Blending type approximation by Stancu-Kantorovich operators associated with the inverse Pólya-Eggenberger distribution

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Abstract

In this paper, we give some approximation properties by Stancu-Kantorovich operators based on inverse Pólya-Eggenberger distribution in the polynomial weighted space introduced in the literature and obtain convergence properties of these operators by using Korovkin's theorem. We discuss the direct result and Voronovskaja type asymptotic formula.

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1 Introduction and preliminaries

The Pólya-Eggenberger (P-E) distribution was introduced by Eggenberger and Pólya [5] in the year 1923. The P-E distribution with parameters (n, A, B, S) is defined as:

$$P(X=k)=\left(\begin{array}{c} n \\ k \end{array}\right)\frac{\prod_{i=0}^{k-1}(A+is)\prod_{i=0}^{n-k-1}(B+is)}{\prod_{i=0}^{n-1}(A+B+is)},\quad k=0,1,...,n \tag{1.1}$$

This gives the probability of getting k white balls out of n drawings from an urn contains A white and B black balls, if each time one ball is drawn at random and then replaced together with S balls of the same color. In literatures, the inverse Pólya-Eggenberger(I-P-E) distribution is defined as:

$$P(X=k)=\left(\begin{array}{c} n+k-1 \\ k \end{array}\right)\frac{\prod_{i=0}^{k-1}(A+is)\prod_{i=0}^{n-1}(B+is)}{\prod_{i=0}^{n+k-1}(A+B+is)},\quad k=0,1,...,n. \tag{1.2}$$

gives the probability that k white balls are drawn preceding the n -th black ball. The details have been given about these two distributions (1.1) and (1.2) in [7].

Using (1.1), Stancu [15] constructed a new class of linear positive operators associated to a real-valued function $f:[0,1]\to\mathbb{R}$ as follows.

$$P_n^{[\alpha]}(f;x)=\sum_{k=0}^{n}p_{n,k}^{[\alpha]}(x)f\left(\frac{k}{n}\right)=\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\frac{\prod_{i=0}^{k-1}(x+i\alpha)\prod_{j=0}^{n-k-1}(1-x+j\alpha)}{(1+\alpha)(1+2\alpha)...(1+(n-1)\alpha)}f\left(\frac{k}{n}\right). \tag{1.3}$$

For $\rho>0$ and $f\in[0,1]$, Kajla and Araci [8] introduced Stancu-Kantorovich type operators arising from Pólya-Eggenberger distribution as follows:

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$$K_{n,\rho}^{[\alpha]}(f;x)=\sum_{k=0}^{n}p_{n,k}^{[\alpha]}(x)\int_0^1 f\left(\frac{k+t^\rho}{n+1}\right)dt, \tag{1.4}$$

where $p_{n,k}^{[\alpha]}(x)=\left(\begin{array}{c} n \\ k \end{array}\right)\frac{1}{1^{[n,-\alpha]}}x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}$ are the known Stancu's fundamental polynomials and $t^{[n,h]}=t(t-h)...(t-(n-1)h)$.

Based on I-P-E (1.2), Stancu [16] studied a generalization of the Baskakov operators for a real-valued function bounded on $[0,\infty)$, defined as:

$$V_n^{[\alpha]}(f;x)=\sum_{k=0}^{\infty}v_{n,k}^{[\alpha]}(x)f\left(\frac{k}{n}\right)=\sum_{k=0}^{\infty}\left(\begin{array}{c} n+k-1 \\ k \end{array}\right)\frac{1^{[n,-\alpha]}x^{[k,-\alpha]}}{(1+x)^{[n+k,-\alpha]}}f\left(\frac{k}{n}\right). \tag{1.5}$$

The operators (1.5) include as a special case ($\alpha=0$), the Baskakov operators [1]

$$V_n(f;x)=\sum_{k=0}^{\infty}\left(\begin{array}{c} n+k-1 \\ k \end{array}\right)\frac{x^k}{(1+x)^{n+k}}f\left(\frac{k}{n}\right). \tag{1.6}$$

In 1989, Razi [12] introduced the Bernstein-Kantorovich operatos based on Pólya-Eggenberger distribution. In [11] was introduced the Bézier variant of genuine-Durrmeyer type operators having Pólya basis functions. Deo et al. [2] considered a Stancu-Kantorovich operators based on inverse Pólya-Eggenberger distribution of the operators (1.5) and established some direct results. Very recently, Dhamija et al. [4] considered the Stancu-Jain type hybrid operator based on inverse Pólya- Eggenberger distribution and studied approximation properties of these operators which include uniform convergence and degree of approximation. In this direction, significant contribution are given in [9, 10, 13, 14].

