Certain results on para-Kenmotsu manifolds equipped with M-projective curvature tensor

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Abstract

The purpose of the article is to study the certain results on para-Kenmotsu manifolds equipped with M-projective curvature tensor. Here we investigate para-Kenmotsu manifolds satisfying some curvature conditions $\widetilde{M}\cdot R=0$, $\widetilde{M}\cdot Q=0$ and $Q\cdot\widetilde{M}=0$, where R, Q and $\widetilde{M}$ respectively denote the Riemannian curvature tensor, Ricci operator and M-projective curvature tensor.

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1 Introduction

In 1976, Sato [22] defined the notions of an almost paracontact Riemannian manifold. An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be of even dimension as well. After that, T. Adati and K. Matsumoto [1] defined and studied p-Sasakian and sp-Sasakian manifolds which are regarded as a special kind of an almost paracontact Riemannian manifolds. Before Sato, Takahashi [23], defined almost contact manifolds (in particular, Sasakian manifolds) equipped with an associated pseudo-Riemannian metric. In 1985, Kaneyuki et al. [15] defined the notion of almost paracontact structure on pseudo-Riemannian manifold of dimension $n=(2m+1)$. Later Zamkovoy [26] showed that any almost paracontact structure admits a pseudo-Riemannian metric with signature $(n+1,n)$. The notion of para-Kenmotsu manifold was introduced by Welyczko [25]. This structure is a analogy of Kenmotsu manifold [14] in paracontact geometry. Para-Kenmotsu (briefly p-Kenmotsu) and special para-Kenmotsu (briefly sp-Kenmotsu) manifolds was studied by Sinha et al. [21], Blaga [2] and Sai Prasad et al. [20] and many others. Apart from the Riemannian curvature tensor, Weyl conformal curvature tensor and concircular curvature tensor, the M-projective curvature tensor is another important tensor from the differential geometric point of view. This curvature tensor bridges the gap between conformal curvature tensor, conharmonic curvature tensor and concircular curvature tensor on one side and H-projective curvature tensor on the other.

The M-projective curvature tensor $\widetilde{M}$ on a $(2n+1)$-dimensional Riemannian manifold is given by [19]

$$\begin{aligned}\widetilde{M}(X,Y)Z &= R(X,Y)Z-\frac{1}{4n}[S(Y,Z)X-S(X,Z)Y \\ &\quad +g(Y,Z)QX-g(X,Z)QY],\end{aligned} \tag{1.1}$$

where R is the Riemannian curvature tensor, S is the Ricci curvature tensor and Q is the Ricci operator defined by $g(QX,Y)=S(X,Y)$, for any $X,Y\in M$. Some properties of M-projective

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curvature in different manifolds have been studied earlier in [5, 8, 9, 10, 11, 17, 18] and many other researchers.

The paper is organized as follows. Section 2 deals with some preliminaries of para-Kenmotsu manifold. In section 3, we study a para-Kenmotsu manifold satisfies $\widetilde{M} \cdot R = 0$, then the manifold is an Einstein manifold. In section 4, we prove that if a para-Kenmotsu manifold satisfies the curvature condition $\widetilde{M} \cdot Q = 0$, then the square of the Ricci tensor S is the linear combination of the Ricci tensor S and the metric tensor g. In the last section, it is shown that if a para-Kenmotsu manifold satisfies the curvature condition $Q \cdot \widetilde{M} = 0$, then the trace of the square of the Ricci operator of a para-Kenmotsu manifold is equal to $-2n$ times of the trace Ricci operator.

2 Preliminaries

In this section, we mention some basic formula and definitions which will be used later.

Let M be an $(2n+1)$-dimensional smooth manifold equipped with an almost paracontact metric structure (φ, ξ, η, g) that is, φ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g is a pseudo-Riemannian metric on M of signature $(n+1, n)$ such that [26]

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.1}$$

$$\varphi^2 = I - \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1, \tag{2.2}$$

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0 \quad \text{and} \quad rank(\varphi) = 2n, \tag{2.3}$$

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.4}$$

φ induces on the $2n$-dimensional distribution $D := ker\eta$ an almost para-complex structure P; i.e. $P^2 = 1$ and the eigen subbundles D^+, D^-, corresponding to the eigenvalues $1, -1$ of P respectively, have equal dimension n; hence $D = D^+ \oplus D^-$.

