Connection problems and matrix representations for certain hybrid polynomials

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Abstract

In this paper, we deal with the connection and duplication problems associated with the hybrid Sheffer family. The hybrid Sheffer polynomials are also studied via matrix approach. The properties of these polynomials are established using simple matrix operations. Examples providing the corresponding results for certain members of the hybrid Sheffer family are considered. This article is first attempt in the direction of obtaining connection and duplication coefficients and matrix representations for the hybrid polynomials.

2010 Mathematics Subject Classification. **33F10**. 68W30, 41A58, 11B83, 15A16
Keywords. Connection coefficients; Duplication coefficients; Hybrid Sheffer polynomials; Creation matrix; Generalized Pascal matrix.

1 Introduction and preliminaries

Special functions comprise a very old branch of mathematics, the origin of their unified and rather complete theory date to the nineteenth century. In numerous branches of mathematics, special polynomials and numbers perform a fundamental role and their exaggeration is always actual. In diverse areas of applied mathematics and mathematical physics, generating functions perform an essential role in the investigation of various useful properties of sequences which they generate. Generating functions are utilized to accomplish numerous properties of formulas for numbers and polynomials in a wide range of research subjects such as modern combinatorics.

The generating functions for the sequence of polynomials can be used in analyzing sequences of functions, in detecting a closed formula for a sequence, in detecting recurrence relations and differential equations, relationships between sequences, asymptotic behavior of sequences, in proving identities involving sequences and in solving enumeration problems in combinatorics and encoding their solutions.

The use of polynomials in many fields of science and engineering is quite remarkable. Throughout this paper, we shall focus on three families of special polynomials. These useful classes are the Appell, Sheffer and hybrid Sheffer polynomials. These polynomials and numbers have applications in many fields such as complex analysis, operator theory, statistics, numerical analysis and data compression *etc.*

The hybrid Sheffer polynomials constructed by Subuhi Khan and M. Riyasat [12] can be viewed as an extension of the Sheffer polynomials. These hybrid polynomials are important due to the fact that they possess important properties such as differential equations, generating functions, series definitions, integral representations *etc.* These polynomials are useful and possess potential for applications in certain problems of number theory, combinatorics, classical and numerical analysis,

Tbilisi Mathematical Journal 11(3) (2018), pp. 81–93.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 17 March 2018.
Accepted for publication: 17 August 2018.

theoretical physics and other fields of pure and applied mathematics. The differential equations satisfied by the hybrid special polynomials may be used to solve new emerging problems in different branches of science.

The hybrid families of special polynomials are introduced as discrete convolution of the known special polynomials. The discrete Appell convolution $f_n^{(A)}(x)$ is defined as [14]:

$$f_n^{(A)}(x) = \sum_{k=0}^{n} \binom{n}{k} A_k f_{n-k}(x).$$

Taking the generic polynomials $f_n(x)$ $(n \in N,\ x \in R)$ as x^n in the above equation, we note that $f_n^{(A)}(x)$ reduces to the Appell polynomials $A_n(x)$ [4], which are defined by either of the following equivalent conditions:

$$\frac{d}{dx} A_n(x) = n A_{n-1}(x),\ n = 1, 2, 3, \cdots, \tag{1.1}$$

in which

$$A_0(x) = a_0,\ a_0 \in \mathbb{R} \setminus \{0\}, \tag{1.2}$$

or, there exists an exponential generating function of the form:

$$a(t) e^{xt} = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!}, \tag{1.3}$$

where $a(t)$ has (at least the formal) expansion:

$$a(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!},\ a_0 \neq 0, \tag{1.4}$$

The Appell polynomials are studied via different approach, see for example [1,7,8]. The hybrid Sheffer polynomials denoted by ${}_sA_n(x)$ are defined as the discrete Appell convolution of the Sheffer polynomials $s_n(x)$ and possess the following generating function [12]:

$$s(t) a(t)\ exp(x G(t)) = \sum_{n=0}^{\infty} {}_sA_n(x) \frac{t^n}{n!}, \tag{1.5}$$

where $a(t)$ has same expansion as in equation (1.4) and $s(t)$ and $G(t)$ have (at least the formal) expansions:

$$s(t) = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!},\ c_0 \neq 0 \tag{1.6}$$

and

$$G(t) = \sum_{n=1}^{\infty} G_n \frac{t^n}{n!},\ G_1 \neq 0, \tag{1.7}$$

respectively.

These hybrid Sheffer polynomials ${}_sA_n(x)$ are also defined by the following series representation:

$$ {}_sA_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k s_{n-k}(x). \tag{1.8} $$

The hybrid Sheffer polynomials ${}_sA_n(x)$ becomes the Sheffer polynomials $s_n(x)$ for $a(t) = 1$, which are defined by the exponential generating function of the form [14, p. 19]:

$$ s(t)\ exp(xG(t)) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \tag{1.9} $$

where $s(t)$ and $G(t)$ have same representations as in equations (1.6) and (1.7). For $G(t) = t$, the Sheffer polynomials $s_n(x)$ reduces to the Appell polynomials $A_n(x)$.

