Semi-slant Riemannian maps from almost contact metric manifolds into Riemannian manifolds

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Abstract

Firstly, a generalization of Riemannian submersions, slant submersions and semi-slant submersions, we introduce semi-slant Riemannian maps from almost contact metric manifolds onto Riemannian manifolds. In this paper, we obtain some results on such maps by taking the vertical structure vector field. Among them, we study integrability of distributions and the geometry of foliations. Further, we find the necessary and sufficient conditions for semi-slant Riemannian maps to be harmonic and totally geodesic. We, also investigate some decomposition theorems and provide some examples to show the existence of the maps.

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1 Introduction

Differentiable maps between Riemannian manifolds play an important role in differential geometry. There are certain kinds of differentiable maps between Riemannian manifolds whose existence influences the geometry of the source manifolds and the target manifolds. These maps between two Riemannian manifolds also play significant role to compare geometric structures defined on both manifolds.

Let f be a differentiable map from a Riemannian manifold (M, g_M) to a Riemannian manifold (N, g_N), where $\dim M = m$ and $\dim N = n$. We know that the map f is harmonic if and only if the tension field $\tau(f) = trace(\nabla f_*) = 0$ [5], and we also know that f is totally geodesic if $(\nabla f_*)(X, Y) = 0$, for all $X, Y \in \Gamma(TM)$ [1].

On the other hand, submersions have been studied widely in differential geometry. Riemannian submersions between Riemannian manifolds were studied by O'Neill [12] and Gray [8]. Such submersions between Riemannian manifolds equipped with an additional structure of almost complex type was firstly studied by Watson in [18]. There are several kinds of Riemannian submersions like:

Almost Hermitian submersion [17], slant submersions from almost Hermitian manifolds [16], semi-Riemannian submersion and Lorentzian submersion [6], etc. As we know, Riemannian submersions are related with physics and have their applications in the Yang-Mills theory ([4], [18]), Supergravity and superstring theories ([9], [11]), Kaluza-Klein theory [10], etc. Semi-slant Riemannian maps into almost Hermitian manifolds was studied by Park and Sahin [14]. It is known that complex techniques in physics have been very effective tools for understanding space time geometry. Now, we study semi-slant Riemannian maps from almost contact metric manifolds into Riemannian manifolds.

A. Fischer introduced a Riemannian map between Riemannian manifolds [7], which unifies and generalizes the notions of an isometric immersion, a Riemannian submersion, and an isometry. Let

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$f:(M,g_M)\to(N,g_N)$ be a differentiable map between Riemannian manifolds such that $0<rank$ $f_*<\min(m,n)$. Then we denote the *kernal* space of f_* by $kerf_*$ such that the orthogonal complementary space $(kerf_*)^\perp$ of $kerf_*$ in TM. Then the TM has the following orthogonal decomposition:

$$TM=\ker f_*\oplus(\ker f_*)^\perp. \tag{1.1}$$

Also, we denote the *range* of f_* by $(rangef_{*f(p)})$, for $p\in M$ and orthogonal complementary space $(rangef_{*f(p)})^\perp$ of $rangef_{*f(p)}$ in $T_{f(p)}N$. Thus the tangent space $T_{f(p)}N$ has the following orthogonal decomposition:

$$T_{f(p)}N=(rangef_{*f(p)})\oplus(rangef_{*f(p)})^\perp. \tag{1.2}$$

Next, a differentiable map $f:(M,g_M)\to(N,g_N)$ is called a Riemannian map at $p\in M$ if the horizontal restriction $f^h_{*p}:(\ker f_{*p})^\perp\to(rangef_{*f(p)})$ is linear isometry between the inner product space $((\ker f_{*p})^\perp,g_M(p)|_{(\ker f_{*p})^\perp})$ and $(rangef_{*f(p)},$
$g_N(f(p))|_{(rangef_{*f(p)})})$. Therefore, Fischer define [7] that a Riemannian map is a map which is as isometric as it can be. In the other hands, a differentiable map $f:(M,g_M)\to(N,g_N)$ between Riemannian manifolds (M,g_M) and (N,g_N) is called a Riemannian map if it satisfies the equation

$$g_N(f_*X,f_*Y)=g_M(X,Y),\ \text{for all } X,Y\in(\ker f_*)^\perp. \tag{1.3}$$

It follows that isometric immersions and Riemannian submersions are particular cases of Riemannian maps with $\ker f_*=\{0\}$ and $(rangef_*)^\perp=\{0\}$ respectively. After that, there are lots of papers on this topic. Further, slant Riemannian maps [15] and semi-slant submersions [13] were studied. As a generalization of slant submersions, semi-slant submersions and slant Riemannian maps, park defined the notion of semi-slant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds [14]. We will study semi-slant Riemannian maps from almost contact metric manifolds into Riemannian manifolds.

In this paper, we study semi-slant Riemannian maps from almost contact metric manifolds into Riemannian manifolds. The paper is organized as follows. In section 2, we collect main notions and formulae which we need for this paper.

In section 3, we introduce semi-slant Riemannian maps from almost contact metric manifolds onto Riemannian manifolds admitting vertical structure vector field. We find necessary and sufficient conditions for semi-slant Riemannian maps to be harmonic and totally geodesic.

2 Preliminaries

An odd-dimensional smooth manifold M is said to have an almost contact structure (φ,ξ,η) if there exist on M, a tensor field φ of type $(1,1)$, a vector field ξ and $1-$form η such that

$$\varphi^2=-I+\eta\otimes\xi,\quad \varphi\xi=0,\ \eta\circ\varphi=0, \tag{2.1}$$

$$\eta(\xi)=1, \tag{2.2}$$

where I denote the identity tensor. The manifold M with an almost contact structure is called almost contact manifold.