For $\rho>0$ and $f\in[0,\infty)$, we introduce Stancu-Baskakov-Kantorovich operators based on inverse Pólya-Eggenberger distribution (1.2), given by

$$B_{n,\rho}^{[\alpha]}(f;x)=\sum_{k=0}^{\infty}v_{n,k}^{[\alpha]}(x)\int_0^1 f\left(\frac{k+t^\rho}{n+1}\right)dt \tag{1.7}$$

where $v_{n,k}^{[\alpha]}(x)=\left(\begin{array}{c} n+k-1 \\ k \end{array}\right)\frac{1^{[n,-\alpha]}x^{[k,-\alpha]}}{(1+x)^{[n+k,-\alpha]}}$.

2 Auxiliary results

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. We recall that the monomials $e_k(x)=x^k$, for $k\in\mathbb{N}_0$ called also test functions, play an important role in uniform approximation by linear positive operators.The computation of the images of test functions by Stancu-Baskakov operators (1.5) was done in [16], in order to prove a theorem concerning the uniform approximation.

In order to calculate the moments of the operators $B_{n,\rho}^{[\alpha]}$ we give a new representation of these operators.

Lemma 2.1. For $\rho > 0$, $\alpha > 0$ and $x \in \mathbb{R}^+$, we get

$$B_{n,\rho}^{[\alpha]}(f;x) \quad = \quad \frac{1}{\beta(\frac{x}{\alpha}, \frac{1}{\alpha})} \int_0^\infty \frac{t^{\frac{x}{\alpha}-1}}{(1+t)^{\frac{1+x}{\alpha}}} K_{n,\rho}(f;t)dt,$$

where

$$K_{n,\rho}(f;t) \quad = \quad \sum_{k=0}^{\infty} \left(\begin{array}{c} n+k-1 \\ k \end{array} \right) \frac{t^k}{(1+t)^{n+k}} \int_0^1 f\left(\frac{k+s^\rho}{n+1}\right) ds,$$

and $\beta(p,q)$, $p,q > 0$ is the Beta function.

Proof. Using the relationship between Euler's functions

$$\beta(p,q) \quad = \quad \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

where $\beta(p,q)$ and $\Gamma(r)$ are beta function of second kind and gamma function, respectively defined by

$$\beta(p,q) \quad = \quad \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}} du, \qquad \Gamma(r) = \int_0^\infty u^{r-1} e^{-u} du,$$

with $\Gamma(r+n) = r(r+1)\hat{A}\Delta...\hat{A}\Delta(r+n-1)\Gamma(r)$, for natural number n, then we get

$$\begin{aligned}
\beta\left(\frac{x}{\alpha}+k, \frac{1}{\alpha}+n\right) \quad &= \quad \frac{\Gamma\left(\frac{x}{\alpha}+k\right)\Gamma\left(\frac{1}{\alpha}+n\right)}{\Gamma\left(\frac{x+1}{\alpha}+n+k\right)} \\
&= \quad \frac{\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)...\left(\frac{x}{\alpha}+k-1\right)\Gamma\left(\frac{x}{\alpha}\right) \cdot \frac{1}{\alpha}\left(\frac{1}{\alpha}+1\right)...\left(\frac{1}{\alpha}+n-1\right)\Gamma\left(\frac{1}{\alpha}\right)}{\left(\frac{x+1}{\alpha}\right)\left(\frac{x+1}{\alpha}+1\right)...\left(\frac{x+1}{\alpha}+n+k-1\right)\Gamma\left(\frac{x+1}{\alpha}\right)} \\
&= \quad \left(\begin{array}{c} n+k-1 \\ k \end{array} \right)^{-1} v_{n,k}^{[\alpha]}(x)\beta\left(\frac{x}{\alpha}, \frac{1}{\alpha}\right).
\end{aligned}$$

Hence

$$v_{n,k}^{[\alpha]}(x) \quad = \quad \left(\begin{array}{c} n+k-1 \\ k \end{array} \right) \beta\left(\frac{x}{\alpha}, \frac{1}{\alpha}\right)^{-1} \beta\left(\frac{x}{\alpha}+k, \frac{1}{\alpha}+n\right)$$

