Laguerre-based Hermite-Bernoulli polynomials associated with bilateral series

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Abstract

In the paper, we define Laguerre-based Hermite-Bernoulli polynomial with its generating function, and investigate certain properties. From this generating function, we derive summation formulas and related bilateral series associated with the newly introduced generating function. Some of whose special cases are also presented. Relevant connections of some results presented here with those involving simpler known partly unilateral and partly bilateral representations are also obtained.

2010 Mathematics Subject Classification. **11B68**. 33C05
Keywords. Hermite polynomials; Laguerre polynomials; Bernoulli polynomials; Laguerre based Hermite-Bernoulli polynomials; Summation formulae; Bilateral series.

1 Introduction

Throughout of the paper we will make use of the following notations: $\mathbb{N} := \{1, 2, 3, \cdots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- := \{-1, -2, -3, \cdots\} = \mathbb{Z}^- \setminus \{0\}$. Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. Let $\exp(z)$ be exponential function given by $\exp(z) := e^z \quad (z \in \mathbb{C}; |z| < 1)$.

Let $L_n(x)$ be classical Laguerre polynomials defined by means of the following generating function:

$$\sum_{n=0}^{\infty} L_n(x)t^n = \frac{1}{1-t}\exp\left(-\frac{xt}{1-t}\right) \qquad (|t| < 1). \tag{1.1}$$

Based on Eq. (1.1), two variables Laguerre polynomials (2VLP) $L_n(x, y)$ are considered as

$$L_n(x, y) := y^n L_n(\frac{x}{y}) \text{ and } L_n(x, 1) := L_n(x) \tag{1.2}$$

representing 2VLP is the same with classical Laguerre polynomials. So, by Eq. (1.2), one can see

$$\sum_{n=0}^{\infty} L_n(x, y)\frac{t^n}{n!} = \exp(yt)J_0(2\sqrt{xt}) \tag{1.3}$$

Tbilisi Mathematical Journal 11(2) (2018), pp. 111–121.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 27 November 2017.
Accepted for publication: 18 March 2018.

where $J_0(x)$ denotes the 0^{th} order Bessel function, and n^{th} order Bessel function $J_n(x)$ are given by the series (see [4],[17],[18]):

$$x^{\frac{n}{2}} J_n\left(2\sqrt{x}\right) = \sum_{n=0}^{\infty} \frac{(-1)^r x^r}{r!(n+r)!} \quad (n \in \mathbb{N}_0). \tag{1.4}$$

The 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x,y)$ are known in [2, 4] as

$$\frac{H_n(x,y)}{n!} = \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!} \text{ and } \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} = e^{xt+yt^2}. \tag{1.5}$$

It is easy to see that $H_n(2x,-1) = H_n(x)$ that stands for classical Hermite polynomials, *cf.* [2].

In [17, 18], the generating function of the 3-variable Laguerre-Hermite polynomials (3VLHP) ${}_LH_n(x,y,z)$ is given by

$$\sum_{n=0}^{\infty} {}_LH_n(x,y,z)\frac{t^n}{n!} = \exp(yt+zt^2)J_0(2\sqrt{xt}). \tag{1.6}$$

The generalized Bernoulli $B_n^{(\alpha)}(x)$, Euler $E_n^{(\alpha)}(x)$ and Genocchi $G_n^{(\alpha)}(x)$ polynomials are also defined by means of the following generating functions

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} \quad (|t| < 2\pi; 1^{\alpha} := 1), \quad \left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!} \quad (|t| < \pi; 1^{\alpha} := 1) \tag{1.7}$$

and

$$\left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x)\frac{t^n}{n!} \quad (|t| < \pi; 1^{\alpha} := 1). \tag{1.8}$$

Obviously that

$B_n^{(1)}(x) = B_n(x),\ E_n^{(1)}(x) = E_n(x)$ and $G_n^{(1)}(x) = G_n(x)$ *cf.* [3, 6, 10, 11, 12, 14, 17, 18].

