p}(\frac{\theta}{k})^p} > \frac{1}{\lambda}.$$

Then,
$$\frac{pk^p}{M^-}\frac{\displaystyle\int_\Omega \max_{|t|\le\theta} F(x,t)dx + \frac{\mu}{\lambda}G^\theta}{\theta^p} < \frac{1}{\lambda}$$
$$< \frac{\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega \max_{|t|\le\theta} F(x,t)dx + \frac{\mu}{\lambda}(G_\eta - G^\theta)}{\frac{M^-}{p}(\frac{\theta}{k})^p}.$$

Moreover, we have
$$\frac{hr}{r\frac{\Psi(w)}{\Phi(w)} - \sup_{u\in\overline{\Phi^{-1}(-\infty,r)}^w}\Psi(u)}$$
$$\le \frac{\frac{hM^-}{p}(\frac{\theta}{k})^p}{\frac{M^-\theta^p}{k^pM^+L\eta^p}\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega \max_{|t|\le\theta} F(x,t)dx + \frac{\mu}{\lambda}(\frac{M^-\theta^p}{k^pM^+L\eta^p}G_\eta - G^\theta)} = \vartheta.$$

Hence, by choosing $u_0 = 0$, $u_1 = w$, taking into account that the weak solutions of the problem $(P^{f,g}_{\lambda,\mu})$ are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$, from Theorem 2.2 it follows that, for each $\lambda \in \Lambda_1$, the problem $(P^{f,g}_{\lambda,\mu})$ possesses at least three weak solutions, and there exist an open interval $\Lambda_2 \subseteq [0,\vartheta]$ and a real positive number σ such that, for each $\lambda \in \Lambda_2$, the problem $(P^{f,g}_{\lambda,\mu})$ possesses at least three weak solutions whose norms in X are less than σ. The proof is complete. Q.E.D.

Remark 3.8. In Theorem 3.7, we observe that
$$\frac{\frac{\eta^p}{p}M^+L}{\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega \max_{|t|\le\theta} F(x,t)dx} < \frac{M^-}{pk^p}\frac{\theta^p}{\displaystyle\int_\Omega \max_{|t|\le\theta} F(x,t)dx}.$$

Because, from (B_2) we have
$$2k^pM^+L\eta^p\int_\Omega \max_{|t|\le\theta} F(x,t)dx < M^-\theta^p\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx$$

and since $\theta < k\sqrt[p]{L}\eta$, we get
$$(M^-\theta^p + k^pM^+L\eta^p)\int_\Omega \max_{|t|\le\theta} F(x,t)dx < M^-\theta^p\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx$$

and so
$$k^pM^+L\eta^p\int_\Omega \max_{|t|\le\theta} F(x,t)dx < M^-\theta^p\Big(\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega \max_{|t|\le\theta} F(x,t)dx\Big).$$

Hence, multiplying by $\frac{1}{pk^p}$ it follows

$$\frac{M^+L\eta^p}{p}\int_\Omega \max_{|t|\le\theta} F(x,t)dx < \frac{M^-\theta^p}{pk^p}\Big(\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega \max_{|t|\le\theta} F(x,t)dx\Big),$$

which concludes

$$\frac{\frac{M^+L\eta^p}{p}}{\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega \max_{|t|\le\theta} F(x,t)dx} < \frac{\frac{M^-\theta^p}{pk^p}}{\int_\Omega \max_{|t|\le\theta} F(x,t)dx}.$$

Therefore, $\Lambda_1' \neq \varnothing$.

We now present the following example to illustrate Theorem 3.7.

Example 3.9. We consider the following problem

$$\begin{cases} \Delta\left(|\Delta u|\Delta u\right) - \left[M\left(\int_\Omega |\nabla u|^3 dx\right)\right]^2 \Delta_p u + \rho|u|u = \lambda f(u) + \mu g(u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega \end{cases} \tag{3.16}$$

where $\Omega = \{(x,y);\ x^2+y^2 < 9\}$, $M(t) = 2+\cos(t)$ for every $t \in [0,+\infty)$, $\rho = 1$, $f(t) = 5t^4e^{-\frac{1}{t}} + t^3e^{-\frac{1}{t}} + 5t^4$ and $g(t) = 3t^2$ for all $t \in \mathbb{R}$. By the expressions of f and g, we have $F(t) = t^5(e^{-\frac{1}{t}}+1)$ and $G(t) = t^3$ for all $t \in \mathbb{R}$. By simple calculations, we obtain $M^- = 1$ and $M^+ = 9$. Choose $x_0 = (0,0)$, $s = 2$, $\theta = 10^{-4}$, and $\eta = 1$. Therefore, since $k = \sqrt[3]{\frac{4}{\pi}}$, $L = 58.18309\pi$, all conditions in Theorem 3.7 are fulfilled. Then, for every

$$\lambda \in \Big(\frac{3L}{\pi(1-9\times 10^{-20}(e^{-10^4}+1))}, \frac{10^{-12}}{108\times 10^{-20}(e^{-10^4}+1)}\Big)$$