and it follows

$$
\begin{aligned}
B_{n,\rho}^{[\alpha]}(f;x) &= \frac{1}{\beta\left(\frac{x}{\alpha},\frac{1}{\alpha}\right)}\sum_{k=0}^{\infty}\begin{pmatrix} n+k-1 \\ k \end{pmatrix}\beta\left(\frac{x}{\alpha}+k,\frac{1}{\alpha}+n\right)\int_0^1 f\left(\frac{k+s^{\rho}}{n+1}\right)ds \\
&= \frac{1}{\beta\left(\frac{x}{\alpha},\frac{1}{\alpha}\right)}\left(\sum_{k=0}^{\infty}\begin{pmatrix} n+k-1 \\ k \end{pmatrix}\int_0^{\infty}\frac{t^{\frac{x}{\alpha}+k-1}}{(1+t)^{\frac{1+x}{\alpha}+n+k}}dt\int_0^1 f\left(\frac{k+s^{\rho}}{n+1}\right)ds\right) \\
&= \frac{1}{\beta(\frac{x}{\alpha},\frac{1}{\alpha})}\int_0^{\infty}\frac{t^{\frac{x}{\alpha}-1}}{(1+t)^{\frac{1+x}{\alpha}}}K_{n,\rho}(f;t)dt.
\end{aligned}
$$

Q.E.D.

Lemma 2.2. Let $e_r(t)=t^r$, $r\in\mathbb{N}\cup\{0\}$. For $\rho>0$, $\alpha>0$, we have

$$
\begin{aligned}
(i)B_{n,\rho}^{[\alpha]}(e_0;x) &= 1 \\
(ii)B_{n,\rho}^{[\alpha]}(e_1;x) &= \frac{n}{(n+1)(1-\alpha)}x+\frac{1}{(n+1)(1+\rho)} \\
(iii)B_{n,\rho}^{[\alpha]}(e_2;x) &= \frac{n}{(n+1)(1-\alpha)(1-2\alpha)}x^2+\frac{n\big((3+\rho)+\alpha(n-8+\rho(2\alpha+n-2))\big)}{(n+1)^2(1+\rho)(1-\alpha)(1-2\alpha)}x \\
&\quad +\frac{1}{(n+1)^2(1+2\rho)}.
\end{aligned}
$$

Lemma 2.3. Taking Lemma 2.2 into the account, we get the following central moments:

$$
\begin{aligned}
(i)B_{n,\rho}^{[\alpha]}(e_1-x;x) &= \left(\frac{n}{(n+1)(1-\alpha)}-1\right)x+\frac{1}{(n+1)(1+\rho)} \\
(ii)B_{n,\rho}^{[\alpha]}((e_1-x)^2;x) &= \frac{\alpha(2\alpha(n+2)+n-3)+1}{(n+1)(1-\alpha)(1-2\alpha)}x^2+\frac{(n-2)(1+\rho)(n\alpha+1)+2\alpha(3-2\alpha+n\alpha(\rho-2))}{(n+1)^2(1+\rho)(1-\alpha)(1-2\alpha)}x \\
&\quad +\frac{1}{(n+1)^2(1+2\rho)}.
\end{aligned}
$$

Lemma 2.4. If $\alpha\to 0$ as $n\to\infty$ and $\lim_{n\to\infty} n\alpha=a\in\mathbb{R}$, then

$$
\begin{aligned}
(i)\lim_{n\to\infty} nB_{n,\rho}^{[\alpha]}(e_1-x;x) &= \frac{1}{1+\rho} \\
(ii)\lim_{n\to\infty} nB_{n,\rho}^{[\alpha]}((e_1-x)^2;x) &= (a+1)x(1+x).
\end{aligned}
$$

3 Approximation in weighted spaces

We now give the Gadzhiev's results in weighted spaces. Therefore we need to introduce the notations of [6]. Let $\rho(x)=1+x^2$, $-\infty<x<\infty$ and $\mathbf{B}_{\rho}$ be the set of all functions f defined on the real axis satisfying the condition $\mid f(x)\mid\leq \mathbf{M}_f\rho(x)$ where $\mathbf{M}_f$ is a constant depending only on f. $\mathbf{B}_{\rho}$ is a normed space with the norm $\parallel f\parallel_{\rho}=\sup_{x\geq 0}\frac{f(x)}{\rho(x)}$, $f\in\mathbf{B}_{\rho}$. $\mathbf{C}_{\rho}$ denotes the subspace of all continuous functions belonging to $\mathbf{B}_\rho$ and $\mathbf{C}_\rho^k$ denotes the subspace of all functions $f \in \mathbf{C}_\rho$ with $\lim_{|x|\to\infty} \frac{f(x)}{\rho(x)} = k$ where k is a constant depending on f.