Recently, Kurt [9] has introduced and investigated the generalized Bernoulli polynomials $B_n^{[\alpha,m-1]}(x)$ $(m \in \mathbb{N})$ defined in a suitable neighborhood of $t = 0$ by means of the following generating function:

$$\sum_{n=0}^{\infty} B_n^{[\alpha,m-1]}(x)\frac{t^n}{n!} = \left(\frac{t^m}{e^t - \sum_{h=0}^{m-1} \frac{t^h}{h!}}\right)^{\alpha} e^{xt}. \tag{1.9}$$

It is clear that if we take $\alpha = 1$ in (1.9), it reduces to known result of Natalini and Bernandini [10].

Let us now recall here an interesting (partly bilateral and partly unilateral) generating function for $L_n^{(\alpha)}(x)$, due to Exton in [5], in the following form (see Pathan and Yasmeen [13]; Srivastava *et al.* [16]):

$$\exp\left(y+z-\frac{xz}{y}\right)=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}L_n^{(m)}(x)\frac{y^m z^n}{(m+n)!} \tag{1.10}$$

where $m^*=\max\{0,-m\}$ with $m\in\mathbb{Z}$.

In this paper, we introduce a new class of generalized Laguerre-based Hermite-Bernoulli polynomials $_{LH}B_n^{[\alpha,m-1]}(x,y,z)$ and develop some elementary properties. We also derive the summation formulae for these generalized polynomials by using different analytical means on their respective generating functions and related bilateral series associated with the newly-introduced generating function. Some of whose special cases are also presented. Finally, relevant connections of some results presented here with those involving simpler known partly unilateral and partly bilateral representations are indicated.

2 A new class of Laguerre-based Hermite-Bernoulli polynomials

Let us now consider the following generating function of the generalized Laguerre-based Hermite-Bernoulli polynomials $_{LH}B_n^{[\alpha,m-1]}(x,y,z)$ given by

$$\sum_{n=0}^{\infty}{}_{LH}B_n^{[\alpha,m-1]}(x,y,z)\frac{t^n}{n!}=\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}J_0(2\sqrt{xt})e^{yt+zt^2} \tag{1.11}$$

defining in a suitable neighborhood of $t=0$.

We readily see from (1.11) that

$${}_{LH}B_n^{[\alpha,m-1]}(x,y,z)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k}^{[m-1]}\,{}_LH_k(x,y,z). \tag{1.12}$$

In the special cases, Eq. (1.11) contains not only generalized Bernoulli polynomials, but also generalization of Laguerre-Hermite polynomials.

Remark 2.1. Setting $m=1$, $z=0$ and y replaced by x in Eq.(1.11), it reduces to known result of Khan *et al.* [7].

Remark 2.2. Setting $x=0$, y replaced by x, and z replaced by y in Eq.(1.11), it reduces to known result of Pathan and Khan [14].

Remark 2.3. For $m=1$, $x=0$, y replaced by x, z replaced by y in Eq.(1.11),one can see that

$$\sum_{n=0}^{\infty}{}_HB_n^{(\alpha)}(x,y)\frac{t^n}{n!}=\left(\frac{t}{e^t-1}\right)^{\alpha}e^{xt+yt^2} \tag{1.13}$$

which is a generalization of the generating function of Dattoli *et al.* [4, Eq (1.6), p.386] in the form:

$$\sum_{n=0}^{\infty} {}_HB_n(x,y)\frac{t^n}{n!} = \left(\frac{t}{e^t-1}\right)e^{xt+yt^2}. \tag{1.14}$$

Let $E_n^{(\alpha)}(x,y)$ be the generalized Hermite-Euler polynomials, and let $G_n^{(\alpha)}(x,y)$ be the generalized Hermite-Genocchi polynomials defined by

$$\sum_{n=0}^{\infty} {}_HE_n^{(\alpha)}(x,y)\frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt+yt^2} \quad (|t|<\pi; 1^{\alpha}:=1) \tag{1.15}$$

$$\sum_{n=0}^{\infty} {}_HG_n^{(\alpha)}(x,y)\frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt+yt^2} \quad (|t|<\pi; 1^{\alpha}:=1). \tag{1.16}$$

In the case when $y=0$ in (1.13), Eqs.(1.15) and (1.16) are generalizations of (1.7) and (1.8).

We now state the following theorem including the connection between Laguerre-Hermite polynomials ${}_LH_n(x,y,z)$ and generalized Bernoulli numbers $B_n^{[\alpha,m-1]}$.

