Certain Subclasses of Bi-univalent Functions Associated with the Chebyshev Polynomials Based on Hohlov Operator

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Abstract

In this paper we introduce and investigate two new subclasses of the function class Σ of bi-univalent functions in the open unit disk, which are associated with the Hohlov operator, and satisfying subordinate conditions. Furthermore, we find estimates on the Taylor-MacLaurin coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses by using Chebyshev polynomials. Several new consequences of these results are also pointed out.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. By $\mathcal{S}$ we will denote the subclass of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. Some of the important and well-investigated subclasses of the class $\mathcal{S}$ include, for example, the class $\mathcal{S}^*(\alpha)$ of *starlike functions of order* α in $\mathbb{U}$, and the class $\mathcal{K}(\alpha)$ of *convex functions of order* α in $\mathbb{U}$, with $0 \leq \alpha < 1$. It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \dots. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if $f(z)$ and $f^{-1}(w)$ are univalent in $\mathbb{U}$, and let Σ denote the class of *bi-univalent functions* in $\mathbb{U}$. The functions $\frac{z}{1-z}$, $-\log(1-z)$, $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ are in the class Σ (see details in [6]). However, the familiar Koebe function is not bi-univalent. Lewin [17] investigated the class of *bi-univalent* functions Σ and obtained a bound $|a_2| \leqq 1.51$. But the coefficient problem for each of the following *Taylor-MacLaurin coefficients* $|a_n|$ $(n \geq 4)$ was an open

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problem (see [4, 5, 6, 17, 20, 33]) until the publication of the article [16] in 2013. The study of *bi-univalent* functions gained momentum mainly due to the work of Srivastava et al. [25]. Due to the pioneering work of Srivastava et al. [25], many researchers (see [1, 2, 3, 12, 13, 19, 25, 26, 28, 31, 32]) investigated several interesting subclasses of the class Σ and found non-sharp estimates on the first two Taylor-MacLaurin coefficients $|a_2|$ and $|a_3|$. Further recently Srivastava et al. [29] defined *m-fold symmetric bi-univalent* function analogues to the concept of *m-fold symmetric univalent* functions and they gave some important results,such as each function $f \in \Sigma$ generates an *m-fold symmetric bi-univalent* function for each $m \in \mathbb{N}$, in their study and found estimates for the first two coefficients of such functions and extend the study to investigate Fekete-Szegö functional problems for functions in these new subclasses(see[27, 30]).

The *convolution* or *Hadamard product* of two functions $f, h \in \mathcal{A}$ is denoted by $f*h$, and is defined by

$$(f*h)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

where f is given by (1.1) and $h(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$. Next, in our present investigation, we need to recall the *convolution operator* $\mathcal{I}_{a,b,c}$ due to Hohlov [14, 15], which is a special case of the *Dziok-Srivastava operator* [10, 9] which itself a special case of the widely-investigated Srivastava-Wright operator [24] (also see[18]).

For the complex parameters a, b and c $(c \neq 0, -1, -2, -3, \dots)$, the *Gaussian hypergeometric function* ${}_2F_1(a, b, c; z)$ is defined as

$$ {}_2F_1(a, b, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = 1 + \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1}} \frac{z^{n-1}}{(n-1)!} \quad (z \in \mathbb{U}), \tag{1.3}$$

where $(\alpha)_n$ is the *Pochhammer symbol* (or the *shifted factorial*) given by

$$(\alpha)_n := \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases} 1, & \text{if} \quad n = 0, \\ \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1), & \text{if} \quad n = 1, 2, 3, \dots . \end{cases}$$

For the real positive values a, b and c, using the Gaussian hypergeometric function (1.3), Hohlov [14, 15] introduced the familiar convolution operator $\mathcal{I}_{a,b,c} : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{I}_{a,b,c} f(z) = [z\, {}_2F_1(a, b, c; z)] * f(z) = z + \sum_{n=2}^{\infty} \varphi_n a_n z^n \quad (z \in \mathbb{U}), \tag{1.4}$$

where

$$\varphi_n = \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (n-1)!}, \tag{1.5}$$

and the function f is of the form (1.1).

Hohlov [14, 15] discussed some interesting geometrical properties exhibited by the operator $\mathcal{I}_{a,b,c}$, and the three-parameter family of operators $\mathcal{I}_{a,b,c}$ contains, as its special cases, most of the known linear integral or differential operators. In particular, if $b = 1$ in (1.4), then $\mathcal{I}_{a,b,c}$ reduces to the *Carlson-Shaffer operator*. Similarly, it is easily seen that the *Hohlov operator* $\mathcal{I}_{a,b,c}$ is also a generalization of the *Ruscheweyh derivative operator* as well as the *Bernardi-Libera-Livingston operator*. It is of interest to note that for $a = c$ and $b = 1$, then $\mathcal{I}_{a,1,a} f = f$, for all $f \in \mathcal{A}$.