Matrices play an important role in all branches of science, engineering and management. They have explicit significance in various fields of mathematics and engineering. There are many special types of matrices such as the Pascal, Vandermonde and Stirling considered in [3], [9] and [10] respectively. These matrices are mostly being used for data classification and to solve other problems using computers. Specifically, the Pascal matrix turns out in combinatorics, numerical analysis, probability and image processing.

In last few years, the matrix approach has attracted the renewed attention of many experts not only in the field of pure mathematics but also in different areas of applied mathematics such as statistics, numerical analysis, computer aided design and combinatorics. Pure mathematics focuses on the existence and uniqueness of solutions, while applied mathematics emphasizes the rigorous justification of the methods for approximating solutions.

Recently, the matrix approach for the Appell and Sheffer polynomials is proposed in [1, 2]. We extend this approach to the hybrid Sheffer polynomials. The matrix approach for the hybrid Sheffer polynomials basically relies on the properties of the creation matrix defined by

$$ (H)_{ij} = \begin{cases} i, & i = j+1, \\ 0, & \text{otherwise}, \end{cases} \qquad i, j = 0, 1, ..., m. \tag{1.10} $$

From [3, p. 232], it follows that

$$ H\mathbf{e}_j = (j+1)\ \mathbf{e}_{j+1}, \qquad j = 0, 1, ..., m, \tag{1.11} $$

where $\mathbf{e}_j$ $(j = 0, 1, 2, \cdots, m)$ denote the standard unit basis vectors in $\mathbb{R}^{m+1}$. Also, we have $\mathbf{e}_j = 0$, whenever $j > m$. This is compatible with the fact that the creation matrix is nilpotent of degree $m+1$, *i.e.*

$$ H^j = 0, \qquad \text{for all } j \geq m+1. \tag{1.12} $$

Replacing t by H in equations (1.4) and (1.6), we find the non-singular matrices:

$$ a(H) = \sum_{n=0}^{m} a_n \frac{H^n}{n!}, \qquad a_0 \neq 0, \tag{1.13} $$

$$ s(H) = \sum_{n=0}^{m} c_n \frac{H^n}{n!}, \qquad c_0 \neq 0, \tag{1.14} $$

Let us, consider the generalized Pascal matrix of the sequence $\{q_n(x)\}_{n\geq 0}$ defined by

$$\big(P[q_n(x)]\big)_{ij} = \begin{cases} \binom{i}{j}\ q_{i-j}(x), & i \geq j, \\ 0, & \text{otherwise}, \end{cases} \qquad i,j = 0,1,...,m. \tag{1.15}$$

Consider the Bell matrix B, whose entries are related to the sequences $\{b_n\}_{n\geq 0}$ with $b_0 = 0$ and $b_1 \neq 0$, as follows

$$\big(B\big)_{ij} = \begin{cases} B_{i,j} := B_{i,j}(b_1, b_2, \ldots b_{i-j+1}), & i \geq j \geq 1, \\ 1, & i = j = 0, \\ 0, & \text{otherwise}, \end{cases} \qquad i,j = 0,1,...,m. \tag{1.16}$$

It is to be noted that this lower triangular matrix is nonsingular due to the fact that diag$(B) = (1, b_1, b_1^2, \ldots b_1^m)$. Also, the vector of monomial powers is defined by

$$\xi(x) = [1\ x\ \ldots\ x^m]^T \tag{1.17}$$

and the transfer matrices $\mathcal{M}$ and N are defined as:

$$(\mathcal{M})_{ij} = \begin{cases} \binom{i}{j}\ a_{i-j}, & i \geq j, \\ 0, & \text{otherwise}, \end{cases} \qquad i,j = 0,1,...,m,\ a_0 \neq 0. \tag{1.18}$$

$$(N)_{ij} = \begin{cases} \binom{i}{j}\ c_{i-j}, & i \geq j, \\ 0, & \text{otherwise}, \end{cases} \qquad i,j = 0,1,...,m,\ c_0 \neq 0. \tag{1.19}$$

From [1, p. 433 (Theorem 3.2)], we have

$$\mathcal{M} = a(H), \qquad N = s(H). \tag{1.20}$$

The study of the connection and duplication problems has gained an increasing interest in the last few years. The computation of the connection and duplication coefficients perform an important role in various situations of pure and applied mathematics, especially in combinatorial analysis. The connection and duplication problems are not only important from a fundamental point of view but also because they are used in computation of physical and chemical properties of quantum-mechanical systems.

The content of this article are inspired by the work under progress in the direction of obtaining connection and duplication coefficients and matrix representations for special polynomials. In this paper, the connection and duplication coefficients and matrix representations for hybrid Sheffer polynomials are established. The corresponding results for certain members belonging to the hybrid Sheffer family are also derived.

2 Connection and duplication coefficients

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$. A polynomial sequence $\{P_n\}_{n\geq 0}$ in $\mathcal{P}$ is called a polynomial set if and only if deg $P_n = n$, for all non-negative integer n.

Given two polynomial sets $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, the connection problem between them is to find the coefficients $C_r(n)$ in the expression:

$$Q_n(x) = \sum_{r=0}^{n} C_r(n) P_r(x). \tag{2.1}$$

A special case of the connection problem, called the duplication problem, asks to find the coefficients $C_r(n, b)$ in the expression:

$$P_n(bx) = \sum_{r=0}^{n} C_r(n, b) P_r(x), \tag{2.2}$$

where b designates a non-zero complex number.