and for every

$$\mu \in \left[0, \min\left\{\min\left\{\frac{\pi(10^{-4} - \lambda\times 12\times 10^{-20}(e^{-10^4}+1))}{12\times 10^{-12}},\right.\right.\right.$$
$$\left.\left.\left.\frac{\lambda\pi(1-9\times 10^{-20}(e^{-10^4}+1)) - 3L}{10^{-12}}\right\}, \frac{1}{108}\right\}\right)$$

the problem (3.16) has at least three weak solutions in $W^{2,3}(\Omega)\bigcap W_0^{1,3}(\Omega)$ and, moreover, for $h = 2$, there exists a positive real number σ such that, for each

$$\lambda \in \Big(0, \frac{10^{-12}}{\frac{10^{-12}\pi}{6L} - 54\times 10^{-20}(e^{-10^4}+1)}\Big),$$

and for every $\mu \in \left[0, \frac{1}{108}\right)$, the problem (3.16) possesses at least three weak solutions in $W^{2,3}(\Omega)\bigcap W_0^{1,3}(\Omega)$ whose norms are less than σ.

Remark 3.10. The statements of Theorems 3.1 and 3.7 depend upon the test function w defined by (3.5). If we take other choices of the test function w we have other statements. For example, if $x^0 \in \Omega$ and we pick $s > 0$ such that $B(x^0, s) \subset \Omega$ where $B(x^0, s)$ denotes the ball with center at x^0 and radius of s, and

$$c_1' := \frac{2^{5P+1}\pi^{N/2}\eta^p}{s^{4p}\Gamma(N/2)} \int_{s/2}^{s} \left|2(N+2)r^2 - 3(N+1)sr + Nr^2\right|^p r^{N+1} dr,$$

$$c_2' := \left(\frac{32}{s^4}\right)^p \int_{B(x^0,s)\setminus B(x^0,\frac{s}{2})} \left[\sum_{i=1}^{N} \left((s-\ell)(s-3\ell)(x_i - x_i^0)\right)^2\right]^{\frac{p}{2}} dx,$$

$$c_3' := \left(\frac{16}{s^4}\right)^p \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \left(\frac{(\frac{s}{2})^N}{N} + \int_{\frac{s}{2}}^{s} \left|r^2(s-r)\right|^p r^{N-1} dr\right)$$

where Γ denotes the Gamma function,

$$L' := c_1' + c_2' + c_3',$$

and we take

$$w(x) := \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus B(x^0, s), \\ 16\frac{\ell^2}{s^4}(s-\ell)^2\eta & \text{if } x \in B(x^0, s) \setminus B(x^0, \frac{s}{2}), \\ \eta & \text{if } x \in B(x^0, \frac{s}{2}) \end{cases} \tag{3.17}$$

as used in [40], with $\ell = dist(x, x^0) = \sqrt{\Sigma_{i=1}^{N}\left(x_i - x_i^0\right)^2}$, then one has

$$\begin{aligned} \frac{\eta^p}{p} M^- L' &\le \frac{1}{p}(c_1'\eta^p + m_0^{p-1}c_2'\eta^p + \rho c_3'\eta^p) \le \Phi(w) = \frac{1}{p}(c_1'\eta^p + \widehat{M}(c_2'\eta^p) + \rho c_3'\eta^p) \\ &\le \frac{1}{p}(c_1'\eta^p + m_1^{p-1}c_2'\eta^p + \rho c_3'\eta^p) \le \frac{\eta^p}{p} M^+ L'. \end{aligned}$$

For calculations see [27, Remark 3.1].
Therefore, Theorem 3.1 takes the following form:
Assume that there exist positive constants $\theta_1, \theta_2, \theta_3$ and η with $\theta_1 < k\sqrt[p]{L'}\eta$, $\max\{\eta, k\sqrt[p]{\frac{M^+L'}{M^-}}\eta\} < \theta_2$ and $\theta_2 < \theta_3$ such that the assumption (A_1) in Theorem 3.1 holds. Furthermore, suppose that

(A_9)

$$\begin{aligned} &\max\left\{\frac{\int_\Omega F(x,\theta_1)dx}{\theta_1^p}, \frac{\int_\Omega F(x,\theta_2)dx}{\theta_2^p}, \frac{\int_\Omega F(x,\theta_3)dx}{\theta_3^p - \theta_2^p}\right\} \\ &< \frac{M^-}{k^p M^+ L'} \frac{\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega F(x,\theta_1)dx}{\eta^p}. \end{aligned}$$

Then, for every

$$\lambda \in \Big(\frac{\frac{\eta^p}{p} M^+ L'}{\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega F(x,\theta_1)dx},$$

$$\frac{M^-}{pk^p}\min\Big\{\frac{\theta_1^p}{\int_\Omega F(x,\theta_1)dx},\frac{\theta_2^p}{\int_\Omega F(x,\theta_2)dx},\frac{\theta_3^p-\theta_2^p}{\int_\Omega F(x,\theta_3)dx}\Big\}\Big)$$

and for every non-negative L^1-Carathéodory function $g:\Omega\times\mathbb{R}\to\mathbb{R}$ there exists $\delta_{\lambda,g}>0$ given by