2 Definitions and Preliminaries

Chebyshev polynomials(see [11]), which is used by us in this paper, play a considerable act in numerical analysis. We know that the Chebyshev polynomials are four kinds. The most of books and research articles related to specific orthogonal polynomials of Chebyshev family, contain essentially results of Chebyshev polynomials of first and second kinds $T_n(x)$ and $U_n(x)$ and their numerous uses in different applications(see Doha [7] and Mason [21]). The well-known kinds of the Chebyshev polynomials are the first and second kinds. In the case of real variable x on $(-1, 1)$, the first and second kinds are defined by

$$T_n(x) = cosn\theta,$$

$$U_n(x) = \frac{sin(n+1)\theta}{sin\theta}$$

where the subscript n denotes the polynomial degree and where $x = cos\theta$. We note that if $t = cos\alpha$, $\alpha \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$, then

$$\Phi(z,t) = \frac{1}{1-2tz+z^2} = 1 + \sum_{n=1}^{\infty} \frac{sin(n+1)\alpha}{sin\alpha} z^n = 1 + 2cos\alpha z + (3cos^2\alpha - sin^2\alpha)z^2 + \cdots \qquad (z \in \mathbb{U}).$$

Thus, we write

$$\Phi(z,t) = 1 + U_1(t)z + U_2(t)z^2 + ... \qquad (z \in \mathbb{U}, t \in (-1,1))$$

where $U_{n-1} = \frac{sin(narccost)}{\sqrt{1-t^2}}$ for $n \in \mathbb{N}$, are the second kind of the Chebyshev polynomials . Also, it is known that

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$$

and

$$U_1(t) = 2t; \qquad U_2(t) = 4t^2 - 1, \qquad U_3(t) = 8t^3 - 4t, \cdots . \tag{1.6}$$

The Chebyshev polynomials $T_n(t), t \in [-1, 1]$, of the first kind have the generating function of the form

$$\sum_{n=0}^{\infty} T_n(t)z^n = \frac{1-tz}{1-2tz+z^2} \qquad (z \in \mathbb{U}).$$

All the same, the Chebyshev polynomials of the first kind $T_n(t)$ and the second kind $U_n(t)$ are well connected by the following relationship

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t),$$

$$T_n(t) = U_n(t) - tU_{n-1}(t),$$

$$2T_n(t) = U_n(t) - U_{n-2}(t).$$

Motivated by the earlier work of Deniz [8], Peng et al. [23] (see also [22, 28]), and Altınkaya and Yalçın [3], in the present paper we introduce new subclasses of the function class Σ, involving

Hohlov operator $\mathcal{I}_{a,b,c}$, and we find estimates on the coefficients $|a_2|$ and $|a_3|$ for the functions that belong to these new subclasses of functions of the class Σ by using Chebyshev polynomials. Several related classes are also considered, and connection to earlier known results are made.

Definition 2.1. For $0 \leq \lambda \leq 1;\ t \in (-1,1)$ a function $f \in \Sigma$ is said to be in the class $\mathcal{G}_{\Sigma}^{a,b,c}(\lambda, \Phi)$ if the following two conditions are satisfied:

$$\frac{z\left(\mathcal{I}_{a,b,c}f(z)\right)'}{(1-\lambda)z + \lambda \mathcal{I}_{a,b,c}f(z)} \prec \Phi(z,t) \tag{1.7}$$

and

$$\frac{w\left(\mathcal{I}_{a,b,c}g(w)\right)'}{(1-\lambda)w + \lambda \mathcal{I}_{a,b,c}g(w)} \prec \Phi(w,t) \tag{1.8}$$

where the function g is given by (1.2), and $z, w \in \mathbb{U}$.

Definition 2.2. For $0 \leq \lambda \leq 1;\ t \in (-1,1)$ a function $f \in \Sigma$ is said to be in the class $\mathcal{M}_{\Sigma}^{a,b,c}(\lambda, \Phi)$ if it satisfies the following two conditions:

$$\frac{z\left(\mathcal{I}_{a,b,c}f(z)\right)' + z^2\left(\mathcal{I}_{a,b,c}f(z)\right)''}{(1-\lambda)z + \lambda z\left(\mathcal{I}_{a,b,c}f(z)\right)'} \prec \Phi(z,t) \tag{1.9}$$

and

$$\frac{w\left(\mathcal{I}_{a,b,c}g(w)\right)' + w^2\left(\mathcal{I}_{a,b,c}g(w)\right)''}{(1-\lambda)w + \lambda w\left(\mathcal{I}_{a,b,c}g(w)\right)'} \prec \Phi(w,t), \tag{1.10}$$

where the function g is given by (1.2), and $z, w \in \mathbb{U}$.

