Double absolute indexed matrix summability with its applications

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Abstract

The well established theory of summability of simple series has been brought to a high degree of development; however the extension of this theory to multiple series is still in its infancy. As regards to the double series, in the proposed paper a result on absolute indexed matrix summability with an additional parameter of doubly infinite lower triangular matrix has been established that generalizes a theorem of E. Savaş and B. E. Rhoades [10] (see E. Savaş and B. E. Rhoades, Double absolute summability factor theorems and applications, Nonlinear Anal. 69 (2008), 189-200). Furthermore, some concluding remarks and applications are presented in support of our result.

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1 Introduction and definitions

Let $A = (a_{mn})$ be a lower-triangular matrix and $\sum a_n$ be an infinite series with sequence of partial sums (s_n) such that the A-transform of the sequence (s_n) is given by,

$$A_n = \sum_{v=0}^{n} a_{nv} s_v.$$

A series $\sum a_n$ is said to be summable $|A|_k \quad (k \geq 1)$ (see [7]) if,

$$\sum_{n=1}^{\infty} n^{k-1} |A_n - A_{n-1}|^k < \infty, \tag{1.1}$$

and is said to be summable $|A, \delta|_k \quad (k \geq 1; \quad \delta \geq 0)$ (see [9]) if,

$$\sum_{n=1}^{\infty} n^{\delta k + k - 1} |A_n - A_{n-1}|^k < \infty. \tag{1.2}$$

Similarly, let $A = (a_{mnjk})$ be a lower-triangular matrix and $\sum \sum a_{mn}$ be a double infinite series with sequence of partial sums (s_{mn}) such that, the mn^{th} term of the A-transform of the sequence (s_{mn}) is defined as,

$$T_{mn} = \sum_{\mu=0}^{m} \sum_{v=0}^{n} a_{mn\mu v} s_{\mu v}.$$

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Note that, a doubly infinite matrix $A = (a_{mnjk})$ is said to be doubly triangular if $a_{mnjk} = 0$ for $j > m$ or $k > n$.

We have, for any double sequence (u_{mn}), Δ_{11} is defined by

$$\Delta_{11} u_{mn} = u_{mn} - u_{m+1,n} - u_{m,n+1} + u_{m+1,n+1}$$

Similarly, for any fourfold sequence (v_{mnij}),

$$\begin{aligned}
\Delta_{11} v_{mnij} &= v_{mnij} - v_{m+1,n,i,j} - v_{m,n+1,i,j} + v_{m+1,n+1,i,j};\\
\Delta_{ij} v_{mnij} &= v_{mnij} - v_{m,n,i+1,j} - v_{m,n,i,j+1} + v_{m,n,i+1,j+1};\\
\Delta_{0j} v_{mnij} &= v_{mnij} - v_{m,n,i,j+1};\\
\Delta_{i0} v_{mnij} &:= v_{mnij} - v_{m,n,i+1,j}.
\end{aligned} \tag{1.3}$$

A series $\sum\sum b_{mn}$, with sequence of partial sum (s_{mn}) is said to be summable $|A|_k$ $(k \geq 1)$ (see [10]) if,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|\Delta_{11} T_{m-1,n-1}|^k < \infty, \tag{1.4}$$

and is said to be summable $|A, \delta|_k$ $(k \geq 1;\ \delta \geq 0)$ if,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{\delta k+k-1}|\Delta_{11} T_{m-1,n-1}|^k < \infty. \tag{1.5}$$

We may associate with A, two doubly triangular matrices $\bar{A}$ and $\hat{A}$ as follows:

$$\bar{a}_{mnij} = \sum_{\mu=i}^{m}\sum_{v=j}^{n} a_{mn\mu v} \quad \text{and} \quad \hat{a}_{m,n,i,j} = \Delta_{11}\bar{a}_{m-1,n-1,i,j} \quad (m, n \in \mathbb{N}_0 =: \{0\} \cup \mathbb{N}). \tag{1.6}$$

Note that $\hat{a}_{0000} = \bar{a}_{0000} = a_{0000}$.

Let (y_{mn}) denote the $(mn)^{th}$ term of the A-transform of $\sum_{\mu=0}^{m}\sum_{v=0}^{n} b_{\mu v}\lambda_{\mu v}$, then we may write,

$$\begin{aligned}
y_{mn} &= \sum_{\mu=0}^{m}\sum_{v=0}^{n} a_{mn\mu v}\sum_{i=0}^{\mu}\sum_{j=0}^{v} b_{ij}\lambda_{ij}\\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}\lambda_{ij}\sum_{\mu=i}^{m}\sum_{v=j}^{n} a_{mn\mu v}\\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}\lambda_{ij}\bar{a}_{mnij}.
\end{aligned}$$

