I-Convergence of triple difference sequence spaces over n-normed space

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Abstract

The main objective of this paper is to study triple difference sequence spaces over n-normed space via the sequence of modulus functions. Some algebraic and topological properties of the newly constructed spaces are also established.

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1 Introduction

A triple sequence (real or complex) is a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ are the set of natural numbers, real numbers, and complex numbers respectively. We denote by ω''' the class of all complex triple sequence (x_{pqr}), where $p, q, r \in \mathbb{N}$. Then under the coordinate wise addition and scalar multiplication ω''' is a linear space. A triple sequence can be represented by a matrix, in case of double sequences we write in the form of a square. In case of triple sequence it will be in the form of a box in three dimensions.

The different types of notions of triple sequences and their statistical convergence were introduced and investigated initially by Sahiner et. al [28]. Later Debnath et.al [3, 4, 7, 8], Esi [10], Esi and Catalbas [11], Esi and Savas [12], Tripathy [30] and many others authors have studied it further and obtained various results.

Kizmaz [20] introduced the notion of difference sequence spaces and defined the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ as follows:

$$Z(\Delta) = \{x = (x_k) \in \omega \colon (\Delta x_k) \in Z\}$$

for $Z = c_0$, c and ℓ_∞, where
$\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$

The difference operator on triple sequence is defined as [2, 5]

$$\Delta x_{mnk} = x_{mnk} - x_{(m+1)nk} - x_{m(n+1)k} - x_{mn(k+1)} + x_{(m+1)(n+1)k} + x_{(m+1)n(k+1)} + x_{m(n+1)(k+1)} - x_{(m+1)(n+1)(k+1)}$$

and $\Delta^0_{mnk} = (x_{mnk})$.

Statistical convergence was introduced by Fast [13] and later on it was studied by Fridy [14, 15] from the sequence space point of view and linked it with summability theory. The notion of

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statistical convergent in double sequence spaces was introduced by Mursaleen and Edely [24] which was further studied by many authors like Debnath and Subramanian [9].

I-convergence is a generalization of the statistical convergence. Kostyrko et. al. [21] introduced the notion of I-convergence of real sequence and studied its several properties. Later Jalal [17, 18, 19], Debnath and Saha [6], Salat et. al. [26] and many other researchers contributed in its study. Sahiner and Tripathy [28] studied I-related properties in triple sequence spaces and showed some interesting results. Tripathy [30] extended the concept of I-convergent to double sequence and later Kumar [22] obtained some results on I-convergent double sequence.

In this paper we define the spaces $c^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$, $c_0^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$, $\ell_\infty^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$, $M_I^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$ and $M_{0I}^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$ by using sequence of modulii function $F = (f_{pqr})$ and also studied some algebraic and topological properties of these new sequence spaces.

2 Definitions and preliminaries

Definition 2.1. Let $X \neq \varphi$. A class $I \subset 2^X$ (power set of X) is said to be an ideal in X if the following conditions hold:

(i) I is additive that is if $A, B \in I$ then $A \cup B \in I$;

(ii) I is hereditary that is if $A \in I$, and $B \subset A$ then $B \in I$.

I is called non-trivial ideal if $X \notin I$

Definition 2.2. [27, 28] A triple sequence (x_{pqr}) is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $\mathbf{N} \in \mathbb{N}$ such that

$$|x_{pqr} - L| < \varepsilon \ , \qquad \text{whenever} \quad p \geq \mathbf{N}, q \geq \mathbf{N}, r \geq \mathbf{N}$$

and write as $\lim_{p,p,r\to\infty} x_{pqr} = L$.

Note: A triple sequence is convergent in Pringsheim's sense may not be bounded [27, 28].
Example Consider the sequence (x_{pqr}) defined by

$$x_{pqr} = \begin{cases} p+q & , \quad \text{for all } p=q \text{ and } r=1 \\ \frac{1}{p^2qr} & , \qquad \text{otherwise.} \end{cases}$$

Then $x_{pqr} \to 0$ in Pringsheim's sense but is unbounded.

Definition 2.3. A triple sequence (x_{pqr}) is said to be I-convergent to a number L if for every $\varepsilon > 0$,

$$\{(p,q,r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{pqr} - L| \geq \varepsilon\} \in I.$$

In this case we write $I - \lim x_{pqr} = L$.