On specializing the parameters λ and $t \in (-1,1)$ one can state the various new subclasses of Σ as illustrated in the following remarks. Thus, taking $\lambda = 1$ in the above two definitions, we obtain:

Remark 2.3. (i) A function $f \in \Sigma$ is said to be in the class $\mathcal{S}_{\Sigma}^{a,b,c}(\Phi)$ if the following conditions are satisfied:

$$\frac{z\left(\mathcal{I}_{a,b,c}f(z)\right)'}{\mathcal{I}_{a,b,c}f(z)} \prec \Phi(z,t) \quad \text{and} \quad \frac{w\left(\mathcal{I}_{a,b,c}g(w)\right)'}{\mathcal{I}_{a,b,c}g(w)} \prec \Phi(w,t)$$

where $g = f^{-1}$ and $z, w \in \mathbb{U}$.

(ii) A function $f \in \Sigma$ is said to be in the class $\mathcal{K}_{\Sigma}^{a,b,c}(\Phi)$ if it satisfies the following conditions:

$$1 + \frac{z\left(\mathcal{I}_{a,b,c}f(z)\right)''}{\left(\mathcal{I}_{a,b,c}f(z)\right)'} \prec \Phi(z,t) \quad \text{and} \quad 1 + \frac{w\left(\mathcal{I}_{a,b,c}g(w)\right)''}{\left(\mathcal{I}_{a,b,c}g(w)\right)'} \prec \Phi(w,t)$$

where $g = f^{-1}$ and $z, w \in \mathbb{U}$.

Taking $\lambda = 0$ in the previous two definitions, we obtain the next special cases:

Remark 2.4. (i) A function $f \in \Sigma$ is said to be in the class $\mathcal{H}_{\Sigma}^{a,b,c}(\Phi)$ if the following conditions are satisfied:

$$\left(\mathcal{I}_{a,b,c}f(z)\right)' \prec \Phi(z,t) \quad \text{and} \quad \left(\mathcal{I}_{a,b,c}g(w)\right)' \prec \Phi(w,t)$$

where $g = f^{-1}$ and $z, w \in \mathbb{U}$.

(ii) A function $f \in \Sigma$ is said to be in the class $\mathcal{Q}_{\Sigma}^{a,b,c}(\Phi)$ if it satisfies the following conditions:

$$\begin{aligned}&(\mathcal{I}_{a,b,c}f(z))' + z\,(\mathcal{I}_{a,b,c}f(z))'' \prec \Phi(z,t) \quad \text{and}\\ &(\mathcal{I}_{a,b,c}g(w))' + w\,(\mathcal{I}_{a,b,c}g(w))'' \prec \Phi(w,t)\end{aligned}$$

where $g = f^{-1}$ and $z, w \in \mathbb{U}$.

In particular, for $a = c$ and $b = 1$, we note that $\mathcal{I}_{a,1,a}f = f$ for all $f \in \mathcal{A}$, and thus, for $\lambda = 1$ and $\lambda = 0$ the classes $\mathcal{S}_{\Sigma}^{a,b,c}(\Phi)$ and $\mathcal{K}_{\Sigma}^{a,b,c}(\Phi)$ reduces to the following subclasses of Σ in Remark 2.5 and Remark 2.6 respectively:

Remark 2.5. (i) A function $f \in \Sigma$ is said to be in the class $\mathcal{S}_{\Sigma}^{*}(\Phi)$ if the following conditions are satisfied:

$$\frac{zf'(z)}{f(z)} \prec \Phi(z,t) \quad \text{and} \quad \frac{wg'(w)}{g(w)} \prec \Phi(w,t),$$

where $g = f^{-1}$ and $z, w \in \mathbb{U}$.

(ii) A function $f \in \Sigma$ is said to be in the class $\mathcal{K}_{\Sigma}(\Phi)$ if the following conditions are satisfied:

$$1 + \frac{zf''(z)}{f'(z)} \prec \Phi(z,t) \quad \text{and} \quad 1 + \frac{wg''(w)}{g'(w)} \prec \Phi(w,t),$$

where $g = f^{-1}$ and $z, w \in \mathbb{U}$.