Thus, we have

$$\Delta_{11}y_{m-1,n-1} = y_{m-1,n-1} - y_{m,n-1} - y_{m-1,n} + y_{mn}$$

$$= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} b_{ij}\lambda_{ij}\bar{a}_{m-1,n-1,i,j} - \sum_{i=0}^{m}\sum_{j=0}^{n-1} b_{ij}\lambda_{ij}\bar{a}_{m,n-1,i,j} - \sum_{i=0}^{m-1}\sum_{j=0}^{n} b_{ij}\lambda_{ij}\bar{a}_{m-1,n,i,j} + \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}\lambda_{ij}\bar{a}_{mnij}$$

$$= \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}\lambda_{ij}\hat{a}_{m,n,i,j} - \sum_{j=0}^{n-1} b_{mj}\lambda_{mj}\bar{a}_{m-1,n-1,m,j} - \sum_{i=0}^{m-1} b_{in}\lambda_{in}\bar{a}_{m-1,n-1,i,n} + \sum_{i=0}^{m} b_{in}\lambda_{in}\bar{a}_{m,n-1,i,n} + \sum_{j=0}^{n} b_{mn}\lambda_{mj}\bar{a}_{m-1,n,m,j}$$

$$= \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}\lambda_{ij}\hat{a}_{mnij}.$$

Since,

$$\bar{a}_{m-1,n-1,m,j} = \bar{a}_{m-1,n-1,i,n} = \bar{a}_{m,n-1,i,n} = \bar{a}_{m-1,n,m,n} = 0,$$

and

$$b_{mn} = s_{m-1,n-1} - s_{m-1,n} - s_{m,n-1} + s_{mn}.$$

So,

$$\begin{aligned}\Delta_{11}y_{m-1,n-1} &= \sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{mnij}\lambda_{ij}(s_{i-1,j-1}-s_{i-1,j}-s_{i,j-1}+s_{ij})\\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\hat{a}_{m,n,i+1,j+1}\lambda_{i+1,j+1}s_{ij}-\sum_{i=0}^{m-1}\sum_{j=0}^{n}\hat{a}_{m,n,i+1,j+1}\lambda_{i+1,j}s_{ij}\\ &\quad -\sum_{i=0}^{m}\sum_{j=0}^{n-1}\hat{a}_{m,n,i,j+1}\lambda_{i,j+1}s_{ij}+\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{mnij}\lambda_{ij}s_{ij}\\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\Delta_{ij}(\hat{a}_{mnij}\lambda_{ij})s_{ij}-\sum_{i=0}^{m-1}\hat{a}_{m,n,i+1,n}\lambda_{i+1,n}s_{in}-\sum_{j=0}^{n-1}\hat{a}_{m,n,m,j+1}\lambda_{m,j+1,n+1}s_{mj}\\ &\quad +\sum_{i=0}^{n}\hat{a}_{mnmj}\lambda_{m,j}s_{mj}+\sum_{i=0}^{m-1}\hat{a}_{mnin}\lambda_{in}s_{in}\\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\Delta_{ij}(\hat{a}_{mnij}\lambda_{ij})s_{ij}+\sum_{i=0}^{m-1}(\Delta_{i0}\hat{a}_{mnin}\lambda_{in})s_{in}\\ &\quad +\sum_{j=0}^{n-1}(\Delta_{0j}\hat{a}_{mnmj}\lambda_{mj})s_{mj}+\hat{a}_{mnmn}\lambda_{mn}s_{mn}. \end{aligned} \tag{1.7}$$

Also, we have

$$\Delta_{i0}\hat{a}_{mnin}\lambda_{in}=\lambda_{in}\Delta_{i0}\hat{a}_{mnin}+\hat{a}_{m,n,i+1,n}\Delta_{i0}\lambda_{in}$$

and

$$\Delta_{0j}\hat{a}_{mnmj}\lambda_{mj}=\lambda_{mj}\Delta_{0j}\hat{a}_{mnmj}+\hat{a}_{m,n,m,j+1}\Delta_{0j}\lambda_{mj}.$$

Clearly,

$$\begin{aligned}\sum_{i=0}^{m-1}(\Delta_{i0}\hat{a}_{mnin}\lambda_{in})s_{in}+\sum_{j=0}^{n-1}(\Delta_{0j}\hat{a}_{mnmj}\lambda_{mj})s_{mj} &= \sum_{i=0}^{m-1}[\lambda_{in}\Delta_{i0}\hat{a}_{mnin}+\hat{a}_{m,n,i+1,n}\Delta_{i0}\lambda_{in}]s_{in}\\ &\quad +\sum_{j=0}^{n-1}[\lambda_{mj}\Delta_{0j}\hat{a}_{mnmj}+\hat{a}_{m,n,m,j+1}\Delta_{0j}\lambda_{mj}]s_{mj}. \end{aligned} \tag{1.8}$$

Next, we present the following Lemma for two dimensional case, which is similar to the one dimensional formula for the first difference of a product of two sequences.

Lemma 1. (see [10]) *Let* $(u_{ij}),(v_{ij})$ *be two double sequences. Then*

$$\Delta_{ij}(u_{ij}v_{ij})=v_{ij}\Delta_{ij}u_{ij}+(\Delta_{0j}u_{i+1,j})(\Delta_{i0}v_{ij})+(\Delta_{i0}u_{i,j+1})(\Delta_{0j}v_{ij})+u_{i+1,j+1}\Delta_{ij}v_{ij}. \tag{1.9}$$

In the year 2008, Savaş [9] has proved a theorem for generalized absolute summability factors. Subsequently, Savaş and Rhoades (see [10]) has proved some inclusion theorems based on double absolute summability factor theorems and applications. Furthermore, in the year 2018, Jena *et al.* [4] has proved on generalized local property of $|A;\delta|_k$-summability of factored Fourier series.