$$\delta_{\lambda,g}:=\min\left\{\frac{1}{pk^p}\min\Big\{\frac{M^-\theta_1^p-\lambda pk^p\displaystyle\int_\Omega F(x,\theta_1)dx}{G^{\theta_1}},\frac{M^-\theta_2^p-\lambda pk^p\displaystyle\int_\Omega F(x,\theta_2)dx}{G^{\theta_2}},\right.$$
$$\frac{M^-(\theta_3^p-\theta_2^p)-\lambda pk^p\displaystyle\int_\Omega F(x,\theta_3)dx}{G^{\theta_3}}\Big\},$$
$$\left.\frac{\frac{\eta^p}{p}M^+L'-\lambda\Big(\displaystyle\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx-\int_\Omega F(x,\theta_1)dx\Big)}{G_\eta-G^{\theta_1}}\right\}$$

such that, for each $\mu\in[0,\delta_{\lambda,g})$, the problem $(P^{f,g}_{\lambda,\mu})$ possesses at least three weak solutions u_1, u_2, and u_3 such that

$$\max_{x\in\Omega}|u_1(x)|<\theta_1,\ \max_{x\in\Omega}|u_2(x)|<\theta_2 \text{ and } \max_{x\in\Omega}|u_3(x)|<\theta_3.$$

Moreover, in this case, Theorem 3.7 takes the following form:
Assume that there exist two positive constants θ and η with $\theta<k\sqrt[p]{L'}\eta$ such that the assumption (B_1) in Theorem 3.7 holds. Furthermore, suppose that

(B$_4$) $\int_\Omega \max_{|t|\le\theta}F(x,t)dx<\frac{1}{2}\frac{M^-\theta^p}{k^pM^+L'\eta^p}\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx$;

(B$_5$) k^p meas$(\Omega)\limsup_{|t|\to\infty}\frac{F(x,t)}{t^p}<\Theta_1$ uniformly with respect to $x\in\Omega$ where

$$\Theta_1:=\max\Big\{\frac{\displaystyle\int_\Omega\max_{|t|\le\theta}F(x,t)dx}{\frac{M^-}{p}(\frac{\theta}{k})^p},\frac{\frac{M^-\theta^p}{k^pM^+L'\eta^p}\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx-\int_\Omega\max_{|t|\le\theta}F(x,t)dx}{\frac{hM^-}{p}(\frac{\theta}{k})^p}\Big\}$$

with $h>1$.

Then, for every

$$\lambda\in\Lambda_1:=\Big(\frac{\frac{\eta^p}{p}M^+L'}{\int_{B(x^0,\frac{s}{2})}F(x,\eta)dx-\int_\Omega\max_{|t|\le\theta}F(x,t)dx},\frac{M^-}{pk^p}\frac{\theta^p}{\displaystyle\int_\Omega\max_{|t|\le\theta}F(x,t)dx}\Big)$$

for every L^1-Carathéodory function $g:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfying the condition (3.13), there exists $\overline{\delta}_{\lambda,g}>0$ given by

$$\overline{\delta}_{\lambda,g}:=\min\left\{\underline{\delta}_{\lambda,g},\ \frac{1}{\max\left\{0,\dfrac{pk^p\text{meas}(\Omega)}{M^-}\limsup\limits_{|t|\to+\infty}\dfrac{\sup_{x\in\Omega}G(x,t)}{t^p}\right\}}\right\}$$

where

$$\underline{\delta}_{\lambda,g} := \min\left\{ \frac{M^-\theta^p - \lambda p k^p \int_\Omega \max_{|t|\le\theta} F(x,t)dx}{pk^p G^\theta}, \right.$$

$$\left. \frac{\frac{\eta^p}{p} M^+ L' - \lambda\Big(\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega \max_{|t|\le\theta} F(x,t)dx\Big)}{G_\eta - G^\theta} \right\}$$

such that, for each $\mu \in [0, \overline{\delta}_{\lambda,g})$, the problem $(P^{f,g}_{\lambda,\mu})$ admits at least three weak solutions in X and, moreover, for each $h > 1$, there exist an open interval

$$\Lambda_2 \subseteq \Big(0, \frac{\frac{hM^-}{p}(\frac{\theta}{k})^p}{\frac{M^-\theta^p}{k^p M^+ L' \eta^p}\int_{B(x^0,\frac{s}{2})} F(x,\eta)dx - \int_\Omega \max_{|t|\le\theta} F(x,t)dx}\Big)$$

and a positive real number σ, for each $\lambda \in \Lambda_2$, and for every L^1-Carathéodory function $g : \Omega\times\mathbb{R} \to \mathbb{R}$ satisfying the condition (3.13), for each

$$\mu \in \left[0, \frac{1}{\max\left\{0, \frac{pk^p \mathrm{meas}(\Omega)}{M^-} \limsup_{|t|\to+\infty} \frac{\sup_{x\in\Omega} G(x,t)}{t^p}\right\}}\right),$$

the problem $(P^{f,g}_{\lambda,\mu})$ possesses at least three weak solutions in X whose norms are less than σ.