Definition 2.4. A triple sequence (x_{pqr}) is said to be I-null if $L = 0$. In this case we write $I - \lim x_{pqr} = 0$.

Definition 2.5. [27, 28] A triple sequence (x_{pqr}) is said to be Cauchy sequence if for every $\varepsilon > 0$, there exists $\mathbf{N} \in \mathbb{N}$ such that

$$|x_{pqr} - x_{lmn}| < \varepsilon \ , \quad \text{whenever} \ \ p \geq l \geq \mathbf{N}, q \geq m \geq \mathbf{N}, r \geq n \geq \mathbf{N}$$

Definition 2.6. A triple sequence (x_{pqr}) is said to be $I-$Cauchy sequence if for every $\varepsilon > 0$, there exists $\mathbf{N} \in \mathbb{N}$ such that

$$\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{pqr} - a_{lmn}| \geq \varepsilon\} \in I$$

whenever $\quad p \geq l \geq \mathbf{N}, q \geq m \geq \mathbf{N}, r \geq n \geq \mathbf{N}$

Definition 2.7. [27, 28] A triple sequence (x_{pqr}) is said to be bounded if there exists $M > 0$, such that $|x_{pqr}| < M$ for all $p, q, r \in \mathbb{N}$.

Definition 2.8. A triple sequence (x_{pqr}) is said to be $I-$bounded if there exists $M > 0$, such that $\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{pqr}| \geq M\} \in I$ for all $p, q, r \in \mathbb{N}$.

Definition 2.9. A triple sequence space E is said to be solid if $(\alpha_{pqr} x_{pqr}) \in E$ whenever $(x_{pqr}) \in E$ and for all sequences (α_{pqr}) of scalars with $|\alpha_{pqr}| \leq 1$, for all $p, q, r \in \mathbb{N}$.

Definition 2.10. Let E be a triple sequence space and $x = (x_{pqr}) \in E$. Define the set $S(x)$ as

$$S(x) = \left\{\left(x_{\pi(pqr)}\right) : \pi \text{ is a permutations of } \mathbb{N}\right\}$$

If $S(x) \subseteq E$ for all $x \in E$, then E is said to be symmetric.

Definition 2.11. A triple sequence space E is said to be convergence free if $(y_{pqr}) \in E$ whenever $(x_{pqr}) \in E$ and $x_{pqr} = 0$ implies $y_{pqr} = 0$ for all $p, q, r \in \mathbb{N}$.

Definition 2.12. A triple sequence space E is said to be sequence algebra if $x \cdot y \in E$, whenever $x = (x_{pqr}) \in E$ and $y = (y_{pqr}) \in E$, that is product of any two sequences is also in the space.

Gähler [16] introduced the notation of 2-normed spaces which was further extended to n-normed space by Misiak [23].

Definition 2.13. [23] (n**-Normed Space**) Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{R}$ of reals of dimension d, where $2 \leq d \leq n$. A real valued function $\| \cdot, ..., \cdot \|$ on X^n satisfying the following four conditions:

(1) $\|x_1, x_2, ..., x_n\| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent in X;

(2) $\|x_1, x_2, ..., x_n\|$ is invariant under permutation;

(3) $\|\alpha x_1, x_2, ..., x_n\| = |\alpha| \|x_1, x_2, ..., x_n\|$ for any $\alpha \in \mathbb{R}$;

(4) $\|x_1 + x_1^{'}, x_2, ..., x_n\| \leq \|x_1, x_2, ..., x_n\| + \|x_1^{'}, x_2, ..., x_n\|$;

is called an n-norm on X and $(X, \| \cdot, ..., \cdot \|)$ is called an $n-$normed space over the field $\mathbb{R}$.
For example $(\mathbb{R}^n, \| \cdot, ..., \cdot \|_E)$ where

$$\|x_1, x_2, ..., x_n\|_E = \text{the volume of the } n\text{-dimensional parallelopiped spanned by the vectors } x_1, x_2, ..., x_n$$