If there exist a Riemannian metric g on an almost contact manifold M satisfying the following conditions;

$$\begin{aligned} g(\varphi X, \varphi Y) &= g(X,Y) - \eta(X)\eta(Y), \\ g(X, \varphi Y) &= -g(\varphi X, Y), \end{aligned} \tag{2.3}$$

$$g(X,\xi) = \eta(X), \tag{2.4}$$

where X, Y are the vector fields on M, then structure (φ, ξ, η, g) is called almost contact metric structure and the manifold M is called an almost contact metric manifold. An almost contact manifold M with almost contact metric structure (φ, ξ, η, g) is denoted by $(M, \varphi, \xi, \eta, g)$. Further, an almost contact structure (φ, ξ, η) is said to be normal if $N + d\eta \otimes \xi = 0$, where N is the Nijenhuis tensor of φ. The fundamental 2-form Φ is defined by $\Phi(X,Y) = g(X, \varphi Y)$.

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be Sasakian manifold [3], if it satisfies the following condition;

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.5}$$

where ∇ denotes the Riemannian connection of metric g on M.

Example 2.1. ([2]) Let R^{2k+1} with cartesian coordinates (x_i, y_i, z) $(i = 1, 2.....k)$ and its usual contact form

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{k} y_i dx_i).$$

The characteristic vector field ξ is given by $2\frac{\partial}{\partial z}$ and its Riemannian metric $g_{R^{2k+1}}$ and tensor field φ are given by

$$g_{R^{2k+1}} = \eta \otimes \eta + \frac{1}{4}(\sum_{i=1}^{k} (dx_i)^2 + (dy_i)^2), \ \varphi = \begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{bmatrix}, \quad i = 1, ..., k.$$

This gives a contact metric structure on R^{2k+1}. The vector fields $E_i = 2\frac{\partial}{\partial y_i}$, $E_{k+i} = 2(\frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z})$ and ξ form a $\varphi-$basis for the contact metric structure. On the other hand, it can be shown that $R^{2k+1}(\varphi, \xi, \eta, g)$ is a Sasakian manifold.

For a Sasakian manifold M, we have

$$R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.6}$$

$$\nabla_X \xi = -\varphi X, \tag{2.7}$$

$$S(X,\xi) = 2n\eta(X), \tag{2.8}$$

for any vector fields X, Y on M.

Let $f : (M, g_M) \to (N, g_N)$ be a differentiable map between Riemannian manifolds. The second fundamental form of f is given by

$$(\nabla f_*)(X,Y) = \nabla^f_X f_* Y - f_*(\nabla_X Y), \quad \text{for } X, Y \in \Gamma TM. \tag{2.9}$$

where ∇^f is the pullback connection and we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N [1] . Recall that f is said to be harmonic if we have the tensor field $\tau(f) = trace(\nabla f_*) = 0$ and we call the map f a totally geodesic map if $(\nabla f_*)(X,Y) = 0$, for $X, Y \in \Gamma TM$. Denote the range of f_* by $rangef_*$ as a subset of the pullback bundle $f^{-1}TN$. With orthogonal complement $(rangef_*)^\perp$ we have the following orthogonal decomposition

$$f^{-1}TN = rangef_* \oplus (range\ f_*)^\perp.$$

We deal with the harmonicity of a Riemannian map f. Given a differentiable map f between Riemannian manifolds, we can naturally define a function $e(f) : M \to [0, \infty]$ given by

$$e(f)(x) := \frac{1}{2} \mid e(f_*)(x) \mid^2, \ x \in M,$$

where $\mid (f_*)(x) \mid$ denotes the Hilbert-Schmidt norm of $(f_*)(x)$ [1]. We call $e(f)$ the energy density of f. Let K be a compact domain of M, i.e., K is the compact closure $\overline{U}$ of a non-empty connected open subset U of M. The energy integral of f over K is the integral of its energy density:

$$E(f;K) = \int_K e(f) v_{g_M} = \frac{1}{2} \int_K \mid (f_*) \mid^2 v_{g_M},$$

where v_{g_M} is the volume form on (M, g_M). Let $C^\infty(M,N)$ denote the space of all differentiable map from M to N. A differentiable map $f : M \to N$ is said to be harmonic if it is a critical point of the energy functional $E(f;K) : C^\infty(M,N) \to R$ for any compact domain $K \subset M$. By the result of J. Eells and J. Sampson [5], we know that the map f is harmonic if and only if the tension field $\tau(f) = trace(\nabla f_*) = 0$.

Let $f : (M, g_M) \to (N, g_N)$ be a Riemannian map. The map f is called a Riemannian map with totally umbilical fibres if

$$T_X Y = g_M(X,Y)H, \quad \text{for } X, Y \in \Gamma(\ker f_*),$$

where H is mean curvature vector field of the fibres.

Given a Riemannian manifold (M, g_M) and distribution D on M call the distribution D autoparallel or totally geodesic foliation if $\nabla_X Y \in \Gamma(D)$, for $X, Y \in \Gamma(D)$. If D is autoparallel, then it is obviously integrable and its leaves are totally geodesic in M. The distribution D is said to be parallel if $\nabla_X Y \in \Gamma(D)$, for $Y \in \Gamma(D)$ and $Z \in \Gamma(TM)$. If D is parallel, then we easily obtain that its orthogonal complementary distribution $D^\perp$ is also parallel. In this situation, M is locally a Riemannian product manifold of the leaves of D and $D^\perp$. It is also easy to show that if the distributions D and $D^\perp$ are simultaneously autoparallel, then they are also parallel.

Lemma 2.2. Let f be a Riemannian map from a Riemannian manifold (M, g_M) to a Riemannian manifold (N, g_N) [7]. Then

$$(\nabla f_*)(X,Y) \in \Gamma((range f_*)^\perp), \quad \text{for } X,Y \in \Gamma((\ker f_*)^\perp).$$

Lemma 2.3. For any X, Y vertical and V, W horizontal vector fields, the tensors $\mathcal{T}$ and $\mathcal{A}$ satisfy [12] :

$$\begin{aligned} \mathcal{T}_X Y &= \mathcal{T}_Y X, \\ \mathcal{A}_V W &= -\mathcal{A}_W V = \frac{1}{2}\mathcal{V}[V,W]. \end{aligned}$$

Lemma 2.4. Let f be a Riemannian map from a Riemannian manifold (M, g_M) to a Riemannian manifold (N, g_N). Then the map f satisfies a generalized eikonal equation [7]

$$2e(f) = \| f_* \|^2 = rank f.$$

As we know, $\| f_* \|^2$ is a continuous function on M and *rank* f is integer-valued so that *rank* f is locally constant. Hence, if M is connected, then *rank* f is a constant function.