Motivated essentially by the above-mentioned works, here based on $|A,\delta|_k$-summability of double infinite lower triangular matrix, we have proved a new theorem that generalizes the result of E. Savaş and B. E. Rhoades (see [10]). Moreover, in the last section we have presented some concluding remarks and corollaries in support of our result.

Moreover, Matrix summability or matrix transformation is very important in the study of summability theory in the sense that it generalizes the different summability methods like Cesàro summability, Nörlund summability, Riesz summability etc. Also, the statistical convergene and statistical summability are more general than the ordinary convergene and ordinary summability. For recent works in this direction, see [1], [2], [3], [5], [6], [8], [11], [12], [13] and [14].

2 Main result

Theorem 1. *Let A be a doubly triangular matrix with non-negative entries satisfying*

(i) $\Delta_{11}a_{m-1,n-1,i,j} \geq 0$;

(ii) $\sum_{v=0}^{n} a_{mniv} = \sum_{v=0}^{n-1} a_{m,n-1,i,v} = b(m,i)$ and $\sum_{\mu=0}^{m} a_{mn\mu,j} = \sum_{\mu=0}^{m-1} a_{m-1,n,\mu,j} = a(n,j)$;

(iii) $mna_{mnmn} = O(1)$;

(iv) $a_{mnij} \geq max\{a_{m,n+1,i,j}a_{m+1,n,i,j}\} \quad (m \geq i,\ n \geq j;\ i,j = 0,1,...,)$;

(v) $\sum_{i=0}^{m}\sum_{j=0}^{n} a_{mnij} = O(1)$;

(vi) $\sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1} (mn)^{\delta k}|\Delta_{ij}\hat{a}_{mnij}| = O((ij)^{\delta k}a_{ijij})$;

(vii) $\sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1} (mn)^{\delta k}\hat{a}_{m,n,i+1,j+1} = O(ij)^{\delta k}$,

let (x_{mn}) be a given double sequence of positive numbers and suppose that $(s_{mn}) = O(X_{mn})$ $(m,n \to \infty)$. If (λ_{mn}) is a double sequence of real numbers satisfying

(viii) $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{\delta k}a_{mnmn}(|\lambda_{mn}|X_{mn})^k < \infty$;

(ix) $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}(ij)^{\delta k}|\Delta_{0j}\lambda_{ij}|X_{ij} = O(1)$;

(x) $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(ij)^{\delta k}|\Delta_{i0}\lambda_{ij}|X_{ij} < \infty$;

(xi) $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}(ij)^{\delta k}|\Delta_{ij}\lambda_{ij}|X_{ij} = O(1);$

(xii) $\sum_{i=0}^{m}\sum_{j=0}^{n}(ij)^{\delta k}(|\lambda_{ij}|X_{ij})^k = O(1),$

then the series $\sum\sum b_{mn}\lambda_{mn}$ is summable $|A,\delta|_k \quad (k \geq 1; \quad 0 \leq \delta \leq 1/k)$.

Proof. In order to prove the theorem, it is required to show that,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{\delta k+k-1}|\Delta_{11}y_{mn}| < \infty.$$

From lemma 1, we have

$$\Delta_{ij}(\hat{a}_{mnij}\lambda_{ij}) = \lambda_{ij}\Delta_{ij}(\hat{a}_{mnij}) + (\Delta_{0j}\hat{a}_{m,n,i+1,j})(\Delta_{i0}\lambda_{ij})$$

$$+(\Delta_{i0}\hat{a}_{m,n,i,j+1})(\Delta_{0j}\lambda_{ij}) + \hat{a}_{m,n,i+1,j+1}\Delta_{ij}\lambda_{ij}. \tag{2.1}$$

Now using (2.1), we get

$$\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\Delta_{ij}(\hat{a}_{mnij}\lambda_{ij})s_{ij} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}[\lambda_{ij}(\Delta_{ij}\hat{a}_{mnij}) + (\Delta_{0j}\hat{a}_{m,n,i+1,j})(\Delta_{i0}\lambda_{ij})$$

$$+(\Delta_{i0}\hat{a}_{m,n,i,j+1})(\Delta_{0j}\lambda_{ij}) + \hat{a}_{m,n,i+1,j+1}(\Delta_{ij}\lambda_{ij})]s_{ij}. \tag{2.2}$$

Further, using (1.7), (1.8) and (2.2), we may, write

$$\Delta_{11}y_{m-1,n-1} = \sum_{r=1}^{9}T_{mnr}.$$

Now using Minkowski's inequality, it is suffices to show that,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{\delta k+k-1}|T_{mnr}|^k < \infty \quad (r = 1, 2, ..., 9).$$