Remark 2.6. (i) A function $f \in \Sigma$ is said to be in the class $\mathcal{H}_{\Sigma}(\Phi)$ if the following conditions are satisfied:

$$f'(z) \prec \Phi(z,t) \quad \text{and} \quad g'(w) \prec \Phi(z,t),$$

where $g = f^{-1}$ and $z, w \in \mathbb{U}$.

(ii) A function $f \in \Sigma$ is said to be in the class $\mathcal{Q}_{\Sigma}(\Phi)$ if the following conditions are satisfied:

$$f'(z) + zf''(z) \prec \Phi(z,t) \quad \text{and} \quad g'(w) + wg''(w) - 1 \prec \Phi(w,t),$$

where $g = f^{-1}$ and $z, w \in \mathbb{U}$.

In order to derive our main results, we shall need the following :

In the following section we find estimates of the coefficients $|a_2|$ and $|a_3|$ for functions of the above-defined subclasses $\mathcal{G}_{\Sigma}^{a,b,c}(\lambda,\Phi)$ and $\mathcal{M}_{\Sigma}^{a,b,c}(\lambda,\Phi)$ of the function class Σ.

3 Coefficient Bounds for the Function Class $\mathcal{G}_{\Sigma}^{a,b,c}(\lambda,\Phi)$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions belonging to the class $\mathcal{G}_{\Sigma}^{a,b,c}(\lambda,\Phi)$.

Supposing that the functions

$$u(z) = c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{U}), \tag{1.11}$$

$$v(w) = d_1 w + d_2 w^2 + \cdots \quad (w \in \mathbb{U}), \tag{1.12}$$

are analytic in $\mathbb{U}$ with $u(0) = 0 = v(0)$ and $|u(z)| < 1$, $|v(w)| < 1$, for all $z, w \in \mathbb{U}$. It is well-known that

$$|u(z)| = |c_1 z + c_2 z^2 + \cdots| < 1 \qquad \text{and} \qquad |v(w)| = |d_1 w + d_2 w^2 + \cdots| < 1, z, w \in \mathbb{U}, \tag{1.13}$$

then for all $k \geq 1$,

$$|c_k| \leq 1, \tag{1.14}$$
$$|d_k| \leq 1 \quad . \tag{1.15}$$

Theorem 3.1. If the function f given by (1.1) belongs to the class $\mathcal{G}_\Sigma^{a,b,c}(\lambda, \Phi)$ and $t \in (0,1)$, then

$$|a_2| \leq \min \left\{ \frac{2t\sqrt{2t}}{\sqrt{\left|(2-\lambda)^2 \varphi_2^2 + [(3-\lambda)\varphi_3 - 2(2-\lambda)\varphi_2^2]4t^2\right|}}; \frac{2t}{(2-\lambda)\varphi_2} \right\} \tag{1.16}$$

and

$$|a_3| \leq \min \left\{ \frac{2t}{(3-\lambda)\varphi_3} + \frac{8t^3}{\left|(2-\lambda)^2 \varphi_2^2 + [(3-\lambda)\varphi_3 - 2(2-\lambda)\varphi_2^2]4t^2\right|}; \right.$$
$$\frac{1}{(3-\lambda)\varphi_3}\left(2t + \left|1 - \frac{8t^2}{(2-\lambda)}\right|\right).;$$
$$\left. \frac{1}{(3-\lambda)\varphi_3}\left(2t + \left|1 + \frac{(\lambda-2)\varphi_2^2 + (3-\lambda)\varphi_3}{(2-\lambda)^2\varphi_2^2} 8t^2\right|\right) \right\}, \tag{1.17}$$

where φ_2 and φ_3 are given by (1.5).

Proof. Since $f \in \mathcal{G}_\Sigma^{a,b,c}(\lambda, \Phi)$, from the definition relations (1.7) and (1.8) it follows that

$$\frac{z\left(\mathcal{I}_{a,b,c} f(z)\right)'}{(1-\lambda)z + \lambda \mathcal{I}_{a,b,c} f(z)} \quad =: 1 + U_1(t)u(z) + U_2(t)u^2(z) + \cdots, \tag{1.18}$$

and

$$\frac{w\left(\mathcal{I}_{a,b,c} g(w)\right)'}{(1-\lambda)w + \lambda \mathcal{I}_{a,b,c} g(w)} \quad =: 1 + U_1(t)v(w) + U_2(t)v^2(w) + \cdots. \tag{1.19}$$

