A sinc-Gauss-Jacobi collocation method for solving Volterra's population growth model with fractional order

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Abstract

A new sinc-Gauss-Jacobi collocation method for solving the fractional Volterra's population growth model in a closed system is proposed. This model is a nonlinear fractional Volterra integro-differential equation where the integral term represents the effects of toxin. The fractional derivative is considered in the Liouville-Caputo sense. In the proposed method, we first convert fractional Volterra's population model to an equivalent nonlinear fractional differential equation, and then the resulting problem is solved using collocation method. The proposed collocation technique is based on sinc functions and Gauss-Jacobi quadrature rule. In this approach, the problem is reduced to a set of algebraic equations. The obtained numerical results of the present method are compared with some well-known results in the literature to show the applicability and efficiency of the proposed method.

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1 Introduction

Fractional calculus, as generalization of integer order integration and differentiation to its non-integer (fractional) order counterpart, is a fast developing field in engineering, physics, biology, quantum, applied mathematics and etc. (e.g., see [14, 23, 28] and references therein). A history of the development of fractional (derivatives an integrals) operators can be found in [16]. Most problems containing fractional derivatives, either do not have closed form solutions or the exact solutions have very complex forms, so numerical techniques for these problems are extensively developed. Among the most recent works concerned with the numerical solution of fractional differential equations and fractional integro-differential equations we can consider papers [1, 4, 6, 7, 8, 13, 19, 22, 29, 31, 32, 33].

The fractional population growth model of a species within a closed system is given in [17, 25] as

$$D^{\beta}p = ap - bp^2 - cp\int_0^{\widetilde{t}} p(x)dx, \qquad p(0) = p_0, \quad 0 < \beta \leq 1. \tag{1.1}$$

Here, $p = p(\widetilde{t})$ denotes the scaled population of identical individuals at time $\widetilde{t}$, $a > 0$ represents the birth rate coefficient, $b > 0$ is the crowding coefficient, $c > 0$ is the toxicity coefficient and p_0 is the initial population. The coefficient c indicates the essential behavior of the population evolution before its level falls to zero in the long term [38]. It is interesting to note that when

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$c = 0$, Eq. (1.1) reduces to the well-known logistic equation. Also, $p\int_0^{\widetilde{t}} p(x)dx$, represented the the "total metabolism" or total amount of toxins accumulated from time zero [38]. Moreover, μ represents the order of fractional derivative. The fractional derivative in Eq. (1.1) is considered in the Liouville-Caputo sense. Introducing the non-dimensional variables by

$$t = \frac{\widetilde{t}c}{b}, \quad u = \frac{pb}{a}, \tag{1.2}$$

Eq. (1.1) is reduced to the non-dimensional problem

$$\kappa D^{\beta}u(t) = u(t) - u^2(t) - u(t)\int_0^t u(x)dx, \qquad u(0) = u_0, \;\; 0 < \beta \leq 1, \tag{1.3}$$

where $\kappa = c/ab$ is a non-dimensional parameter. It is important to notice that when $\beta = 1$, Eq. (1.3) reduces to the classical logistic growth model, and the numerical methods for this equation have been extensively studied by many authors (e.g., see [3, 12, 18, 24, 26] and references therein). The analytical solution for classical logistic growth model is [26]

$$u(t) = u_0 \exp\left(\frac{1}{\kappa}\int_0^t \left[1 - u(\tau) - \int_0^{\tau} u(x)dx\right] d\tau\right). \tag{1.4}$$

Some researchers have worked on problem(1.3); For instance, Parand and Delkhosh [25] by generalized fractional order Chebyshev functions, Parand and Nikarya [27] by the Bessel collocation method, Maleki et al. [17] by multi-domain pseudospectral method, Momani and Qaralleh [20] by Adomian decomposition method and Pade approximation, Yüzbaşi [41] by Bessel collocation method and Xu [39] and Ghasemi et al. [9] by homotopy analysis method.

In the past three decades or so, many researchers used sinc approximation in various problems such as boundary value problems [34, 40], squeezing flow [30], fractional convection-diffusion equations [29], time-fractional order telegraph equation [37], fractional diffusion equation [11], Troesch's problem [21], Bagley-Torvik equation [5] and fractional order boundary value problem [2].