We begin by recalling a result giving the connection coefficients between two σ-Appell polynomials. That is to say $\sigma A_n = n A_{n-1},\ n = 0, 1, \cdots, n$, where σ is a linear operator not depending on n and is a lowering operator.

Result 2.1 *([7, Corollary 3.4]). Let $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$be two σ-Appell polynomial sets of transfer power series, respectively, A_1 and A_2. Then*

$$Q_n(x) = \sum_{m=0}^{n} \frac{n!}{m!}\ \alpha_{n-m}\ P_m(x), \qquad \textit{where } \frac{A_2(t)}{A_1(t)} = \sum_{k=0}^{\infty} \alpha_k\ t^k. \tag{2.3}$$

First, we establish the connection formula for the hybrid Sheffer polynomials ${}_sA_n(x)$ in the form of following result:

Theorem 2.1. Let $\{{}_sA_n^{(1)}(x)\}_{n\geq 0}$ be the hybrid Sheffer polynomial set of transfer power series $s_1(t),\ a_1(t)$ and $\{{}_sA_n^{(2)}(x)\}_{n\geq 0}$ be another hybrid Sheffer polynomial set of transfer power series $s_2(t),\ a_2(t)$. Then the following connection formula holds true:

$${}_sA_n^{(2)}(x) = \sum_{r=0}^{n} C_r(n)\ {}_sA_r^{(1)}(x), \tag{2.4}$$

where $C_r(n) = \sum\limits_{k=0}^{n-r} \binom{n}{r}\binom{n-r}{k} \alpha_{n-r-k}\gamma_k$ and $\mathrm{h(t)} = \frac{s_2(t)a_2(t)}{s_1(t)a_1(t)} = \sum\limits_{n,k=0}^{\infty} \alpha_n \gamma_k \frac{t^{n+k}}{n!\ k!}$.

Proof. From generating function (1.5), we have

$$s_2(t)a_2(t)\exp(xG(t)) = \sum_{n=0}^{\infty} {}_sA_n^{(2)}(x)\frac{t^n}{n!},$$

which can be written as

$$\frac{s_2(t)a_2(t)}{s_1(t)a_1(t)}\ s_1(t)a_1(t)\exp(xG(t)) = \sum_{n=0}^{\infty} {}_sA_n^{(2)}(x)\frac{t^n}{n!}. \tag{2.5}$$

Let

$$h(t)=\frac{s_2(t)a_2(t)}{s_1(t)a_1(t)}=\sum_{n,k=0}^{\infty}\alpha_n\gamma_k\frac{t^{n+k}}{n!\ k!},$$

which on applying the Cauchy-product rule becomes

$$h(t)=\frac{s_2(t)a_2(t)}{s_1(t)a_1(t)}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\alpha_{n-k}\gamma_k\frac{t^n}{n!}. \tag{2.6}$$

In view of equations (1.5), (2.5) and (2.6), it follows that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\alpha_{n-k}\gamma_k\frac{t^n}{n!}\sum_{r=0}^{\infty}{}_sA_r^{(1)}(x)\frac{t^r}{r!}=\sum_{n=0}^{\infty}{}_sA_n^{(2)}(x)\frac{t^n}{n!}. \tag{2.7}$$

Again, applying the Cauchy-product rule in the l.h.s. of equation (2.7) and then equating the coefficients of identical powers of t on both sides of the resultant equation assertion (2.4) follows.

Q.E.D.

Next, the duplication formula involving the hybrid Sheffer polynomials ${}_sA_n(x)$ is obtained in the form of following result:

Theorem 2.2. The hybrid Sheffer polynomial set $\{{}_sA_n(x)\}_{n\geq 0}$ consisting of transfer power series $s(t)$ and $a(t)$ possess the following duplication formula:

$${}_sA_n(bx)=\sum_{r=0}^{n}C_r(n,b)\ {}_sA_r(x), \tag{2.8}$$

where $C_r(n,b)=\sum\limits_{k=0}^{n-r}\binom{n}{r}\binom{n-r}{k}b^r\ \ \alpha_{n-r-k}(b)\gamma_k(b)$ and g(t) $=\frac{s(t)a(t)}{s(bt)a(bt)}=\sum\limits_{n,k=0}^{\infty}\alpha_n(b)\gamma_k(b)\frac{t^{n+k}}{n!\ k!}$.