We have, for $r = 1$, and by using Hölder's inequality,

$$\begin{aligned} I_1 &= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k+k-1}|T_{mn1}|^k \\ &= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k+k-1}\left(\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{ij}\hat{a}_{mnij}||\lambda_{ij}|X_{ij}\right)^k \\ &= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k+k-1}\left(\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{ij}\hat{a}_{mnij}||\lambda_{ij}|^k|X_{ij}|^k\right)\left(\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{ij}\hat{a}_{mnij}|\right)^{k-1}. \end{aligned}$$

Also, from (1.6),

$$\hat{a}_{mnij} = \Delta_{11}\bar{a}_{m-1,n-1,i,j}$$

$$= \bar{a}_{m-1,n-1,i,j} - \bar{a}_{m,n-1,i,j} - \bar{a}_{m-1,n,i,j} + \bar{a}_{mnij}$$

$$= \sum_{\mu=i}^{m-1}\sum_{v=j}^{n-1} a_{m-1,n-1,i,j} - \sum_{\mu=i}^{m}\sum_{v=j}^{n-1} a_{m,n-1,i,j} - \sum_{\mu=i}^{m-1}\sum_{v=j}^{n} a_{m-1,n,i,j} + \sum_{\mu=i}^{m}\sum_{v=j}^{n} a_{mnij}.$$

Again since,

$$a_{m-1,n,m,v} = a_{m,n-1,\mu,n} = 0.$$

So, by using (1.3) and by property (ii),

$$\hat{a}_{mnij} = \sum_{\mu=i}^{m}\sum_{v=j}^{n}(a_{m-1,n-1,\mu,v} - a_{m,n-1,\mu,v} - a_{m-1,n,\mu,v} + a_{m,n,\mu,v})$$

$$= \sum_{\mu=i}^{m-1}[b(m-1,\mu) - \sum_{v=0}^{j-1} a_{m-1,n-1,\mu,v} - b(m,\mu) + \sum_{v=0}^{j-1} a_{m,n-1,\mu,v}$$

$$-b(m-1,\mu) + \sum_{v=0}^{j-1} a_{m-1,n,\mu,v} + b(m,\mu) - \sum_{v=0}^{j-1} a_{m,n,\mu,v}]$$

$$= \sum_{\mu=i}^{m-1}\sum_{v=j}^{n-1}(-a_{m-1,n-1,\mu,v} + a_{m,n-1\mu,v} + a_{m-1,n,\mu,v} - a_{m,n,\mu,v})$$

$$= \sum_{v=0}^{j-1}\sum_{\mu=i}^{m-1}(-a_{m-1,n-1,\mu,v} + a_{m,n-1\mu,v} + a_{m-1,n,\mu,v} - a_{m,n,\mu,v})$$

$$= \sum_{v=0}^{j-1}[-a(m-1,v) + \sum_{\mu=0}^{j-1} a_{m-1,n-1,\mu,v} + a(m,v)$$

$$-\sum_{\mu=0}^{i-1} a_{m,n-1,\mu,v} + a(m-1,v) - \sum_{\mu=0}^{i} a_{m-1,n,\mu,v} - a(m,v) + \sum_{\mu=0}^{i} a_{m,n,\mu,v}]$$

$$= \sum_{\mu=0}^{i-1}\sum_{v=0}^{j-1} \Delta_{11} a_{m-1,n-1,\mu,v} \geq 0. \tag{2.3}$$

Further, using (1.3) and (2.3),

$$\Delta_{ij}\hat{a}_{mnij} = \left(\sum_{\mu=0}^{i-1}\sum_{v=0}^{j-1} - \sum_{\mu=0}^{i}\sum_{v=0}^{j-1} - \sum_{\mu=0}^{i-1}\sum_{v=0}^{j} + \sum_{\mu=0}^{i}\sum_{v=0}^{j}\right)\Delta_{11}a_{m-1,n-1,\mu,v}$$

$$= -\sum_{v=0}^{j-1}\Delta_{11}a_{m-1,n-1,i,v} + \sum_{v=0}^{j}\Delta_{11}a_{m-1,n-1,i,v}$$

$$= \Delta_{11}a_{m-1,n-1,i,j}. \tag{2.4}$$

Again, from condition (ii),

$$\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\Delta_{ij}\hat{a}_{mnij} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}(a_{m-1,n-1,i,j} - a_{m,n-1,i,j} - a_{m-1,n,i,j} + a_{mnij})$$

$$= \sum_{i=0}^{m-1}(b(m-1,i) - b(m,i) - b(m-1,i) + a_{m-1,n,i,n} + b(m,i) - a_{mnin})$$

$$= \sum_{i=0}^{m-1}(a_{m-1,n,i,n} - a_{mnin})$$

$$= a(n,n) - a(n,n) + a_{mnnm}.$$

Now using condition (iii), we get

$$I_1 = O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mna_{mnmn})^{k-1}(mn)^{\delta k}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{ij}\hat{a}_{mnij}||\lambda_{ij}|^k X_{ij}^k$$

$$= O(1)\sum_{m=1}^{M}\sum_{n=1}^{N}(|\lambda_{ij}|X_{ij})^k\sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}(mn)^{\delta k}|\Delta_{ij}\hat{a}_{mnij}|.$$

