Modified Apostol-Euler numbers and polynomials of higher order

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Abstract

The purpose of this work is to give modified definitions of Apostol-Euler polynomials and numbers of higher order. We establish their elementary properties, sums, explicit relations, integrals and differential relations. A number of new results which introduced are generalization of known results and their special cases lead to the corresponding formulas of the classical Euler numbers and polynomials.

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1 Introduction and definitions

The classical Euler polynomials $E_n(a)$, Euler polynomials of higher order $E_n^{(\mu)}(a)$ and Apostol-Euler polynomials $\mathcal{E}_n(a;x)$ are defined by the following generating function(see [13, p. 63-66, Eq (39),(40),(64)]), also (see [14]):

$$\frac{2e^{az}}{e^z+1} = \sum_{n=0}^{\infty} E_n(a)\frac{z^n}{n!}, \qquad (|z| < \pi), \tag{1.1}$$

$$\left(\frac{2}{e^z+1}\right)^{\mu} e^{az} = \sum_{n=0}^{\infty} E_n^{(\mu)}(a)\frac{z^n}{n!}, \qquad (|z| < \pi), \tag{1.2}$$

$$\frac{2e^{az}}{xe^z+1} = \sum_{n=0}^{\infty} \mathcal{E}_n(a;x)\frac{z^n}{n!}, \qquad (|z+\log x| < \pi), \tag{1.3}$$

respectively.
In [7] Luo introduced analogous definitions for the classical Euler numbers and polynomials of higher order by using Apostol's idea as follows (see [2, p. 165 (3.1)]):

Definition 1.1. (Luo [7]). Apostol-Euler polynomials of higher order $\mathcal{E}_n^{(\mu)}(a;x)$ are defined by means of the generating function:

$$\left(\frac{2}{xe^z+1}\right)^{\mu} e^{az} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\mu)}(a;x)\frac{z^n}{n!}, \qquad (|z+\log x| < \pi), \tag{1.4}$$

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with, of course,

$$E_n^{(\mu)}(a) = \mathcal{E}_n^{(\mu)}(a;1),\ E_n(a) = \mathcal{E}_n^{(1)}(a;1) \ \ and \ \ \mathcal{E}_n(a;x) := \mathcal{E}_n^{(1)}(a;x), \tag{1.5}$$

Definition 1.2. Apostol-Euler numbers of higher order $\mathcal{E}_n^{(\mu)}(x)$ are defined by means of the generating function:

$$\left(\frac{2e^z}{xe^{2z}+1}\right)^{\mu} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\mu)}(x)\frac{z^n}{n!}, \qquad (|2z+\log x| < \pi). \tag{1.6}$$

with, of course,

$$E_n^{(\mu)} = \mathcal{E}_n^{(\mu)}(1),\ E_n = \mathcal{E}_n^{(1)}(1) \ \ and \ \ \mathcal{E}_n(x) := \mathcal{E}_n^{(1)}(x), \tag{1.7}$$

where $E_n^{(\mu)}$, E_n and $\mathcal{E}_n(x)$ denote the so-called Euler numbers of higher order, classical Euler numbers and Apostol-Euler numbers respectively.
Certain properties of the Apostol-Euler polynomials such as asymptotic estimates, fourier expansions, multiplication formulas etc. are studied by several researchers, see for example [3, 5, 7, 8, 9, 11]. In this work we similarly give the analogous and modified definitions for the classical Euler numbers and polynomials of higher order by using Apostol's idea as follows

Definition 1.3. A Modified Apostol-Euler polynomials of higher order $\mathcal{E}_n^{(\mu;\lambda)}(a;x)$ are defined by means of the generating function:

$$\left(\frac{2}{xe^{\lambda z}+1}\right)^{\mu} e^{az} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\mu;\lambda)}(a;x)\frac{z^n}{n!}, \qquad (\lambda \neq 0; |z+\log x| < \pi), \tag{1.8}$$

It easily follows from (1.8) that

$$\mathcal{E}_n^{(\mu;\lambda)}(a;x) = 2^{\mu}\sum_{k=0}^{\infty}(\mu)_k(a+\lambda k)^n\frac{(-x)^k}{k!} \tag{1.9}$$

In its special cases when $\lambda = 1$, (1.8) reduces to (1.4).
Whereas, for $\mu = \lambda = 1$, (1.8) reduces to (1.3). For $x = \mu = \lambda = 1$, (1.8) reduces to (1.1). For $x = \lambda = 1$, (1.8) reduces to (1.2).