Proof. From generating function (1.5), we have

$$s(t)a(t)\exp(bxG(t))=\sum_{n=0}^{\infty}{}_sA_n(bx)\frac{t^n}{n!},$$

or equivalently

$$\frac{s(t)a(t)}{s(bt)a(bt)}\ s(bt)a(bt)\exp(bxG(t))=\sum_{n=0}^{\infty}{}_sA_n(bx)\frac{t^n}{n!}. \tag{2.9}$$

Let

$$g(t)=\frac{s(t)a(t)}{s(bt)a(bt)}=\sum_{n,k=0}^{\infty}\alpha_n(b)\gamma_k(b)\frac{t^{n+k}}{n!\ k!},$$

which on applying the Cauchy-product rule becomes

$$g(t)=\frac{s(t)a(t)}{s(bt)a(bt)}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\alpha_{n-k}(b)\gamma_k(b)\frac{t^n}{n!}. \tag{2.10}$$

In view of equations (1.5), (2.9) and (2.10), it follows that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\alpha_{n-k}(b)\gamma_k(b)\frac{t^n}{n!}\sum_{r=0}^{\infty}{}_sA_r(x)\frac{b^rt^r}{r!}=\sum_{n=0}^{\infty}{}_sA_n(bx)\frac{t^n}{n!}. \tag{2.11}$$

Again, applying the Cauchy-product rule in the l.h.s. of equation (2.11) and then equating the coefficients of identical powers of t on both sides of the resultant equation, we get assertion (2.8). Q.E.D.

In the next section, certain properties of the hybrid Sheffer polynomials are established via matrix approach.

3 Matrix representations

In order to utilize the Pascal and transfer matrices, the vector form of the hybrid Sheffer polynomial sequence is required.

The hybrid Sheffer vector denoted by ${}_{\mathbf{s}}\mathbf{A}(x)$ is defined as:

$${}_{\mathbf{s}}\mathbf{A}(x)=[{}_sA_0(x)\ {}_sA_1(x)\ .\ .\ .\ {}_sA_m(x)]^T, \tag{3.1}$$

where ${}_sA_n(x)$ are the hybrid Sheffer polynomials defined by generating function (1.5).

The following theorem is proved to establish the matrix representation of the hybrid Sheffer polynomials:

Theorem 3.1. Let $\mathbf{A}(x),\mathbf{s}(x)$ and ${}_{\mathbf{s}}\mathbf{A}(x)$ are the Appell, Sheffer and hybrid Sheffer vectors for the corresponding Appell, Sheffer and hybrid Sheffer polynomials $\{A_n(x)\}_{n\geq 0},\{s_n(x)\}_{n\geq 0}$ and $\{{}_sA_n(x)\}_{n\geq 0}$ respectively. Then ${}_{\mathbf{s}}\mathbf{A}(x)=N\mathcal{M}B\xi(x)$ is the matrix representation of the hybrid Sheffer vector ${}_{\mathbf{s}}\mathbf{A}(x)$.

Proof. Let $\mathcal{M}$ and N be the transfer matrices for the Appell and Sheffer vectors $\mathbf{A}(x)$ and $\mathbf{s}(x)$ respectively. From [1, p. 432 (3.9)], we have

$$\mathbf{A}(x)=\mathcal{M}\xi(x) \tag{3.2}$$

and from [2, p. 94 (20)], we find

$$\mathbf{s}(x)=NB\xi(x). \tag{3.3}$$

Replacing the powers of x *i.e.*, $x^0,x^1,x^2,\cdots,x^n$ by $A_0(x),A_1(x),\cdots,A_n(x)$, in equation (3.3) and denoting the resulting polynomials by ${}_sA_0(x),{}_sA_1(x),\cdots,{}_sA_n(x)$ respectively and their corresponding vectors by ${}_{\mathbf{s}}\mathbf{A}(x)=[{}_sA_0(x)\ {}_sA_1(x)\cdots,{}_sA_n(x)]^T$, it follows that

$${}_{\mathbf{s}}\mathbf{A}(x)=NB\mathbf{A}(x).$$

Using relation (3.2) and commutativity of the matrices B and $\mathcal{M}$, we find

$$ {}_{\mathbf{s}}\mathbf{A}(x) = N\mathcal{M}B\xi(x). \tag{3.4} $$

It can be easily seen from the above equation the ${}_{\mathbf{s}}\mathbf{A}(x)$ is the hybrid Sheffer vector for the pair $(N\mathcal{M}, B)$.

Q.E.D.

Note. It is important to observe that relation (3.4) for the hybrid Sheffer polynomials ${}_{s}A_n(x)$ is different from relation [2, p. 94 (20)] for the Sheffer polynomials $s_n(x)$.
Remark 3.1 Using the transfer matrix and some properties of the generalized Pascal matrix, the matrix representation of the hybrid Sheffer identity:

$$ {}_{s}A_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)\ {}_{s}A_{n-k}(y), \qquad \forall\ n \in \mathbb{N}_0,\ x, y \in \mathbb{R}, \tag{3.5} $$

is obtained as:

$$ {}_{\mathbf{s}}\mathbf{A}(x+y) = P\big([p_n(y)]\big)\ {}_{\mathbf{s}}\mathbf{A}(x). \tag{3.6} $$

Remark 3.2 Setting $y = (m-1)x$ in identity (3.6), the following multiplication formula for the hybrid Sheffer vector ${}_{\mathbf{s}}\mathbf{A}(x)$ is obtained:

$$ {}_{\mathbf{s}}\mathbf{A}(mx) = P\big([p_n((m-1)x)]\big)\ {}_{\mathbf{s}}\mathbf{A}(x). \tag{3.7} $$