which can also be written as

$$\|x_1, x_2, ..., x_n\|_E = |\det(x_{ij})|$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, \|\cdot, ..., \cdot\|)$ be an $n-$normed space of dimension $2 \leq n \leq d$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $\|\cdot, ..., \cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, ..., x_{n-1}\|_\infty = \max\{\|x_1, x_2, ..., x_{n-1}, a_i\| : i = 1, 2, ..., n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, ..., a_n\}$.
The standard n-norm on X, a real inner product space of dimension $d \leq n$ is defined as follows:

$$\|x_1, x_2, \cdots, x_n\|_S = \begin{vmatrix} \langle x_1, x_1\rangle & \cdot & \cdot & \cdot & \langle x_1, x_n\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle x_n, x_1\rangle & \cdot & \cdot & \cdot & \langle x_n, x_n\rangle \end{vmatrix}^{\frac{1}{2}}$$

where $\langle\cdot, \cdot\rangle$ denotes the inner product on X. For $n = 1$ this n-norm is the usual norm $\|x\| = \langle x_1, x_1\rangle^{\frac{1}{2}}$.
A sequence (x_k) in a n-normed space $(X, \|\cdot, ..., \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} \|x_k - L, z_1, ..., z_{n-1}\| = 0 \qquad \text{for every} \quad z_1, ..., z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, \|\cdot, ..., \cdot\|)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} \|x_k - x_p, z_1, ..., z_{n-1}\| = 0 \qquad \text{for every} \quad z_1, ..., z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-complete n-normed space is said to be n-Banach space. The n-normed space has been studied in stretch [1, 12, 19, 25, 29].

Definition 2.14. (**Modulus Function**) A function $f : [0, \infty) \to [0, \infty)$ is called a modulus function if it satisfies the following conditions

(i) $f(x) = 0$ if and only if $x = 0$.

(ii) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$ and $y \geq 0$.

(iii) f is increasing.

(iv) f is continuous from the right at 0.

Since $|f(x) - f(y)| \leq f(|x - y|)$, it follows from condition (iv) that f is continuous on $[0, \infty)$. Furthermore, from condition (2) we have $f(nx) \leq nf(x)$, for all $n \in \mathbb{N}$, and so
$f(x) = f\big(nx(\frac{1}{n})\big) \leq nf\big(\frac{x}{n}\big)$.
Hence $\frac{1}{n}f(x) \leq f(\frac{x}{n})$ for all $n \in \mathbb{N}$.

Let I be an admissible ideal, $F = (f_{pqr})$ be a sequence of modulus functions and $(X, \|\cdot, \ldots, \cdot\|)$ be a n-normed space. By $\omega^{'''}(n - X)$ we denote the space of all triple sequences defined over $(X, \|\cdot, \ldots, \cdot\|)$. In the present paper we define the following sequence spaces

$$c^3[\Delta, F, \|\cdot,\ldots,\cdot\|]^I = \Big\{x = (x_{pqr}) \in \omega'''(n-X) : \forall\, \varepsilon > 0, \text{ the set } \Big\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : f_{pqr}\left(\|\Delta x_{pqr} - L, z_1, \cdots, z_{n-1}\|\right) \geq \varepsilon, \text{ for some } L \in \mathbb{C} \text{ and } z_1,\ldots,z_{n-1} \in X\Big\} \in I\Big\}$$

$$c_0^3[\Delta, F, \|\cdot,\ldots,\cdot\|]^I = \Big\{x = (x_{pqr}) \in \omega'''(n-X) : \forall\, \varepsilon > 0, \text{ the set } \Big\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : f_{pqr}\left(\|\Delta x_{pqr}, z_1, \cdots, z_{n-1}\|\right) \geq \varepsilon,\ z_1,\ldots,z_{n-1} \in X\Big\} \in I\Big\}$$

$$\ell_\infty^3[\Delta, F, \|\cdot,\ldots,\cdot\|]^I = \Big\{x = (x_{pqr}) \in \omega'''(n-X) : \exists\, K > 0 \text{ such that } \Big\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \sup_{p,q,r\geq 1}\left\{f_{pqr}\left(\|\Delta x_{pqr}, z_1, \cdots, z_{n-1}\|\right)\right\} \geq K,\ z_1,\ldots,z_{n-1} \in X\Big\} \in I\Big\}$$

and

$$M^3[\Delta, F, \|\cdot,\ldots,\cdot\|]^I = c^3[\Delta, F, \|\cdot,\ldots,\cdot\|]^I \cap \ell_\infty^3[\Delta, F, \|\cdot,\ldots,\cdot\|]^I$$
$$M_0^3[\Delta, F, \|\cdot,\ldots,\cdot\|]^I = c_0^3[\Delta, F, \|\cdot,\ldots,\cdot\|]^I \cap \ell_\infty^3[\Delta, F, \|\cdot,\ldots,\cdot\|]^I$$