Moreover, using the condition (vi) and (vii), we obtain

$$I_1 = O(1)\sum_{i=0}^{M}\sum_{j=0}^{N}(ij)^{\delta k}a_{ijij}(|\lambda_{ij}|X_{ij})^k = O(1).$$

Next, for $r = 2$ and by using Hölder's inequality,

$$I_2 = \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k+k-1}|T_{mn2}|^k$$

$$= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k+k-1}\left|\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}(\Delta_{0j}\hat{a}_{m,n,i+1,j})(\Delta_{i0}\lambda_{ij})s_{ij}\right|^k$$

$$= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k+k-1}| \left[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{0j}\hat{a}_{m,n,i+1,j}|\Delta_{i0}\lambda_{ij}|X_{ij}\right]$$

$$\cdot\left[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{0j}\hat{a}_{m,n,i+1,j}|\Delta_{i0}\lambda_{ij}|X_{ij}\right]^{k-1}.$$

Using (2.3) and by property (ii), we have

$$\begin{aligned}0 \leq \hat{a}_{m,n,i+1,j} &= \sum_{\mu=0}^{i}\sum_{v=0}^{j-1}\Delta_{11}a_{m-1,n-1,\mu,v}\\ &\leq \sum_{\mu=0}^{m-1}\sum_{v=0}^{n-1}(a_{m-1,n-1,\mu,v} -_{m,n-1,\mu,v} -a_{m-1,n,\mu,v} + a_{m,n,\mu,v})\\ &= \sum_{\mu=0}^{m-1}(b(m-1,\mu) - b(m,\mu) - b(m-1,\mu) + a_{m-1,n,\mu,n} + b(m,\mu) - a_{mn\mu v})\\ &= \sum_{\mu=0}^{m-1}(a_{m-1,n,\mu,n} - a_{mn\mu v})\\ &= a(n,n) - a(n,n) + a_{mnmn}.\end{aligned}$$

Again, since

$$|\Delta_{0j}\hat{a}_{m,n,i+1,j}| \leq \hat{a}_{m,n,i+1,j} + \hat{a}_{m,n,i+1,j+1},$$

so by using properties (iii), (vii) and (x), we obtain

$$\begin{aligned}I_2 &= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mna_{mnmn})^{k-1}(mn)^{\delta k}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{0j}\hat{a}_{m,n,i+1,j}|\Delta_{i0}\lambda_{ij}|X_{ij}\\ &= O(1)\sum_{m=1}^{M}\sum_{n=1}^{N}|\Delta_{i0}\lambda_{ij}|X_{ij}\sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}(ij)^{\delta k}|\Delta_{0j}\hat{a}_{m,n,i+1,j}|\\ &= O(1)\sum_{m=1}^{M}\sum_{n=1}^{N}|\Delta_{i0}\lambda_{ij}|X_{ij}\sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}(ij)^{\delta k}(\hat{a}_{m,n,i+1,j} + \hat{a}_{m,n,i+1,j+1})\\ &= O(1).\end{aligned}$$

In the similar lines, it can be proved that

$$I_3 = O(1).$$

Next, for $r = 4$ and by using Hölder's inequality,

$$
\begin{aligned}
I_4 &= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k+k-1}|T_{mn4}|^k \\
&= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k+k-1}\left(\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\hat{a}_{m,n,i+1,j+1}||\Delta_{ij}\lambda_{ij}|X_{ij}\right)^k \\
&= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k+k-1}\left[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\hat{a}_{m,n,i+1,j+1}||\Delta_{ij}\lambda_{ij}|X_{ij}\right] \\
&\quad \cdot\left[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\hat{a}_{m,n,i+1,j+1}||\Delta_{ij}\lambda_{ij}|X_{ij}\right]^{k-1}.
\end{aligned}
$$

Also, From (2.3) and by property (ii),

$$
\begin{aligned}
0 \leq \hat{a}_{m,n,i+1,j+1} &= \sum_{\mu=0}^{i}\sum_{v=0}^{j}\Delta_{11}a_{m-1,n-1,\mu,v} \\
&\leq \sum_{\mu=0}^{m-1}\sum_{v=0}^{n-1}(a_{m-1,n-1,\mu,v} -_{m,n-1,\mu,v} -a_{m-1,n,\mu,v} + a_{m,n,\mu,v}) \\
&= \sum_{\mu=0}^{m-1}(b(m-1,\mu) - b(m,\mu) - b(m-1,\mu) + a_{m-1,n,\mu,n} + b(m,\mu) - a_{mn\mu v}) \\
&= \sum_{\mu=0}^{m-1}(a_{m-1,n,\mu,n} - a_{mn\mu v}) \\
&= a(n,n) - a(n,n) + a_{mnmn}.
\end{aligned}
$$