$$\begin{aligned} &1 + (2-\lambda)\varphi_2 a_2 z + \left[(\lambda^2 - 2\lambda)\varphi_2^2 a_2^2 + (3-\lambda)\varphi_3 a_3\right] z^2 + \ldots \\ &\quad =: 1 + U_1(t)c_1 z + [U_1(t)c_2 + U_2(t)c_1^2]z^2 + \cdots, \end{aligned} \tag{1.20}$$

$$\begin{aligned} &1 - (2-\lambda)\varphi_2 a_2 w + \left[(\lambda^2 - 2\lambda)\varphi_2^2 a_2^2 + (3-\lambda)\varphi_3 \left(2a_2^2 - a_3\right)\right] w^2 + \ldots \\ &\quad =: 1 + U_1(t)d_1 w + [U_1(t)d_2 + U_2(t)d_1^2]w^2 + \cdots. \end{aligned} \tag{1.21}$$

Now, equating the coefficients in (1.20) and (1.21), we get

$$(2-\lambda)\varphi_2 a_2 = U_1(t)c_1, \tag{1.22}$$
$$(\lambda^2-2\lambda)\varphi_2^2 a_2^2 + (3-\lambda)\varphi_3 a_3 = U_1(t)c_2 + U_2(t)c_1^2, \tag{1.23}$$

and

$$-(2-\lambda)\varphi_2 a_2 = U_1(t)d_1, \tag{1.24}$$
$$(\lambda^2-2\lambda)\varphi_2^2 a_2^2 + (3-\lambda)\varphi_3\left(2a_2^2 - a_3\right) = U_1(t)d_2 + U_2(t)d_1^2. \tag{1.25}$$

From (1.22) and (1.24), we find that

$$a_2 = \frac{U_1(t)c_1}{(2-\lambda)\varphi_2} = \frac{-U_1(t)d_1}{(2-\lambda)\varphi_2}, \tag{1.26}$$

which implies

$$c_1 = -d_1$$

further

$$a_2^2 = \frac{U_1^2(t)[c_1^2+d_1^2]}{2(2-\lambda)^2\varphi_2^2}. \tag{1.27}$$

By using (1.6), (1.14) and (1.15), we get

$$|a_2| \leq \frac{2t}{(2-\lambda)\varphi_2}, \tag{1.28}$$

Adding (1.23) and (1.25), by using (1.27) we obtain

$$\left[2\left(\lambda^2-2\lambda\right)\varphi_2^2 + 2(3-\lambda)\varphi_3\right]a_2^2 = U_1(t)(c_2+d_2) + U_2(t)(c_1^2+d_1^2).$$

$$\left[2\left(\lambda^2-2\lambda\right)\varphi_2^2 + 2(3-\lambda)\varphi_3 - \frac{2(2-\lambda)^2\varphi_2^2 U_2(t)}{U_1^2(t)}\right]a_2^2 = U_1(t)(c_2+d_2).$$

Now, by using (1.6),(1.14) and (1.15), we get

$$|a_2|^2 \leq \frac{8t^3}{\left|(2-\lambda)^2\varphi_2^2 + [(3-\lambda)\varphi_3 - 2(2-\lambda)\varphi_2^2]4t^2\right|}, \tag{1.29}$$

hence

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{\left|(2-\lambda)^2\varphi_2^2 + [(3-\lambda)\varphi_3 - 2(2-\lambda)\varphi_2^2]4t^2\right|}},$$

which gives the bound on $|a_2|$ as asserted in (1.16).

Next, in order to find the upper-bound for $|a_3|$, by subtracting (1.25) from (1.23), we get

$$2(3-\lambda)\varphi_3 a_3 = U_1(t)(c_2-d_2) + 2(3-\lambda)\varphi_3 a_2^2. \tag{1.30}$$

It follows from (1.14),(1.15), (1.29) and (1.30), that

$$|a_3| \leq \frac{2t}{(3-\lambda)\varphi_3} + \frac{8t^3}{\left|(2-\lambda)^2\varphi_2^2 + [(3-\lambda)\varphi_3 - 2(2-\lambda)\varphi_2^2]4t^2\right|}.$$

From (1.22) and (1.23) we have

$$a_3 = \frac{1}{(3-\lambda)\varphi_3}\left(U_1(t)c_2 + U_2(t)c_1^2 - \frac{(\lambda^2-2\lambda)U_1^2(t)}{(2-\lambda)^2}c_1^2\right),$$

hence

$$|a_3| \leq \frac{1}{(3-\lambda)\varphi_3}\left(2t + \left|1 - \frac{8t^2}{(2-\lambda)}\right|\right).$$

Further, from (1.22) and (1.25) we deduce that

$$|a_3| \leq \frac{1}{(3-\lambda)\varphi_3}\left(2t + \left|1 + \frac{(\lambda-2)\varphi_2^2 + (3-\lambda)\varphi_3}{(2-\lambda)^2\varphi_2^2}8t^2\right|\right),$$

and thus we obtain the conclusion (1.17) of our theorem.