In this work, we intend to extend the application of sinc functions to solve the problem (1.3), for $0 < \beta < 1$. It is worthy to mention here that, in [26], a collocation approach using sinc functions is applied to problem (1.3) for $\beta = 1$. Our method consists of reducing the solution of fractional population growth model to a set of algebraic equations by expanding the candidate function in terms of sinc functions with unknown coefficients. The collocation method, the properties of sinc functions and the Gauss-Jacobi quadrature rule are then used to evaluate the unknown coefficients and find the solution of problem(1.3). To the best knowledge of the authors, until now, such approach has not been employed for solving problem(1.3), for $0 < \beta < 1$.

This paper is arranged as follows: in the next section, some preliminary results of fractional calculus, sinc functions and Gauss-Jacobi quadrature rule are given. In Section 3, the new method proposed in the current work is presented. Results and discussion of the proposed method is shown in Section 4. Also, we compare our achievements with existing results in other published works in the literature. Section 5 contains a brief conclusion.

2 Preliminaries and notations

In this section, we present some basic definitions and preliminary materials which will be used throughout the paper.

2.1 The fractional derivative in the Liouville-Caputo sense

Definition 1. *Let $\beta > 0$ and $n = [\beta] + 1$, then the Liouville-Caputo fractional derivative of order β is defined as [14, 28]*

$$D^{\beta} f(x) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\beta+1-n}} dt, & n-1 < \beta < n, \ \ n \in \mathbb{N}, \\ \frac{d^n}{dx^n} f(x), & \beta = n \in \mathbb{N}. \end{cases} \tag{1.5}$$

where β is the order of the derivative, $\Gamma(.)$ is the Gamma function and $[\beta]$ denoting the integer part of β. Some properties of Liouville-Caputo's derivative are mentioned below [14].

- $D^{\beta} C = 0, \quad$ (C is a constant).
- For any $\beta > 0$ and any nonnegative integer m, we have the relation

$$D^{\beta} \left(D^m f(x) \right) = D^{m+\beta} f(x). \tag{1.6}$$

- The Liouville-Caputo's derivative of $f(x) = x^m, m \in \mathbb{N}$ is given as

$$D^{\beta} x^m = \begin{cases} 0, & m < \lceil \beta \rceil, \\ \frac{\Gamma(m+1)}{\Gamma(m+1-\beta)} x^{m-\beta}, & m \geq \lceil \beta \rceil, \end{cases} \tag{1.7}$$

  where $\lceil \beta \rceil$ denoting the smallest integer greater than or equal to β.
- Liouville-Caputo's fractional differentiation is a linear operator, i.e.,

$$D^{\beta}(a_1 f_1(x) + a_2 f_2(x)) = a_1 D^{\beta} f_1(x) + a_2 D^{\beta} f_2(x), \tag{1.8}$$

  where a_1 and a_2 are constants.

2.2 Sinc function approximation

A general review of sinc function approximation is given in [15, 36]. We recall here the main properties of sinc functions which will be used in the sequel. As is well known, the Whittaker cardinal (sinc) function is defined on $-\infty < x < \infty$, by

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Also, for any $h > 0$ and $k = 0, \pm 1, \pm 2, \ldots$, the translated sinc functions with evenly spaced nodes are given by

$$S(k,h)(x) = \operatorname{sinc}\left(\frac{x - kh}{h} \right) = \begin{cases} \frac{\sin[\frac{\pi}{h}(x-kh)]}{\frac{\pi}{h}(x-kh)}, & x \neq kh, \\ 1, & x = kh. \end{cases} \tag{1.9}$$

It is easy to see that

$$S(k,h)(jh) = \delta_{kj} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}$$

Let f be an analytic function defined on the real axis. Then the series

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh)\,\mathrm{Sinc}\left(\frac{x-kh}{h}\right),$$

is called the Whittaker cardinal expansion of f whenever this series converges. Most properties of the Whittaker cardinal expansion may be found in [15]. To construct approximations on the interval $(0,\infty)$, which is used in this paper, we consider the conformal mappings

$$w = \varphi(z) = \ln(\sinh(z)),$$

which transforms the eye-shaped domain, D_E, in the z-plane, onto the infinite strip in the complex w-plane, D_S, where for $d>0$,

$$D_E = \left\{ z = x+iy \ : |\, arg(\sinh(z))| < d \leq \frac{\pi}{2} \right\},$$

and

$$D_S = \left\{ w = t+is \ : |s| < d \leq \frac{\pi}{2} \right\}.$$

Thus the basis sinc functions over $(0,\infty)$ are given by

$$S_k(x) = S(k,h)\circ\varphi(x) = \mathrm{sinc}\left(\frac{\varphi(x)-kh}{h}\right), \tag{1.10}$$