Definition 1.4. A Modified Apostol-Euler numbers of higher order $\mathcal{E}_n^{(\mu;\lambda)}(x)$ are defined by means of the generating function:

$$\left(\frac{2e^z}{xe^{2\lambda z}+1}\right)^{\mu} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\mu;\lambda)}(x)\frac{z^n}{n!}, \qquad (\lambda \neq 0, |2z+\log x| < \pi). \tag{1.10}$$

It easily follows from (1.10) that

$$\mathcal{E}_n^{(\mu;\lambda)}(x) = 2^{\mu+n}\sum_{k=0}^{\infty}(\mu)_k\left(\frac{\mu}{2}+\lambda k\right)^n\frac{(-x)^k}{k!} \tag{1.11}$$

which, in the special case when $\lambda = 1$, (1.10) reduces to (1.6).
The principal object of this paper is to present a natural further step toward the mathematical properties and presentations concerning the modified Apostol-Euler polynomials and numbers of higher order defined in (1.8) and (1.10). Some elementary properties, sums, explicit relations and expansions involving classical functions and hypergeometric series for these functions are established. Some particular cases and consequences of our main results are also considered.

2 Some elementary properties

In the present section we can readily proof of the following results in a straightforwad way by using the formulas (1.8)-(1.11).

Theorem 2.1. For the Modified Apostol-Euler polynomials and numbers defined by (1.8) and (1.10) the following result holds

$$\mathcal{E}_n^{(\mu;\lambda)}(x) = 2^n \mathcal{E}_n^{(\mu;\lambda)}\left(\frac{\mu}{2}; x\right). \tag{2.1}$$

where $\mu \neq 0$.
Proof. If in formula (1.8), we replace a by $\frac{\mu}{2}$, and z by $2z$, then in view of definition (1.10) we get (2.1).□
In its special cases when $\lambda = 1$, (2.1) reduces to the formula (see [7, p. 918 (3)]):

$$\mathcal{E}_n^{(\mu)}(x) = 2^n \mathcal{E}_n^{(\mu)}\left(\frac{\mu}{2}; x\right). \tag{2.2}$$

It is easily seen that

$$\mathcal{E}_n^{(0;\lambda)}(a; x) = a^n. \tag{2.3}$$

Corollary 2.1.

$$\int_\alpha^\beta \mathcal{E}_n^{(\mu;\lambda)}(a; x)\, da = \frac{\mathcal{E}_{n+1}^{(\mu;\lambda)}(\beta; x) - \mathcal{E}_{n+1}^{(\mu;\lambda)}(\alpha; x)}{n+1}. \tag{2.4}$$

Proof. Integrating both the sides of equation (1.9) with respect to a, from α to β, we get (2.4).□
Theorem 2.2. (*Addition formula*). If n is positive integer, then

$$\mathcal{E}_n^{(\mu+\nu;\lambda)}(a+b; x) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{E}_k^{(\mu;\lambda)}(a; x)\mathcal{E}_{n-k}^{(\nu;\lambda)}(b; x). \tag{2.5}$$

Proof. The generating relation (1.8) yields

$$\sum_{n=0}^{\infty} \mathcal{E}_n^{(\mu+\nu;\lambda)}(a+b; x)\frac{z^n}{n!} = \left(\frac{2}{xe^{\lambda z}+1}\right)^{\mu+\nu} e^{(a+b)z}$$

$$= \left(\frac{2}{xe^{\lambda z}+1}\right)^{\mu} e^{az}. \left(\frac{2}{xe^{\lambda z}+1}\right)^{\nu} e^{bz}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathcal{E}_k^{(\mu;\lambda)}(a;x)\mathcal{E}_n^{(\nu;\lambda)}(b;x)\frac{z^{n+k}}{n!k!}$$

$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\mathcal{E}_k^{(\mu;\lambda)}(a;x)\mathcal{E}_{n-k}^{(\nu;\lambda)}(b;x)\right]\frac{z^n}{n!}. \tag{2.6}$$

We now compare the coefficients of $\frac{z^n}{n!}$ on both side of (2.6), and we obtain the formula (2.5) immediately.□

For $\lambda = 1$, (2.5) reduces to the formula (see [7, p. 919 (9)]):

$$\mathcal{E}_n^{(\mu+\nu)}(a+b;x) = \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\mathcal{E}_k^{(\mu)}(a;x)\mathcal{E}_{n-k}^{(\nu)}(b;x). \tag{2.7}$$

Corollary 2.2. If n is a positive integer. Then

$$\mathcal{E}_n^{(\mu;\lambda)}(a+b;x) = \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right) b^{n-k}\mathcal{E}_k^{(\mu;\lambda)}(a;x). \tag{2.8}$$

For $\lambda = 1$, (2.10) reduces to the formula (see [14, p. 95 (38)]):

$$\mathcal{E}_n^{(\mu)}(a+b;x) = \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right) b^{n-k}\mathcal{E}_k^{(\mu)}(a;x). \tag{2.9}$$

Theorem 2.3. (*Difference equation*). Let $a \neq -\lambda; \lambda \neq 0$. Then

$$x\mathcal{E}_n^{(\mu;\lambda)}(a+\lambda;x) + \mathcal{E}_n^{(\mu;\lambda)}(a;x) = 2\mathcal{E}_n^{(\mu-1;\lambda)}(a;x). \tag{2.10}$$