For the special cases $\lambda = 1$ and $\lambda = 0$, the Theorem 3.1 reduces to the following corollaries, respectively:

Corollary 3.2. If the function f given by (1.1) belongs to the class $\mathcal{S}_{\Sigma}^{a,b,c}(\Phi)$, then

$$|a_2| \leq \min\left\{\frac{2t\sqrt{2t}}{\sqrt{|\varphi_2^2 + [\varphi_3 - \varphi_2^2]8t^2|}}; \frac{2t}{\varphi_2}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{t}{\varphi_3} + \frac{8t^3}{|\varphi_2^2 + (\varphi_3 - \varphi_2^2)8t^2|}; \frac{1}{2\varphi_3}\left(2t + \left|1 - 8t^2\right|\right); \frac{1}{2\varphi_3}\left(2t + \left|1 + \frac{2\varphi_3 - \varphi_2^2}{\varphi_2^2}8t^2\right|\right)\right\}$$

where $t \in (0,1)$, φ_2 and φ_3 are given by (1.5).

Corollary 3.3. If the function f given by (1.1) belongs to the class $\mathcal{H}_{\Sigma}^{a,b,c}(\Phi)$, then

$$|a_2| \leq \min\left\{\frac{t\sqrt{2t}}{\sqrt{|\varphi_2^2 + [3\varphi_3 - 4\varphi_2^2]t^2|}}; \frac{t}{\varphi_2}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{2t}{3\varphi_3} + \frac{2t^3}{|\varphi_2^2 + (3\varphi_3 - 4\varphi_2^2)t^2|}; \frac{1}{3\varphi_3}\left(2t + \left|1 - 4t^2\right|\right); \frac{1}{3\varphi_3}\left(2t + \left|1 + \frac{3\varphi_3 - 2\varphi_2^2}{\varphi_2^2}2t^2\right|\right)\right\},$$

where $t \in (0,1)$, φ_2 and φ_3 are given by (1.5).

4 Coefficient Bounds for the Function Class $\mathcal{M}_{\Sigma}^{a,b,c}(\lambda,\Phi)$

Theorem 4.1. If the function f given by (1.1) belongs to the class $\mathcal{M}_{\Sigma}^{a,b,c}(\lambda,\Phi)$, then

$$|a_2| \leq \min\left\{\frac{t\sqrt{2t}}{\sqrt{(2-\lambda)^2\varphi_2^2+[3(3-\lambda)\varphi_3-8(2-\lambda)\varphi_2^2]t^2}};\frac{t}{(2-\lambda)\varphi_2}\right\} \tag{1.31}$$

and

$$\begin{aligned}|a_3| \leq \min\Bigg\{&\frac{1}{3(3-\lambda)\varphi_3}\left(2t+\left|1-\frac{8t^2}{(2-\lambda)}\right|\right);\\ &\frac{2t}{3(3-\lambda)\varphi_3}+\frac{2t^3}{|(2-\lambda)^2\varphi_2^2+[3(3-\lambda)\varphi_3-8(2-\lambda)\varphi_2^2]t^2|};\\ &\frac{1}{3(3-\lambda)\varphi_3}\left(1+\frac{3(3-\lambda)\varphi_3-2(2-\lambda)\varphi_2^2}{2(2-\lambda)^2\varphi_2^2}2t^2\right)+\frac{2t}{3(3-\lambda)\varphi_3}\Bigg\},\end{aligned} \tag{1.32}$$

where $t\in(0,1)$,φ_2 and φ_3 are given by (1.5).

Proof. For $f\in\mathcal{M}_{\Sigma}^{a,b,c}(\lambda,\Phi)$, from the definition relations (1.9) and (1.10) it follows that

$$\begin{aligned}&\frac{z\left(\mathcal{I}_{a,b,c}f(z)\right)'+z^2\left(\mathcal{I}_{a,b,c}f(z)\right)''}{(1-\lambda)z+\lambda z\left(\mathcal{I}_{a,b,c}f(z)\right)'}=1+U_1(t)u(z)+U_2(t)u^2(z)+\cdots,\\ &1+2(2-\lambda)\varphi_2a_2z+\left[4(\lambda^2-2\lambda)\varphi_2^2a_2^2+3(3-\lambda)\varphi_3a_3\right]z^2+\dots\\ &\qquad=1+U_1(t)c_1z+[U_1(t)c_2+U_2(t)c_1^2]z^2+\cdots,\end{aligned} \tag{1.33}$$