where $S(k,h)\circ\varphi(x)$ is defined by $S(k,h)(\varphi(x))$. The inverse map of $w=\varphi(z)$ is

$$z = \varphi^{-1}(w) = \ln(e^w + \sqrt{e^{2w}+1}).$$

We define the range of $\psi = \varphi^{-1}$ on the real line as

$$\Gamma = \{\psi(t) \in D_E \ : -\infty < t < \infty\} = (0,\infty).$$

Also, the image of the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ is denoted by

$$x_k = \psi(kh) = \ln(e^{kh} + \sqrt{e^{2kh}+1}), \qquad k = 0,\pm1,\pm2,\dots \tag{1.11}$$

Definition 2. *Let $B(D_E)$ be the class of functions F which are analytic in D_E and satisfy*

- $\int_{\psi(t+L)} |F(z)dz| \longrightarrow 0, \qquad t \longrightarrow \pm\infty,$
- $N(F) = \int_{\partial D_E} |F(z)dz| < \infty.$

where $L = \left\{ iv \ : |v| < d \leq \frac{\pi}{2} \right\}$ and ∂D_E is the boundary of D_E.
The following theorem, whose proof can be found in [36] provide interpolation formulas for function in $B(D_E)$.

Theorem 1. *If $\varphi' F \in B(D_E)$ then for all $x \in \Gamma$*

$$\left| F(x) - \sum_{k=-\infty}^{\infty} F(x_k) S(k,h)\circ\varphi(x) \right| \leq \frac{N(F\varphi')}{2\pi d \sinh(\pi d/h)}$$

$$\leq \frac{2N(F\varphi')}{\pi d} e^{-\pi d/h}.$$

Moreover, if $|F(x)| \leq Ce^{-\alpha|\varphi(x)|}$, $\quad x \in \Gamma$, *for some positive constants* C *and* α, *and if the selection* $h = \sqrt{\pi d/\alpha N} \leq 2\pi d/\ln 2$, *then*

$$\left| F(x) - \sum_{k=-N}^{N} F(x_k) S(k,h) \circ \varphi(x) \right| \leq C_2 \sqrt{N} \exp(-\sqrt{\pi d \alpha N}), \quad x \in \Gamma,$$

where C_2 *depends only on* F, d *and* α.

As seen in above theorem, the sinc interpolation in $B(D_E)$ converge exponentially [36]. We also require derivatives of composite sinc functions evaluated at the nodes. Technical calculations provide the following results [36].

$$\delta_{k,j}^{(0)} = [S(k,h) \circ \varphi(x)]|_{x=x_j} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \tag{1.12}$$

$$\delta_{k,j}^{(1)} = h \frac{d}{d\varphi} [S(k,h) \circ \varphi(x)]|_{x=x_j} = \begin{cases} 0, & k = j, \\ \frac{(-1)^{j-k}}{j-k}, & k \neq j. \end{cases} \tag{1.13}$$

2.3 Gauss-Jacobi quadrature

Denote $P_m^{(\lambda,\mu)}(x); \lambda > -1, \mu > -1$ as the m-th order Jacobi polynomial defined on $[-1,1]$. These polynomials are given explicitly by [35]

$$P_m^{(\lambda,\mu)}(x) = 2^{-m} \sum_{k=0}^{m} \binom{m+\mu}{m-k} \binom{m+\lambda}{k} (x-1)^{m-k} (x+1)^k.$$

The Jacobi polynomials are orthogonal on the interval $(-1,1)$ with respect to the weight function $\rho^{(\lambda,\mu)}(x) = (1-x)^\lambda (1+x)^\mu$, i.e.,

$$\int_{-1}^{1} P_n^{(\lambda,\mu)}(x) P_m^{(\lambda,\mu)}(x) \rho^{(\lambda,\mu)}(x) dx = \begin{cases} \frac{2^{\lambda+\mu+1}}{\lambda+\mu+2n+1} \frac{\Gamma(\lambda+n+1)\Gamma(\mu+n+1)}{n!\Gamma(\lambda+\mu+n+1)}, & n = m, \\ 0, & n \neq m. \end{cases}$$

All the zeros of $P_m^{(\lambda,\mu)}(x)$ are simple and they belong to the interval $(-1,1)$ [35].