Theorem 3.1. Let (B_n) be the sequence of positive linear operators which act from $\mathbf{C}_\rho$ to $\mathbf{B}_\rho$ such that

$$\lim_{n\to\infty} \|B_n(t^i;x) - x^i\|_\rho = 0, \quad i \in \{0,1,2\}.$$

Then for any function $f \in C_\rho^k$

$$\lim_{n\to\infty} \|B_n f - f\|_\rho = 0.$$

and there exists a function $f^* \in C_\rho \backslash C_\rho^k$

$$\lim_{n\to\infty} \|B_n f^* - f^*\|_\rho \geq 1.$$

Theorem 3.2. Let $B_{n,\rho}^{[\alpha]}$ be the sequence of positive linear operators defined by (1.7) and $\rho(x) = 1+x^2$, $\alpha \in \mathbb{N}_0$ depending on $n \in \mathbb{N}$, with $\alpha \to 0$, as $n \to \infty$. Then for each $f \in C_\rho^k$

$$\lim_{n\to\infty} \|B_{n,\rho}^{[\alpha]}(f;x) - f(x)\|_\rho = 0.$$

Proof. It is enough to prove that the conditions of the weighted Korovkin type theorem given by Theorem 3.1 are satisfied. From Lemma 2.2 (i), it is immediate that

$$\lim_{n\to\infty} \|B_{n,\rho}^{[\alpha]}(e_0;x) - 1\|_\rho = 0. \tag{3.1}$$

Using Lemma 2.2 (ii), we have

$$\|B_{n,\rho}^{[\alpha]}(e_1;x) - x\|_\rho = \left(\frac{n}{(n+1)(1-\alpha)} - 1\right) \sup_{x\in R_0} \frac{x}{1+x^2} + \frac{1}{(n+1)(1+\rho)} \sup_{x\in R_0} \frac{1}{1+x^2}.$$

Hence we obtain

$$\lim_{n\to\infty} \|B_{n,\rho}^{[\alpha]}(e_1;x) - x\|_\rho = 0. \tag{3.2}$$

By means of Lemma 2.2 (iii), we get

$$\begin{aligned}\|B_{n,\rho}^{[\alpha]}(e_2;x) - x^2\|_\rho \quad &= \quad \left(\frac{\alpha(2\alpha(n+2)+n-3)+1}{(n+1)(1-\alpha)(1-2\alpha)}\right) \sup_{x\in R_0} \frac{x^2}{1+x^2} \\ &+ \left(\frac{(n-2)(1+\rho)(n\alpha+1) + 2\alpha(3-2\alpha+n\alpha(\rho-2))}{(n+1)^2(1+\rho)(1-\alpha)(1-2\alpha)}\right) \sup_{x\in R_0} \frac{x}{1+x^2} \\ &+ \frac{1}{(n+1)^2(1+2\rho)} \sup_{x\in R_0} \frac{1}{1+x^2}\end{aligned}$$

$$\lim_{n\to\infty} \|B_{n,\rho}^{[\alpha]}(e_2;x) - x^2\|_\rho = 0. \tag{3.3}$$

From (3.1), (3.2) and (3.3), for $i \in \{0,1,2\}$, we have

$$\lim_{n\to\infty} \|B_n^*(t^i;x) - x^i\|_\rho = 0.$$

Applying Theorem 3.1, we obtain the desired result. Q.E.D.

4 Direct results

In this section we discuss the direct result and Voronovskaja type asymptotic formula for the operators $B_{n,\rho}^{[\alpha]}$. By $C_B[0,\infty)$, we denote the space of real-valued continuous and bounded functions f defined on the interval $[0,\infty)$. The norm $\|\cdot\|$ on the space $C_B[0,\infty)$ is given by

$$\| f \| = \sup_{0\le x<\infty} | f(x) | .$$

Further let us consider the following K-functional:

$$K_2(f,\delta) = \inf_{g\in W^2}\{\| f-g \| + \delta \| g^{''} \|\},$$

Where $\delta > 0$ and $W^2 = \{g \in C_B[0,\infty) : g^{'}, g^{''} \in C_B[0,\infty)\}$. By Theorem 2.4 of [3], there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \le C\omega_2(f,\sqrt{\delta}) \tag{4.1}$$

where

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\le\sqrt{\delta}} \sup_{x\in[0,\infty)} | f(x+2h) - 2f(x+h) + f(x) |$$

is the second order modulus of smoothness of $f \in C_B[0,\infty)$. The usual modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$w(f,\delta) = \sup_{0<h\le\delta} \sup_{x\in[0,\infty)} | f(x+h) - f(x) | .$$

Theorem 4.1. For $f \in C_B[0,\infty)$, we have

$$| B_{n,\rho}^{[\alpha]}(f;x) - f(x) | \le \mathcal{C}\omega_2(f,\delta) + \omega_1\Bigg(f, \Bigg|\left(\frac{n}{(n+1)(1-\alpha)} - 1\right)x + \frac{1}{(n+1)(1+\rho)}\Bigg|\Bigg),$$

where

$$\delta = \sqrt{B_{n,\rho}^{[\alpha]}((e_1-x)^2;x) + \left(\left(\frac{n}{(n+1)(1-\alpha)} - 1\right)x + \frac{1}{(n+1)(1+\rho)}\right)^2}$$