Theorem 2.4. For $n \in \mathbb{N}_0$, we have

$$ {}_LH_n(x,y) = \frac{1}{n+1}\left(LHB_{n+1}^{[1,1]}(x,y+1,z) - LHB_{n+1}^{[1,1]}(x,y,z)\right). \tag{1.17}$$

Proof. Consider Eq.(1.11), we have

$$\begin{aligned} e^{yt+zt^2}C_0(xt) &= \frac{e^t-1}{t}\left(\frac{t}{e^t-1}\right)e^{yt+zt^2}J_0(2\sqrt{xt}) \\ &= \frac{1}{t}\left(\left(\frac{t}{e^t-1}\right)e^{(y+1)t+zt^2}J_0(2\sqrt{xt}) - \left(\frac{t}{e^t-1}\right)e^{yt+zt^2}J_0(2\sqrt{xt})\right). \end{aligned}$$

Then, by using the definition of Kampé de Fériet generalization of the Laguerre-Hermite polynomials ${}_LH_n(x,y)$ and Laguerre-based Hermite-Bernoulli polynomials $LHB_n^{[\alpha,m-1]}(x,y,z)$, we get

$$\sum_{n=0}^{\infty} {}_LH_n(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{n+1}\left(LHB_{n+1}^{[1,1]}(x,y+1,z) - LHB_{n+1}^{[1,1]}(x,y,z)\right)\frac{t^n}{n!}.$$

Finally, comparing the coefficients of $\frac{t^n}{n!}$, we arrive at the desired identity given in (1.17). Q.E.D.

Theorem 2.5. For $n \in \mathbb{N}_0$, we have

$$LHB_n^{[\alpha+\beta,m-1]}(x,y+w,z) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}^{[\alpha,m-1]}(w)LHB_k^{[\beta,m-1]}(x,y,z). \tag{1.18}$$

Proof. By Definition (1.11), we have

$$\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha+\beta}\exp\left((y+w)t+zt^2\right)J_0(2\sqrt{xt}) = \sum_{n=0}^{\infty}LHB_n^{[\alpha+\beta,m-1]}(x,y+w,z)\frac{t^n}{n!}$$

$$=\left(\sum_{n=0}^{\infty}B_n^{[\alpha,m-1]}(w)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}LHB_n^{[\beta,m-1]}(x,y,z)\frac{t^n}{n!}\right).$$

Now replacing n by $n-k$ in the RHS of above equation, and comparing the coefficients of $\frac{t^n}{n!}$, we complete the proof. Q.E.D.

Theorem 2.6. For $n\in\mathbb{N}_0$, we have

$$LHB_n^{[\alpha,m-1]}(x,y,z)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k}^{[m-1]}LHB_k^{[\alpha-1,m-1]}(x,y,z). \tag{1.19}$$

Proof. Using (1.11), we can write

$$\sum_{n=0}^{\infty}LHB_n^{[\alpha,m-1]}(x,y,z)\frac{t^n}{n!} = \frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha-1}\exp(yt+zt^2)J_0(2\sqrt{xt})$$

$$= \left(\sum_{n=0}^{\infty}B_n^{[m-1]}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}LHB_n^{[\alpha-1,m-1]}(x,y,z)\frac{t^n}{n!}\right).$$

On replacing n by $n-k$ in the RHS of above equation and comparing the coefficients of $\frac{t^n}{n!}$, we arrive at the desired result (2.9).

Q.E.D.

3 Summation formulae for Laguerre-based Hermite-Bernoulli polynomials

We now give the interesting summation properties for $LHB_n^{[\alpha,m-1]}(x,y,z)$ by using series manipulation methods. The obtained results here are corresponding generalization of some known special polynomial which we stated in this part.

Theorem 3.1. The following summation formula holds true:

$$LHB_{q+l}^{[\alpha,m-1]}(x,w,z) = \sum_{n,p=0}^{q,l}\binom{q}{n}\binom{l}{p}(w-y)^{n+p}LHB_{q+l-n-p}^{[\alpha,m-1]}(x,y,z) \tag{1.20}$$

in which we have used $\sum\limits_{n,p=0}^{q,l}=\sum\limits_{n=0}^{q}\sum\limits_{p=0}^{l}$.