Remark 3.11. By choosing w as given in [32, 33] which is

$$w(x) := \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus B(x^0, s_2), \\ \frac{3(\ell^4 - s_2^4) - 4(s_1+s_2)(\ell^3 - s_2^3) + 6 s_1 s_2 (\ell^2 - s_2^2)}{(s_2-s_1)^3(s_1+s_2)}\eta & \text{if } x \in B(x^0, s_2) \setminus B(x^0, s_1), \\ \eta & \text{if } x \in B(x^0, s_1) \end{cases}$$

where $\ell = \mathrm{dist}(x, x^0) = \sqrt{\Sigma_{i=1}^N (x_i - x_i^0)^2}$ and $s_1, s_2 \in \mathbb{R}$ with $s_2 > s_1 > 0$, we have another statement of Theorems 3.1 and 3.7.

Remark 3.12. If in Theorem 3.7, either $f(x,0) \neq 0$ or $g(x,0) \neq 0$ for all $x \in \Omega$, or both hold true, then the ensured weak solutions in the above results are obviously non-trivial. On the other hand, the non-triviality of the weak solutions can be achieved also in the case $f(x,0) = 0$ and $g(x,0) = 0$ for all $x \in \Omega$ requiring the extra condition at zero of f, that is there are a non-empty open set $D \subseteq \Omega$ and a set $B \subset D$ of positive Lebesgue measure such that

$$\limsup_{\xi\to 0^+} \frac{ess\inf_{x\in B} F(x,\xi)}{|\xi|^p} = +\infty$$

and

$$\liminf_{\xi\to 0^+} \frac{ess\inf_{x\in D} F(x,\xi)}{|\xi|^p} > -\infty$$

(see [1, Remark 3.8]).

We now present a particular case of Theorem 3.7, in which the function f has separated variables.

Theorem 3.13. Let $f_1 \in L^1(\Omega)$ and $f_2 \in C(\mathbb{R})$ be two functions. Put $\tilde{F}(t) = \int_0^t f_2(\xi)d\xi$ for all $t \in \mathbb{R}$ and assume that there exist two positive constants θ and η with $\theta < k\sqrt[p]{L}\eta$ such that

(B$_6$) $f_1(x) \geq 0$ for each $x \in \Omega \backslash B(x_0, \frac{s}{2})$ and $f_2(t) \geq 0$ for each $t \in [0, \eta]$;

(B$_7$) $\max_{|t|\leq\theta} \tilde{F}(t) \int_\Omega f_1(x)dx < \frac{\tilde{F}(\eta)}{2} \frac{M^-\theta^p}{k^p M^+ L\eta^p} \int_{B(x^0,\frac{s}{2})} f_1(x)dx$;

(B$_8$) k^p meas$(\Omega) \limsup_{|t|\to\infty} \frac{\tilde{F}(t)}{t^p} < \Theta_2$ where

$$\Theta_2 := \max\Big\{ \frac{\max\limits_{|t|\leq\theta} \tilde{F}(t) \displaystyle\int_\Omega f_1(x)dx}{\frac{M^-}{p}(\frac{\theta}{k})^p},$$

$$\frac{\frac{M^-\theta^p}{k^p M^+ L\eta^p} \tilde{F}(\eta) \int_{B(x^0,\frac{s}{2})} f_1(x)dx - \max_{|t|\leq\theta} \tilde{F}(t) \int_\Omega f_1(x)dx}{\frac{hM^-}{p}(\frac{\theta}{k})^p} \Big\}$$

with $h > 1$.

Then, for every

$$\lambda \in \Lambda_1' := \Big(\frac{\frac{\eta^p}{p} M^+ L}{\tilde{F}(\eta) \int_{B(x^0,\frac{s}{2})} f_1(x)dx - \max_{|t|\leq\theta} \tilde{F}(t) \int_\Omega f_1(x)dx},$$

$$\frac{M^-}{pk^p} \frac{\theta^p}{\max\limits_{|t|\leq\theta} \tilde{F}(t) \displaystyle\int_\Omega f_1(x)dx} \Big)$$

and for every L^1-Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the condition (3.13), there exists $\overline{\delta}'_{\lambda,g} > 0$ given by

$$\overline{\delta}'_{\lambda,g} := \min \left\{ \underline{\delta}'_{\lambda,g}, \ \frac{1}{\max\left\{0, \dfrac{pk^p \text{meas}(\Omega)}{M^-} \limsup\limits_{|t|\to+\infty} \dfrac{\sup_{x\in\Omega} G(x,t)}{t^p}\right\}} \right\}$$

where

$$\underline{\delta}'_{\lambda,g} := \min \Big\{ \frac{M^-\theta_1^p - \lambda pk^p \max\limits_{|t|\leq\theta} \tilde{F}(t) \displaystyle\int_\Omega f_1(x)dx}{pk^p G^\theta},$$

$$\frac{\frac{\eta^p}{p} M^+ L - \lambda\big(\tilde{F}(\eta) \displaystyle\int_{B(x^0,\frac{s}{2})} f_1(x)dx - \max\limits_{|t|\leq\theta} \tilde{F}(t) \displaystyle\int_\Omega f_1(x)dx\big)}{G_\eta - G^\theta} \Big\},$$

such that, for each $\mu \in [0, \overline{\delta}'_{\lambda,g})$, the problem (3.9) possesses at least three weak solutions in X and, moreover, for each $h > 1$, there exist an open interval

$$\Lambda_2' \subseteq \left(0, \frac{\frac{hM^-}{p}(\frac{\theta}{k})^p}{\frac{M^-\theta^p}{k^p M^+ L\eta^p}\tilde{F}(\eta)\int_{B(x^0,\frac{s}{2})} f_1(x)dx - \max_{|t|\le\theta}\tilde{F}(t)\int_\Omega f_1(x)dx}\right)$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2''$, and for every L^1-Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the condition (3.13), for each

$$\mu \in \left[0, \frac{1}{\max\left\{0, \frac{pk^p \mathrm{meas}(\Omega)}{M^-}\limsup_{|t|\to+\infty}\frac{\sup_{x\in\Omega} G(x,t)}{t^p}\right\}}\right),$$

the problem (3.9) possesses at least three weak solutions in X whose norms are less than σ.