3 Semi-slant Riemannian maps admitting vertical structure vector field

In this section we define semi-slant Riemannian maps from almost contact metric manifolds to Riemannian manifolds. We investigate integrability of distributions and harmonicity conditions for semi-slant Riemannian maps. Further, we find the conditions for a semi-slant Riemannian map to be totally geodesic and prove some decomposition theorems. Throughout this section, we have taken semi-slant Riemannian maps admitting vertical structure vector field and give as.

Definition 3.1. Let $(M, \varphi, \xi, \eta, g_M)$ be an almost contact metric manifold and (N, g_N) be a Riemannian manifold. A Riemannian map $f : (M, \varphi, \xi, \eta, g_M) \to (N, g_N)$ is called a semi-slant Riemannian map if there are two distributions $D_1, D_2 \subset \ker f_*$ such that

$$\ker f_* = D_1 \oplus D_2 \oplus <\xi>, \varphi(D_1) = D_1,$$

and the angle $\theta = \theta(X)$ between φX and the space $(D_2)_p$ is constant for non-zero vector fields $X \in (D_2)_p$ and $p \in M$, where $D_1 \oplus D_2 \oplus <\xi>$ is an orthogonal decomposition of $\ker f_*$.

We call the angle θ a semi$-$slant angle.

Note that given a Euclidean space R^{2n+1} with coordinates $(x_1, x_2, ..., x_{2n}, x_{2n+1})$ we can canonically choose an almost contact metric structure (φ, ξ, η, g) on R^{2n+1} as follows:

$$\begin{aligned} & \varphi(a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + \dots\dots\dots\dots + a_{2n-1}\frac{\partial}{\partial x_{2n-1}} + a_{2n}\frac{\partial}{\partial x_{2n}} + a_{2n+1}\frac{\partial}{\partial x_{2n+1}}) \\ = \;& (-a_2\frac{\partial}{\partial x_1} + a_1\frac{\partial}{\partial x_2} + \dots\dots\dots\dots\dots\dots - a_{2n}\frac{\partial}{\partial x_{2n-1}} + a_{2n-1}\frac{\partial}{\partial x_{2n}}), \end{aligned}$$

where $\xi = \frac{\partial}{\partial x_{2n+1}}$ and $a_1, a_2, a_{3,}...., a_{2n}, a_{2n+1}$ are $C^\infty-$real valued functions in R^{2n+1}. Let $\eta = dx_{2n+1}$ is $1-$form on R^{2n+1} and let $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n+1}}, ..., \frac{\partial}{\partial x_{2n}}, \frac{\partial}{\partial x_{2n+1}}\}$ is orthonormal basis of vector fields on R^{2n+1}. Let $g_{R^{2n+1}}$ is a Euclidean metric on R^{2n+1}.

Example 3.2. Let R^{11} has almost contact metric structure with metric g_{11} as defined above. Let $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11})$ be coordinate system in R^{11} and $(y_1, y_2, y_3, y_4, y_5, y_6, y_7)$ be coordinate system in R^7. Let g_7 be Euclidean metric on R^7. Define a map $f : R^{11} \to R^7$ by

$$f(x_1, x_2, ..., x_{11}) = (c, 0, \frac{-x_3 + x_5}{\sqrt{2}}, x_4, d, \frac{x_7 + x_9}{\sqrt{2}}, \frac{x_8 + x_{10}}{\sqrt{2}}),$$

with $c, d \in R$. Then the map f is a semi-slant Riemannian map such that

$$\begin{aligned}
D_1 &= < \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_7} - \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_8} - \frac{\partial}{\partial x_{10}} >, \\
D_2 &= < \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6} >, \ \xi = \frac{\partial}{\partial x_{11}}, \\
(\ker f_*)^\perp &= < V_1 = \frac{\partial}{\partial x_4}, V_2 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}, V_3 = \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_9}, \\
V_4 &= \frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_{10}} >, \\
\omega(D_2) &= < \frac{\partial}{\partial x_4}, \ \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5} >, \ \mu =< \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_{10}} >, \\
f_* V_1 &= \frac{\partial}{\partial y_4}, f_* V_2 = \sqrt{2}\frac{\partial}{\partial y_3}, f_* V_3 = \sqrt{2}\frac{\partial}{\partial y_6}, f_* V_4 = \sqrt{2}\frac{\partial}{\partial y_7},
\end{aligned}$$

with the semi-slant angle $\theta = \frac{\pi}{4}$.

Let $f : (M, \varphi, \xi, \eta, g_M) \to (N, g_N)$ be a semi-slant Riemannian map. Then there are distributions $D_1, D_2 \subset \ker f_*$ such that

$$\ker f_* = D_1 \oplus D_2 \oplus < \xi >, \varphi(D_1) = D_1,$$

and the angle $\theta = \theta(X)$ between φX and space $(D_2)_p$ is constant for non-zero vector fields $X \in (D_2)_p$ and $p \in M$, where $D_1 \oplus D_2 \oplus < \xi >$ is an orthogonal decomposition of a $\ker f_*$. Then for $X \in \Gamma(\ker f_*)$, we get

$$X = PX + QX + \eta(X)\xi, \tag{3.1}$$

where $PX \in \Gamma(D_1)$ and $QX \in \Gamma(D_2)$.

For $X \in \Gamma(\ker f_*)$, we write

$$\varphi X = \psi X + \omega X, \tag{3.2}$$

where $\psi X \in \Gamma(\ker f_*)$ and $\omega X \in \Gamma((\ker f_*)^\perp)$.

For $Z \in \Gamma((\ker f_*)^\perp)$, we write

$$\varphi Z = BZ + CZ, \tag{3.3}$$

where $BZ \in \Gamma(\ker f_*)$ and $CZ \in \Gamma((\ker f_*)^\perp)$.