Remark 3.3 Let

$$ (\mathcal{M})^{-1}_{ij} = \begin{cases} \binom{i}{j}\ \gamma_{i-j}, & i \geq j, \\ 0, & \text{otherwise}, \end{cases} \qquad i, j = 0, 1, ..., m,\ \gamma_0 \neq 0. \tag{3.8} $$

In view of transfer matrices defined by equations (1.18) and (3.8), it follows that

$$ \gamma_0 = \frac{1}{a_0} \quad \text{and } \gamma_k = -\frac{1}{a_0} \sum_{s=0}^{k-1} \binom{k}{s} a_{k-s}\gamma_s, \qquad k = 1, 2, \cdots, m. \tag{3.9} $$

Using relation (3.9), we obtain the following recurrence relation for the hybrid Sheffer polynomials ${}_{s}A_n(x)$:

$$ {}_{s}A_n(x) = \frac{1}{\gamma_0}\Big(s_n(x) - \sum_{k=0}^{n-1} \binom{n}{k} \gamma_{n-k}\ {}_{s}A_k(x)\Big), \qquad n = 0, 1, \cdots \quad . \tag{3.10} $$

The forthcoming section aims at presenting certain results for the polynomials belonging to the hybrid Sheffer family.

4 Examples

The members of the Appell family can be obtained by making suitable selections of the function $a(t)$ in generating function (1.3). Some of these members are listed in Table 1.

Table 1. Certain members belonging to the Appell family

S. No.	$a(t)$	Generating function	Name of the polynomials and related numbers
I.	$a(t)=\frac{t}{e^t-1}$	$\frac{t}{e^t-1}\,e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}$ $B_n:=B_n(0)$	The Bernoulli polynomials and numbers [15]
II.	$a(t)=\frac{2}{e^t+1}$	$\frac{2}{e^t+1}\,e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}$ $E_n:=E_n(0)$	The Euler polynomials and numbers [15]
III.	$a(t)=e^{-t^2/4}$	$e^{xt-t^2/4}=\sum_{n=0}^{\infty}\hat{H}_n(x)\frac{t^n}{n!}$ $\hat{H}_n:=\hat{H}_n(0)$	The monic Hermite polynomials and numbers [5]

Different members belonging to the Sheffer family defined by equation (1.9) can be obtained by selecting the appropriate pair $(s(t), G(t))$. Some of these members are listed in Table 2.

Table 2. Certain members belonging to the Sheffer family

S. No.	$G(t);\ s(t)$	Generating function	Name of the polynomials
I.	$G(t)=\nu t;\ s(t)=e^{-t^m}$	$exp(\nu xt-t^m)=\sum_{n=0}^{\infty}H_{n,m,\nu}(x)\frac{t^n}{n!}$	Generalized Hermite polynomials $H_{n,m,\nu}(x)$ [13]
II.	$G(t)=\frac{t}{t-1};\ s(t)=(1-t)^{-\alpha-1}$	$\frac{1}{(1-t)^{\alpha+1}}\,exp(\frac{xt}{t-1})=\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)\frac{t^n}{n!}$	Generalized Laguerre polynomials $n!L_n^{(\alpha)}(x)$ [14]
III.	$G(t)=1-e^t;\ s(t)=exp(\beta t)$	$exp(\beta t+x(1-e^t))=\sum_{n=0}^{\infty}a_n^{(\beta)}(x)\frac{t^n}{n!}$	Actuarial polynomials $a_n^{(\beta)}(x)$ [6]
IV.	$G(t)=ln\left(1+\frac{t}{a}\right);\ s(t)=exp(-t)$	$e^{-t}(1+\frac{t}{a})^x=\sum_{n=0}^{\infty}c_n(x;a)\frac{t^n}{n!}$	Poisson-Charlier polynomials $c_n(x;a)$ [11]

The advantage of the extended Sheffer family defined by equation (1.5) is that it allows to consider mixed type polynomials as its members. These hybrid polynomials can be obtained by taking the pair $(s(t), G(t))$ of the polynomials belonging to the Sheffer class and $a(t)$ of the polynomials belonging to the Appell class. Due to this fact, it is justified to call these polynomials as the hybrid Sheffer polynomials for the pair $(s(t)a(t), G(t))$. Thus, by making suitable selections for the pair $(s(t)a(t), G(t))$, the corresponding member belonging to the hybrid Sheffer family can be obtained. These hybrid polynomials are listed in Table 3.

Table 3. Certain members belonging to the hybrid Sheffer family

S. No.	$G(t);\ s(t);\ a(t)$	Generating function	Name of the polynomials
I.	$G(t)=\nu t;\ s(t)=e^{-t^m};$ $a(t)=\left(\frac{t}{e^t-1}\right)$	$\left(\frac{t}{e^t-1}\right)exp(\nu xt-t^m)=\sum_{n=0}^{\infty}{}_HB_{n,m,\nu}(x)\frac{t^n}{n!}$	Generalized Hermite-Bernoulli polynomials ${}_HB_{n,m,\nu}(x)$
II.	$G(t)=\nu t;\ s(t)=e^{-t^m};$ $a(t)=\left(\frac{2}{e^t+1}\right)$	$\left(\frac{2}{e^t+1}\right)exp(\nu xt-t^m)=\sum_{n=0}^{\infty}{}_HE_{n,m,\nu}(x)\frac{t^n}{n!}$	Generalized Hermite-Euler polynomials ${}_HE_{n,m,\nu}(x)$