For $F(x) = x$ we have

$$c^3[\Delta, \|\cdot,\ldots,\cdot\|]^I = \Big\{x = (x_{pqr}) \in \omega'''(n-X) : \forall\, \varepsilon > 0, \text{ the set } \Big\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|\Delta x_{pqr} - L, z_1, \cdots, z_{n-1}\| \geq \varepsilon, \text{ for some } L \in \mathbb{C} \text{ and } z_1,\ldots,z_{n-1} \in X\Big\} \in I\Big\}$$

$$c_0^3[\Delta, \|\cdot,\ldots,\cdot\|]^I = \Big\{x = (x_{pqr}) \in \omega'''(n-X) : \forall\, \varepsilon > 0, \text{ the set } \Big\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|\Delta x_{pqr}, z_1, \cdots, z_{n-1}\| \geq \varepsilon,\ z_1,\ldots,z_{n-1} \in X\Big\} \in I\Big\}$$

$$\ell_{\infty I}^3[\Delta, \|\cdot,\ldots,\cdot\|]^I = \Big\{x = (x_{pqr}) \in \omega'''(n-X) : \exists\, K > 0 \text{ such that } \Big\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \sup_{p,q,r\geq 1}\left(\|\Delta x_{pqr}, z_1, \cdots, z_{n-1}\|\right) \geq K,\ z_1,\ldots,z_{n-1} \in X\Big\} \in I\Big\}$$

and

$$M^3[\Delta, \|\cdot,\ldots,\cdot\|]^I = c^3[\Delta, \|\cdot,\ldots,\cdot\|]^I \cap \ell_\infty^3[\Delta, \|\cdot,\ldots,\cdot\|]^I$$
$$M_0^3[\Delta, \|\cdot,\ldots,\cdot\|]^I = c_0^3[\Delta, \|\cdot,\ldots,\cdot\|]^I \cap \ell_\infty^3[\Delta, \|\cdot,\ldots,\cdot\|]^I$$

3 Algebraic and Topological Properties of the new Sequence spaces

Theorem 3.1. Let $F = (f_{pqr})$ be a sequence of modulus functions then the triple sequence spaces $c_0^3[\Delta, F, \|\cdot,\ldots,\cdot\|]^I$, $c^3[\Delta, F, \|\cdot,\ldots,\cdot\|]^I$, $\ell_\infty^3[\Delta, F, \|\cdot,\ldots,\cdot\|]^I$, $M^3[\Delta, F, \|\cdot,\ldots,\cdot\|]^I$ and $M_0^3[\Delta, F, \|\cdot,\ldots,\cdot\|]^I$ all linear over the field $\mathbb{C}$ of complex numbers.

Proof. We prove the result for the sequence space $c^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I$.
Let $x=(x_{pqr}), y=(y_{pqr}) \in c^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I$ and $\alpha, \beta \in \mathbb{C}$, then there exist positive integers m_α and n_β such that $|\alpha| \le m_\alpha$ and $|\beta| \le n_\beta$, then for $z_1, z_2, \dots, z_{n-1} \in X$

$$I - \lim f_{pqr}\left(\|\Delta x_{pqr} - L_1, z_1, \dots, z_{n-1}\|\right) = 0, \quad \text{for some } L_1 \in \mathbb{C}.$$
$$I - \lim f_{pqr}\left(\|\Delta x_{pqr} - L_2, z_1, \dots, z_{n-1}\|\right) = 0, \quad \text{for some } L_2 \in \mathbb{C}.$$

Now for a given $\varepsilon > 0$ we set

$$C_1 = \left\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : f_{pqr}(\|\Delta x_{pqr} - L_1, z_1, \dots, z_{n-1}\|) > \frac{\varepsilon}{2}\right\} \in I \qquad (3.1)$$

$$C_2 = \left\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : f_{pqr}(\|\Delta y_{pqr} - L_2, z_1, \dots, z_{n-1}\|) > \frac{\varepsilon}{2}\right\} \in I \qquad (3.2)$$