For U$\in \Gamma(TM)$, we obtain

$$U = \mathcal{V}U + \mathcal{H}U, \tag{3.4}$$

where $\mathcal{V}U \in \Gamma(\ker f_*)$ and $\mathcal{H}U \in \Gamma((\ker f_*)^\perp)$.

For $W \in \Gamma(f^{-1}TN)$, we have

$$W = \overline{P}W + \overline{Q}W, \tag{3.5}$$

where $\overline{P}W \in \Gamma(range\ f_*)$ and $\overline{Q}W \in \Gamma(range\ f_*)^\perp$.

Then

$$(\ker f_*)^\perp = \omega D_2 \oplus \mu,$$

where μ is the orthogonal complement of ωD_2 in $(\ker f_*)^\perp$ and is invariant under φ.

Thus, we have

$$TM = (\ker f_*) \oplus (\ker f_*)^\perp.$$

Futher more, we have

$$\begin{aligned} \psi D_1 &= D_1,\ \omega D_1 = 0 \ \Leftrightarrow\ \varphi D_1 = D_1, \\ \psi D_2 &\subset D_2,\ B((\ker f_*)^\perp) = D_2, \\ \psi^2 + B\omega &= -id,\ C^2 + \omega B = -id, \\ \omega\psi + C\omega &= 0,\ BC + \varphi B = 0, \psi\xi = 0, \omega\xi = 0. \end{aligned}$$

Define tensors $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \tag{3.6}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{3.7}$$

for $E, F \in \Gamma(TM)$, where ∇ is the Levi-Civita connection of (M, g_M).

For $X, Y \in \Gamma(\ker f_*)$, define

$$\widehat{\nabla}_X Y = \mathcal{V}\nabla_X Y, \tag{3.8}$$

$$(\nabla_X \psi)Y = \widehat{\nabla}_X \psi Y - \psi \widehat{\nabla}_X Y, \tag{3.9}$$

$$(\nabla_X \omega)Y = \mathcal{H}\nabla_X \omega Y - \omega \widehat{\nabla}_X Y. \tag{3.10}$$

On the other hand, from equations (3.6) and (3.7), we have

$$\nabla_X Y = \mathcal{T}_X Y + \widehat{\nabla}_X Y, \tag{3.11}$$

$$\nabla_X Z = \mathcal{H}\nabla_X Z + \mathcal{T}_X Z, \tag{3.12}$$

$$\nabla_Z X = \mathcal{A}_Z X + \mathcal{V}\nabla_Z X, \tag{3.13}$$

$$\nabla_Z W = \mathcal{H}\nabla_Z W + \mathcal{A}_Z W, \tag{3.14}$$

for $X, Y \in \Gamma(\ker f_*)$ and $Z, W \in \Gamma((\ker f_*)^\perp)$.

Lemma 3.3. Let $(M, \varphi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. Let $f : (M, \varphi, \xi, \eta, g_M) \to (N, g_N)$ be a semi-slant Riemannian map. Then

(1)

$$(\nabla_X \psi)Y = B\mathcal{T}_X Y - \mathcal{T}_X \omega Y + R(\xi, X)Y, \tag{3.15}$$

$$(\nabla_X \omega)Y = C\mathcal{T}_X Y - \mathcal{T}_X \psi Y, \tag{3.16}$$

for $X, Y \in \Gamma(\ker f_*)$.

(2)

$$\mathcal{V}\nabla_Z BW + \mathcal{A}_Z CW = \psi\mathcal{A}_Z W + B\mathcal{H}\nabla_Z W + g(Z, W)\xi, \tag{3.17}$$

$$\mathcal{A}_Z BW + \mathcal{H}\nabla_Z CW = \omega\mathcal{A}_Z W + C\mathcal{H}\nabla_Z W, \tag{3.18}$$

for $Z, W \in \Gamma((\ker f_*)^\perp)$.

(3)

$$\widehat{\nabla}_X BZ + \mathcal{T}_X CZ = \psi\mathcal{T}_X Z + B\mathcal{H}\nabla_X Z, \tag{3.19}$$

$$\mathcal{T}_X BZ + \mathcal{H}\nabla_X CZ = C\mathcal{H}\nabla_X Z + \omega\mathcal{T}_X Z, \tag{3.20}$$

$$\mathcal{V}\nabla_Z \psi X + \mathcal{A}_Z \omega X = \psi\mathcal{V}\nabla_Z X + B\mathcal{A}_Z X, \tag{3.21}$$

$$\mathcal{A}_Z \psi X + \mathcal{H}\nabla_Z \omega X + \eta(X)Z = \omega\mathcal{V}\nabla_Z X + C\mathcal{A}_Z X, \tag{3.22}$$

for $X \in \Gamma(\ker f_*)$ and $Z \in \Gamma((\ker f_*)^\perp)$.

Theorem 3.4. Let f be a semi-slant Riemannian map from an almost contact metric manifold $(M, \varphi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N) with the semi-slant angle θ. Then

$$\psi^2 X = -\cos^2\theta . X, \ \ for X \in \Gamma(D_2).$$

Proof. Let f be a semi-slant Riemannian map from an almost contact metric manifold $(M, \varphi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N) with the semi-slant angle θ. Then for a non-vanishing vector field $X \in \Gamma(D_2)$, we have

$$\cos\theta = \frac{|\ \psi X\ |}{|\ \varphi X\ |}, \tag{3.23}$$

and

$$\cos\theta = \frac{g_M(\varphi X, \psi X)}{|\ \varphi X\ ||\ \psi X\ |}.$$

By using equation (3.2), we have

$$\cos\theta = \frac{g_M(\psi X, \psi X)}{\mid \varphi X \mid\mid \psi X \mid},$$

$$\cos\theta = -\frac{g_M(X, \psi^2 X)}{\mid \varphi X \mid\mid \psi X \mid}. \tag{3.24}$$

From equations (1.1), (3.23) and (3.24), we get

$$\psi^2 X = -\cos^2\theta.X, \quad \text{for } X \in \Gamma(D_2).$$

Q.E.D.