Proof. Set $f(x,u) = f_1(x)f_2(u)$ for each $(x,u) \in \Omega \times \mathbb{R}$. Since

$$F(x,t) = f_1(x)\tilde{F}(t) \tag{3.18}$$

from (B_6) and (B_7) we get (B_1) and (B_2) fulfilled, respectively. From (3.18) and (B_8) we have

$$F(x,t) \le |f_1(x)\tilde{F}(t)| \le |f_1(x)||\tilde{F}(t)|$$

for each $(x,u) \in \Omega \times \mathbb{R}$, so the condition (B_3) is satisfied. Then, Theorem 3.7 yields the conclusion.

Q.E.D.

We have the following result as a direct consequence of Theorem 3.7.

Theorem 3.14. Assume that there exist two positive constants θ and η with $\theta < k\sqrt[p]{L}\eta$ such that

(B_9) $f(t) \ge 0$ for each $t \in [0,\eta]$;

(B_{10})

$$\mathrm{meas}(\Omega)\max_{|t|\le\theta} F(t) < \frac{\mathrm{meas}(B(x^0,\frac{s}{2}))}{2}\frac{M^-\theta^p}{k^p M^+ L\eta^p}F(\eta);$$

(B_{11}) $k^p\ \mathrm{meas}(\Omega)\limsup_{|t|\to\infty}\frac{F(t)}{t^p} < \Theta_3$ where

$$\Theta_3 := \max\left\{\frac{\mathrm{meas}(\Omega)\max\limits_{|t|\le\theta} F(t)}{\frac{M^-}{p}(\frac{\theta}{k})^p},\right.$$

$$\left.\frac{\frac{M^-\theta^p}{k^p M^+ L\eta^p}\mathrm{meas}(B(x^0,\frac{s}{2}))F(\eta) - \mathrm{meas}(\Omega)\max\limits_{|t|\le\theta} F(t)}{\frac{hM^-}{p}(\frac{\theta}{k})^p}\right\}$$

with $h > 1$.

Then, for every

$$\lambda \in \Lambda_1'' := \Big(\frac{\frac{\eta^p}{p}M^+L}{\mathrm{meas}(B(x^0,\frac{s}{2}))F(\eta)-\mathrm{meas}(\Omega)\max_{|t|\le\theta}F(t)}, \frac{M^-}{pk^p}\frac{\theta^p}{\mathrm{meas}(\Omega)\max\limits_{|t|\le\theta}F(t)}\Big)$$

and for every L^1-Carathéodory function $g:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfying the condition (3.13), there exists $\overline{\delta}''_{\lambda,g}>0$ given by

$$\overline{\delta}''_{\lambda,g} := \min\left\{\underline{\delta}''_{\lambda,g},\ \frac{1}{\max\left\{0, \dfrac{pk^p\mathrm{meas}(\Omega)}{M^-}\limsup\limits_{|t|\to+\infty}\dfrac{\sup_{x\in\Omega}G(x,t)}{t^p}\right\}}\right\}$$

where

$$\underline{\delta}''_{\lambda,g} := \min\left\{\frac{M^-\theta_1^p-\lambda pk^p\mathrm{meas}(\Omega)\max\limits_{|t|\le\theta}F(t)}{pk^pG^\theta}, \frac{\frac{\eta^p}{p}M^+L-\lambda\big(\mathrm{meas}(B(x^0,\dfrac{s}{2}))F(\eta)-\mathrm{meas}(\Omega)\max\limits_{|t|\le\theta}F(t)\big)}{G_\eta-G^\theta}\right\},$$

such that, for each $\mu\in[0,\overline{\delta}''_{\lambda,g})$, the problem (3.10) admits at least three weak solutions in X and, moreover, for each $h>1$, there exist an open interval

$$\Lambda_2''\subseteq\Big(0, \frac{\frac{hM^-}{p}(\frac{\theta}{k})^p}{\frac{M^-\theta^p}{k^pM^+L\eta^p}\mathrm{meas}(B(x^0,\frac{s}{2}))F(\eta)-\mathrm{meas}(\Omega)\max_{|t|\le\theta}F(t)}\Big)$$

and a positive real number σ such that, for each $\lambda\in\Lambda_2''$, and for every L^1-Carathéodory function $g:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfying the condition (3.13), for each

$$\mu\in\left[0, \frac{1}{\max\left\{0, \dfrac{pk^p\mathrm{meas}(\Omega)}{M^-}\limsup\limits_{|t|\to+\infty}\dfrac{\sup_{x\in\Omega}G(x,t)}{t^p}\right\}}\right)$$

the problem (3.10) possesses at least three weak solutions in X whose norms are less than σ.