So, by using properties (iii), (xi), and (vii), we fairly obtain

$$I_4 = O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mna_{mnmn})^{k-1}(mn)^{\delta k}\left[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\hat{a}_{m,n,i+1,j+1}||\Delta_{ij}\lambda_{ij}|X_{ij}\right]$$

$$\cdot\left[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{ij}\lambda_{ij}|X_{ij}\right]^{k-1}$$

$$= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k}\left[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\hat{a}_{m,n,i+1,j+1}||\Delta_{ij}\lambda_{ij}|X_{ij}\right]$$

$$= O(1)\sum_{i=0}^{M}\sum_{j=0}^{N}|\Delta_{ij}\lambda_{ij}|X_{ij}\sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}(mn)^{\delta k}|\hat{a}_{m,n,i+1,j+1}|$$

$$= O(1)\sum_{i=0}^{m-1}\sum_{j=0}^{N}(mn)^{\delta k}|\Delta_{ij}\lambda_{ij}|X_{ij}$$

$$= O(1).$$

Next, for $r = 5$ and by using Hölder's inequality, we have

$$I_5 = \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k+k-1}|T_{mn5}|^k$$

$$= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k+k-1}\left|\sum_{i=0}^{m-1}\lambda_{in}\Delta_{i0}\hat{a}_{mnin}s_{in}\right|^k$$

$$= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k+k-1}\left(\sum_{i=0}^{m-1}\lambda_{in}|\Delta_{i0}\hat{a}_{mnin}|X_{in}\right)^k$$

$$= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k+k-1}\left[\sum_{i=0}^{m-1}|\Delta_{i0}\hat{a}_{mnin}|(|\lambda_{in}|X_{in})^k\right]\cdot\left[\sum_{i=0}^{m-1}|\Delta_{i0}\hat{a}_{mnin}|\right]^{k-1}.$$

Also, from (1.6),

$$\begin{aligned}\Delta_{i0}\hat{a}_{mnin} &= \Delta_{i0}(\Delta_{11}\bar{a}_{m-1,n-1,i,n})\\ &= \Delta_{i0}(\bar{a}_{m-1,n-1,i,n} - \bar{a}_{m,n-1,i,n} - \bar{a}_{m-1,n,i,n} + \bar{a}_{mnin})\\ &= \Delta_{i0}\left(-\sum_{\mu=i}^{m-1} a_{m-1,n,v,n} + \sum_{\mu=i}^{m} a_{mn\mu n}\right)\\ &= a_{m-1,n,i,n} + a_{mnin} \leqq 0.\end{aligned}$$

Again, by property (ii),

$$\begin{aligned}\sum_{i=0}^{m-1} |\Delta_{i0}\hat{a}_{mnin}| &= \sum_{i=0}^{m-1}(a_{m-1,n-1,i,n} - a_{mnin})\\ &= a(n,n) - a(n,n) + a_{mnmn}.\end{aligned}$$

Thus, by using property (iii), (vi) and (viii), we get

$$\begin{aligned}I_5 &= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mna_{mnmn})^{k-1}(mn)^{\delta k}\left[\sum_{i=0}^{m-1}|\Delta_{i0}\hat{a}_{mnin}|(|\lambda_{in}|X_{in})^k\right]\\ &= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k}\left(\sum_{i=0}^{m-1}|\Delta_{i0}\hat{a}_{mnin}|(|\lambda_{in}|X_{in})^k\right)\\ &= O(1)\sum_{n=1}^{N+1}\sum_{i=0}^{M}(|\lambda_{in}|X_{in})^k\left(\sum_{i=0}^{m-1}(mn)^{\delta k}|\Delta_{i0}\hat{a}_{mnin}|\right)\\ &= O(1).\end{aligned}$$

Further, for $r = 6$ and by using Hölder's inequality, we have

$$I_6 = \sum_{m=1}^{M+1} \sum_{n=1}^{N+1} (mn)^{\delta k+k-1} |T_{mn6}|^k$$

$$= \sum_{m=1}^{M+1} \sum_{n=1}^{N+1} (mn)^{\delta k+k-1} \left| \sum_{i=0}^{m-1} \hat{a}_{m,n,i+1,n} (\Delta_{i0}\lambda_{in}) s_{in} \right|^k$$

$$= O(1) \sum_{m=1}^{M+1} \sum_{n=1}^{N+1} (mn)^{\delta k+k-1} \left(\sum_{i=0}^{m-1} |\hat{a}_{m,n,i+1,n}||(\Delta_{i0}\lambda_{in})| X_{in} \right)^k$$

$$= O(1) \sum_{m=1}^{M+1} \sum_{n=1}^{N+1} (mn)^{\delta k+k-1} \left[\sum_{i=0}^{m-1} |\hat{a}_{m,n,i+1,n}||(\Delta_{i0}\lambda_{in})| X_{in} \right]$$

$$\cdot \left[\sum_{i=0}^{m-1} |\hat{a}_{m,n,i+1,n}||(\Delta_{i0}\lambda_{in})| X_{in} \right]^{k-1}.$$