Proof. Replacing t by $t + u$ in (1.11), and then using the formula [15, p. 52 (2)]

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!} \tag{1.21}$$

we see that

$$\left(\frac{(t+u)^m}{e^{t+u}-\sum_{h=0}^{m-1}\frac{(t+u)^h}{h!}}\right)^{\alpha} e^{z(t+u)^2}J_0(2\sqrt{x\,(t+u)}) = e^{-y(t+u)}\sum_{q,l=0}^{\infty} LHB_{q+l}^{[\alpha,m-1]}(x,y,z)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{1.22}$$

Replacing y by w in Eq. (1.22), we find

$$\exp((w-y)(t+u))\sum_{q,l=0}^{\infty} LHB_{q+l}^{[\alpha,m-1]}(x,y,z)\frac{t^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} LHB_{q+l}^{[\alpha,m-1]}(x,w,z)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{1.23}$$

Expanding exponential function in Eq.(1.23) gives

$$\sum_{N=0}^{\infty}\frac{[(w-y)(t+u)]^N}{N!}\sum_{q,l=0}^{\infty} LHB_{q+l}^{[\alpha,m-1]}(x,y,z)\frac{t^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} LHB_{q+l}^{[\alpha,m-1]}(x,w,z)\frac{t^q}{q!}\frac{u^l}{l!} \tag{1.24}$$

which on using formula (1.21) in the first summation on the LHS becomes

$$\sum_{n,p=0}^{\infty}\frac{(w-y)^{n+p}t^n u^p}{n!p!}\sum_{q,l=0}^{\infty} LHB_{q+l}^{[\alpha,m-1]}(x,y,z)\frac{t^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} LHB_{q+l}^{[\alpha,m-1]}(x,w,z)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{1.25}$$

Now replacing q by $q-n$, l by $l-p$ and using the lemma ([15, p.100 (1)])

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A(n,k) = \sum_{k=0}^{\infty}\sum_{n=0}^{k} A(n,k-n) \tag{1.26}$$

in the LHS of (1.25), we derive

$$\begin{aligned} &\sum_{q,l=0}^{\infty}\sum_{n,p=0}^{q,l}\frac{(w-y)^{n+p}}{n!p!} LHB_{q+l-n-p}^{[\alpha,m-1]}(x,y,z)\frac{t^q}{(q-n)!}\frac{u^l}{(l-p)!} \\ =\;& \sum_{q,l=0}^{\infty} LHB_{q+l}^{[\alpha,m-1]}(x,w,z)\frac{t^q}{q!}\frac{u^l}{l!}. \end{aligned} \tag{1.27}$$

Finally, on equating the coefficients of the like powers of t^q and u^l in the above equation, we complete the proof. Q.E.D.

Taking $l = 0$ in assertion (1.20), we have the following corollary.

Corollary 3.2.

$$LHB_q^{[\alpha,m-1]}(x,w,z) \;=\; \sum_{n=0}^{q}\binom{q}{n}(w-y)^n LHB_{q-n}^{[\alpha,m-1]}(x,y,z). \tag{1.28}$$

The following theorem is the product of two Laguerre-based Hermite-Bernoulli polynomials.

Theorem 3.3. For $n \in \mathbb{N}_0$ and $s \in \mathbb{N}_0$, we have

$$\begin{aligned} &LHB_n^{[\alpha,m-1]}(x,w,u)LHB_s^{[\alpha,m-1]}(X,W,U) \qquad (1.29)\\ &= \sum_{r,k=0}^{n,s}\binom{n}{r}\binom{s}{k}H_r(w-y,u-z)H_k(W-Y,U-Z)LHB_{n-r}^{[\alpha,m-1]}(x,y,z)LHB_{s-k}^{[\alpha,m-1]}(X,Y,Z). \end{aligned}$$