In the following, we give a direct application of Theorem 3.7.

Theorem 3.15. Assume that $F(\eta)>0$ for some $\eta>0$ and $F(\xi)\ge0$ in $[0,\eta]$ and

$$\liminf_{\xi\to0}\frac{F(\xi)}{\xi^p}=\limsup_{\xi\to\infty}\frac{F(\xi)}{\xi^p}=0.$$

Then, there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ and for every L^1-Carathéodory function $g : \Omega\times\mathbb{R} \to \mathbb{R}$ satisfying the condition (3.13) such that, for each $\mu \in [0, \delta^*_{\lambda,g})$, the problem (3.10) possesses at least three weak solutions.

Proof. Fix $\lambda > \lambda^* := \frac{\frac{\eta^p}{p}M^+L}{\text{meas}(B(x^0,\frac{s}{2}))F(\eta)-\text{meas}(\Omega)\max_{|t|\leq\theta}F(t)}$ for some $\eta > 0$. From the condition

$$\liminf_{\xi\to 0}\frac{F(\xi)}{\xi^p} = 0,$$

there is a sequence $\{\theta_n\} \subset]0, +\infty[$ such that $\lim\limits_{n\to\infty}\theta_n = 0$ and

$$\lim_{n\to\infty}\frac{\sup\limits_{|\xi|\leq\theta_n}F(\xi)}{\theta_n^p} = 0.$$

Indeed, one has

$$\lim_{n\to\infty}\frac{\sup\limits_{|\xi|\leq\theta_n}F(\xi)}{\theta_n^p} = \lim_{n\to\infty}\frac{F(\xi_{\theta_n})}{\xi^p_{\theta_n}}\frac{\xi^p_{\theta_n}}{\theta_n^p} = 0$$

where $F(\xi_{\theta_n}) = \sup\limits_{|\xi|\leq\theta_n}F(\xi)$. Hence, there exists $\overline{\theta} > 0$ such that

$$\frac{\sup\limits_{|\xi|\leq\overline{\theta}}F(\xi)}{\overline{\theta}^p} < \min\left\{\frac{M^-\text{meas}(B(x^0,\frac{s}{2}))F(\eta)}{k^pM^+L\eta^p\text{meas}(\Omega)},\ \frac{M^-}{\lambda pk^p\text{meas}(\Omega)}\right\}$$

and $\theta < k\sqrt[p]{L}\eta$. Applying Theorem 3.7 we have the conclusion. Q.E.D.

We here give an immediate consequence of Theorem 3.7 as follows.

Theorem 3.16. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative bounded continuous function such that

$$\int_0^{10^{-3}} f(t)dt < \frac{10^{-9}}{648(58.18309)}\int_0^1 f(t)dt.$$

Then, for every

$$\lambda \in \Lambda_1^* := \left(\frac{3(58.18309)}{\int_0^1 f(t)dt - 9\int_0^{10^{-3}} f(t)dt}, \frac{10^{-9}}{108\int_0^{10^{-3}} f(t)dt}\right)$$

and for every L^1-Carathéodory function $g : \Omega\times\mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\limsup_{|t|\to\infty}\frac{\sup_{x\in\Omega}\int_0^t g(x,s)ds}{t^3} < \infty, \tag{3.19}$$

there exists $\delta^{**}_{\lambda,g} > 0$, such that, for each $\mu \in [0, \delta^{**}_{\lambda,g})$, the problem

$$\begin{cases} \Delta\left(|\Delta u|\Delta u\right) - \left[M\left(\int_\Omega|\nabla u|^3dx\right)\right]^2\Delta_p u + \rho|u|u = \lambda f(u) + \mu g(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.20}$$

admits at least three weak solutions in $W^{2,3}(\Omega)\bigcap W_0^{1,3}(\Omega)$ and, moreover, there exist an open interval

$$\Lambda_2^* \subseteq \Big(0, \frac{10^{-9}}{\frac{10^{-9}}{6(58.18309)}\int_0^1 f(t)dt - 54\int_0^{10^{-3}} f(t)dt}\Big)$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2^*$, and for every L^1-Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the condition (3.19), there exists $\dot{\delta}_{\lambda,g}^{**} > 0$, such that, for each $\mu \in [0, \dot{\delta}_{\lambda,g}^{**})$, the problem (3.20) possesses at least three weak solutions in $W^{2,3}(\Omega)\bigcap W_0^{1,3}(\Omega)$ whose norms are less than σ.

Proof. Taking into account that $\max_{|t|\le\theta} F(t) = F(\theta)$ and $\int_0^{\eta} F(t)dt > 0$, the conclusion follows from Theorem 3.7, by choosing $\Omega = \{(x,y);\ x^2+y^2 < 9\}$, $M(t) = 2+\sin(t)$ for every $t \in [0,+\infty)$, $\rho = 1$, $x_0 = (0,0)$, $s = 2$, $\theta = 10^{-3}, \eta = 1$ and $h = 2$. Q.E.D.

Now, we conclude this paper by giving the following example to illustrate Theorem 3.16.