In this case, $(M, \varphi, \xi, \eta, g)$ is called an almost paracontact metric manifold, φ the structure endomorphism, ξ the characteristic vector field, η the paracontact form and g compatible metric.

Note that the canonical distribution D is φ-invariant since $D = Im\varphi$. Moreover, ξ is orthogonal to D and therefore the tangent bundle splits orthogonally: $TM = D \oplus \langle\xi\rangle$.

An analogue of the Kenmotsu manifold [14] in paracontact geometry will be further considered.

If, moreover:

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{2.5}$$

where ∇ denotes the Levi-Civita connection of g, then the almost paracontact metric structure (φ, η, ξ, g) is called para-Kenmotsu manifold. The para-Kenmotsu structure for 3-dimensional normal almost paracontact metric structures was introduced by Welyczko [25]. Examples of almost paracontact metric structures can be found in [7] and [6]. From the definition it follows that is the g-dual of ξ:

$$g(X, \xi) = \eta(X), \tag{2.6}$$

ξ is a unitary vector field:

$$g(\xi, \xi) = 1, \tag{2.7}$$

and φ is a g-skew-symmetric operator:

$$g(X, \varphi Y) = -g(\varphi X, Y) \tag{2.8}$$

for any vector fields X and Y on M.

Moreover, the curvature tensor R and the Ricci tensor S satisfy

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y), \tag{2.9}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.10}$$

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi \tag{2.11}$$

$$\eta(R(X,Y)Z) = \eta(Y)X - g(X,Y)\xi, \tag{2.12}$$

$$\eta(R(X,Y)\xi) = 0, \tag{2.13}$$

$$S(X,\xi) = -2n\eta(X), \tag{2.14}$$

$$QX = -2nX, \tag{2.15}$$

$$Q\xi = -2n\xi, \tag{2.16}$$

for any vector fields X, Y, Z on M.

A para-Kenmotsu manifold is said to be Einstein if [24]

$$S(X,Y) = ag(X,Y), \tag{2.17}$$

where S is the Ricci tensor of type $(0,2)$ and a is a constant.

3 Para-Kenmotsu manifolds satisfying the curvature condition $\widetilde{M} \cdot R = 0$

In this section we consider para-Kenmotsu manifold satisfying the curvature condition $\widetilde{M} \cdot R = 0$. Therefore

$$(\widetilde{M}(X,Y) \cdot R)(U,V)W = 0. \tag{3.1}$$

for all vector fields X, Y, U, V and W. The explicit form of the above equation is

$$\begin{aligned} & \widetilde{M}(X,Y)R(U,V)W - R(\widetilde{M}(X,Y)U,V)W \\ & -R(U,\widetilde{M}(X,Y)V)W - R(U,V)\widetilde{M}(X,Y)W \\ = \; & 0. \end{aligned} \tag{3.2}$$

Putting $X = U = \xi$ in (3.2), we have

$$\begin{aligned} & \widetilde{M}(\xi,Y)R(\xi,V)W - R(\widetilde{M}(\xi,Y)\xi,V)W \\ & -R(\xi,\widetilde{M}(\xi,Y)V)W - R(\xi,V)\widetilde{M}(\xi,Y)W \\ = \; & 0. \end{aligned} \tag{3.3}$$

Using $(1.1), (2.11)$ and (2.14), we obtain

$$\begin{aligned} \widetilde{M}(\xi,Y)R(\xi,V)W = \; & \eta(W)\eta(V)Y - g(V,W)Y - g(Y,V)\eta(W)\xi \\ & +g(V,W)\eta(Y)\xi - \frac{1}{4n}[S(Y,V)\eta(W)\xi \\ & +2ng(V,W)\eta(Y)\xi + 4n\eta(V)\eta(W)Y \\ & -4ng(V,W)Y - 2ng(Y,V)\eta(W)\xi \\ & +2ng(V,W)\eta(Y)\xi]. \end{aligned} \tag{3.4}$$