Proof. The generating relation (1.8) yields

$$\sum_{n=0}^{\infty}\left[x\mathcal{E}_n^{(\mu;\lambda)}(a+\lambda;x) + \mathcal{E}_n^{(\mu;\lambda)}(a;x)\right]\frac{z^n}{n!} = x\left(\frac{2}{xe^{\lambda z}+1}\right)^{\mu} e^{(a+\lambda)z} + \left(\frac{2}{xe^{\lambda z}+1}\right)^{\mu} e^{az}$$

$$= \left(\frac{2}{xe^{\lambda z}+1}\right)^{\mu} e^{az}\left(xe^{\lambda z}+1\right)$$

$$= 2\left(\frac{2}{xe^{\lambda z}+1}\right)^{\mu-1} e^{az} = \sum_{n=0}^{\infty} 2\mathcal{E}_n^{(\mu-1;\lambda)}(a;x)\frac{z^n}{n!}. \tag{2.11}$$

We now compare the coefficients of $\frac{z^n}{n!}$ on both side of (2.11), and we obtain the formula (2.10) immediately. □

For $\lambda = 1$, (2.10) reduces to the formula (see [7, p. 918 (5)]):

$$x\mathcal{E}_n^{(\mu)}(a+1;x) + \mathcal{E}_n^{(\mu)}(a;x) = 2\mathcal{E}_n^{(\mu-1)}(a;x). \tag{2.12}$$

Next, by combining (2.8) and (2.10) (with $b = \lambda$), we find that

$$\mathcal{E}_n^{(\mu-1;\lambda)}(a;x) = \frac{1}{2}\left[x\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\lambda^{n-k}\mathcal{E}_k^{(\mu;\lambda)}(a;x) + \mathcal{E}_n^{(\mu;\lambda)}(a;x)\right]. \tag{2.13}$$

which, in the special cases when $\lambda = 1$, (2.13) reduces to the formula (see [14, p. 96 (42)]):

$$\mathcal{E}_n^{(\mu-1)}(a;x) = \frac{1}{2}\left[x\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k^{(\mu)}(a;x) + \mathcal{E}_n^{(\mu)}(a;x)\right]. \tag{2.14}$$

For $\mu = \lambda = 1$, (2.13) reduces to the formula (see [14, p. 96 (43)]):

$$a^n = \frac{1}{2}\left[x\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k(a;x) + \mathcal{E}_n(a;x)\right]. \tag{2.15}$$

Theorem 2.4. (*Two recursion formulas*). Let $a \neq -\lambda; \lambda \neq 0$. Then

$$\mathcal{E}_{n+1}^{(\mu;\lambda)}(a;x) = a\mathcal{E}_n^{(\mu;\lambda)}(a;x) - \frac{\lambda\mu x}{2}\mathcal{E}_n^{(\mu+1;\lambda)}(a+\lambda;x), \tag{2.16}$$

$$\mathcal{E}_n^{(\mu+1;\lambda)}(a;x) = \frac{2}{\lambda\mu}\mathcal{E}_{n+1}^{(\mu;\lambda)}(a;x) + \frac{2(\lambda\mu - a)}{\lambda\mu}\mathcal{E}_n^{(\mu+1;\lambda)}(a;x). \tag{2.17}$$

Proof. Denote, for the convenience, the right-hand side of formula (2.16) by I. Then in view of formula (1.9), it is easily seen that:

$$I = a2^\mu\sum_{k=0}^{\infty}(\mu)_k(a+\lambda k)^n\frac{(-x)^k}{k!} + \lambda 2^\mu\sum_{k=0}^{\infty}(\mu)_{k+1}\left(a+\lambda(k+1)\right)^n\frac{(-x)^{k+1}}{k!}. \tag{2.18}$$

Now, let $k \to k-1$ in the second summation of (2.18) we get

$$I = 2^\mu\sum_{k=0}^{\infty}(\mu)_k(a+\lambda k)^{n+1}\frac{(-x)^k}{k!}.$$

Then in view of formula (1.9), we are finally led to the left-hand side of formula (2.16). Similarly, one can prove the formula (2.17). □

For $\lambda = 1$, (2.16) and (2.17) reduce to the formulas (see [7, p. 919 (12), (13)]):

$$\mathcal{E}_{n+1}^{(\mu)}(a;x) = a\varepsilon_n^{(\mu)}(a;x) - \frac{\mu x}{2}\mathcal{E}_n^{(\mu+1)}(a+1;x), \tag{2.19}$$

$$\mathcal{E}_n^{(\mu+1)}(a;x) = \frac{2}{\mu}\mathcal{E}_{n+1}^{(\mu)}(a;x) + \frac{2(\mu - a)}{\mu}\mathcal{E}_n^{(\mu+1)}(a;x), \tag{2.20}$$

respectively.