Proof. Introduce auxiliary operators as follows:

$$\widehat{B}_{n,\rho}^{[\alpha]}(f;x) \quad = \quad B_{n,\rho}^{[\alpha]}(f;x) - f\left(\frac{n}{(n+1)(1-\alpha)}x + \frac{1}{(n+1)(1+\rho)}\right) + f(x).$$

These operators are linear and preserve the linear functions in view of Lemma 2.2. Let $g \in W_\infty^2$ and $x, t \in [0,\infty)$. Then from the Taylor's expansion, we get

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\mathrm{d}u.$$

Applying $\widehat{B}_{n,\rho}^{[\alpha]}(\cdot;x)$, we get

$$\widehat{B}_{n,\rho}^{[\alpha]}(g;x) - g(x) = g'(x)\widehat{B}_{n,\rho}^{[\alpha]}(t-x;x) + \widehat{B}_{n,\rho}^{[\alpha]}\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right),$$

and hence

$$\begin{aligned}
\left|\widehat{B}_{n,\rho}^{[\alpha]}(g;x) - g(x)\right| &\leq \widehat{B}_{n,\rho}^{[\alpha]}\left(\left|\int_x^t \mid t-u \mid \ \mid g''(u) \mid \mathrm{d}u\right|;x\right)\\
&\leq B_{n,\rho}^{[\alpha]}\left(\left|\int_x^t \mid t-x \mid\mid g''(u) \mid \mathrm{d}u\right|;x\right)\\
&\quad + \left|\int_x^{\frac{n}{(n+1)(1-\alpha)}x + \frac{1}{(n+1)(1+\rho)}} \left(\frac{n}{(n+1)(1-\alpha)}x + \frac{1}{(n+1)(1+\rho)} - u\right)g''(u)\mathrm{d}u\right|\\
&\leq B_{n,\rho}^{[\alpha]}\left((t-x)^2;x\right)\parallel g'' \parallel + \left(\left(\frac{n}{(n+1)(1-\alpha)} - 1\right)x + \frac{1}{(n+1)(1+\rho)}\right)^2 \|g''\|\\
&\leq \left[B_{n,\rho}^{[\alpha]}\left((t-x)^2;x\right) + \left(\left(\frac{n}{(n+1)(1-\alpha)} - 1\right)x + \frac{1}{(n+1)(1+\rho)}\right)^2\right] \parallel g'' \parallel\\
&= \delta^2 \ \|g''\| .
\end{aligned}$$

Since

$$\mid B_{n,\rho}^{[\alpha]}(f;x) \mid \quad \leq \quad \sum_{k=0}^{\infty} v_{n,k}^{[\alpha]}(x) \int_0^1 f\left(\frac{k+t^\rho}{n+1}\right)dt \leq \parallel f \parallel,$$

it follows,

$$\begin{aligned}
\mid B_{n,\rho}^{[\alpha]}(f;x) - f(x) \mid \quad &\leq \quad \mid \widehat{B}_{n,\rho}^{[\alpha]}(f-g;x) - (f-g)(x) \mid + \mid \widehat{B}_{n,\rho}^{[\alpha]}(g;x) - g(x) \mid\\
&\quad + \left|f\left(\frac{n}{(n+1)(1-\alpha)}x + \frac{1}{(n+1)(1+\rho)}\right) - f(x)\right|\\
&\leq \quad \parallel f-g \parallel + \delta^2 \parallel g'' \parallel + \omega_1\left(f, \left|\left(\frac{n}{(n+1)(1-\alpha)} - 1\right)x + \frac{1}{(n+1)(1+\rho)}\right|\right).
\end{aligned}$$

Taking infimum over all $g \in W^2$, we get

$$\mid B_{n,\rho}^{[\alpha]}(f;x) - f(x) \mid \leq \ \mathcal{C}K_2(f,\delta^2) + \omega_1\left(f, \left|\left(\frac{n}{(n+1)(1-\alpha)} - 1\right)x + \frac{1}{(n+1)(1+\rho)}\right|\right).$$

In view of (4.1),

$$| B_{n,\rho}^{[\alpha]}(f;x) - f(x) | \leq \mathcal{C}\omega_2(f,\delta) + \omega_1\left(f, \left|\left(\frac{n}{(n+1)(1-\alpha)} - 1\right)x + \frac{1}{(n+1)(1+\rho)}\right|\right).$$

This completes the proof. Q.E.D.