Again using $(1.1), (2.11)$ and (2.14), we get

$$R(\widetilde{M}(\xi, Y)\xi, V)W = 0, \tag{3.5}$$

and

$$R(\xi, \widetilde{M}(\xi, Y)V)W = 0. \tag{3.6}$$

Finally, using $(1.1), (2.11)$ and (2.14), we have

$$\begin{aligned} R(\xi, V)\widetilde{M}(\xi, Y)W \quad = \quad & \eta(Y)\eta(W)V - g(Y, W)V \\ & -\frac{1}{4n}[S(Y, W)V + 4n\eta(Y)\eta(W)V \\ & -2ng(Y, W)V] - g(V, Y)\eta(W)\xi \\ & +g(Y, W)\eta(V)\xi + \frac{1}{4n}[S(Y, W)\eta(V) \\ & +4ng(Y, V)\eta(W) - 2ng(Y, W)\eta(V)]\xi \end{aligned} \tag{3.7}$$

Putting $(3.4) - (3.7)$ in (3.3), we have

$$\begin{aligned} & \eta(W)\eta(V)Y - g(V, W)Y + g(V, W)\eta(Y)\xi \\ & -\eta(Y)\eta(W)V + g(Y, W)V - g(Y, W)\eta(V)\xi \\ & -\frac{1}{4n}[S(Y, V)\eta(W)\xi + 4ng(V, W)\eta(Y)\xi \\ & +4n\eta(V)\eta(W)Y - 4ng(V, W)Y + 2ng(Y, V)\eta(W)\xi \\ & +S(Y, W)\eta(V)\xi - 2ng(Y, W)\eta(V)\xi - S(Y, W)V \\ & -4n\eta(Y)\eta(W)V + 2ng(Y, W)V] \\ = \quad & 0 \end{aligned} \tag{3.8}$$

Substituting $W = \xi$ in (3.8) and using (2.14), we get

$$S(Y, V)\xi + 2ng(Y, V)\xi = 0. \tag{3.9}$$

Taking inner product of (3.9) with ξ, we have

$$S(Y, V) = -2ng(Y, V), \tag{3.10}$$

which implies the manifold is an Einstein manifold.

Thus, we conclude the following:

Theorem 3.1. If a para-Kenmotsu manifold satisfies $\widetilde{M} \cdot R = 0$ then the manifold is an Einstein manifold.

4 Para-Kenmotsu manifolds satisfying the curvature condition $\widetilde{M} \cdot Q = 0$

This section is devoted to study para-Kenmotsu manifold satisfying the curvature condition $\widetilde{M} \cdot Q = 0$. It means

$$\widetilde{M}(X, Y)QZ - Q(\widetilde{M}(X, Y)Z) = 0, \tag{4.1}$$

for all vector fields X, Y and Z.

Putting $Y = \xi$ in (4.2), we obtain

$$\widetilde{M}(X,\xi)QZ - Q(\widetilde{M}(X,\xi)Z) = 0. \tag{4.2}$$

Using $(1.1), (2.11)$ and (2.14), we get

$$\begin{aligned} \widetilde{M}(X,\xi)QZ &= g(X,QZ)\xi - \eta(QZ)X \\ &\quad -\frac{1}{4n}[-S(X,QZ)\xi - 4n\eta(QZ)X + 2ng(X,QZ)\xi], \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} Q(\widetilde{M}(X,\xi)Z) &= -2ng(X,Z)\xi + 2n\eta(Z)X \\ &\quad -\frac{1}{4n}[2nS(X,Z)\xi + 8n^2\eta(Z)X - 4n^2g(X,Z)\xi]. \end{aligned} \tag{4.4}$$

Substituting (4.3) and (4.4) in (4.2), we obtain

$$S(X,QZ)\xi = -6nS(X,Z)\xi - 8n^2g(X,Z)\xi. \tag{4.5}$$

Taking inner product with ξ, the relation (4.5) becomes:

$$S^2(X,Z) = -6nS(X,Z) - 8n^2g(X,Z), \tag{4.6}$$

where $S^2(X,Z) = S(X,QZ)$.

Thus, we conclude the following:

Theorem 4.1. If a para-Kenmotsu manifold satisfies the curvature condition $\widetilde{M}\cdot Q = 0$, then the square of the Ricci tensor S is the linear combination of the Ricci tensor S and the metric tensor g.