Theorem 2.5. Let $a \neq \lambda\mu$. Then

$$\mathcal{E}_n^{(\mu;\lambda)}(\lambda\mu - a;x) = \frac{(-1)^n}{x^\mu}\mathcal{E}_n^{(\mu;\lambda)}(a;x^{-1}). \tag{2.21}$$

Proof. The generating relation (1.8) yields

$$\sum_{n=0}^{\infty}\mathcal{E}_n^{(\mu;\lambda)}(\lambda\mu-a;x)\frac{z^n}{n!}=\left(\frac{2}{xe^{\lambda z}+1}\right)^{\mu}e^{(\lambda\mu-a)z}$$
$$=\left(\frac{2e^{\lambda z}}{xe^{\lambda z}+1}\right)^{\mu}e^{-az}$$
$$=\frac{1}{x^{\mu}}\left(\frac{2}{x^{-1}e^{-\lambda z}+1}\right)^{\mu}e^{-az}$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{x^{\mu}}\mathcal{E}_n^{(\mu;\lambda)}(a;x^{-1})\frac{z^n}{n!}. \tag{2.22}$$

We now compare the coefficients of $\frac{z^n}{n!}$ on both side of (2.22), and we obtain the formula (2.21) immediately. □

For $\lambda=1$, (2.21) reduces to the formula (see [7, p. 919 (10)]):

$$\mathcal{E}_n^{(\mu)}(\mu-a;x)=\frac{(-1)^n}{x^{\mu}}\mathcal{E}_n^{(\mu)}(a;x^{-1}). \tag{2.23}$$

3 Explicit representations

In this section, we geve some explicit formulas for the modified Apostol-Euler polynomials and numbers of higher order.

Theorem 3.1. If n, is a positive integer and μ and x are arbitrary real or complex parameters, then we have

$$\mathcal{E}_n^{(\mu;\lambda)}(a;x)=\sum_{k=0}^{n}\binom{n}{k}\frac{\mathcal{E}_k^{(\mu;\lambda)}(x)}{2^k}\left(a-\frac{\mu}{2}\right)^{n-k}. \tag{3.1}$$

Proof. Denote, for the convenience, the right-hand side of formula (3.1) by I. Then in view of Theorem 2.1 and formula (1.9), it is easily seen that:

$$I=\sum_{m=0}^{\infty}2^{\mu}(\mu)_m\frac{(-x)^m}{m!}\sum_{k=0}^{n}\binom{n}{k}\left(a-\frac{\mu}{2}\right)^{n-k}\left(\frac{\mu}{2}+\lambda m\right)^{k}$$
$$=\sum_{m=0}^{\infty}2^{\mu}(\mu)_m(a+\lambda m)^n\frac{(-x)^m}{m!}.$$

Hence, the left-hand side of formula (3.1) follows. □

For $\lambda=1$, (3.1) reduces to the formula (see [7, p. 918 (3)]):

$$\mathcal{E}_n^{(\mu)}(a;x)=\sum_{k=0}^{n}\binom{n}{k}\frac{\mathcal{E}_k^{(\mu)}(x)}{2^k}\left(a-\frac{\mu}{2}\right)^{n-k}. \tag{3.2}$$

Theorem 3.2. *For* $\{\mu, n\} \in \mathbb{N}$, we have

$$\mathcal{E}_{n-1}^{(n;\lambda)}(a;x) = \frac{1}{2^{(\mu-1)n}} \sum_{k=0}^{(\mu-1)n} \left(\begin{array}{c} (\mu-1)n \\ k \end{array} \right) x^k \mathcal{E}_{n-1}^{(\mu n;\lambda)}(a+\lambda k;x). \tag{3.3}$$

Proof. Denote, for the convenience, the right-hand side of formula (3.3) by I. Then in view of formula (1.9), it is easily seen that:

$$I = \sum_{m=0}^{\infty} \sum_{k=0}^{(\mu-1)n} (-1)^k \left(\begin{array}{c} (\mu-1)n \\ k \end{array} \right) 2^n (\mu n)_m (a+\lambda(k+m))^{n-1} \frac{(-x)^{m+k}}{m!}. \tag{3.4}$$

Now, letting $m \to m-k$ in (3.4) using the formulas
$\left(\begin{array}{c} \mu \\ k \end{array} \right) = \frac{(-\mu)_k}{(-1)^k k!}$; $(\mu)_{m-k} = \frac{(-1)^k (\mu)_m}{(1-\mu-m)_k}$ and $(-m)_k = \frac{(-1)^k m!}{(m-k)!}$ we get

$$I = \sum_{m=0}^{\infty} 2^n (\mu n)_m (a+\lambda m)^{n-1} \frac{(-x)^m}{m!} \sum_{k=0}^{(\mu-1)n} \frac{(n-\mu n)_k (-m)_k}{(1-\mu n-m)_k} \frac{1^m}{m!}.$$