and

$$\begin{aligned}&\frac{w\left(\mathcal{I}_{a,b,c}g(w)\right)'+w^2\left(\mathcal{I}_{a,b,c}g(w)\right)''}{(1-\lambda)w+\lambda z\left(\mathcal{I}_{a,b,c}g(w)\right)'}=1+U_1(t)v(w)+U_2(t)v^2(w)+\cdots,\\ &1-2(2-\lambda)\varphi_2a_2w+\left[4(\lambda^2-2\lambda)\varphi_2^2a_2^2+3(3-\lambda)\varphi_3(2a_2^2-a_3)\right]w^2+\dots\\ &\qquad=:1+U_1(t)d_1w+[U_1(t)d_2+U_2(t)d_1^2]w^2+\cdots.\end{aligned} \tag{1.34}$$

Now, equating the coefficients in (1.33) and (1.34), we get

$$2(2-\lambda)\varphi_2a_2=U_1(t)c_1, \tag{1.35}$$

$$4(\lambda^2-2\lambda)\varphi_2^2a_2^2+3(3-\lambda)\varphi_3a_3=U_1(t)c_2+U_2(t)c_1^2, \tag{1.36}$$

and

$$-2(2-\lambda)\varphi_2a_2=U_1(t)d_1, \tag{1.37}$$

$$4(\lambda^2-2\lambda)\varphi_2^2a_2^2+3(3-\lambda)(2a_2^2-a_3)\varphi_3=U_1(t)d_2+U_2(t)d_1^2. \tag{1.38}$$

From (1.35)and (1.37) we get

$$a_2=\frac{U_1(t)c_1}{2(2-\lambda)\varphi_2}=-\frac{U_1(t)d_1}{2(2-\lambda)\varphi_2}, \tag{1.39}$$

which implies

$$c_1 = -d_1$$

further

$$a_2^2 = \frac{U_1^2(t)[c_1^2 + d_1^2]}{8(2-\lambda)^2\varphi_2^2}. \tag{1.40}$$

According to (1.6),(1.14) and (1.15), we easily deduce

$$|a_2| \leq \frac{t}{(2-\lambda)\varphi_2}.$$

By adding (1.36) and (1.38),

$$8(\lambda^2 - 2\lambda)\varphi_2^2 a_2^2 + 6(3-\lambda)\varphi_3 a_2^2 = (c_2 + d_2)U_1(t) + (c_1^2 + d_1^2)U_2(t)$$

and using (1.40) we get

$$[8(\lambda^2 - 2\lambda)\varphi_2^2 + 6(3-\lambda)\varphi_3]a_2^2 = (c_2 + d_2)U_1(t) + \frac{8(2-\lambda)^2\varphi_2^2 a_2^2\, U_2(t)}{U_1^2(t)}$$

According to (1.6),(1.14) and (1.15), we easily deduce

$$|a_2|^2 \leq \frac{2t^3}{|(2-\lambda)^2\varphi_2^2 + [3(3-\lambda)\varphi_3 - 8(2-\lambda)\varphi_2^2]t^2|}. \tag{1.41}$$

which yields

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|(2-\lambda)^2\varphi_2^2 + [3(3-\lambda)\varphi_3 - 8(2-\lambda)\varphi_2^2]|t^2}}.$$

Next, in order to find the upper-bound for $|a_3|$, from (1.36), by using (1.39) we have

$$|a_3| \leq \frac{1}{3(3-\lambda)\varphi_3}\left(2t + \left|1 - \frac{8t^2}{2-\lambda}\right|\right).$$

Subtracting (1.38) and (1.36) we obtain

$$6(3-\lambda)\varphi_3 a_3 - 6(3-\lambda)a_2^2\varphi_3 = U_1(t)(c_2 - d_2),$$

and using (1.41) we deduce

$$|a_3| \leq \frac{2t}{3(3-\lambda)\varphi_3} + \frac{2t^3}{|(2-\lambda)^2\varphi_2^2 + [3(3-\lambda)\varphi_3 - 8(2-\lambda)\varphi_2^2]t^2|}.$$

Finally, from (1.38) and using (1.39) we get

$$|a_3| \leq \frac{1}{3(3-\lambda)\varphi_3}\left(1 + \frac{3(3-\lambda)\varphi_3 - 2(2-\lambda)\varphi_2^2}{(2-\lambda)^2\varphi_2^2}2t^2\right) + \frac{2t}{3(3-\lambda)\varphi_3}.$$