For a given positive integer m we denote the Gauss-Jacobi points by $\{\xi_i^{(\lambda,\mu)}\}_{i=1}^m$. In fact, these points are zeros of the polynomial $P_m^{(\lambda,\mu)}(x)$. An m-point Gauss-Jacobi quadrature rule, with parameters λ and μ, is based on Gauss-Jacobi points $\{\xi_i^{(\lambda,\mu)}\}_{i=1}^m$ and can be used to approximate the integral of a function over the range $[-1,1]$ with weight $\rho^{(\lambda,\mu)}(x)$ as

$$\int_{-1}^{1} f(x) \rho^{(\lambda,\mu)}(x) dx \approx \sum_{i=1}^{m} \omega_i^{(\lambda,\mu)} f(\xi_i^{(\lambda,\mu)}), \tag{1.14}$$

where the Gauss-Jacobi weights $\{\omega_i^{(\lambda,\mu)}\}_{i=1}^m$ are given by [10]

$$\omega_i^{(\lambda,\mu)} = \frac{\Gamma(\lambda+m+1)\Gamma(\mu+m+1)}{m!\Gamma(\lambda+\mu+m+1)} \frac{2^{\lambda+\mu+1}}{\left(1 - \left(\xi_i^{(\lambda,\mu)}\right)^2\right) \left[P_m^{(\lambda,\mu)\prime}(\xi_i^{(\lambda,\mu)})\right]^2}. \tag{1.15}$$

Also, the error is

$$\frac{f^{2m}(\eta)}{(2m)!}\int_{-1}^{1}\left[\prod_{i=1}^{m}(\xi-\xi_i^{(\lambda,\mu)})\right]^2\rho^{(\lambda,\mu)}(\xi)d\xi,\quad \eta\in(-1,1).$$

Thus, the Gauss-Jacobi quadrature rule has a degree of exactness of $2m-1$, i.e., it is exact for polynomials of degree up to $2m-1$.

3 The sinc-Gauss-Jacobi collocation method

In this section, we solve the fractional Volterras population model of the form (1.3) by using the collocation method, based on modified sinc functions, in combination with the Gauss-Jacobi quadrature formulae. First of all, we reformulate the problem (1.3) to an equivalent nonlinear fractional differential equation.

3.1 Reformulation of the problem

Let

$$y(x)=\int_0^x u(t)dt, \tag{1.16}$$

this transformation readily leads to

$$y'(x)=u(x). \tag{1.17}$$

Using Eqs. (1.6) and (1.17) yield

$$D^{\beta}u=D^{\beta}D^{1}y=D^{\beta+1}y. \tag{1.18}$$

Substituting Eqs. (1.16)-(1.18) into Eq. (1.3) yields the nonlinear fractional differential equation

$$\kappa D^{\beta+1}y(x)=y'(x)-\left(y'(x)\right)^2-y'(x)y(x), \tag{1.19}$$

with the initial conditions

$$y(0)=0,\qquad y'(0)=u_0. \tag{1.20}$$

3.2 Solving fractional Volterra's population model

The sinc basis functions in Eq. (1.10) are not differentiable when x tends to zero. Following [26], we modify the sinc basis functions as

$$xS_k(x). \tag{1.21}$$

A straightforward calculation reveals that the derivative of the modified sinc basis functions are defined and are equal to zero as x approaches zero. To approximate the solution of Eq. (1.19) with initial conditions (1.20), first of all, we construct a polynomial

$$q(x)=\theta x^2+u_0x, \tag{1.22}$$

that satisfies Eq. (1.20). Here, θ is a constant to be determined. Now, the approximate solution for $y(x)$ in Eq. (1.19) with initial conditions in Eq. (1.20) is represented by

$$y_N(x)=z_N(x)+q(x), \tag{1.23}$$

where

$$z_N(x) = \sum_{k=-N}^{N} c_k x S_k(x). \tag{1.24}$$

It is worth pointing out that the approximate solution $y_N(x)$ satisfies the initial conditions in Eq. (1.20), since

$$\lim_{x\to 0} z_N(x) = \lim_{x\to 0} z'_N(x) = 0. \tag{1.25}$$

The $2N+1$ coefficients, $\{c_k\}_{k=-N}^{N}$ and the unknown θ are determined by substituting $y_N(x)$ into Eq. (1.19) and evaluating the result at the sinc collocation points:

$$x_j = \ln(e^{jh} + \sqrt{e^{2jh}+1})\,, \qquad j = -N-1, \ldots, N. \tag{1.26}$$