Now, using the formulas

$$\sum_{k=0}^{n} \frac{(a)_k (b)_k}{(c)_k} \frac{1^k}{k!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}; \qquad \frac{\Gamma(a-k)}{\Gamma(a)} = (a)_{-k}$$

and $(a)_{-k} = \frac{(-1)^k}{(1-a)_k}$, gives us the left-hand side of formula (3.3). □
For $s = \lambda = 1$, (3.3) reduces to the formula (see [6, p. 8 (2.9)]):

$$\mathcal{E}_{n-1}^{(n)}(a;x) = \frac{1}{2^{(\mu-1)n}} \sum_{k=0}^{(\mu-1)n} \left(\begin{array}{c} (\mu-1)n \\ k \end{array} \right) x^k \mathcal{E}_{n-1}^{(\mu n)}(a+k;x). \tag{3.5}$$

Theorem 3.3. If n is a positive integer and μ and x are arbitrary real or complex parameters, then we have

$$\mathcal{E}_n^{(\mu;\lambda)}\left(\frac{a}{2};x\right) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \left(-\frac{a}{2}\right)^{n-k} \mathcal{E}_k^{(\mu;\lambda)}(a;x). \tag{3.6}$$

Proof. The generating relation (1.8) yields

$$\sum_{n=0}^{\infty} \mathcal{E}_n^{(\mu;\lambda)}(\frac{a}{2};x)\frac{z^n}{n!} = \left(\frac{2}{xe^{\lambda z}+1}\right)^{\mu} e^{\frac{a}{2}z}$$

$$= \left(\frac{2}{xe^{\lambda z}+1}\right)^{\mu} e^{az}.e^{-\frac{a}{2}z}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{a}{2}\right)^{n}\sum_{k=0}^{\infty}\mathcal{E}_{k}^{(\mu;\lambda)}(a;x)\frac{z^{n+k}}{n!k!}.$$

$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{a}{2}\right)^{n-k}\mathcal{E}_{k}^{(\mu;\lambda)}(a;x)\right]\frac{z^{n}}{n!}. \tag{3.7}$$

We now compare the coefficients of $\frac{z^n}{n!}$ on both side of (3.7), and we obtain the formula (3.6) immediately. □

For $\lambda = 1$, (3.6) reduces to the formula (see [7, p. 922 (27)]):

$$\mathcal{E}_{n}^{(\mu)}\left(\frac{a}{2};x\right) = \sum_{k=0}^{n}\binom{n}{k}\left(-\frac{a}{2}\right)^{n-k}\mathcal{E}_{k}^{(\mu)}(a;x). \tag{3.8}$$

Theorem 3.4. *For* $k \in \mathbb{N}_0$, we have

$$\sum_{k=0}^{\infty}\mathcal{E}_{n+k}^{(\mu;\lambda)}(a;x)\frac{w^{k}}{k!} = e^{aw}\mathcal{E}_{n}^{(\mu;\lambda)}(a;xe^{\lambda w}). \tag{3.9}$$

Proof. Denote, for the convenience, the right-hand side of formula (3.9) by I. Then, in view of formula (1.9), it is easily seen that:

$$I = \sum_{m=0}^{\infty}2^{\mu}(\mu)_{m}(a+\lambda m)^{n}\frac{(-x)^{m}}{m!}e^{(a+\lambda m)w}$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}2^{\mu}(\mu)_{m}(a+\lambda m)^{n+k}\frac{(-x)^{m}}{m!}\frac{w^{k}}{k!}. \tag{3.10}$$

Now, with the help of definition (1.9), equation (3.10) gives us the left-hand side of formula (3.9). □

In its special cases when $\lambda = 1$, (3.9) reduces to the formula

$$\sum_{k=0}^{\infty}\mathcal{E}_{n+k}^{(\mu)}(a;x)\frac{w^{k}}{k!} = e^{aw}\mathcal{E}_{n}^{(\mu)}(a;xe^{w}). \tag{3.11}$$

Theorem 3.5. If n is a positive integer and μ and x are arbitrary real or complex parameters, then we have

$$\mathcal{E}_{n}^{(\mu;\lambda)}(a;x) = 2^{\mu}\sum_{l=0}^{n}\binom{n}{l}\binom{\mu+l-1}{l}(\lambda x)^{l}(x+1)^{-\mu-l}$$

$$\times\sum_{k=0}^{l}(-1)^{k}\binom{l}{k}k^{l}(a+\lambda k)^{n-l}{}_{2}F_{1}\left[l-n,l;l+1;\frac{\lambda k}{(a+\lambda k)}\right], \tag{3.12}$$

where ${}_2F_1$ denotes the Gaussian hypergeometric funtion [13, p.44 (4)].
Proof. we differentiate both side of the generating relation (1.8) with respect to the variable z. By using Leibniz's rule, we get

$$\mathcal{E}_n^{(\mu;\lambda)}(a;x) = D_z^n \left\{ \left(\frac{2}{xe^{\lambda z}+1} \right)^{\mu} e^{az} \right\}\Big|_{z=0}, \qquad D_z = \frac{d}{dz}.$$

$$= 2^{\mu} \sum_{s=0}^{n} \binom{n}{s} a^{n-s} D_z^s \left\{ \left[(x+1) + x(e^{\lambda z}-1) \right]^{-\mu} \right\}\Big|_{z=0}. \tag{3.13}$$