Let $B_{x^2}[0,\infty)= \{f$: for every $x \in [0,\infty)$, $| f(x) | \leq M_f(1+x^2)$, M_f being a constant depending on $f\}$.

We denote by $C_{x^2}[0,\infty)$, the space of all continuous functions on $[0,\infty)$ belonging to $B_{x^2}[0,\infty)$.

Our next result in this section is the Voronovskaja type asymptotic formula:

Theorem 4.2. For any function $f \in C_{x^2}[0,\infty)$ such that $f', f'' \in C_{x^2}[0,\infty)$, $\alpha \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} n\alpha = a \in \mathbb{R}$, we have

$$\lim_{n\to\infty} n[B_{n,\rho}^{[\alpha]}(f;x) - f(x)] = \frac{1}{1+\rho}f'(x) + \frac{(a+1)x(1+x)}{2}f''(x),$$

for every $x \geq 0$.

Proof. Let $f, f', f'' \in C_2^*[0,\infty)$ and $x \in [0,\infty)$ be fixed. By Taylor expansion we can write

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + r(t,x)(t-x)^2,$$

where $r(t,x)$ is the Peano form of the remainder, $r(t,x) \in C_B[0,\infty)$ and $\lim_{t\to x} r(t,x) = 0$. Applying $B_{n,\rho}^{[\alpha]}$, we get

$$n[B_{n,\rho}^{[\alpha]}(f;x) - f(x)] = f'(x)nB_{n,\rho}^{[\alpha]}(t-x;x) + \frac{f''(x)}{2!}nB_{n,\rho}^{[\alpha]}((t-x)^2;x) + nB_{n,\rho}^{[\alpha]}(r(t,x)(t-x)^2;x).$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} n[B_{n,\rho}^{[\alpha]}(f;x) - f(x)] &= f'(x)\lim_{n\to\infty} nB_{n,\rho}^{[\alpha]}(t-x;x) + \frac{f''(x)}{2!}\lim_{n\to\infty} nB_{n,\rho}^{[\alpha]}((t-x)^2;x) \\ &\quad + \lim_{n\to\infty} nB_{n,\rho}^{[\alpha]}(r(t,x)(t-x)^2;x) \\ &= \frac{1}{1+\rho}f'(x) + \frac{(a+1)x(1+x)}{2}f''(x) + \lim_{n\to\infty} nB_{n,\rho}^{[\alpha]}(r(t,x)(t-x)^2;x) \\ &= \frac{1}{1+\rho}f'(x) + \frac{(a+1)x(1+x)}{2}f''(x) + E.\end{aligned}$$

By Cauchy-Schwarz inequality, we have

$$| E | \leq \lim_{n\to\infty} nB_{n,\rho}^{[\alpha]}(r^2(t,x);x)^{\frac{1}{2}} B_{n,\rho}^{[\alpha]}((t-x)^4;x)^{\frac{1}{2}}. \tag{4.2}$$

Observe that $r^2(x,x)=0$ and $r^2(\cdot,x)\in C_2^*[0,\infty)$. Then, it follows that

$$\lim_{n\to\infty} nB_{n,\rho}^{[\alpha]}(r^2(t,x);x)=(r^2(x,x);x)=0 \tag{4.3}$$

uniformly with respect to $x\in[0,A]$. Now from (4.2) and (4.3) it follows

$$\lim_{n\to\infty} nB_{n,\rho}^{[\alpha]}(r(t,x)(t-x)^2;x) \quad = \quad 0.$$

Hence, $E=0$. Thus, we have

$$\lim_{n\to\infty} n[B_{n,\rho}^{[\alpha]}(f;x)-f(x)] \quad = \quad \frac{1}{1+\rho}f'(x)+\frac{(a+1)x(1+x)}{2}f''(x),$$

which completes the proof. Q.E.D.

Theorem 4.3. If $f\in C_E[0,\infty)$, then $B_{n,\rho}^{[\alpha]}$ operators verify the following inequality:

$$|B_{n,\rho}^{[\alpha]}(f;x)-f(x)|\le 2\omega\Big(f,\delta_n(a)\Big)$$

where

$$\begin{aligned}\delta^2:=\delta_n^2(a) \quad = \quad & \frac{\alpha(2\alpha(n+2)+n-3)+1}{(n+1)(1-\alpha)(1-2\alpha)}a^2+\frac{(n-2)(1+\rho)(n\alpha+1)+2\alpha(3-2\alpha+n\alpha(\rho-2))}{(n+1)^2(1+\rho)(1-\alpha)(1-2\alpha)}a \\ & +\frac{1}{(n+1)^2(1+2\rho)}.\end{aligned}$$