We have, from (1.6) and by condition (ii),

$$\hat{a}_{m,n,i+1,n} = \bar{a}_{m-1,n-1,i+1,n} - \bar{a}_{m,n-1,i+1,n} - \bar{a}_{m-1,n,i+1,n} + \bar{a}_{m,n,i+1,n}$$

$$= -\sum_{\mu=i+1}^{m-1} a_{m-1,n,\mu,n} + \sum_{\mu=i+1}^{m} a_{m,n,\mu,n}$$

$$= -a(n,n) + \sum_{\mu=0}^{i} a_{m-1,n,\mu,n} + a(n,n) - \sum_{\mu=0}^{i} a_{m,n,\mu,n} \geq 0$$

$$\leq \sum_{\mu=0}^{m-1} (a_{m-1,n,\mu,n} - a_{m,n,\mu,n})$$

$$= a(n,n) - a(n,n) + a_{mnmn}.$$

Clearly, using conditions (iii), (vii) and (x), we get

$$I_6 = O(1) \sum_{m=1}^{M+1} \sum_{n=1}^{N+1} (mna_{mnmn})^{k-1} (mn)^{\delta k} \left[\sum_{i=0}^{m-1} |\hat{a}_{m,n,i+1,n}||(\Delta_{i0}\lambda_{in})|X_{in} \right] \cdot \left[\sum_{i=0}^{m-1} |\Delta_{i0}\lambda_{in}|X_{in} \right]^{k-1}$$

$$= O(1) \sum_{m=1}^{M+1} \sum_{n=1}^{N+1} (mn)^{\delta k} \left[\sum_{i=0}^{m-1} |\hat{a}_{m,n,i+1,n}||(\Delta_{i0}\lambda_{in})|X_{in} \right]$$

$$= O(1) \sum_{m=1}^{M} \sum_{n=1}^{N+1} |\Delta_{i0}\lambda_{in}|X_{in} \sum_{m=i+1}^{M+1} (mn)^{\delta k} |\hat{a}_{m,n,i+1,n}|$$

$$= O(1) \sum_{m=1}^{M} \sum_{n=1}^{N+1} (mn)^{\delta k} |\Delta_{i0}\lambda_{in}|X_{in}$$

$$= O(1).$$

Next, for $r = 7$ and by Hölder's inequality, we have

$$I_7 = \sum_{m=1}^{M+1} \sum_{n=1}^{N+1} (mn)^{\delta k + k - 1} |T_{mn7}|^k$$

$$= \sum_{m=1}^{M+1} \sum_{n=1}^{N+1} (mn)^{\delta k + k - 1} \left| \sum_{j=0}^{n-1} \lambda_{mj} (\Delta_{0j}\hat{a}_{mnmj}) s_{mj} \right|^k$$

$$= O(1) \sum_{m=1}^{M+1} \sum_{n=1}^{N+1} (mn)^{\delta k + k - 1} \left(\sum_{j=0}^{n-1} |\lambda_{mj}||(\Delta_{0j}\hat{a}_{mnmj})|X_{mj} \right)^k$$

$$= O(1) \sum_{m=1}^{M+1} \sum_{n=1}^{N+1} (mn)^{\delta k + k - 1} \left[\sum_{j=0}^{n-1} |\Delta_{0j}\hat{a}_{mnmj} (|\lambda_{mj}||X_{mj})^k \right] \left[\sum_{j=0}^{n-1} |\Delta_{0j}\hat{a}_{mnmj} \right]^{k-1}.$$

Also, from (1.3),

$$\hat{a}_{mnmj} = \bar{a}_{m-1,n-1,m,j} - \bar{a}_{m,n-1,m,j} - \bar{a}_{m-1,n,m,j} + \bar{a}_{m,n,m,j}$$

$$= -\sum_{v=j}^{n-1} a_{m,n-1,m,j} + \sum_{v=j}^{n} a_{m,n,m,j}.$$

Again, since

$$\Delta_{0j}\hat{a}_{mnmj} = -a_{m,n-1,m,j} + a_{m,m,m,j},$$

so, properties (iv) and (ii), yields

$$\sum_{j=0}^{n-1} |\Delta_{0j}\hat{a}_{mnmj}| = \sum_{j=0}^{n-1} (a_{m,n-1,m,j} - a_{m,n,m,j})$$

$$= b(m,m) - b(m,m) + a_{mnmn}.$$

Clearly, using properties (iii), (vi) and (ix), we get

$$I_7 = O(1) \sum_{m=1}^{M+1} \sum_{n=1}^{N+1} (mna_{mnmn})^{k-1} (mn)^{\delta k} \sum_{j=0}^{n-1} |\Delta_{0j}\hat{a}_{mnmj}| (|\lambda_{mj}| X_{mj})^k$$

$$= O(1) \sum_{m=1}^{M+1} \sum_{n=1}^{N+1} (|\lambda_{mj}| X_{mj})^k \sum_{n=j+1}^{N+1} (mn)^{\delta k} |\Delta_{0j}\hat{a}_{mnmj}|$$