Applying the series expansion:

$$(x+\omega)^{-\mu} = \sum_{l=0}^{\infty} \binom{\mu+l-1}{l} x^{-\mu-l}(-\omega)^l, \qquad (|\omega|<|x|), \tag{3.14}$$

and the well-known formula (see [13, p.58 (15)])

$$(e^{\lambda z}-1)^l = l! \sum_{r=l}^{\infty} S(r,l) \frac{(\lambda z)^r}{r!}, \tag{3.15}$$

we have

$$\mathcal{E}_n^{(\mu;\lambda)}(a;x) = 2^{\mu} \sum_{s=0}^{n} \binom{n}{s} a^{n-s} \sum_{l=0}^{s} \binom{\mu+l-1}{l} (x+1)^{-\mu-l}(-x)^l \lambda^s l! S(s,l). \tag{3.16}$$

We now change the order of summation in the above double series, and make use of the following formula:(see [13, p.58 (20)])

$$S(s,l) = \frac{1}{l!} \sum_{k=0}^{l} (-1)^{l-k} \binom{l}{k} k^s, \tag{3.17}$$

so that

$$\mathcal{E}_n^{(\mu;\lambda)}(a;x) = 2^{\mu} \sum_{l=0}^{n} \binom{n}{l} \binom{\mu+l-1}{l} (\lambda x)^l (x+1)^{-\mu-l} a^{n-l}$$

$$\times \sum_{k=0}^{l} (-1)^k \binom{l}{k} k^l {}_2F_1 \left[l-n, 1; l+1; \frac{-\lambda k}{a} \right], \tag{3.18}$$

in terms of the Gaussian hypergeometric funtion. Finally, if we apply the known transformation [1, p.284 (23)]:

$${}_2F_1[a,b;c;z] = (1-z)^{-a} {}_2F_1 \left[a, c-b; c; \frac{z}{(z-1)} \right]$$

in (3.18), we are led immediately to the explicit formula (3.12) asserted by Theorem 3.5. □
For $\lambda = 1$, (3.12) reduces to the formula (see [7, p. 920 (16)]):

$$\mathcal{E}_n^{(\mu)}(a;x) = 2^{\mu} \sum_{l=0}^{n} \binom{n}{l} \binom{\mu+l-1}{l} x^l (x+1)^{-\mu-l}$$

$$\times \sum_{k=0}^{l}(-1)^k \binom{l}{k} k^l (a+k)^{n-l} {}_2F_1\left[l-n, l; l+1; \frac{k}{(a+k)}\right]. \tag{3.19}$$

Corollary 3.1. If n is a positive integer and μ and x are arbitrary real or complex parameters, then we have

$$\mathcal{E}_n^{(\mu;\lambda)}(x) = 2^{l+\mu} \sum_{l=0}^{n} \binom{n}{l} \binom{\mu+l-1}{l} (\lambda x)^l (x+1)^{-\mu-l}$$

$$\times \sum_{k=0}^{l}(-1)^k \binom{l}{k} k^l (\mu+2\lambda k)^{n-l} {}_2F_1\left[l-n, l; l+1; \frac{2\lambda k}{(\mu+2\lambda k)}\right]. \tag{3.20}$$

Proof. For $a = \frac{\mu}{2}$ in Theorem 3.5, then in view of Theorem 2.1, we obtain the formula (3.20) immediately. □

The following extended Hurwitz-Larch Zeta function introduced and studied earlier by Bin-Saad [4, p.271 (2.7)]

$$\Phi^*_{\mu,\lambda}(x,z,a) = \sum_{m=0}^{\infty} \frac{(\mu)_n}{(a+\lambda m)^z} \frac{x^m}{m!}, \qquad \mu \in \mathbb{C}\backslash\mathbb{Z}_0^-; a \in \mathbb{C}\backslash\{(-\lambda n)\}; |x| < 1. \tag{3.21}$$

Below we give the following relationships between the modified Apostol-Euler polynomials defined by (1.8) and Bin-Saad-Hurwitz-Larch Zeta function (3.21).