Since $F=(f_{pqr})$ is a modulus function, so it is non-decreasing and convex, hence we get

$$\begin{aligned} & f_{pqr}(\|(\alpha\Delta x_{pqr} + \beta\Delta y_{pqr}) - (\alpha L_1 + \beta L_2), z_1, \dots, z_{n-1}\|) \\ &\quad = f_{pqr}(\|(\alpha\Delta x_{pqr} - \alpha L_1) + (\beta\Delta y_{pqr} - \beta L_2), z_1, \dots, z_{n-1}\|) \\ &\quad \le f_{pqr}(|\alpha|\|\Delta x_{pqr} - L_1, z_1, \dots, z_{n-1}\|) + f_{pqr}(|\beta|\|\Delta y_{pqr} - L_2, z_1, \dots, z_{n-1}\|) \\ &\quad = |\alpha| f_{pqr}(|\Delta x_{pqr} - L_1|) + |\beta| f_{pqr}(|\Delta y_{pqr} - L_2|) \\ &\quad \le m_\alpha f_{pqr}(\|\Delta x_{pqr} - L_1, z_1, \dots, z_{n-1}\|) + n_\beta f_{pqr}(\|\Delta y_{pqr} - L_2, z_1, \dots, z_{n-1}\|). \end{aligned}$$

From (3.1) and (3.2) we can write

$$\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : f_{pqr}(\|(\alpha\Delta x_{pqr} + \beta\Delta y_{pqr}) - (\alpha L_1 + \beta L_2), z_1, \dots, z_{n-1}\|) > \varepsilon\} \subseteq C_1 \cup C_2.$$

Thus $\alpha x + \beta y \in c^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I$.
Therefore $c^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I$ is a linear space.
In the same way we can show that other spaces are linear as well. Q.E.D.

Theorem 3.2. Let $F = (f_{pqr})$ be a sequence of modulus functions then the inclusions $c_0^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I \subset c^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I \subset \ell_\infty^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I$ holds .

Proof. The inclusion $c_0^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I \subset c^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I$ is obvious.
We prove $c^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I \subset \ell_\infty^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I$.
Let $x=(x_{pqr}) \in c^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I$ then there exists $L \in \mathbb{C}$ such that $I - \lim f_{pqr}(\|\Delta x_{pqr} - L, z_1, \dots, z_{n-1}\|) = 0$, $z_1, \dots, z_{n-1} \in X$.
Since $F=(f_{pqr})$ is a sequence of modulus functions so

$$f_{pqr}(\|\Delta x_{pqr}, z_1, \dots, z_{n-1}\|) \le f_{pqr}(\|\Delta x_{pqr} - L, z_1, \dots, z_{n-1}\|) + f_{pqr}(\|L, z_1, \dots, z_{n-1}\|).$$

On taking supremum over p, q and r on both sides gives
$x=(x_{pqr}) \in \ell_\infty^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I$
Hence the inclusion $c_0^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I \subset c^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I$
$\subset \ell_\infty^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I$ holds. Q.E.D.

Theorem 3.3. The triple difference sequence $c_0^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$ and $M_0^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$ are solid.

Proof. We prove the result for $c_0^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$.
Consider $x = (x_{pqr}) \in c_0^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$, then $I - \lim_{p,q,r} f_{pqr}(\|\Delta x_{pqr}, z_1, \ldots, z_{n-1}\|) = 0$.
Consider a sequence of scalar (α_{pqr}) such that $|\alpha_{pqr}| \leq 1$ for all $p, q, r \in \mathbb{N}$.
Then we have

$$\begin{aligned} I - \lim_{p,q,r} f_{pqr}(|\Delta\alpha_{pqr}(x_{pqr}), z_1, \ldots, z_{n-1}\|) &\leq I - |\alpha_{pqr}| \lim_{p,q,r} f_{pqr}(\|\Delta x_{pqr}, z_1, \ldots, z_{n-1}\|) \\ &\leq I - \lim_{p,q,r} f_{pqr}(\|\Delta x_{pqr}, z_1, \ldots, z_{n-1}\|) \\ &= 0 \end{aligned}$$

Hence $I - \lim_{p,q,r} f_{pqr}(\|\Delta\alpha_{pqr} x_{pqr}, z_1, \ldots, z_{n-1}\|) = 0$ for all $p, q, r \in \mathbb{N}$.
Which gives $(\alpha_{pqr} x_{pqr}) \in c_0^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$.
Hence the sequence space $c_0^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$ is solid.
The result for $M_0^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$ can be similarly proved. Q.E.D.