Example 3.17. Define the function

$$f(t) = \begin{cases} 5t^4, & \text{if } |t| \le 1, \\ \frac{3t^2+2}{t^2}, & \text{if } |t| > 1. \end{cases}$$

By simple computations, we obtain

$$F(t) = \begin{cases} t^5, & \text{if } |t| \le 1, \\ 3t - \frac{2}{t}, & \text{if } |t| > 1. \end{cases}$$

Clearly, we see that f satisfies the assumptions of Theorem 3.16. Hence, Theorem 3.16 follows that for every

$$\lambda \in \Big(\frac{3(58.18309)}{1-9\times 10^{-15}}, \frac{10^{-9}}{108\times 10^{-15}}\Big),$$

there exists $\hat{\delta}_{\lambda,g} > 0$, such that, for each $\mu \in [0, \hat{\delta}_{\lambda,g})$, the problem (3.20) admits at least three weak solutions in $W^{2,3}(\Omega)\bigcap W_0^{1,3}(\Omega)$ and, moreover, there exists a positive real number σ such that for each

$$\lambda \in \Big(0, \frac{10^{-9}}{\frac{10^{-9}}{6(58.18309)} - 54\times 10^{-15}}\Big),$$

there exists $\hat{\delta}_{\lambda,g} > 0$, such that, for each $\mu \in [0, \hat{\delta}_{\lambda,g})$, the problem (3.20) possesses at least three weak solutions in $W^{2,3}(\Omega)\bigcap W_0^{1,3}(\Omega)$ whose norms are less than σ.

References

[1] G.A. Afrouzi, G. Caristi, S. Moradi, D. Barilla, *One solution for nonlocal fourth order equations,* Boletim Sociedade Paranaense de Matemtica, to appear.

[2] G. Autuori, F. Colasuonno, P. Pucci, *Blow up at infinity of solutions of polyharmonic Kirchhoff systems,* Complex Var. Elliptic Equ. **57** (2012) 379-395.

[3] G. Autuori, F. Colasuonno, P. Pucci, *Lifespan estimates for solutions of polyharmonic Kirchhoff systems,* Math. Mod. Meth. Appl. Sci. **22** (2012) 1150009 [36 pages].

[4] G. Autuori, F. Colasuonno, P. Pucci, *On the existence of stationary solutions for higher-order p-Kirchhoff problems,* Commun. Contemp. Math. **16** (2014) 1450002 [43 pages].

[5] G. Bonanno, *A critical points theorem and nonlinear differential problems,* J. Global Optim. **28** (2004) 249-258.

[6] G. Bonanno, P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities,* J. Differ. Equ. **244** (2008) 3031-3059.

[7] G. Bonanno, B. Di Bella, *A boundary value problem for fourth-order elastic beam equations,* J. Math. Anal. Appl. **343** (2008) 1166-1176.

[8] A. Cabada, J.A. Cid, L. Sanchez, *Positivity and lower and upper solutions for fourth-order boundary value problems,* Nonlinear Anal. TMA **67** (2007) 1599-1612.

[9] M. Chipot, B. Lovat, *Some remarks on non local elliptic and parabolic problems,* Nonlinear Anal. TMA **30** (1997) 4619-4627.

[10] F. Colasuonno, P. Pucci, *Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations,* Nonlinear Anal. TMA **74** (2011) 5962-5974.

[11] M. Ferrara, S. Khademloo, S. Heidarkhani, *Multiplicity results for perturbed fourth-order Kirchhoff type elliptic problems,* Appl. Math. Comput. **234** (2014) 316-325.

[12] J.R. Graef, S. Heidarkhani, L. Kong, *A variational approach to a Kirchhoff-type problem involving two parameters,* Results. Math. **63** (2013) 877-889.

[13] J.R. Graef, S. Heidarkhani, L. Kong, *Multiple solutions for a class of $(p_1, \ldots, p_n)$-biharmonic systems,* Commun. Pure Appl. Anal. (CPAA) **12** (2013) 1393-1406.

[14] X. He, W. Zou, *Infinitely many positive solutions for Kirchhoff-type problems,* Nonlinear Anal. TMA **70** (2009) 1407-1414.

[15] S. Heidarkhani, *Existence of non-trivial solutions for systems of n fourth order partial differential equations,* Math. Slovaca **64** (2014) 1249-1266.

[16] S. Heidarkhani, *Existence of solutions for a two-point boundary-value problem of a fourth order Sturm-Liouville type,* Electronic J. Differ. Equ., Vol. **2012** (2012), No. 84, pp. 1-15.

[17] S. Heidarkhani, G.A. Afrouzi, *Some multiplicity results to the existence of three solutions for a dirichlet boundary value problem involving the p-Laplacian,* Math. Model. Anal. **16** (2011) 390-400.

[18] S. Heidarkhani, G.A. Afrouzi, M. Ferrara, S. Moradi, *Variational approaches to impulsive elastic beam equations of Kirchhoff type,* Complex Var. Elliptic Equ. **61** (2016) 931-968.

[19] S. Heidarkhani, G. A. Afrouzi, S. Moradi, G. Caristi, *Existence of three solutions for multi-point boundary value problems,* J. Nonlinear Funct. Anal. 2017, Art. ID 47, 119.