$$= O(1).$$

Next, for $r = 8$ and by Hölder inequality, we have

$$I_8 = \sum_{m=1}^{M+1} \sum_{n=1}^{N+1} (mn)^{\delta k + k - 1} |T_{mn8}|^k$$

$$= \sum_{m=1}^{M+1} \sum_{n=1}^{N+1} (mn)^{\delta k + k - 1} \left| \sum_{j=0}^{n-1} \hat{a}_{m,n,m,j+1} (\Delta_{0j}\lambda_{mj}) s_{mj} \right|^k$$

$$= O(1) \sum_{m=1}^{M+1} \sum_{n=1}^{N+1} (mn)^{\delta k + k - 1} \left(\sum_{j=0}^{n-1} \hat{a}_{m,n,m,j+1} (\Delta_{0j}\lambda_{mj}) X_{mj} \right)^k$$

$$= O(1) \sum_{m=1}^{M+1} \sum_{n=1}^{N+1} (mn)^{\delta k + k - 1} \left[\sum_{j=0}^{n-1} \hat{a}_{m,n,m,j+1} (\Delta_{0j}\lambda_{mj}) X_{mj} \right]$$

$$\cdot \left[\sum_{j=0}^{n-1} \hat{a}_{m,n,m,j+1} (\Delta_{0j}\lambda_{mj}) X_{mj} \right]^{k-1}.$$

Now in the similar lines as in the proof of I_6 and by using properties (iii), (vii), and (ix), we get

$$I_8 = O(1).$$

Finally, for $r = 9$ and from properties (ii), (iii), (v) and (xii), together with (1.7) and under the consideration of $\hat{a}_{mnmn} = a_{mnmn}$, we have

$$\begin{aligned} I_9 &= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k+k-1}|T_{mn9}|^k \\ &= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k+k-1}(a_{mnmn}|\lambda_{mn}|X_{mn})^k \\ &= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mna_{mnmn})^{k-1}(mn)^{\delta k}a_{mnmn}(|\lambda_{mn}|X_{mn})^k \\ &= O(1). \end{aligned}$$

Which completes proof of Theorem 1.

3 Concluding remarks and applications

In the concluding section, we recall here the criterion for double triangular matrix $(\bar{N}, p, q)$ (see [10]) and accordingly we say, a factorable double weighted mean matrix, written as $(\bar{N}, p, q, \delta)$ is a double triangular matrix with entries,

$$a_{mnij} = \frac{p_i q_j}{P_m Q_n},$$

where (p_m), (q_n) are non-negative sequences with $p_0, q_0 > 0$, and

$$P_m = \sum_{i=0}^{m} p_i \to \infty; \quad Q_n = \sum_{j=0}^{n} p_j \to \infty.$$

Now, we present below the following corollaries and remarks for the sake of applications of our result demonstrated in this paper.

Corollary 1. *Suppose that* $(\bar{N}, p, q, \delta)$ *satisfies*

(i) $\frac{mnp_m q_n}{P_m Q_n} = O(1)$;

(ii) $\displaystyle\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k}\left|\frac{p_m q_n}{P_m Q_n P_{m-1} Q_{n-1}}\right| = O\left(\frac{(ij)^{\delta k}}{P_{ij} Q_{ij}}\right)$,

let (X_{mn}) *be a given double sequence of positive numbers and suppose that* $(s_{mn}) = O(X_{mn})$ $(m, n \to \infty)$*. If* (λ_{mn}) *is a double sequence of real numbers satisfying*

(iii) $\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mn)^{\delta k}\frac{p_m q_n}{P_m Q_n}(|\lambda_{mn}|X_{mn})^k < \infty,$

and condition (ix)-(xii) of Theorem 2,

then the series $\sum\sum b_{mn}\lambda_{mn}$ is summable $|\bar{N}, p, q, \delta|_k \quad (k \geq 0)$.

Proof. Conditions (i), (ii), (iv) and (v) of Theorem 1 are automatically satisfied. Condition (iii) becomes condition (i) of Corollary 1, and conditions (vi) and (vii) of Theorem 1 become conditions (ii) of Corollary 1. Finally condition (iii) of Corollary 1 is condition (viii) of Theorem 1. So it is fairly obvoius.

Remark 1. Let $s_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}$, define

$$A_k = \left\{ s_{mn} : \sum_{m=1}^{\infty}\sum_{m=1}^{\infty}(mn)^{\delta k+k-1}|b_{mn}|^k \leqq \infty \right\}.$$

A double infinite matrix (A, δ) is said to be a bounded operator on A_k, written as $(A, \delta) \in B(A_k)$, if every sequence in A_k is summable $|A, \delta|_k$.

Corollary 2. *Let A satisfy properties (i)-(vii) of Theorem 1. Then $(A, \delta) \in B(A_k)$.*

Proof. By setting each $\lambda_{ij} = 1$ and assuming $X_{ij} = |b_{ij}|$ in Theorem 1, the conditions (ix)-(xi) are automatically satisfied. Condition (iii) implies that (a_{mnmn}) is bounded, thus (viii) holds true.

Remark 2. The result obtained in this paper is more general in the sense that by taking $\delta = 0$, the double absolute $|A|_k$-summability can be obtained from Theorem 1.

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