III.	$G(t)=\nu t;\ s(t)=e^{-t^m};$ $a(t)=e^{-t^2/4}$	$exp(\nu xt-t^m-t^2/4)=\sum_{n=0}^{\infty}{}_H\hat{H}_{n,m,\nu}(x)\frac{t^n}{n!}$	Generalized Hermite-monic Hermite polynomials ${}_H\hat{H}_{n,m,\nu}(x)$
IV.	$G(t)=\frac{t}{t-1};\ s(t)=(1-t)^{-\alpha-1};$ $a(t)=\left(\frac{t}{e^t-1}\right)$	$\frac{1}{(1-t)^{\alpha+1}}\left(\frac{t}{e^t-1}\right)exp(\frac{xt}{t-1})=\sum_{n=0}^{\infty}{}_LB_n^{(\alpha)}(x)\frac{t^n}{n!}$	Generalized Laguerre-Bernoulli polynomials $n!\ {}_LB_n^{(\alpha)}(x)$
V.	$G(t)=\frac{t}{t-1};\ s(t)=(1-t)^{-\alpha-1};$ $a(t)=\left(\frac{2}{e^t+1}\right)$	$\frac{1}{(1-t)^{\alpha+1}}\left(\frac{2}{e^t+1}\right)exp(\frac{xt}{t-1})=\sum_{n=0}^{\infty}{}_LE_n^{(\alpha)}(x)\frac{t^n}{n!}$	Generalized Laguerre-Euler polynomials $n!\ {}_LE_n^{(\alpha)}(x)$
VI.	$G(t)=\frac{t}{t-1};\ s(t)=(1-t)^{-\alpha-1};$ $a(t)=e^{-t^2/4}$	$\frac{1}{(1-t)^{\alpha+1}}\ exp(\frac{xt}{t-1}-t^2/4)=\sum_{n=0}^{\infty}{}_L\hat{H}_n^{(\alpha)}(x)\frac{t^n}{n!}$	Generalized Laguerre-monic Hermite polynomials $n!\ {}_L\hat{H}_n^{(\alpha)}(x)$
VII.	$G(t)=1-e^t;\ s(t)=exp(\beta t);$ $a(t)=\left(\frac{t}{e^t-1}\right)$	$\left(\frac{t}{e^t-1}\right)exp(\beta t+x(1-e^t))=\sum_{n=0}^{\infty}{}_aB_n^{(\beta)}(x)\frac{t^n}{n!}$	Actuarial-Bernoulli polynomials ${}_aB_n^{(\beta)}(x)$
VIII.	$G(t)=1-e^t;\ s(t)=exp(\beta t);$ $a(t)=\left(\frac{2}{e^t+1}\right)$	$\left(\frac{2}{e^t+1}\right)exp(\beta t+x(1-e^t))=\sum_{n=0}^{\infty}{}_aE_n^{(\beta)}(x)\frac{t^n}{n!}$	Actuarial-Euler polynomials ${}_aE_n^{(\beta)}(x)$
IX.	$G(t)=1-e^t;\ s(t)=exp(\beta t);$ $a(t)=e^{-t^2/4}$	$exp(\beta t-t^2/4+x(1-e^t))=\sum_{n=0}^{\infty}{}_a\hat{H}_n^{(\beta)}(x)\frac{t^n}{n!}$	Actuarial-monic Hermite polynomials ${}_a\hat{H}_n^{(\beta)}(x)$
X.	$G(t)=ln\left(1+\frac{t}{a}\right);\ s(t)=e^{-t};$ $a(t)=\left(\frac{t}{e^t-1}\right)$	$\left(\frac{t}{e^t-1}\right)e^{-t}(1+\frac{t}{a})^x=\sum_{n=0}^{\infty}{}_cB_n(x;a)\frac{t^n}{n!}$	Poisson-Charlier-Bernoulli polynomials ${}_cB_n(x;a)$
XI.	$G(t)=ln\left(1+\frac{t}{a}\right);\ s(t)=e^{-t};$ $a(t)=\left(\frac{2}{e^t+1}\right)$	$\left(\frac{2}{e^t+1}\right)e^{-t}(1+\frac{t}{a})^x=\sum_{n=0}^{\infty}{}_cE_n(x;a)\frac{t^n}{n!}$	Poisson-Charlier-Euler polynomials ${}_cE_n(x;a)$
XII.	$G(t)=ln\left(1+\frac{t}{a}\right);\ s(t)=e^{-t};$ $a(t)=e^{-t^2/4}$	$e^{-(t+t^2/4)}(1+\frac{t}{a})^x=\sum_{n=0}^{\infty}{}_c\hat{H}_n(x;a)\frac{t^n}{n!}$	Poisson-Charlier-monic Hermite polynomials ${}_c\hat{H}_n(x;a)$

In view of Theorem 2.1, the connection formulas for certain members belonging to the hybrid Sheffer family mentioned in Table 3 are established. We present these formulas in Table 4.