Theorem 3.6. Let $n \in \mathbb{N}$; $-1 < x \le 1$; $\mu \in \mathbb{C}\backslash\mathbb{Z}_0^-$; $\lambda \neq 0$. Then

$$\mathcal{E}_n^{(\mu;\lambda)}(a;x) = 2^\mu \Phi^*_{\mu,\lambda}(-x,-n,a). \tag{3.22}$$

Proof. By (1.8) and the generalized binomial theorem, we have

$$\sum_{n=0}^{\infty} \mathcal{E}_n^{(\mu;\lambda)}(a;x)\frac{z^n}{n!} = \left(\frac{2}{xe^{\lambda z}+1}\right)^{\mu} e^{az}$$

$$= 2^\mu \sum_{k=0}^{\infty} \frac{(\mu)_k}{k!}(-x)^k e^{(a+\lambda k)z}$$

$$= \sum_{n=0}^{\infty}\left[2^\mu \sum_{k=0}^{\infty} \frac{(\mu)_k}{k!}(-x)^k (a+\lambda k)^n\right]\frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left[2^\mu \sum_{k=0}^{\infty} \frac{(\mu)_k}{(a+\lambda k)^{-n}} \frac{(-x)^k}{k!}\right]\frac{z^n}{n!}. \tag{3.23}$$

Hence the result (3.22) follows. □

4 Differential relations

The modified Apostol-Euler polynomials of higher order $\mathcal{E}_n^{(\mu;\lambda)}(a;x)$ satisfies some differential recurrence relation. Fortunately these properties of $\mathcal{E}_n^{(\mu;\lambda)}(a;x)$ can be developed directly from the formula (1.8). First, by recalling the familiar derivative formula from calculus in terms of the gamma function [10]

$$D_x^m x^n = \frac{\Gamma(n+1)}{\Gamma(n-m+1)} x^{n-m}, \quad n-m \geq 0, \ \ D_x = \frac{d}{dx}, \tag{4.1}$$

where $\{m,n\} \in \mathbb{N}$, we aim now to derive the following differential relation .

Theorem 4.1. Let $\lambda \in \mathbb{C}\backslash\{0\}$. Then

$$\frac{\partial}{\partial\lambda}\mathcal{E}_n^{(\mu;\lambda)}(a;x) = -\frac{nx\mu}{2}\mathcal{E}_{n-1}^{(\mu+1;\lambda)}(a+\lambda;x). \tag{4.2}$$

Proof. The generating relation (1.8) yields

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial\lambda}\mathcal{E}_n^{(\mu;\lambda)}(a;x)\frac{z^n}{n!} = 2^\mu e^{az}\frac{\partial}{\partial\lambda}\left(xe^{\lambda z}+1\right)^{-\mu}$$

$$= -\frac{zx\mu}{2}\left(\frac{2}{xe^{\lambda z}+1}\right)^{\mu+1} e^{(a+\lambda)z}$$

$$= -\frac{x\mu}{2}\sum_{n=0}^{\infty}\mathcal{E}_n^{(\mu+1;\lambda)}(a+\lambda;x)\frac{z^{n+1}}{n!}. \tag{4.3}$$

Now, letting, $n \to n-1$ in the right-hand side of (4.3) and comparing the coefficients of $\frac{z^n}{n!}$ on both side (4.3) we get connection formula (4.2) immediately. □

Corollary 4.1. Let $k \in \mathbb{N}$. Then

$$\frac{\partial^k}{\partial a^k}\mathcal{E}_n^{(\mu;\lambda)}(a;x) = \frac{n!}{(n-k)!}\mathcal{E}_{n-k}^{(\mu;\lambda)}(a;x). \tag{4.4}$$

For $\lambda = 1$, (4.4) reduces to the formula (see [7, p. 918 (6)]):

$$\frac{\partial^k}{\partial a^k}\mathcal{E}_n^{(\mu)}(a;x) = \frac{n!}{(n-k)!}\mathcal{E}_{n-k}^{(\mu)}(a;x). \tag{4.5}$$

Theorem 4.2. Let $k \in \mathbb{N}$. Then

$$D_x^k\mathcal{E}_n^{(\mu;\lambda)}(a;x) = \left(\frac{-1}{2}\right)^k (\mu)_k\mathcal{E}_n^{(\mu+k;\lambda)}(a+\lambda k;x). \tag{4.6}$$

Proof. By starting from the left-hand side of (4.6) and in view of (1.9) and by using the relation (4.1), we get :

$$D_x^k \mathcal{E}_n^{(\mu;\lambda)}(a;x) = 2^\mu \sum_{m=0}^{\infty} (-1)^m (\mu)_m (a+\lambda m)^n \frac{x^{m-k}}{(m-k)!}. \tag{4.7}$$

Now, letting $m \to m+k$ in (4.7), using the formula $(\mu)_{m+k} = (\mu)_k(\mu+k)_m$ and considering the formula (1.9), we get the right-hand side of formula (4.6). $\square$

For $\lambda = 1$, (4.6) reduces to the formula

$$D_x^k \mathcal{E}_n^{(\mu)}(a;x) = \left(\frac{-1}{2}\right)^k (\mu)_k \mathcal{E}_n^{(\mu+k)}(a+k;x). \tag{4.8}$$