For symmetric $(0,2)$-tensor fields A and B on M define Kulkarni-Nomizu product $A\overline{\Lambda}B$ of A and B by $([4], p-47)$

$$\begin{aligned} A\overline{\Lambda}B(X_1,...,X_4) &= A(X_1,X_4)B(X_2,X_3) - A(X_1,X_3)B(X_2,X_4) \\ &\quad + A(X_2,X_3)B(X_1,X_4) - A(X_2,X_4)B(X_1,X_3). \end{aligned}$$

We recall the following lemma:

Lemma 4.2. [12]Let A be symmetric $(0,2)$-tensor at point x of a semi Riemannian manifold (M,g), dim $M \geq 3$, and let $T = g\overline{\Lambda}A$ be the Kulkarni-Nomizu product of g and A. Then the relation

$$T\cdot T = \alpha Q(g,T), \alpha \in R$$

is satisfied at x if and only if the condition

$$A^2 = \alpha A + \lambda g, \alpha \in R$$

holds at x.

Therefore, we have the following corollary:

Corollary 4.3. Let M be a para-Kenmotsu manifold satisfying the condition $\widetilde{M}\cdot Q = 0$ then $T\cdot T = \alpha Q(g,T)$, where $T = g\overline{\Lambda}S$ and $\alpha = -6n$.

5 Para-Kenmotsu manifolds satisfying the curvature condition $Q \cdot \widetilde{M} = 0$

In this section we study para-Kenmotsu manifold satisfying the curvature condition $Q \cdot \widetilde{M} = 0$. Therefore

$$(Q \cdot \widetilde{M})(X, Y)Z = 0, \tag{5.1}$$

for all vector fields X, Y and Z.

Equation (5.1) can be written as

$$Q(\widetilde{M}(X, Y)Z) - \widetilde{M}(QX, Y)Z - \widetilde{M}(X, QY)Z - \widetilde{M}(X, Y)QZ = 0, \tag{5.2}$$

Putting $Y = \xi$ in (5.2), we get

$$Q(\widetilde{M}(X, \xi)Z) - \widetilde{M}(QX, \xi)Z - \widetilde{M}(X, Q\xi)Z - \widetilde{M}(X, \xi)QZ = 0. \tag{5.3}$$

Using (1.1) and (2.14) in (2.16), we obtain

$$\begin{aligned} Q(\widetilde{M}(X, \xi)Z) &= -2ng(X, Z)\xi + 2n\eta(Z)X \\ &\quad -\frac{1}{2}[S(X, Z)\xi + 4n\eta(Z)X - 2ng(X, Z)\xi]. \end{aligned} \tag{5.4}$$

By virtue of (1.1) and (2.11), we get

$$\begin{aligned} \widetilde{M}(QX, \xi)Z &= g(QX, Z)\xi - \eta(Z)QX \\ &\quad -\frac{1}{4n}[-S(QX, Z)\xi - 4n\eta(Z)QX + 2ng(QX, Z)\xi]. \end{aligned} \tag{5.5}$$

Using (1.1), (2.11) and (2.14), we have

$$\begin{aligned} \widetilde{M}(X, Q\xi)Z &= -2n[g(X, Z)\xi - \eta(Z)X] - \frac{1}{4n}[-2nS(\xi, Z)X \\ &\quad +2nS(X, Z)\xi - 2n\eta(Z)QX - 4n^2 g(X, Z)\xi]. \end{aligned} \tag{5.6}$$

Now, by virtue of (1.1) and (2.11), we get

$$\begin{aligned} \widetilde{M}(X, \xi)QZ &= g(X, ZQ)\xi - \eta(QZ)X \\ &\quad -\frac{1}{4n}[-S(X, QZ)\xi - 4n\eta(QZ)X + 2ng(X, QZ)\xi]. \end{aligned} \tag{5.7}$$

Putting (5.4) – (5.7) in (5.3), we have

$$S(QX, Z)\xi + 2ng(QX, Z)\xi = 0. \tag{5.8}$$

Taking inner product with ξ of above equation and using (2.11), we obtain

$$g(Q^2X, Z) = -2ng(QX, Z). \tag{5.9}$$

Let $\{e_i\}, i = 1, 2, ..., (2n + 1)$ be a local orthonormal basis of the tangent space at a point of the manifold M. Then by substituting $X = Z = e_i$ in (5.9) and taking summation over i, $1 \le i \le (2n + 1)$, we get

$$Tr(Q^2) = \sum_{i=1}^{2n+1} g(Q^2 e_i, e_i) = -2n \sum_{i=1}^{2n+1} g(Qe_i, e_i) = -2nTr(Q). \tag{5.10}$$

Thus, we are in a position to state the following:

Theorem 5.1. If a para-Kenmotsu manifold satisfies the curvature condition $Q \cdot \widetilde{M} = 0$, then the trace of square of the Ricci operator of a para-Kenmotsu manifold is equal to $-2n$ times trace of the Ricci operator.