Remark 3.5. From above theorem, it is easy to see that

$$g_M(\psi X, \psi Y) = \cos^2\theta g_M(X, Y),$$

$$g_M(\omega X, \omega Y) = \sin^2\theta g_M(X, Y),$$

for $X, Y \in \Gamma(D_2)$, when $\theta \in (0, \frac{\pi}{2})$. We can locally choose an orthonormal frame $\{e_1, \psi e_1, ..., e_k, \psi e_k, f_1, \sec\theta\psi f_1, \csc\theta\omega f_1, ..., f_s, \sec\theta\psi f_s, \csc\theta\omega f_s, \xi, g_1, \psi g_1, ..., g_t, \psi g_t\}$ of TM such that $\{e_1, \psi e_1, ..., e_k, \psi e_k\}$ is an orthonormal frame of D_1, $\{f_1, \sec\theta\psi f_1\ ,, ..., f_s, \sec\theta\psi f_s\}$ an orthonormal frame of D_2, $< \xi >$ an orthogonal D_1 and D_2 in $\Gamma(\ker f_*)$, $\{\csc\theta\omega f_1, ..., \csc\theta\omega f_s\}$ an orthonormal frame of ωD_2, and $\{g_1, \psi g_1, ..., g_t, \psi g_t\}$ an orthonormal frame of μ.

Lemma 3.6. Let f be a semi-slant Riemannian map from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N) with the semi-slant angle θ. If tensor ω is parallel, then

$$\mathcal{T}_{\psi X}\psi X = -\cos^2\theta.\mathcal{T}_X X, \qquad for X \in \Gamma(D_2). \tag{3.25}$$

Proof. If the tensor ω is parallel such that

$$(\nabla_X \omega)Y = 0. \tag{3.26}$$

From equation (3.16), we have

$$\mathcal{T}_Y \psi X = \mathcal{T}_X \psi Y.$$

Replace $Y \to \psi Y$, we have

$$\mathcal{T}_{\psi X}\psi X = -\cos^2\theta.\mathcal{T}_X X, \qquad for X \in \Gamma(D_2).$$

Q.E.D.

Proposition 3.7. Let f be a semi-slant Riemannian map from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then the slant distribution D_1 is not integrable when dimension distribution D_1 greater than or equal to 1.

Proof. For $X \in \Gamma D_1$, since g_M is Riemannian metric and using equations (2.3), and (2.7), we get

$$g_M([X, \varphi X], \xi) \neq 0.$$

So D_1 is not integrable. Q.E.D.

Theorem 3.8. Let f be a semi-slant Riemannian map from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then the slant distribution $D_1 \oplus < \xi >$ is integrable if and only if

$$\omega(\widehat{\nabla}_X Y - \widehat{\nabla}_Y X) = 0,$$
$$\psi(\widehat{\nabla}_X Y - \widehat{\nabla}_Y X) = 0,$$

for $X, Y \in \Gamma D_1 \oplus < \xi >$.

Proof. For $X, Y \in \Gamma D_1 \oplus < \xi >$ and $W \in \Gamma(D_2)$, since $[X, Y] \in \Gamma((\ker f_*)$. Using equations (2.3), (2.4), (2.5), (2.7) and (3.2), we get

$$\begin{aligned} g_M([X,Y], W) &= g_M(\varphi[X,Y], \varphi W), \\ &= g_M(\varphi(\nabla_X Y - \nabla_Y X), \varphi W), \\ &= g_M(\varphi \nabla_X Y - \varphi \nabla_Y X, \varphi W). \end{aligned}$$

Again using equations (3.2), (3.3), (3.11) and (3.16), we have

$$\begin{aligned} g_M([X,Y], W) &= g_M(B\mathcal{T}_X Y + C\mathcal{T}_X Y + \psi \widehat{\nabla}_X Y + \omega \widehat{\nabla}_X Y \\ &\quad - B\mathcal{T}_Y X - C\mathcal{T}_Y X - \psi \widehat{\nabla}_Y X - \omega \widehat{\nabla}_Y X, \psi W + \omega W), \\ &= g_M(B\mathcal{T}_X Y + \psi \widehat{\nabla}_X Y - B\mathcal{T}_Y X - \psi \widehat{\nabla}_Y X, \psi W) \\ &\quad + g_M(C\mathcal{T}_X Y + \omega \widehat{\nabla}_X Y - C\mathcal{T}_Y X - \omega \widehat{\nabla}_Y X, \omega W), \end{aligned}$$

$$g_M([X,Y], W) = g_M(\psi \widehat{\nabla}_X Y - \psi \widehat{\nabla}_Y X, \psi W) + g_M(\omega \widehat{\nabla}_X Y - \omega \widehat{\nabla}_Y X, \omega W).$$

Hence, $\Gamma D_1 \oplus < \xi >$ is integrable $\Leftrightarrow \psi(\widehat{\nabla}_X Y - \widehat{\nabla}_Y X) = 0$ and $\omega(\widehat{\nabla}_X Y - \widehat{\nabla}_Y X) = 0$, for $X, Y \in \Gamma D_1 \oplus < \xi >$. Q.E.D.

Theorem 3.9. Let f be a semi-slant Riemannian map from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then the slant distribution $D_2 \oplus < \xi >$ is integrable if and only if

$$P(\psi(\widehat{\nabla}_X Y - \widehat{\nabla}_Y X)) = 0,$$

for $X, Y \in \Gamma(D_2) \oplus < \xi >$.

Proof. For $X, Y \in \Gamma D_2 \oplus <\xi>$ and $W \in \Gamma(D_1)$, since $[X,Y] \in \Gamma((\ker f_*)$. Using equations $(3.2), (3.3), (3.11)$ and (3.16), we have

$$\begin{aligned} g_M([X,Y],\varphi W) &= g_M([X,Y],\varphi W) \\ &= -g_M(\varphi\nabla_X Y - \varphi\nabla_Y X, W), \\ &= -g_M(B\mathcal{T}_X Y + C\mathcal{T}_X Y + \psi\widehat{\nabla}_X Y + \omega\widehat{\nabla}_X Y \\ &\quad -B\mathcal{T}_Y X - C\mathcal{T}_Y X - \psi\widehat{\nabla}_Y X - \omega\widehat{\nabla}_Y X, W), \\ &= g_M(\psi\widehat{\nabla}_Y X - \psi\widehat{\nabla}_X Y, W). \end{aligned}$$

Hence, $D_2 \oplus <\xi>$ is integrable $\Leftrightarrow P(\psi(\widehat{\nabla}_X Y - \widehat{\nabla}_Y X)) = 0$, for $X, Y \in \Gamma D_2 \oplus <\xi>$. Q.E.D.