Obviously by using Eqs. (1.12) and (1.23) we have

$$y_N(x_j) = c_j x_j + q(x_j), \qquad j = -N-1, \ldots, N, \tag{1.27}$$

where we used $c_{-N-1} = 0$. To compute $y'_N(x_j)$, we first differentiate Eq. (1.21) as

$$\frac{d}{dx}\left[x S_k(x)\right] = S_k(x) + x\varphi'(x)\frac{d}{d\varphi}\left[S(k,h)\circ\varphi(x)\right]. \tag{1.28}$$

Now, using Eqs. (1.12), (1.13), (1.23) and (1.28) we obtain

$$y'_N(x_j) = \sum_{k=-N}^{N} c_k \left\{\delta_{kj}^{(0)} + \frac{1}{h} x_j \varphi'(x_j)\delta_{kj}^{(1)}\right\} + q'(x_j), \qquad j = -N-1, \ldots, N. \tag{1.29}$$

Lemma 1. *Let ξ_i and w_i be the nodes and the corresponding weights of the Gauss-Jacobi quadrature formula given in Eq. (1.14), respectively. Also, let $1 < \gamma < 2$ and x_j be sinc collocation points given in Eq. (1.26). Then the following relation holds:*

$$D^{\gamma}(xS_k(x))|_{x=x_j} \approx \frac{(\frac{x_j}{2})^{2-\gamma}}{\Gamma(2-\gamma)} \sum_{i=1}^{m} \omega_i^{(1-\gamma,0)} \left\{2S_k^{(1)}(\widehat{x_{j,i}}) + \widehat{x_{j,i}} S_k^{(2)}(\widehat{x_{j,i}})\right\}, \tag{1.30}$$

where, $\widehat{x_{j,i}} = \frac{x_j}{2}(1+\xi_i^{(1-\gamma,0)})$.
Proof. Using (1.5), it follows that

$$D^{\gamma}(xS_k(x))|_{x=x_j} = \frac{1}{\Gamma(2-\gamma)} \int_0^{x_j} (x_j - t)^{1-\gamma}\{2S_k^{(1)}(t) + tS_k^{(2)}(t)\}dt. \tag{1.31}$$

Now, we employ the Gauss-Jacobi quadrature rule to approximate the integral in the right-hand side of Eq. (1.31). First, the affine transformation $\tau = \frac{2}{x_j}t - 1$ is used to change the t-interval $[0, x_j]$ into τ- interval $[-1, 1]$. The following form can be obtained

$$D^{\gamma}(xS_k(x))|_{x=x_j} = \frac{(\frac{x_j}{2})^{2-\gamma}}{\Gamma(2-\gamma)} \int_{-1}^{1} (1-\tau)^{1-\gamma} \left\{2S_k^{(1)}\left(\frac{x_j}{2}(1+\tau)\right)\right.$$

$$+\left(\frac{x_j}{2}(1+\tau)\right) S_k^{(2)}\left(\frac{x_j}{2}(1+\tau)\right)\Big\} d\tau.$$

Employing the m-point Gauss-Jacobi quadrature rule (1.14), with parameters $\lambda = 1-\gamma$ and $\mu = 0$, the proof is clear. □

In the following theorem we approximate the fractional derivative of $y_N(x_j)$.

Theorem 2. *For, $1 < \gamma < 2$ the following relation holds:*

$$D^{\gamma}(y_N(x))|_{x=x_j} \approx \sum_{k=-N}^{N} c_k \delta_{kj}^{(\gamma)} + \frac{2\theta}{\Gamma(3-\gamma)} x_j^{2-\gamma}, \tag{1.32}$$

where $\delta_{kj}^{(\gamma)}$ is given by

$$\delta_{kj}^{(\gamma)} = \frac{(\frac{x_j}{2})^{2-\gamma}}{\Gamma(2-\gamma)} \sum_{i=1}^{m} \omega_i^{(1-\gamma,0)} \left\{ 2S_k^{(1)}(\widehat{x_{j,i}}) + \widehat{x_{j,i}} S_k^{(2)}(\widehat{x_{j,i}}) \right\}.$$

Proof. Since the Liouville-Caputo's fractional differentiation is a linear operation we have

$$D^{\gamma}(y_N(x)) = \sum_{k=-N}^{N} c_k D^{\gamma}(x S_k(x)) + D^{\gamma}(q(x)). \tag{1.33}$$