Taking $\lambda = 1$ and $\lambda = 0$ in Theorem 3.1, we obtain the following corollaries, respectively:

Corollary 4.2. If the function f given by (1.1) belongs to the class $\mathcal{K}_{\Sigma}^{a,b,c}(\Phi)$, then

$$|a_2| \leq \min\left\{\frac{t\sqrt{2t}}{\sqrt{\varphi_2^2+[6\varphi_3-8\varphi_2^2]t^2}}; \frac{t}{\varphi_2}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{t}{3\varphi_3}+\frac{1}{6\varphi_3}\left|1-8t^2\right|;\ \frac{t}{3\varphi_3}+\frac{2t^3}{\varphi_2^2+[6\varphi_3-8\varphi_2^2]t^2};\ \frac{1}{6\varphi_3}\left(1+\frac{6\varphi_3-2\varphi_2^2}{\varphi_2^2}2t^2\right)+\frac{t}{3\varphi_3}\right\},$$

where $t \in (0,1)$, φ_2 and φ_3 are given by (1.5).

Corollary 4.3. If the function f given by (1.1) belongs to the class $\mathcal{Q}_{\Sigma}(\Phi)$, then

$$|a_2| \leq \min\left\{\frac{t\sqrt{2t}}{\sqrt{4\varphi_2^2+[9\varphi_3-16\varphi_2^2]t^2}}; \frac{t}{2\varphi_2}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{1}{9\varphi_3}\left(2t+\left|1-4t^2\right|\right);\ \frac{2t}{9\varphi_3}+\frac{2t^3}{4\varphi_2^2+[9\varphi_3-16\varphi_2^2]t^2};\ \frac{1}{9\varphi_3}\left(1+\frac{9\varphi_3-4\varphi_2^2}{2\varphi_2^2}t^2\right)+\frac{2t}{9\varphi_3}\right\},$$

where $t \in (0,1)$, φ_2 and φ_3 are given by (1.5).

5 Corollaries and its Consequence

Remark 5.1. For $a = c;\ b = 1$, we have $\varphi_n = 1$ for all $n \geq 1$ and $t \in (0,1)$ in Corollary 3.2 and Corollary 3.3 we obtain more accurate results for the classes in Remarks 2.5 and 2.6.

Corollary 5.2. If the function f given by (1.1) belongs to the class $\mathcal{S}_{\Sigma}^{*}(\Phi)$ and $t \in (0,1)$, then

$$|a_2| \leq 2t\sqrt{2t}$$

and

$$|a_3| \leq \min\left\{t+8t^3; t+\frac{1}{2}\left|1-8t^2\right|; t+\frac{1}{2}\left|1+8t^2\right|\right\}$$

where φ_2 and φ_3 are given by (1.5).

Corollary 5.3. If the function f given by (1.1) belongs to the class $\mathcal{H}_{\Sigma}(\Phi)$,and for $t \in (0,1)$ then

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|1-t^2|}}$$

and

$$|a_3| \leq \min\left\{\frac{2t}{3}+\frac{2t^3}{|1-t^2|}; \frac{1}{3}\left(2t+\left|1-4t^2\right|\right); \frac{1}{3}\left(2t+\left|1+2t^2\right|\right)\right\},$$

where φ_2 and φ_3 are given by (1.5).

Corollary 5.4. If the function f given by (1.1) belongs to the class $\mathcal{K}_\Sigma(\Phi)$ and for $t \neq \frac{1}{\sqrt{2}}$, then

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|1-2t^2|}}$$

and

$$|a_3| \leq \min\left\{\frac{t}{3} + \frac{1}{6}|1-8t^2|;\ \frac{t}{3} + \frac{2t^3}{|1-2t^2|};\ \frac{t}{3} + \frac{1}{6}\left(1+8t^2\right)\right\},$$

where φ_2 and φ_3 are given by (1.5).

Corollary 5.5. If the function f given by (1.1) belongs to the class $\mathcal{Q}_\Sigma(\Phi)$ and for $t \neq \frac{2}{\sqrt{7}}$, then

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|4-7t^2|}}$$

and

$$|a_3| \leq \min\left\{\frac{1}{9}\left(2t + \left|1-4t^2\right|\right);\ \frac{2t}{9} + \frac{2t^3}{|4-7t^2|};\ \frac{2t}{9} + \frac{1}{9}\left(1+\frac{5}{2}t^2\right)\right\},$$

where φ_2 and φ_3 are given by (1.5).

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