[20] S. Heidarkhani, G.A. Afrouzi, D. O'Regan, *Existence of three solutions for a Kirchhoff-type boundary-value problem,* Electronic J. Differ. Equ. Vol. **2011** (2011), No. 91, pp. 1-11.

[21] S. Heidarkhani, A. Cabada, G.A. Afrouzi, S. Moradi, G. Caristi, *A variational approach to perturbed impulsive fractional differential equations,* J. Comput. Appl. Math. **341** (2018) 42-60.

[22] S. Heidarkhani, A.L.A. De Araujo, G.A. Afrouzi, S. Moradi, *Multiple solutions for Kirchhoff-type problems with variable exponent and nonhomogeneous Neumann conditions,* Math. Nachr. **291** (2018) 326-342.

[23] S. Heidarkhani, M. Ferrara, S. Khademloo, *Nontrivial solutions for one-dimensional fourth-order Kirchhoff-type equations,* Mediterr. J. Math. **13** (2016) 217-236.

[24] S. Heidarkhani, J. Henderson, *Infinitely many solutions for nonlocal elliptic system of* $(p_1, \ldots, p_n)$*-Kirchhoff type,* Electronic J. Differ. Equ., Vol. **2012** (2012), No. 69, pp. 1-15.

[25] S. Heidarkhani, S. Khademloo, A. Solimaninia, *Multiple solutions for a perturbed fourth-order Kirchhoff type elliptic problem,* Portugal. Math. (N.S.) **71** (2014) 39-61.

[26] S. Heidarkhani, S. Moradi, S.A. Tersian, *Three solutions for second-order boundary-value problems with variable exponents,* Electron. J. Qual. Theory Differ. Equ. 2018, No. 33, 1-19.

[27] S. Heidarkhani, A. Salari, G. Caristi, D. Barilla, *Perturbed nonlocal fourth order equations of Kirchhoff type with Navier boundary conditions,* Bound. Value Probl. (2017) 2017:86, 1-20.

[28] S. Heidarkhani, Y. Tian, C.-L. Tang, *Existence of three solutions for a class of* $(p_1, \ldots, p_n)$*-biharmonic systems with Navier boundary conditions,* Ann. Polon. Math. **104** (2012) 261-277.

[29] L. Jiang, Z. Zhou, *Three solutions to Dirichlet boundary value problems for p-Laplacian difference equations,* Adv. Differ. Equ. **2008** (2007) 1-10.

[30] G. Kirchhoff, *Vorlesungen uber mathematische Physik: Mechanik,* Teubner, Leipzig (1883).

[31] L. Kong, *Existence of solutions to boundary value problems arising from the fractional advection dispersion equation,* Electronic J. Diff. Equ., Vol. **2013** (2013), No. 106, pp. 1-15.

[32] L. Li, C.-L. Tang, *Existence of three solutions for* (p, q)*-biharmonic systems,* Nonlinear Anal. TMA **73** (2010) 796-805.

[33] C. Li, C.-L. Tang, *Three solutions for a Navier boundary value problem involving the p-biharmonic,* Nonlinear Anal. TMA **72** (2010) 1339-1347.

[34] J. Liu, S.X. Chen, X. Wu, *Existence and multiplicity of solutions for a class of fourth-order elliptic equations in* $\mathbb{R}^N$*,* J. Math. Anal. Appl. **395** (2012) 608-615.

[35] H. Liu, N. Su, *Existence of three solutions for a p-biharmonic problem,* Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **15** (2008) 445-452.

[36] A. Mao, Z. Zhang, *Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition,* Nonlinear Anal. TMA **70** (2009) 1275-1287.

[37] M. Massar, E.M. Hssini, N. Tsouli, M. Talbi, *Infinitely many solutions for a fourth-order Kirchhoff type elliptic problem,* J. Math. Comput. Sci. **8** (2014) 33-51.

[38] G. Molica Bisci, V. Rădulescu, *Applications of local linking to nonlocal Neumann problems,* Commun. Contemp. Math. **17** (2014) 1450001 [17 pages].

[39] G. Molica Bisci, V. Rădulescu, *Mountain pass solutions for nonlocal equations,* Annales AcademiæScientiarum FennicæMathematica **39** (2014) 579-592.

[40] G. Molica Bisci, D. Repovš, *Multiple solutions of p-biharmonic equations with Navier boundary conditions,* Complex Var. Elliptic Equ. **59** (2014) 271-284.

[41] K. Perera, Z.T. Zhang, *Nontrivial solutions of Kirchhoff-type problems via the Yang index,* J. Differ. Equ. **221** (2006) 246-255.

[42] B. Ricceri, *On an elliptic Kirchhoff-type problem depending on two parameters,* J. Global Optim. **46** (2010) 543-549.

[43] F. Wang, Y. An, *Existence and multiplicity of solutions for a fourth-order elliptic equation,* Bound. Value Probl. (2012) 2012:6, 1-9.

[44] F. Wang, T. An, Y. An, *Existence of solutions for fourth order elliptic equations of Kirchhoff type on* $\mathbb{R}^N$, Electron. J. Qual. Theory Differ. Equ. **39** (2014) 1-11.

[45] F. Wang, M. Avci, Y. An, *Existence of solutions for fourth order elliptic equations of Kirchhoff-type,* J. Math. Anal. Appl. **409** (2014) 140-146.