Also, using Eqs. (1.7) and (1.22) we get

$$D^{\gamma}(q(x)) = \frac{2\theta}{\Gamma(3-\gamma)} x^{2-\gamma}. \tag{1.34}$$

A combination of Lemma 1 and Eqs. (1.33), (1.34) leads to the desired result. □

We are now ready to solve problem (1.19)-(1.20). Substituting Eqs. (1.27), (1.29) and (1.32) in Eq. (1.19) we obtain

$$\kappa D^{\beta+1} y(x_j) = y'(x_j) - (y'(x_j))^2 - y'(x_j) y(x_j), \qquad j = -N-1, \ldots, N. \tag{1.35}$$

The $2N+2$ nonlinear algebraic equations (1.35) can be solved for the unknown coefficients c_k and θ by using the an iterative method. Consequently, $y_N(x)$ given in Eq. (1.23) can be calculated. Throughout this paper, we use the Maple's fsolve command with the initial approximation ($c_k = 0.5, k = -N, \cdots, N,$ and $\theta = 0.5$) to find unknown coefficients c_k and θ from the nonlinear system (1.35).

4 Numerical results and discussion

This section is devoted to computational results. We applied the method presented in this paper for solving the fractional Volterra's population model (1.3) with $u_0 = 0.1$. Throughout this section, we choose $\alpha = 1$ and $d = \pi/2$ which leads to $h = \pi/\sqrt{2N}$. Also, the number of Gauss-Jacobi points is set to be $m = 15$.

The obtained solution of Eq. (1.3) for $\beta = 0.5, 0.75$ and 0.99 for different values of κ are shown in Figures 1-3, respectively. From these figures, it is clear that the maximum of $u(t)$ decreases as

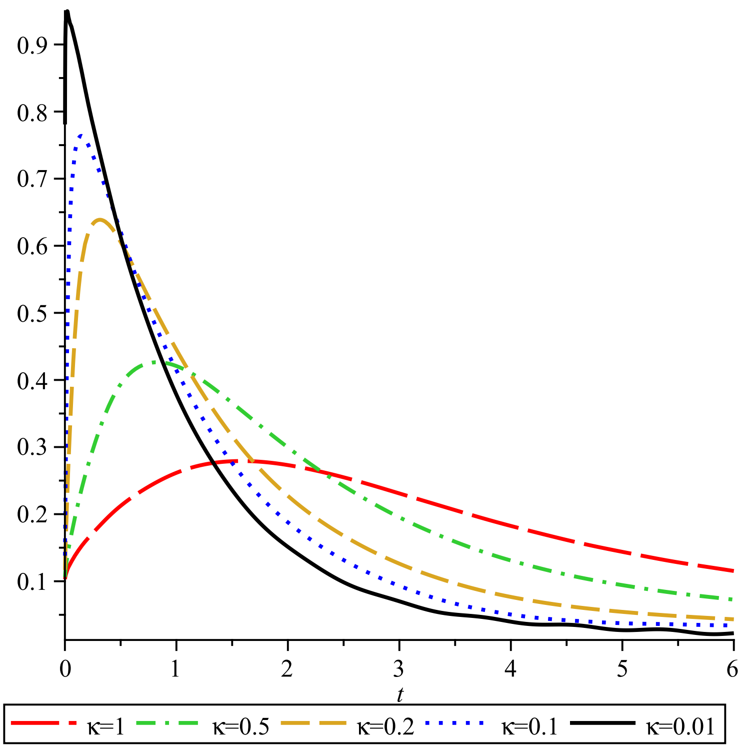

FIGURE 1. Resulting graphs of $u(t)$, for $\beta = 0.5$ with $N = 30$.

κ increases. This tendency is similar to that observed in [26, 24] for the case $\beta = 1$. Moreover, Figures 1-3, show a rapid rise along the logistic curve and then a fast exponential decay to zero for small values of κ. We also present the behavior of the solution of Eq. (1.3) for $\kappa = 0.2$ and $\beta = 0.6, 0.7, 0.8, 0.9$ and 0.99 in Figure 4. According to Figure 4, we find that the solution falls slowly when the the value of β increases. Figures 1-4 show a very good agreement between the results obtained by the present method and those obtained by Maleki et al. [17], Parand et al. [25] and Momani et al. [20]. In addition, for the purpose of comparison in Table 1, we report the values of $u(t)$ for $\kappa = 2.5$ and $\beta = 0.5, 0.75, 0.9$ together with the results given in [25]. Results show that the methods are in a good agreement with each other. Finally, in Tables 2, we give the values of $u_{\max}$ for $\beta = 0.75$ and 0.9 with different values of N.