References

[1] T. Adati and K. Matsumoto, *On conformally recurrent and conformally symmetric P-Sasakian manifolds*, TRU Math., 13 (1977), 25-32.

[2] A. M. Blaga, *η-Ricci solitons on para-Kenmotsu manifolds*, Balkan Journal of Geometry and Its Applications, 20(1) (2015), 1-13.

[3] D. E. Blair, *Two remarks on contact metric structures*, Tohoku Math. J., 29 (1977), 319-324.

[4] A. L. Besse, *Einstein manifolds*, Springer-verlag, Berlin-Heidelberg (1987).

[5] S. K. Chaubey and R. H. Ojha, *On the M-projective curvature tensor on Kenmotsu manifolds*, Diff. Goem. Dynam. Syst., 12 (2010), 52-60.

[6] P. Dacko and Z. Olszak, *On weakly para-cosymplectic manifolds of dimension 3*. J. Geom. Phys. 57 (2007), 561-570.

[7] S. Ivanov, D. Vassilev and S. Zamkovoy, *Conformal paracontact curvature and the local flatness theorem*. Geom. Dedicata 144 (2010), 79-100.

[8] U. C. De and S. Samui, *E-Bochner curvature tensor on (k, μ)-contact metric manifolds*, Int. Electron. J. Geom., 7(1) (2014), 143-153.

[9] U. C. De and P. Pal, *On generalized M-projectively recurrent manifolds*, Ann. Univ. Paedagog. Crac. Stud. Math., 13 (2014), 77-101.

[10] U. C. De and S. Mallick, *m-Projective curvature tensor on $N(k)$-quasi-Einstein manifolds*, Diff. Geom. Dynam. Syst., 16 (2014), 98-112.

[11] U. C. De and A. Haseeb, *On generalized Sasakian-space-forms with M-projective curvature tensor*, Adv. Pure Appl. Math., 9(1) (2018), 67–73.

[12] R. Deszcz, L. Verstraelen and S. Yaprak, *Warped products realizing a certain conditions of pseudosymmetry type imposed on the Weyl curvature tensor*, Chin. J. Math. 22 (1994), 139-157.

[13] A. Ghosh, T. Koufogiorgos and R. Sharma, *Conformally flat contact metric manifolds*, J. Geom., 70 (2001), 66-76.

[14] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J., 24 (1972), 93-103.

[15] S. Kaneyuki and F. L. Williams, *Almost paracontact and parahodge structure on manifolds*, Nagoya Math. J., 99 (1985), 173-187.

[16] P. Majhi and G. Ghosh, *On a classification of Para-Sasakian manifolds*, Facta Universitatis, Ser. Math. Inform., 32(5) (2017), 781–788.

[17] R. H. Ojha, *A note on the M-projective curvature tensor,*, Indian J. Pure Appl. Math., 8 (12) (1975), 1531-1534.

[18] R. H. Ojha, *M-projectively flat Sasakian manifolds*, Indian J. Pure Appl. Math., 17(4) (1986), 481-484.

[19] G. P. Pokhariyal and R. S. Mishra, *Curvature tensor and their relativistic significance II*, Yokohama Math. J., 19 (1971), 97-103.

[20] K. L. Sai Prasad and T. Satyanarayan, *On para-Kenmotsu manifold,* Int. J. Pure Appl. Math., 90(1) (2014), 35-41.

[21] B. B. Sinha and K. L. Prasad, *A class of Almost paracontact metric manifold*, Bull. Calcutta Math, Soc., 87 (1995), 307-312.

[22] I. Sato, *On a structure similar to the almost contact structure I*, Tensor N. S., 30 (1976), 219-224.

[23] T. Takahashi, *Sasakian manifold with pseudo-Riemannian metric*, Thoku Math. J., 21(2) (1969), 644-653.

[24] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Mathematics, World Scientific Publishing Co., Singapore 3 (1984).

[25] J. Welyczko, *Slant curves in* 3*-dimensional normal almost paracontact metric manifolds*, Mediter, J. Math., (2013).

[26] S. Zamkovoy, *Canonical connections on paracontact manifolds*, Ann. Glob. Anal. Geom. 36(1) (2009), 37-60.