Proof. Knowing that Stancu-Kantorovich operators (1.7) preserve constants and using the well-known property of modulus of continuity

$$|f(x)-f(y)| \quad \le \quad \omega(f,|\,x-y\,|)\le\left(1+\frac{1}{\delta}\,|\,x-y\,|\,\right)\omega(f,\delta),$$

it follows

$$\begin{aligned}|B_{n,\rho}^{[\alpha]}(f;x)-f(x)| \quad & \le \quad \sum_{k=0}^{\infty} v_{n,k}^{[\alpha]}(x)\int_0^1\left|f\left(\frac{k+t^\rho}{n+1}\right)-f(x)\right|dt \\ & \le \quad \left\{1+\frac{1}{\delta}\sum_{k=0}^{\infty} v_{n,k}^{[\alpha]}(x)\int_0^1\left|\frac{k+t^\rho}{n+1}-x\right|dt\right\}\omega(f,\delta).\end{aligned}$$

Using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}|B_{n,\rho}^{[\alpha]}(f;x)-f(x)| \quad & \le \quad \left\{1+\frac{1}{\delta}\left(\sum_{k=0}^{\infty} v_{n,k}^{[\alpha]}(x)\int_0^1\left(\frac{k+t^\rho}{n+1}-x\right)^2dt\right)^{\frac{1}{2}}\right\}\omega(f,\delta) \\ & = \quad \left\{1+\frac{1}{\delta}\sqrt{B_{n,\rho}^{[\alpha]}\big((e_1-x)^2;x\big)}\right\}\omega(f,\delta).\end{aligned} \tag{4.4}$$

By means of Lemma 2.3 (ii), for $0 \le x \le a$, one gets

$$B_{n,\rho}^{[\alpha]}\big((e_1-x)^2;x\big) \;\le\; \frac{\alpha(2\alpha(n+2)+n-3)+1}{(n+1)(1-\alpha)(1-2\alpha)}a^2+\frac{(n-2)(1+\rho)(n\alpha+1)+2\alpha(3-2\alpha+n\alpha(\rho-2))}{(n+1)^2(1+\rho)(1-\alpha)(1-2\alpha)}a + \frac{1}{(n+1)^2(1+2\rho)}. \quad (4.5)$$

Using (4.5) and taking $\delta^2 := \delta_n^2(a)$ in (4.4), we obtain the desired result. Q.E.D.

Theorem 4.4. If f is a differentiable function on $[0,\infty)$ and $f \in C_B[0,\infty)$, then for any $x \in [0,\infty)$ and $\delta > 0$, it follows

$$|B_{n,\rho}^{[\alpha]}(f;x)-f(x)| \;=\; \left|\left(\frac{n}{(n+1)(1-\alpha)}-1\right)x+\frac{1}{(n+1)(1+\rho)}\right|f'(x)+2\delta\omega(f',\delta),$$

with $\delta = \left(B_{n,\rho}^{[\alpha]}\big((e_1-x)^2;x\big)\right)^{\frac{1}{2}}$.

Proof. Starting with the identity

$$f(t)-f(x) = f'(x)(t-x)+f(t)-f(x)-f'(x)(t-x), \quad (4.6)$$

we get for ξ between t and x

$$|f(t)-f(x)-f'(x)(t-x)| \;=\; |f'(\xi)-f'(x)||t-x|,$$

using the Lagrange mean value theorem ($f(t)-f(x) = f'(\xi)(t-x)$, with ξ between t and x). Because $|\xi - x| \le |t-x|$, it follows

$$|f'(\xi)-f'(x)| \;\le\; \omega(f',|\,t-x\,|) \le \left(1+\frac{1}{\delta}\,|\,t-x\,|\right)\omega(f',\delta),$$

and

$$|f(t)-f(x)-f'(x)(t-x)| \;\le\; \left(\,|\,t-x\,|+\frac{1}{\delta}(t-x)^2\right)\omega(f',\delta).$$

Applying the linear positive Stancu-Baskakov-Kantorovich operators (1.7) to the inequality

$$|f(t)-f(x)| \;\le\; |f'(x)(t-x)|+\left(\,|\,t-x\,|+\frac{1}{\delta}(t-x)^2\right)\omega(f',\delta),$$

obtained from (4.6) and the above relations, it follows

$$|B_{n,\rho}^{[\alpha]}(f;x)-f(x)| \leq f'(x)\left|B_{n,\rho}^{[\alpha]}(e_1-x;x)\right| + \left(B_{n,\rho}^{[\alpha]}(e_1-x;x)+\frac{1}{\delta}B_{n,\rho}^{[\alpha]}((e_1-x)^2;x)\right)\omega(f',\delta). \quad (4.7)$$