Table 4. Connection formulas for certain members belonging to the hybrid Sheffer family

S. No.	${}_sA_r^{(1)}(x);\ {}_sA_n^{(2)}(x)$	Expression for $h(t)$	Connection formulas
I.	${}_H\hat{H}_{r,m,\nu}(x);\ {}_a\hat{H}_n^{(\beta)}(x)$	$e^{\beta t}\ e^{t^m}$	${}_a\hat{H}_n^{(\beta)}(x)=\sum_{r=0}^{n}\sum_{k=0}^{[\frac{n-r}{m}]}\binom{n}{r}\frac{\beta^{n-r-mk}(n-r)!}{k!\ (n-r-mk)!}\ {}_H\hat{H}_{r,m,\nu}(x)$
II.	${}_a\hat{H}_r^{(\beta)}(x);\ {}_c\hat{H}_n(x;a)$	$e^{(\beta+1)t}$	${}_c\hat{H}_n(x;a)=\sum_{r=0}^{n}\binom{n}{r}(\beta+1)^{n-r}\ {}_a\hat{H}_r^{(\beta)}(x)$
III.	${}_LB_r^{(\alpha)}(x);\ {}_LE_n^{(\alpha)}(x)$	$\frac{2(e^t-1)}{t(e^t+1)}$	${}_LE_n^{(\alpha)}(x)=\sum_{r=0}^{n}\sum_{k=0}^{n-r}\binom{n-r}{k}\frac{E_{n-r-k}}{k+1}\ \frac{1}{(n-r)!}\ {}_LB_r^{(\alpha)}(x)$
IV.	${}_H\hat{H}_{r,m,\nu}(x);\ {}_HB_{n,m,\nu}(x)$	$\frac{t}{e^t-1}\exp(\frac{t^2}{4})$	${}_HB_{n,m,\nu}(x)=\sum_{r=0}^{n}\sum_{k=0}^{[\frac{n-r}{2}]}\binom{n}{r}\frac{B_{n-r-2k}(n-r)!}{k!\ (n-r-2k)!\ 4^k}\ {}_H\hat{H}_{r,m,\nu}(x)$
V.	${}_a\hat{H}_r^{(\alpha)}(x);\ {}_a\hat{H}_n^{(\beta)}(x)$	$\exp((\beta-\alpha)t)$	${}_a\hat{H}_n^{(\beta)}(x)=\sum_{r=0}^{n}\binom{n}{r}(\beta-\alpha)^{n-r}\ {}_a\hat{H}_r^{(\alpha)}(x)$

Similarly, in view of Theorem 2.2, the duplication formulas for certain members belonging to the hybrid Sheffer family mentioned in Table 3 are established. We list these formulas in Table 5.

Table 5. Duplication formulas for certain members belonging to the hybrid Sheffer family

S. No.	${}_sA_n(x)$	Expression for $g(t)$	Duplication formulas
I.	${}_c\hat{H}_n(x;a)$	$\exp(t(b-1))\ \exp\left(\frac{t^2(b^2-1)}{4}\right)$	${}_c\hat{H}_n(bx;a) = \sum_{r=0}^{n}\sum_{k=0}^{[\frac{n-r}{2}]}\binom{n}{r}\frac{b^r\ (b-1)^{n-r-2k}(b^2-1)^k(n-r)!}{(n-r-2k)!\ k!\ 4^k}\ {}_c\hat{H}_r(x;a)$
II.	${}_a\hat{H}_n^{(\beta)}(x)$	$\exp(\beta t(1-b))\ \exp\left(\frac{t^2(b^2-1)}{4}\right)$	${}_a\hat{H}_n^{(\beta)}(bx) = \sum_{r=0}^{n}\sum_{k=0}^{[\frac{n-r}{2}]}\binom{n}{r}\frac{\beta^{n-r-2k}(1-b)^{n-r-2k}(b^2-1)^k b^r(n-r)!}{(n-r-2k)!\ 4^k\ k!}\ {}_a\hat{H}_r^{(\beta)}(x)$
III.	${}_cB_n(x;a)$	$\left(\frac{t}{e^t-1}\right)\left(\frac{e^{bt}-1}{bt}\right)e^{t(1-b)}$	${}_cB_n(bx;a) = \sum_{r=0}^{n}\sum_{k,p=0}^{k+p\leq n-r}\binom{n}{r}\frac{B_{n-r-k-p}\ b^{k+r}(1-b)^p\ (n-r)!}{(n-r-k-p)!\ (k+1)!\ p!}\ {}_cB_r(x;a)$
IV.	${}_aB_n^{(\beta)}(x)$	$\left(\frac{t}{e^t-1}\right)\left(\frac{e^{bt}-1}{bt}\right)e^{\beta t(1-b)}$	${}_aB_n^{(\beta)}(bx) = \sum_{r=0}^{n}\sum_{k,p=0}^{k+p\leq n-r}\binom{n}{r}\frac{B_{n-r-k-p}\ b^{k+r}\beta^p\ (1-b)^p\ (n-r)!}{(n-r-k-p)!\ (k+1)!\ p!}\ {}_aB_r^{(\beta)}(x)$
V.	${}_HB_{n,m,\nu}(x)$	$\left(\frac{t}{e^t-1}\right)\left(\frac{e^{bt}-1}{bt}\right)$ $\times\exp(t^m(b^m-1))$	${}_HB_{n,m,\nu}(bx) = \sum_{r=0}^{n}\sum_{k,p=0}^{k+mp\leq n-r}\binom{n}{r}\frac{B_{n-r-k-mp}\ b^{k+r}\ (b^m-1)^p\ (n-r)!}{(n-r-k-mp)!\ (k+1)!\ p!}\ {}_HB_{r,m,\nu}(x)$