Theorem 3.10. Let f be a semi-slant Riemannian map from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N) such that $D_1 \oplus <\xi>$ is integrable. Then f is harmonic if and only if $trace(\nabla f_*) = 0$ on D_2 and $\overline{H}$, where $\overline{H}$ denotes the mean curvature vector field of $range f_*$.

Proof. Using Lemma 1, we get $trace(\nabla f_*)|_{\ker f_*} \in (range\ f_*)$ and $trace$ $(\nabla f_*)|_{(\ker f_*)^\perp} \in (range\ f_*)^\perp$ so that

$$trace(\nabla f_*) = 0 \Leftrightarrow trace(\nabla f_*)|_{\ker f_*} = 0,$$

and

$$trace(\nabla f_*)|_{(\ker f_*)^\perp} = 0.$$

Since D_1 is invariant under φ, we can choose locally orthonormal frame $\{e_1, \varphi e_1, ...$ $....., e_k, \varphi e_k, \xi\}$ of $D_1 \oplus <\xi>$. Using equations $(2.1), (2.5), (2.7)$ and (2.9), we have

$$\begin{aligned} (\nabla f_*)(\varphi e_i, \varphi e_i) &= -f_*(\nabla_{\varphi e_i}\varphi e_i), \\ &= f_*(\nabla_{e_i} e_i), \\ &= -(\nabla f_*)(e_i, e_i), \quad \text{for } 1 \le i \le k, \\ where\ \ f_*(\nabla_{\varphi e_i}\varphi)\varphi e_i &= 0. \end{aligned}$$

and

$$\begin{aligned} (\nabla f_*)(\xi, \xi) &= -f_*(\nabla_\xi \xi), \\ &= 0. \end{aligned}$$

Using the integrability of the distribution $D_1 \oplus <\xi>$, we have

$$(\nabla f_*)(\varphi e_i, \varphi e_i) + (\nabla f_*)(e_i, e_i) + (\nabla f_*)(\xi, \xi) = 0.$$

Thus,

$$trace(\nabla f_*)|_{(\ker f_*)} = 0 \Leftrightarrow trace(\nabla f_*)|_{D_2} = 0 .$$

Moreover, it is easy to see that

$$trace(\nabla f_*)|_{(\ker f_*)^\perp} = l\overline{H}, \quad for\ l = \dim(\ker f_*)^\perp,$$

so that

$$trace(\nabla f_*)|_{(\ker f_*)^\perp} = 0 \Leftrightarrow \overline{H} = 0.$$

Q.E.D.

Theorem 3.11. Let f be a semi-slant Riemannian map from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then f is a totally geodesic map if and only if

$$\omega(\widehat{\nabla}_X \psi Y + \mathcal{T}_X \omega Y) + \eta(Y)\omega X) + C(\mathcal{T}_X \psi Y + \mathcal{H}\nabla_X \omega Y) = 0,$$

$$\omega(\widehat{\nabla}_X BZ + \mathcal{T}_X CZ) + C(\mathcal{T}_X BZ + \mathcal{H}\nabla_X CZ) = 0,$$

$$\overline{Q}(\nabla^f_{Z_1} f_* Z_2) = 0,$$

for $X, Y \in \Gamma(\ker f_*)$ and $Z, Z_1, Z_2 \in \Gamma((\ker f_*)^\perp)$.

Proof. If $Z_1, Z_2 \in \Gamma((\ker f_*)^\perp$, then by Lemma 1, we get

$$(\nabla f_*)(Z_1, Z_2) = 0 \Leftrightarrow \overline{Q}((\nabla f_*)(Z_1, Z_2)) = \overline{Q}(\nabla^f_{Z_1} f_* Z_2) = 0.$$

For $X, Y \in \Gamma(\ker f_*)$, using equations $(2.1), (3.2), (3.3), (3.11)$ and (3.12), we have

$$\begin{aligned}
(\nabla f_*)(X, Y) &= -f_*(\nabla_X Y), \\
&= f_*(\varphi(\widehat{\nabla}_X \psi Y + \mathcal{T}_X \psi Y + \mathcal{T}_X \omega Y + \mathcal{H}\nabla_X \omega Y) + \eta(Y)\varphi X), \\
&= f_*(\psi \widehat{\nabla}_X \psi Y + \omega \widehat{\nabla}_X \psi Y + B\mathcal{T}_X \psi Y + C\mathcal{T}_X \psi Y + \psi \mathcal{T}_X \omega Y \\
&\quad + \omega \mathcal{T}_X \omega Y + B\mathcal{H}\nabla_X \omega Y + C\mathcal{H}\nabla_X \omega Y + \eta(Y)\psi X + \eta(Y)\omega X).
\end{aligned}$$

Hence

$$(\nabla f_*)(X, Y) = 0$$

$$\Leftrightarrow \omega(\widehat{\nabla}_X \psi Y + \mathcal{T}_X \omega Y + \eta(Y)\omega X) + C(\mathcal{T}_X \omega Y + \mathcal{H}\nabla_X \omega Y) = 0.$$

If $X \in \Gamma(\ker f_*)$ and $Z \in \Gamma((\ker f_*)^\perp)$, since the tensor ∇f_* is symmetric. Using equations $(2.1), (3.2), (3.3), (3.11)$ and (3.12), we get

$$\begin{aligned}
(\nabla f_*)(X, Z) &= -f_*(\nabla_X Z), \\
&= f_*(\varphi(\widehat{\nabla}_X BZ + \mathcal{T}_X BZ + \mathcal{T}_X CZ + \mathcal{H}\nabla_X CZ)), \\
&= f_*(\psi \widehat{\nabla}_X BZ + \omega \widehat{\nabla}_X BZ + B\mathcal{T}_X BZ + C\mathcal{T}_X BZ + \psi \mathcal{T}_X CZ \\
&\quad + \omega \mathcal{T}_X CZ + B\mathcal{H}\nabla_X CZ + C\mathcal{H}\nabla_X CZ).
\end{aligned}$$

Thus,

$$(\nabla f_*)(X, Z) = 0 \Leftrightarrow \omega(\widehat{\nabla}_X BZ + \mathcal{T}_X CZ) + C(\mathcal{T}_X BZ + \mathcal{H}\nabla_X CZ) = 0.$$

Q.E.D.