Proof. Consider the product of two Laguerre-based Hermite-Bernoulli polynomials (1.11) in the following form:

$$\begin{aligned} &\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}\exp(yt+zt^2)J_0(2\sqrt{xt})\left(\frac{T^m}{e^T-\sum\limits_{h=0}^{m-1}\frac{T^h}{h!}}\right)^{\alpha}\exp(YT+ZT^2)J_0(2\sqrt{XT})\\ &= \left(\sum_{n=0}^{\infty}LHB_n^{[\alpha,m-1]}(x,y,z)\frac{t^n}{n!}\right)\left(\sum_{s=0}^{\infty}LHB_s^{[\alpha,m-1]}(X,Y,Z)\frac{T^s}{s!}\right). \qquad (1.30) \end{aligned}$$

Replacing y by w, z by u, Y by W and Z by U in (1.30), we find

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}LHB_n^{[\alpha,m-1]}(x,w,u)LHB_s^{[\alpha,m-1]}(X,W,U)\frac{t^n}{n!}\frac{T^s}{s!}\\ =\;& \exp((w-y)t+(u-z)t^2)\exp((W-Y)T+(U-Z)T^2)\\ &\times\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}LHB_n^{[\alpha,m-1]}(x,y,z)LHB_s^{[\alpha,m-1]}(X,Y,Z)\frac{t^n}{n!}\frac{T^s}{s!}, \end{aligned}$$

which on using the generating function (1.26) in the exponential on the RHS, it becomes

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}LHB_n^{[\alpha,m-1]}(x,w,u)LHB_s^{[\alpha,m-1]}(X,W,U)\frac{t^n}{n!}\frac{T^s}{s!}\\ =\;& \sum_{n,r=0}^{\infty}\sum_{s,k=0}^{\infty}H_r(w-y,u-z)LHB_n^{[\alpha,m-1]}(x,y,z)\frac{t^{n+r}}{n!r!}H_k(W-Y,U-Z)LHB_s^{[\alpha,m-1]}(X,Y,Z)\frac{T^{s+k}}{s!k!}. \end{aligned}$$

Finally, replacing n by $n-r$ and s by $s-k$, and matching the coefficients of like powers of t^n and T^s, our assertion follows. Q.E.D.

Changing u to z and U to Z in Eq. (1.29), we have the following corollary.

Corollary 3.4. We have

$$LHB_n^{[\alpha,m-1]}(x,w,z)LHB_s^{[\alpha,m-1]}(X,W,Z) = \sum_{r,k=0}^{n,s}\binom{n}{r}\binom{s}{k}(w-y)^r LHB_{n-r}^{[\alpha,m-1]}(x,y,z) \times(W-Y)^k LHB_{s-k}^{[\alpha,m-1]}(X,Y,Z). \quad (1.31)$$

Now also, we have the following summation formula for $LHB_n^{[\alpha,m-1]}(z,w,y)$.

Theorem 3.5. For, we have

$$LHB_{k+l}^{[\alpha,m-1]}(z,w,y) = \sum_{n,p=0}^{k,l}\binom{k}{n}\binom{l}{p}{}_H B_{l+k-n-p}^{[\alpha,m-1]}(x,y)_q L_{n+r}(w,z-x).$$

Proof. The following identity is derived in [14]

$${}_H B_{k+l}^{[\alpha,m-1]}(z,y) = \sum_{n,p=0}^{k,l}\binom{k}{n}\binom{l}{p}(z-x)^{n+p}{}_H B_{l+k-n-p}^{[\alpha,m-1]}(x,y). \quad (1.32)$$

Based on this identity, and applying $\exp\left(D_w^{-1}\frac{\delta^q}{\delta z^q}\right)$ to both sides of Eq.(1.32), we have

$$\exp\left(D_w^{-1}\frac{\delta^q}{\delta z^q}\right){}_H B_{k+l}^{[\alpha,m-1]}(z,y) \quad (1.33)$$

$$= \sum_{n,p=0}^{k,l}\binom{k}{n}\binom{l}{p}{}_H B_{l+k-n-p}^{[\alpha,m-1]}(x,y)\exp\left(D_w^{-1}\frac{\delta^q}{\delta z^q}\right)(z-x)^{n+p}.$$

Using the operational definitions (see [8]) in the LHS and RHS of Eq.(1.33) completes the proof.

Q.E.D.