5 Conclusion

In this study, the properties of sinc functions and Gauss-Jacobi quadrature rule are used to reduce the solution of Volterra's population model of fractional order to the solution of system of algebraic equations. The obtained results are shown in different graphs and tables and the ability of sinc-Gauss-Jacobi collocation method in solving fractional Volterra's population model is presented. This method can be easily implemented and is simple. One issue of future work is to develop

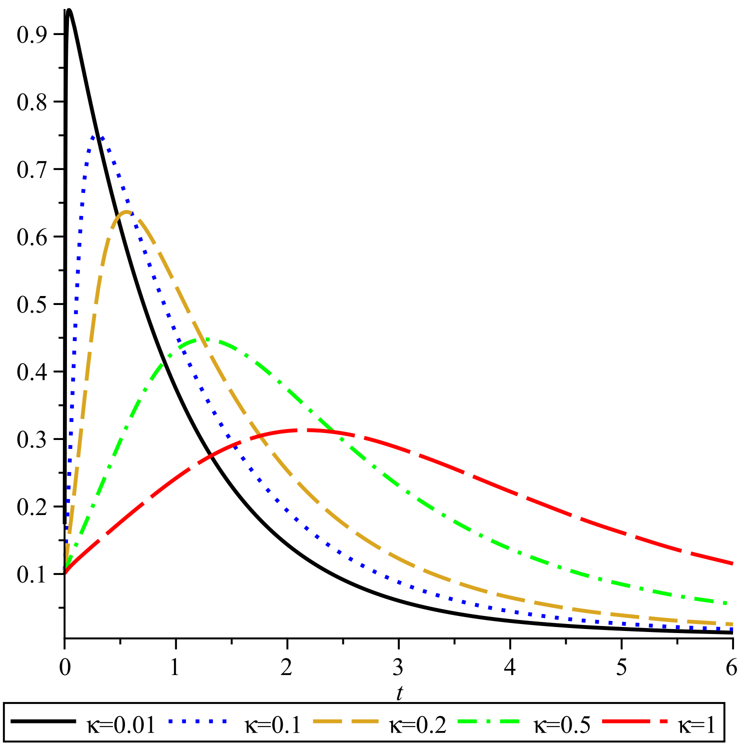

FIGURE 2. Resulting graphs of $u(t)$, for $\beta = 0.75$ with $N = 30$.

TABLE 1. Comparison of the values of $u(t)$ for $\kappa = 2.5$ with $N = 20$.

	$\beta = 0.5$		$\beta = 0.75$		$\beta = 0.9$	
t	Present method	Method of [25]	Present method	Method of [25]	Present method	Method of [25]
0.25	0.12354	0.12292	0.11471	0.11458	0.11117	0.11117
0.50	0.13349	0.13344	0.12559	0.12545	0.12141	0.12137
0.75	0.14224	0.14148	0.13528	0.13526	0.13134	0.13142
1.00	0.14929	0.14795	0.14443	0.14434	0.14128	0.14134
1.25	0.15454	0.15320	0.15297	0.15275	0.15117	0.15108
1.50	0.15847	0.15741	0.16060	0.16050	0.16047	0.16053
1.75	0.16159	0.16070	0.16759	0.16755	0.16941	0.16958
2.00	0.16408	0.16316	0.17404	0.17383	0.17817	0.17811
2.25	0.16583	0.16487	0.17954	0.17931	0.18615	0.18600
2.50	0.16691	0.16590	0.18397	0.18395	0.19298	0.19314
2.75	0.16750	0.16632	0.18773	0.18772	0.19917	0.19940
3.00	0.16759	0.16620	0.19088	0.19060	0.20487	0.20471