Theorem 3.4. The triple difference sequence spaces $c_0^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$, $c^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$, $\ell_\infty^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$, $M^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$ and $M_0^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$ are sequence algebras.

Proof. We prove the result for $c_0^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$.
Let $x = (x_{pqr}), y = (y_{pqr}) \in c_0^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$.
Then we have $I - \lim f_{pqr}(\|\Delta x_{pqr}, z_1, \ldots, z_{n-1}\|) = 0$
and

$$I - \lim f_{pqr}(\|\Delta y_{pqr}, z_1, \ldots, z_{n-1}\|) = 0.$$

Now $I - \lim f_{pqr}(\|\Delta(x_{pqr} \cdot y_{pqr}), z_1, \ldots, z_{n-1}\|) = 0$ as

$$\begin{aligned} \Delta(x_{pqr} \cdot y_{pqr}) =& x_{pqr} \cdot y_{pqr} - x_{(p+1)qr} \cdot y_{(p+1)qr} - x_{p(q+1)r} \cdot y_{p(q+1)r} - x_{pq(r+1)} \cdot \\ & y_{pq(r+1)} + x_{(p+1)(q+1)r} \cdot y_{(p+1)(q+1)r} + x_{(p+1)q(r+1)} \cdot y_{(p+1)q(r+1)} + \\ & x_{p(q+1)(r+1)} \cdot y_{p(q+1)(r+1)} - x_{(p+1)(q+1)(r+1)} \cdot y_{(p+1)(q+1)(r+1)}. \end{aligned}$$

It implies that $(x_{pqr} \cdot y_{pqr}) \in c_0^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$
Hence the proof.
The result can be proved for the spaces $c^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$, $\ell_\infty^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$, $M^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$ and $M_0^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$ in the same way. Q.E.D.

Theorem 3.5. In general the sequence spaces $c_0^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$, $c^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$ and $\ell_\infty^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$ are not convergence free.

Proof. We prove the result for the sequence space $c^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$ using an example.
Example: Let $I = I_f$ define the triple sequence $x = (x_{pqr})$ as

$$x_{pqr} = \begin{cases} 0 & , \quad \text{if } p = q = r \\ 1 & , \quad \text{otherwise.} \end{cases}$$

Then if $f_{pqr}(x) = (x_{pqr}) \ \ \forall\, p, q, r \in \mathbb{N}$, we have $x = (x_{pqr}) \in c^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I$.
Now define the sequence $y = y_{pqr}$ as

$$y_{pqr} = \begin{cases} 0 & , \quad \text{if } r \text{ is odd , and } \ p, q \in \mathbb{N} \\ lmn & , \quad \text{otherwise.} \end{cases}$$

Then for $f_{pqr}(x) = (x_{pqr})\ \ \forall\ p,q,r \in \mathbb{N}$, it is clear that $y = (y_{pqr}) \notin c^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I$
Hence the sequence spaces $c^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I$ is not convergence free.
The space $c^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I$ and $\ell^3_\infty[\Delta, F, \|\cdot,\dots,\cdot\|]^I$ are not convergence free in general can be proved in the same fashion. Q.E.D.

Theorem 3.6. In general the triple difference sequences $c^3_0[\Delta, F, \|\cdot,\dots,\cdot\|]^I$ and $c^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I$ are not symmetric if I is neither maximal nor $I = I_f$.