Theorem 3.12. Let f be a semi-slant Riemannian map from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then $(M, \varphi, \xi, \eta, g_M)$ is locally a Riemannian product manifold of the leaves of $\Gamma(\ker f_*)$ and $\Gamma((\ker f_*)^\perp)$ if and only if

$$\omega\widehat{\nabla}_X\psi Y + \omega\mathcal{T}_X\omega Y - \eta(Y)\omega X + C(\mathcal{T}_X\psi Y + \mathcal{H}\nabla_X\omega Y) = 0,$$

for $X, Y \in \Gamma(\ker f_*)$ and

$$\psi(\mathcal{V}\nabla_Z BW + \mathcal{A}_Z CW) + B(\mathcal{A}_Z BW + \mathcal{H}\nabla_Z CW) + \eta(\nabla_Z W)\xi = 0,$$

for $Z, W \in \Gamma((\ker f_*)^\perp)$.

Proof. For $X, Y \in \Gamma(\ker f_*)$, using equation (2.1), (3.2), (3.3), (3.11) and (3.12), we get

$$\begin{aligned}
\nabla_X Y &= -(-\nabla_X Y),\\
&= -\varphi(\widehat{\nabla}_X\psi Y + \mathcal{T}_X\psi Y + \mathcal{T}_X\omega Y + \mathcal{H}\nabla_X\omega Y) - \eta(Y)\varphi X + \eta(\nabla_X Y)\xi,\\
&= -(\psi\widehat{\nabla}_X\psi Y + \omega\widehat{\nabla}_X\psi Y + B\mathcal{T}_X\psi Y + C\mathcal{T}_X\psi Y + \psi\mathcal{T}_X\omega Y + \omega\mathcal{T}_X\omega Y\\
&\quad + B\mathcal{H}\nabla_X\omega Y + C\mathcal{H}\nabla_X\omega Y) - \eta(Y)\psi X - \eta(Y)\omega X + \eta(\nabla_X Y)\xi.
\end{aligned}$$

Thus,

$$\nabla_X Y \in \Gamma(\ker f_*)$$

$$\Leftrightarrow \omega\widehat{\nabla}_X\psi Y + \omega\mathcal{T}_X\omega Y - \eta(Y)\omega X + C(\mathcal{T}_X\omega Y + \mathcal{H}\nabla_X\omega Y) = 0.$$

For $Z, W \in \Gamma((\ker f_*)^\perp)$, using equations (2.1), (3.2), (3.3), (3.13) and (3.14), we have

$$\begin{aligned}
\nabla_Z W &= -(-\nabla_Z W),\\
&= -\varphi(\mathcal{V}\nabla_Z BW + \mathcal{A}_Z BW + \mathcal{A}_Z CW + \mathcal{H}\nabla_Z CW),\\
&= -(\psi\mathcal{V}\nabla_Z BW + \omega\mathcal{V}\nabla_Z BW + B\mathcal{A}_Z BW + C\mathcal{A}_Z BW\\
&\quad + \psi\mathcal{A}_Z CW + \omega\mathcal{A}_Z CW + B\mathcal{H}\nabla_Z CW + C\mathcal{H}\nabla_Z CW) + \eta(\nabla_Z W)\xi.
\end{aligned}$$

Hence
$\nabla_Z W \in \Gamma((\ker f_*)^\perp$
$\Leftrightarrow \psi(\mathcal{V}\nabla_Z BW + \mathcal{A}_Z CW) + B(\mathcal{A}_Z BW + \mathcal{H}\nabla_Z CW) + \eta(\nabla_Z W)\xi = 0.$ Q.E.D.

Theorem 3.13. Let f be a semi-slant Riemannian map from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ to a Riemannian manifold (N, g_N). Then the fibers of f are locally Riemannian product manifolds of the leaves of D_1 and D_2 if and only if

$$Q(\psi\widehat{\nabla}_U\psi V + B\mathcal{T}_U\psi V + g(\varphi U, V)\xi) = 0,$$

$$\omega\widehat{\nabla}_U\psi V + C\mathcal{T}_U\psi V = 0,$$

for $U, V \in \Gamma(D_1)$,
and

$$P(\psi(\widehat{\nabla}_X\psi Y + \mathcal{T}_X\omega Y) + B(\mathcal{T}_X\psi Y + \mathcal{H}\nabla_X\omega Y) + g(\varphi X, Y)\xi) + \eta(\nabla_X Y)\xi = 0,$$

$$\omega(\widehat{\nabla}_X\psi Y + \mathcal{T}_X\omega Y) + C(\mathcal{T}_X\psi Y + \mathcal{H}\nabla_X\omega Y) = 0,$$

for $X, Y \in \Gamma(D_2)$.