The Cauchy-Schwarz inequality for linear positive operators leads to

$$B_{n,\rho}^{[\alpha]}(\mid e_1-x\mid;x) \leq \left(B_{n,\rho}^{[\alpha]}(e_0;x)\right)^{\frac{1}{2}} \cdot \left(B_{n,\rho}^{[\alpha]}((e_1-x)^2;x)\right)^{\frac{1}{2}}. \quad (4.8)$$

Using the relation (4.8) and the result presented Lemma 2.3, the inequality (4.7) become

$$|B_{n,\rho}^{[\alpha]}(f;x)-f(x)| \quad = \quad \left|\left(\frac{n}{(n+1)(1-\alpha)}-1\right)x+\frac{1}{(n+1)(1+\rho)}\right|f'(x)+2\delta\omega(f',\delta),$$

with $\delta = \left(B_{n,\rho}^{[\alpha]}\big((e_1-x)^2;x\big)\right)^{\frac{1}{2}}$.

Now we present ordinary approximation in terms of Lipschitz constant defined by

$$lip_M(\beta) = \left\{f \in C_B[0,\infty) :\mid f(t)-f(x)\mid \leq M\frac{\mid t-x\mid^{\beta}}{(t+x)^{\frac{\beta}{2}}}\right\}, \quad (4.9)$$

where M is a positive constant and $0<\beta\leq 1$. Q.E.D.

Theorem 4.5. Let $f \in C_B[0,\infty)$. Then for any $x \in [0,\infty)$, the following inequality holds:

$$|B_{n,\rho}^{[\alpha]}(f;x)-f(x)| \quad \leq \quad M\left(\frac{\varphi_{n,\rho}^{[\alpha]}}{x}\right)^{\frac{\beta}{2}},$$

where $\varphi_{n,\rho}^{[\alpha]} = B_{n,\rho}^{[\alpha]}((e_1-x)^2;x)$.

Proof. Let us prove the theorem for the case $0<\beta\leq 1$, applying Holder's inequality with $p=\frac{2}{\beta}$,

$q = \frac{2}{2-\beta}$,

$$\begin{aligned}
|B^{[\alpha]}_{n,\rho}(f;x) - f(x)| &\leq \sum_{k=0}^{\infty} v^{[\alpha]}_{n,k}(x) \int_0^1 \Big| f\Big(\frac{k+t^\rho}{n+1}\Big) - f(x)\Big| dt \\
&\leq \sum_{k=0}^{\infty} v^{[\alpha]}_{n,k}(x) \Big(\int_0^1 \Big| f\Big(\frac{k+t^\rho}{n+1}\Big) - f(x)\Big|^{\frac{2}{\beta}} dt\Big)^{\frac{\beta}{2}} \\
&\leq \Big(\sum_{k=0}^{\infty} v^{[\alpha]}_{n,k}(x) \int_0^1 \Big| f\Big(\frac{k+t^\rho}{n+1}\Big) - f(x)\Big|^{\frac{2}{\beta}} dt\Big)^{\frac{\beta}{2}} \Big(\sum_{k=0}^{\infty} v^{[\alpha]}_{n,k}(x)\Big)^{\frac{2-\beta}{2}} \\
&= \Big(\sum_{k=0}^{\infty} v^{[\alpha]}_{n,k}(x) \int_0^1 \Big| f\Big(\frac{k+t^\rho}{n+1}\Big) - f(x)\Big|^{\frac{2}{\beta}} dt\Big)^{\frac{\beta}{2}} \\
&\leq M\Big(\sum_{k=0}^{\infty} v^{[\alpha]}_{n,k}(x) \int_0^1 \frac{(\frac{k+t^\rho}{n+1} - x)^2}{(\frac{k+t^\rho}{n+1} + x)} dt\Big)^{\frac{\beta}{2}} \\
&\leq \frac{M}{x^{\frac{\beta}{2}}}\Big(\sum_{k=0}^{\infty} v^{[\alpha]}_{n,k}(x) \int_0^1 \Big(\frac{k+t^\rho}{n+1} - x\Big)^2 dt\Big)^{\frac{\beta}{2}} \\
&= \frac{M}{x^{\frac{\beta}{2}}} B^{[\alpha]}_{n,\rho}\Big((e_1 - x)^2; x\Big)^{\frac{\beta}{2}} = M\Big(\frac{\varphi^{[\alpha]}_{n,\rho}}{x}\Big)^{\frac{\beta}{2}}.
\end{aligned}$$

Therefore, the proof is completed. Q.E.D.

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