Proof. We prove the result for the sequence space $c^3_0[\Delta, F, \|\cdot,\dots,\cdot\|]^I$ using an example.
Example: Define the triple sequence $x = (x_{pqr})$ as

$$x_{pqr} = \begin{cases} 0 \quad , & \text{if } r = 1, \quad \text{for all } p,q \in \mathbb{N} \\ 1 \quad , & \text{otherwise.} \end{cases}$$

Then if $f_{pqr}(x) = (x_{pqr})\ \ \forall\ p,q,r \in \mathbb{N}$, we have $x = (x_{pqr}) \in c^3_0[\Delta, F, \|\cdot,\dots,\cdot\|]^I$.
Now if $x_{\pi(pqr)}$ be a rearrangement of $x = (x_{pqr})$ defined as

$$x_{\pi(pqr)} = \begin{cases} 1 \quad , & \text{for } \ p,q,r \text{ even} \in K \\ 0 \quad , & \text{otherwise.} \end{cases}$$

Then $\{x_{\pi(p,q,r)}\} \notin c^3_0[\Delta, F, \|\cdot,\dots,\cdot\|]^I$ as $\Delta x_{\pi(pqr)} = 1$.
Hence the sequence spaces $c^3_0[\Delta, F, \|\cdot,\dots,\cdot\|]^I$ is not symmetric in general.
The space $c^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I$ is not symmetric in general can be proved in the same fashion. Q.E.D.

Theorem 3.7. Let $F = (f_{pqr})$ and $G = (g_{pqr})$ be two sequences of modulus functions. Then

$$Z^3[\Delta, F, \|\cdot,\dots,\cdot\|]^I \cap Z^3[\Delta, G, \|\cdot,\dots,\cdot\|]^I \subseteq Z^3[\Delta, F+G, \|\cdot,\dots,\cdot\|]^I$$

where $Z = c_0$, c and ℓ_∞.

Proof. We prove the result for $Z = \ell_\infty$. Let $x = (x_{pqr}) \in \ell^3_\infty[\Delta, F, \|\cdot,\dots,\cdot\|]^I \cap \ell^3_\infty[\Delta, G, \|\cdot,\dots,\cdot\|]^I$.
Then for $z_1,\dots,z_{n-1} \in X$ we have

$$\left\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \sup_{p,q,r\geq 1} \{f_{pqr}\left(\|\Delta x_{pqr}, z_1,\cdots,z_{n-1}\|\right)\} \geq K_1\right\} \in I \qquad \text{for some } K_1 > 0$$

and

$$\left\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \sup_{p,q,r\geq 1} \{g_{pqr}\left(\|\Delta x_{pqr}, z_1,\cdots,z_{n-1}\|\right)\} \geq K_2\right\} \in I \qquad \text{for some } K_2 > 0.$$

Now since

$$\begin{aligned} \sup_{p,q,r\geq 1}\left\{(f_{pqr}+g_{pqr})\left(\left\|\Delta x_{pqr}, z_1,\cdots,z_{n-1}\right\|\right)\right\} &= \sup_{p,q,r\geq 1}\left\{f_{pqr}\left(\left\|\Delta x_{pqr}, z_1,\cdots,z_{n-1}\right\|\right) + g_{pqr}\left(\left\|\Delta x_{pqr}, z_1,\cdots,z_{n-1}\right\|\right)\right\} \\ &\leq \sup_{p,q,r\geq 1}\left\{f_{pqr}\left(\left\|\Delta x_{pqr}, z_1,\cdots,z_{n-1}\right\|\right)\right\} + \sup_{p,q,r\geq 1}\left\{g_{pqr}\left(\left\|\Delta x_{pqr}, z_1,\cdots,z_{n-1}\right\|\right)\right\}. \end{aligned}$$

Hence for $K = \max\{K_1, K_2\}$ we have

$$\left\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \sup_{p,q,r\geq 1} \left\{(f_{pqr}+g_{pqr})\left(\|\Delta x_{pqr}, z_1,\cdots,z_{n-1}\|\right)\right\} \geq K\right\} \in I.$$

Therefore $x = (x_{pqr}) \in \ell_\infty^3[\Delta, F+G, \|\cdot, \ldots, \cdot\|]^I$.
Hence
$$\ell_\infty^3[\Delta, F, \|\cdot, \ldots, \cdot\|]^I \cap \ell_\infty^3[\Delta, G, \|\cdot, \ldots, \cdot\|]^I \subseteq \ell_\infty^3[\Delta, F+G, \|\cdot, \ldots, \cdot\|]^I.$$
In the same way the inclusion for $Z = c_0, c$ can be proved. Q.E.D.

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