Theorem 4.3. Let $\Re(\nu) > 0$. Then

$$D_x^{\nu-\mu}\left[x^{\nu-1}\mathcal{E}_n^{(\mu;\lambda)}(a;x)\right] = \frac{\Gamma(\nu)}{\Gamma(\mu)} 2^{\mu-\nu} x^{\mu-1} \mathcal{E}_n^{(\nu;\lambda)}(a;x). \tag{4.9}$$

Proof. By starting from the left-hand side of formula (4.9) and in view of (1.9) and by using the relation (4.1), we get :

$$\begin{aligned} D_x^{\nu-\mu}\left[x^{\nu-1}\mathcal{E}_n^{(\mu;\lambda)}(a;x)\right] &= 2^\mu \sum_{m=0}^{\infty} (-1)^m (\mu)_m (a+\lambda m)^n \frac{\Gamma(\nu+m)}{\Gamma(\mu+m)} \frac{x^{m+\mu-1}}{m!} \\ &= \frac{\Gamma(\nu)}{\Gamma(\mu)} 2^{\mu-\nu} x^{\mu-1} 2^\nu \sum_{m=0}^{\infty} (\nu)_m (a+\lambda m)^n \frac{(-x)^m}{m!} \\ &= \frac{\Gamma(\nu)}{\Gamma(\mu)} 2^{\mu-\nu} x^{\mu-1} \mathcal{E}_n^{(\nu;\lambda)}(a;x). \end{aligned}$$

$\square$

For $\lambda = 1$, (4.9) reduces to the formula

$$D_x^{\nu-\mu}\left[x^{\nu-1}\mathcal{E}_n^{(\mu)}(a;x)\right] = \frac{\Gamma(\nu)}{\Gamma(\mu)} 2^{\mu-\nu} x^{\mu-1} \mathcal{E}_n^{(\nu)}(a;x). \tag{4.10}$$

Closely associated with the derivative of the gamma function is the digamma function defined by [1,p.74(2.51)]:

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x \neq 0, -1, -2, ... \; . \tag{4.11}$$

Now, we wish to establish the derivative of the function $\mathcal{E}_n^{(\mu)}(a;x)$ with respect to the parameter μ.

Theorem 4.4. Let $\mu \in \mathbb{C}\backslash\mathbb{Z}_0^-$. Then

$$\frac{\partial}{\partial\mu}\mathcal{E}_n^{(\mu;\lambda)}(a;x) = 2^\mu \sum_{m=0}^{\infty} (\mu)_m (a+\lambda m)^n \frac{(-x)^m}{m!} [\log 2 + \psi(\mu+m) - \psi(\mu)]. \tag{4.12}$$

Proof. By starting from the left-hand side of formula (4.12), then according to the result:

$$\frac{d}{d\mu}\left[2^{\mu}(\mu)_m\right] = (\mu)_m \frac{d}{d\mu}2^{\mu} + 2^{\mu}\frac{d}{d\mu}\left[\frac{\Gamma(\mu+m)}{\Gamma(\mu)}\right] = 2^{\mu}(\mu)_m[\log 2 + \psi(\mu+m) - \psi(\mu)],$$

we obtain the right-hand side of formula (4.12). □

Theorem 4.5. Let $n \in \mathbb{N}$. Then

$$\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{n-k}^{(\mu;\lambda)}(a;x)y^k = e^{y\frac{\partial}{\partial a}}\mathcal{E}_n^{(\mu;\lambda)}(a;x). \tag{4.13}$$

$$= \mathcal{E}_n^{(\mu;\lambda)}(a+y;x). \tag{4.14}$$

Proof. Denote, for convenience, the right-hand side of equation (4.13) by I. Then

$$I = e^{y\frac{\partial}{\partial a}}\mathcal{E}_n^{(\mu;\lambda)}(a;x)$$

$$= \sum_{k=0}^{\infty}\frac{y^k}{k!}\frac{\partial}{\partial a}\mathcal{E}_n^{(\mu;\lambda)}(a;x).$$

Upon using (4.4), we led finally to the left-hand side of the formula (4.13). Moreover, if we apply the formula (2.8) (replace b by y and k by $n-k$), we get (4.14). □

Theorem 4.6. Let $n \in \mathbb{N}$. Then

$$\sum_{k=0}^{n}\binom{n}{k}(\alpha)_k\mathcal{E}_{n-k}^{(\mu;\lambda)}(a;x)y^k = \left(1 - y\frac{\partial}{\partial a}\right)^{-\alpha}\mathcal{E}_n^{(\mu;\lambda)}(a;x), \tag{4.15}$$

$$\sum_{k=0}^{n}(\alpha)_k(\beta)_k\mathcal{E}_{n-k}^{(\mu;\lambda)}(a;x)\frac{y^k}{k!} = {}_2F_1\left(\alpha,\beta;-n;-y\frac{\partial}{\partial a}\right)\mathcal{E}_n^{(\mu;\lambda)}(a;x), \tag{4.16}$$

where ${}_2F_1$ is the Gaussian hypergeometric function (see [12] and [13]).
Proof. We refer to the proof of Theorem 4.5. □

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