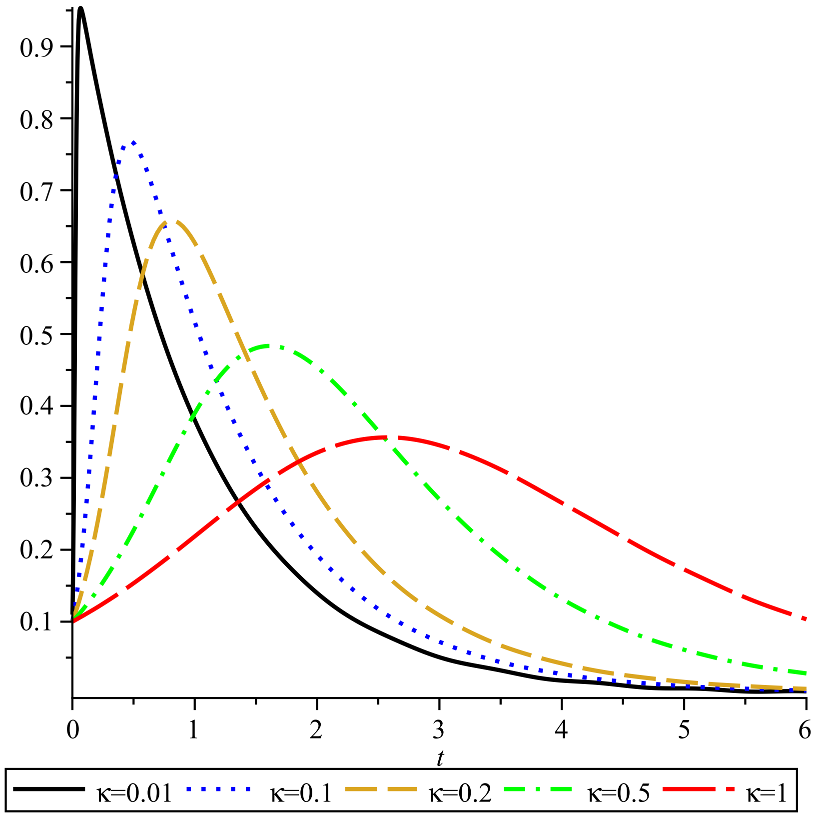

FIGURE 3. Resulting graphs of $u(t)$, for $\beta = 0.99$ with $N = 30$.

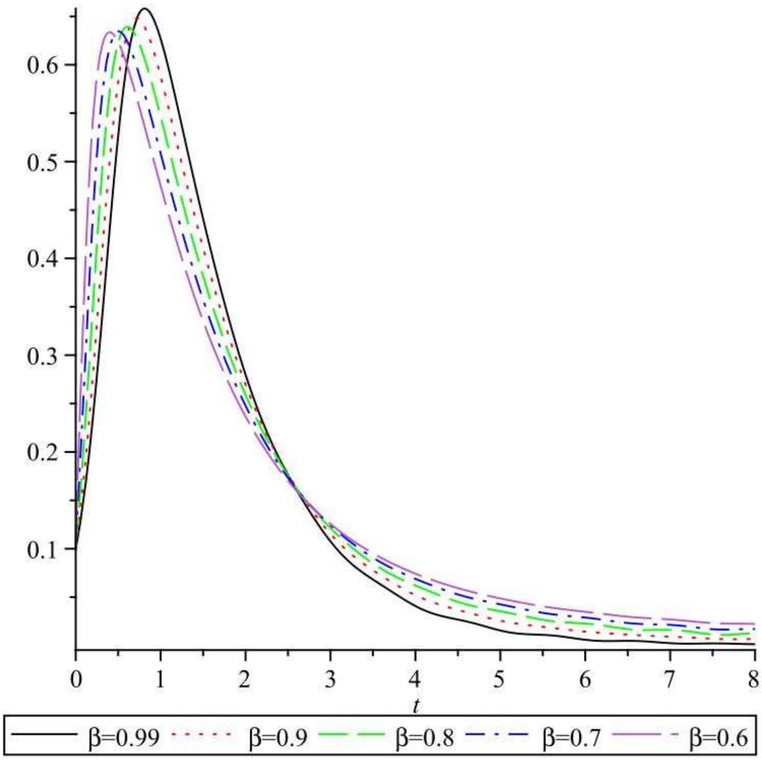

FIGURE 4. Resulting graphs of $u(t)$, for $\kappa = 0.2$ with $N = 20$.

TABLE 2. Values of $u_{\max}$ for $\kappa = 0.2$ with different values of N

β	$N = 15$	$N = 20$	$N = 25$	$N = 30$	$N = 40$
0.75	0.636185	0.636443	0.636468	0.636442	0.636402
0.90	0.648661	0.647911	0.647676	0.647578	0.647579

similar technique to solve some interesting fractional differential equations.

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