4 Generating functions for the Laguerre-based Hermite-Bernoulli polynomials involving bilateral series

Set

$$V^{(\alpha,m)}(x,y,z,w;s,t) = \left(\frac{t^m}{e^t - \sum_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{s-\frac{wt}{s}+yt+zt^2}J_0(2\sqrt{xt}). \quad (1.34)$$

Expanding $\exp(s-\frac{wt}{s})$ in the series form, and then by using (1.11), we get

$$V^{(\alpha,m)} = \sum_{M=0}^{\infty}\frac{s^M}{M!}\sum_{K=0}^{\infty}\left(\frac{-wt}{s}\right)^K\frac{1}{K!}\sum_{N=0}^{\infty}LHB_N^{[\alpha,m-1]}(x,y,z)\frac{t^N}{N!}. \quad (1.35)$$

Upon replacing the summation indices M and N in (1.35) by $K+N=n$ and $M-K=m$, respectively, rearranging the summation series gives

$$V^{(\alpha,m)} = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}s^m t^n\sum_{K=0}^{n}\frac{(-w)^K}{K!(m+K)!(n-K)!}LHB_{n-K}^{[\alpha,m-1]}(x,y,z) \quad (1.36)$$

which can be justified by absolute convergence of the series involved. From here, we have

$$e^{s-\frac{wt}{s}+yt+zt^2}C_0(xt) = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} s^m t^n \sum_{K=0}^{n} \frac{(-w)^K}{K!(m+K)!(n-K)!}\,{}_LH_{n-K}(x,y,z). \tag{1.37}$$

Now we list some special cases of the result (1.36) as follows.
(i) Setting $x=0$, $y=1$ and using $L_n(0,1)=1$ reduces to

$$\begin{aligned}&\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{s-\frac{wt}{s}+zt^2}\\ =\ &\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} s^m t^n \sum_{K=0}^{n} \frac{(-w)^K}{K!(m+K)!(n-K)!} BHL_{n-K}^{[\alpha,m-1]}(0,1,z).\end{aligned}$$

(ii) Taking $s=t=\frac{w}{2}$, $\alpha=x=0$ and $y=1$ becomes

$$e^{w^2z/4} = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} (\frac{w}{2})^{m+n} \sum_{K=0}^{n} \frac{(-w)^K}{K!(m+K)!(n-K)!} LHB_{n-K}^{[\alpha,m-1]}(0,1,z).$$

(iii) Substituting $s=t=\frac{w}{2}$, $x=1$ and $\alpha=y=0$, $z=\frac{2}{w}$ in (1.21), we get a new representation of Bessel Function $J_0(2\sqrt{x})$.

$$J_0(2\sqrt{\frac{2}{w}}) = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} (\frac{w}{2})^{m+n} \sum_{K=0}^{n} \frac{(-w)^K}{K!(m+K)!(n-K)!} LHB_{n-K}^{[\alpha,m-1]}(1,0,\frac{2}{w}).$$

(iv) Choosing $m=1, x=1$ and $y=0$, we obtain

$$\begin{aligned}&\left(\frac{t}{e^t-1}\right)^{\alpha} e^{s-\frac{wt}{s}+zt^2} J_0\left(2\sqrt{t}\right)\\ =\ &\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} s^m t^n \sum_{K=0}^{n} \frac{(-w)^K}{K!(m+K)!(n-K)!} LHB_{n-K}^{[\alpha,0]}(1,0,z).\end{aligned}$$

(v) Letting $\alpha=m=1$ and $y=0$, we see

$$\begin{aligned}&\left(\frac{t}{e^t-1}\right) e^{s-\frac{wt}{s}+zt^2} J_0\left(2\sqrt{xt}\right)\\ =\ &\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} s^m t^n \sum_{K=0}^{n} \frac{(-w)^K}{K!(m+K)!(n-K)!} LHB_{n-K}(1,0,z).\end{aligned}$$

5 Concluding Remarks

In the paper, we have established the generating functions for the Laguerre-based Hermite-Bernoulli polynomials involving 0^{th} order Bessel function and the generating function of Laguerre polynomials. The equivalent forms of these generating functions can be derived by using Equations (1.6) and (1.9). We have also used the concepts and the formalism associated with Laguerre polynomials in order to derive the Laguerre-based Hermite-Bernoulli polynomials and established their properties.

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