Proof. For $U, V \in \Gamma D_1$, using equations (2.1), (3.2), (3.3) and (3.11), we have

$$\begin{aligned} \nabla_U V &= -(-\nabla_U V), \\ &= -\varphi(\widehat{\nabla}_U\psi V + \mathcal{T}_U\psi V) + \eta(\nabla_U V)\xi, \\ &= -(\psi\widehat{\nabla}_U\psi V + \omega\widehat{\nabla}_U\psi V + B\mathcal{T}_U\psi V + C\mathcal{T}_U\psi V) + g(\varphi U, V)\xi. \end{aligned}$$

Hence

$$\nabla_U V \in \Gamma D_1 \Leftrightarrow Q(\psi\widehat{\nabla}_U\varphi V + B\mathcal{T}_U\varphi V - g(\varphi U, V)\xi) = 0,$$

and

$$\omega\widehat{\nabla}_U\varphi V + C\mathcal{T}_U\varphi V = 0.$$

For $X, Y \in \Gamma D_2$, using equations (2.1), (3.2), (3.3), (3.11) and (3.12), we have

$$\begin{aligned} \nabla_X Y &= -(-\nabla_X Y), \\ &= -(\psi\widehat{\nabla}_X\psi Y + \omega\widehat{\nabla}_X\psi Y + B\mathcal{T}_X\psi Y + C\mathcal{T}_X\psi Y + \psi\mathcal{T}_X\omega Y \\ &\quad +\omega\mathcal{T}_X\omega Y + B\mathcal{H}\nabla_X\omega Y + C\mathcal{H}\nabla_X\omega Y + \eta(\nabla_X Y)\xi. \end{aligned}$$

Hence $\nabla_X Y \in \Gamma D_2 \Leftrightarrow P(\psi\widehat{\nabla}_X\psi Y + \psi\mathcal{T}_X\omega Y + B\mathcal{T}_X\psi Y + B\mathcal{H}\nabla_X\omega Y) + \eta(\nabla_X Y)\xi = 0$
and $\omega\widehat{\nabla}_X\psi Y + \omega\mathcal{T}_X\omega Y + C\mathcal{T}_X\psi Y + C\mathcal{H}\nabla_X\omega Y = 0$. Q.E.D.

Example 3.14. Let R^9 has got a Sasakian structure as in Example 1, for $k = 4$. Let $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z)$ be coordinate system in R^9 and $(z_1, z_2, z_3, z_4, z_5)$ be coordinate system in R^5. Let the Riemannian metric on R^5 is $g_{R^5} = \frac{1}{4}(dz_1^2 + 2dz_2^2 + dz_3^2 + dz_4^2 + dz_5^2)$. Define a map $f : R^9 \to R^5$ by

$$f(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z) = (x_2, \frac{x_3 - y_4}{\sqrt{2}}, 0, y_3, y_2).$$

Then the map f is a semi-slant Riemannian map such that

$$\begin{aligned}
\xi &= E_9,\ D_1 =< E_1, E_5 >, \\
D_2 &= < E_8, \frac{1}{\sqrt{2}}(E_4 + E_7) >, \\
(\ker f_*)^{\perp} &= < V_1 = E_2, V_2 = E_3, V_3 = \frac{1}{\sqrt{2}}(E_4 - E_7), V_4 = E_6 >, \\
\omega(D_2) &= < E_3, \frac{1}{\sqrt{2}}(E_4 - E_7) >, \quad \mu =< E_2, E_6 >, \\
f_* V_1 &= 2\frac{\partial}{\partial z_5}, f_* V_2 = 2\frac{\partial}{\partial z_4}, f_* V_3 = \sqrt{2}\frac{\partial}{\partial z_2}, f_* V_4 = 2\frac{\partial}{\partial z_1}.
\end{aligned}$$

Here $g_{R^9}(V_i, V_i) = 1$ for $i = 1, 2, 3, 4$ and $g_{R^5}(f_* V_i, f_* V_i) = 1$ for $i = 1, 2, 3, 4$. So f is Riemannian map with the semi-slant angle $\theta = \frac{\pi}{4}$. Here equation (1.3) is satisfying.

References

[1] P. Baird and JC. Wood, Harmonic Morphisms between Riemannian Manifolds, Oxford science publications (2003).

[2] D. E. Blair, Contact manifolds in Riemannian geometry, Lectures Notes in Mathematics **509**, Springer-Verlag, Berlin (1976), 146 p.

[3] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Progress in Mathematics. 203, Birkhauser Boston, Basel, Berlin (2002).

[4] J. P. Bourguignon and H. B. Lawson, Stability and isolation phenomena for Yang-mills fields, Comm. Math. Phys. **79**(2) (1981), 189–230.

[5] J. Eells and J. M. Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. Math. **86** (1964), 109–160.

[6] M. Falcitelli, S. Ianus and A. M. Pastore, Riemannian Submersions and Related Topics, World Scientic Publishing Co. (2004).

[7] A. E. Fischer, Riemannian maps between Riemannian manifolds, Contemporary Math. **132** (1992), 331–366.

[8] A. Gray, Pseudo-Riemannian almost product manifolds and submersions, J. Math. Mech. **16** (1967), 715–737.

[9] S. Ianus and M. Visinescu, Space-time compactification and Riemannian submersions, In: Rassias, G.(ed.) The Mathematical Heritage of C. F. Gauss. World Scientic, River Edge. (1991), 358–371.

[10] S. Ianus and M. Visinescu, Kaluza-Klen theory with scalar fields and generalized Hopf manifold, Classical Quantum Grav. **4**(5) (1987), 1317–1325.

[11] M. T. Mustafa, Applications of harmonic morphisms to gravity, J. Math. Phys. **41**(10) (2000), 6918–6929.

[12] B. O'Neill, The fundamental equations of a submersion, Mich. Math. J. **13** (1966), 458–469.

[13] K. S. Park and R. Prasad, Semi-slant submersions, Bull. Korean Math. Soc. **50**(3) (2013), 951–962.

[14] K. S. Park, Semi-slant Riemannian maps, Czechoslovak mathematical Journal, **64** (139) (2014), 1045–1061.

[15] B. Sahin, Slant Riemannian maps from almost Hermitian manifolds, Quaest. Math. **36** (2013), 449–461.

[16] B. Sahin, Slant submersions from almost Hermitian manifolds, Bull. Math. Soc. Sci. Math. Roumanie Tome. 54: **102**(1) (2011), 93–105.

[17] B. Watson, Almost Hermitian submersions, J. Dierential Geom. **11**(1) (1976), 147–165.

[18] B. Watson, G$^{|}$-Riemannian submersions and nonlinear gauge field equations of general relativity. In: Rassias, T. (ed.) Global Analysis - Analysis on manifolds, dedicated M. Morse, Teubner-Texte Math. **57,** 324–349 (1983). Teubner, Leipzig