In view of Theorem 3.1, it is evident that the knowledge of the pair $(N\mathcal{M}, B)$ is sufficient for a concrete matrix representation formula for any hybrid Sheffer sequence. Therefore, the following matrix representations of certain polynomials listed in Table 3 are established.

Table 6. Matrix representations of certain members belonging to the hybrid Sheffer family

S. No.	Name and notation of the polynomials	Matrix representation
I.	Generalized Hermite-Bernoulli polynomials $\{_HB_{n,m,\nu}(x)\}_{0\leq n\leq m}$	$\left(e^{-H^m}\left(\sum_{n=0}^{m}\frac{H^n}{(n+1)!}\right)^{-1}, D(\nu)\right)$
II.	Generalized Hermite-Euler polynomials $\{_HE_{n,m,\nu}(x)\}_{0\leq n\leq m}$	$\left(e^{-H^m}2(e^H+1)^{-1}, D(\nu)\right)$
III.	Generalized Hermite-monic Hermite polynomials $\{_H\hat{H}_{n,m,\nu}(x)\}_{0\leq n\leq m}$	$\left(e^{-H^m}e^{-H^2/4}, D(\nu)\right)$
IV.	Generalized Laguerre-Bernoulli polynomials $\{n!\ _LB_n^{(\alpha)}(x)\}_{0\leq n\leq m}$	$\left((1-H)^{-\alpha-1}\left(\sum_{n=0}^{m}\frac{H^n}{(n+1)!}\right)^{-1}, \mathcal{L}D(-1)\right)$
V.	Generalized Laguerre-Euler polynomials $\{n!\ _LE_n^{(\alpha)}(x)\}_{0\leq n\leq m}$	$\left((1-H)^{-\alpha-1}\ 2(e^H+1)^{-1}, \mathcal{L}D(-1)\right)$
VI.	Generalized Laguerre-monic Hermite polynomials $\{n!\ _L\hat{H}_n^{(\alpha)}(x)\}_{0\leq n\leq m}$	$\left((1-H)^{-\alpha-1}\ e^{-H^2/4}, \mathcal{L}D(-1)\right)$
VII.	Actuarial-Bernoulli polynomials $\{_aB_n^{(\beta)}(x)\}_{0\leq n\leq m}$	$\left(P[\beta^n]\left(\sum_{n=0}^{m}\frac{H^n}{(n+1)!}\right)^{-1}, \mathcal{S}D(-1)\right)$
VIII.	Actuarial-Euler polynomials $\{_aE_n^{(\beta)}(x)\}_{0\leq n\leq m}$	$\left(P[\beta^n]\ 2(e^H+1)^{-1}, \mathcal{S}D(-1)\right)$
IX.	Actuarial-monic Hermite polynomials $\{_a\hat{H}_n^{(\beta)}(x)\}_{0\leq n\leq m}$	$\left(P[\beta^n]\ e^{-H^2/4}, \mathcal{S}D(-1)\right)$
X.	Poisson-Charlier-Bernoulli polynomials $\{_cB_n(x;a)\}_{0\leq n\leq m}$	$\left(P[(-1)^n]\left(\sum_{n=0}^{m}\frac{H^n}{(n+1)!}\right)^{-1}, (\mathcal{S}D(a))^{(-1)}\right)$
XI.	Poisson-Charlier-Euler polynomials $\{_cE_n(x;a)\}_{0\leq n\leq m}$	$\left(P[(-1)^n]\ 2(e^H+1)^{-1}, (\mathcal{S}D(a))^{(-1)}\right)$
XII.	Poisson-Charlier-monic Hermite polynomials $\{_c\hat{H}_n(x;a)\}_{0\leq n\leq m}$	$\left(P[(-1)^n]\ e^{-H^2/4}, (\mathcal{S}D(a))^{(-1)}\right)$

We note that other results such as the recurrence relations and multiplication theorem for the special polynomials belonging to the hybrid Sheffer family can also be obtained. The proposed matrix approach is interesting for its remarkable simplicity and sets highlights that the hybrid Sheffer polynomials extend both the Sheffer and Appell polynomials. Further, the connection and duplication formulas for the hybrid special polynomials introduced in this article are important from the point of view of applications in several areas of science. The approach presented in this article is general and can be extended to other hybrid classes of special polynomials.

Acknowledgements. This work has been done under Senior Research Fellowship (Award letter No. F./2014-15/NFO-2014-15-OBC-UTT-24168/(SA-III/Website)) awarded to the second author by the University Grants Commission, Government of India, New Delhi.

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