Approximate n-dimensional additive functional equation in various Banach Spaces

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Abstract

In this paper, we modify the Cauchy additive functional equation and find all solutions of this new functional equation. Then, we study generalized Ulam-Hyers stability of such functional equation in various Banach spaces via Hyers' method.

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1 Introduction

During the last two decades, several stability problems of a large variety of functional equations in miscellaneous spaces have been extensively studied and generalized by a number of mathematician. The problem that for the first time was proposed by Ulam [23] in 1940 as follows: When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true? In the next year, Hyers [13] solved this stability problem for additive mappings subject to the Hyers' condition

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

on approximately additive mappings $f : \mathcal{X} \longrightarrow \mathcal{Y}$ for a fixed $\delta \geq 0$ and all $x, y \in \mathcal{X}$ where $\mathcal{X}$ is a real normed space and $\mathcal{Y}$ a real Banach space. In 1950, Aoki [1] generalized the Hyers theorem for additive mappings. In 1978, Th. M. Rassias [20] provided a generalized version of the Hyers theorem which permitted the Cauchy difference to become unbounded. By regarding a large influence of Ulam, Hyers and Rassias on the investigation of stability problems of functional equations the stability phenomenon that was introduced and proved by Rassias [20] is called the Hyers-Ulam-Rassias stability. Some results regarding to the stability of various forms of the miscellaneous functional equations have been investigated by a number of authors. In fact, the generalized Ulam-Hyers stability problems for functional equations have been broadly investigated by these chronological [2, 3, 4, 5, 6, 7, 8, 12, 14, 19, 22] backgrounds.

The mass renowned functional equation is the Cauchy additive functional equation

$$f(x+y) = f(x) + f(y). \tag{1.1}$$

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In this paper, the authors write the general solution and generalized Ulam-Hyers stability of a additive functional equation

$$\begin{aligned} f\left(u_{1}+u_{2}\right)+f\left(u_{2}+u_{3}\right)+\cdots+f\left(u_{n-1}+u_{n}\right) \\ =f\left(u_{1}\right)+2 f\left(u_{2}\right)+\cdots+2 f\left(u_{n-1}\right)+f\left(u_{n}\right) \end{aligned} \tag{1.2}$$

where n is a positive integer with $n>1$ in various Banach spaces via Hyers' method.

2 Stability Results : Banach Space

We firstly investigate the general solution of the additive functional equation (1.2). To find general solution, assume A_1 and A_2 are real vector spaces.

Theorem 2.1. If $f: A_1 \longrightarrow A_2$ is a function fulfilling the functional equation *(1.1)* for all $x, y \in A_1$ if and only if f is a function agreeable the functional equation *(1.2)* for all $u_1, u_2, \cdots, u_n \in A_1$.

Proof. Assume that $f: A_1 \longrightarrow A_2$ satisfies the functional equation (1.1). Putting $x=y=0$ in (1.1), we have $f(0)=0$. Substituting (x, y) by (u_1, u_2),(u_2, u_3),(u_3, u_4), $\cdots$, (u_{n-2}, u_{n-1}),(u_{n-1}, u_n) in (1.1), respectively, we get

$$\begin{aligned} f(u_1+u_2) &= f(u_1)+f(u_2) \\ f(u_2+u_3) &= f(u_2)+f(u_3) \\ f(u_3+u_4) &= f(u_3)+f(u_4) \\ \vdots \quad & \vdots \quad \vdots \\ f(u_{n-1}+u_n) &= f(u_{n-1})+f(u_n) \end{aligned}$$

for all $u_1, u_2, \cdots, u_n \in A_1$. Adding all the preceding equations, we arrive the equation (1.2).

Conversely, suppose that $f: A_1 \longrightarrow A_2$ satisfies the functional equation (1.2) for all $u_1, u_2, \cdots, u_n \in A_1$. Replacing all u_j by zero in (1.2), we get $f(0)=0$. This implies that by interchanging $(u_1, u_2, u_3 \cdots, u_n)$ into $(x, y, 0, \cdots, 0)$, one can obtain the desired result. Q.E.D.

Here and subsequently, for notational convenience, for a mapping $f: \mathcal{A} \longrightarrow \mathcal{B}$, we define

$$\begin{aligned} \mathfrak{F}(u_1, \cdots, u_n) :=& f\left(u_{1}+u_{2}\right)+f\left(u_{2}+u_{3}\right)+\cdots+f\left(u_{n-1}+u_{n}\right) \\ & -\left[f\left(u_{1}\right)+2 f\left(u_{2}\right)+\cdots+2 f\left(u_{n-1}\right)+f\left(u_{n}\right)\right] \end{aligned}$$

for all $u_1, \cdots, u_n \in \mathcal{A}$.

Throughout this paper, for a vector space A, we denote $\overbrace{A \times A \times \ldots \times A}^{n-times}$ by A^n. In this section, we establish the generalized Ulam-Hyers stability of the additive functional equation (1.2) in Banach spaces. For this, we assume $\mathcal{A}$ is normed space and $\mathcal{B}$ is a Banach space.

Theorem 2.2. Let $j \in \{1,-1\}$. Let $f: \mathcal{A} \longrightarrow \mathcal{B}$ be a mapping satisfying the functional inequality

$$\left\|\mathfrak{F}(u_1, \cdots, u_n)\right\| \leq \delta(u_1, \cdots, u_n) \tag{2.1}$$

where $\delta : \mathcal{A}^n \longrightarrow [0,\infty)$ is a function such that

$$\lim_{m\to\infty} \frac{\delta\left(2^{mj}u_1,\cdots,2^{mj}u_n\right)}{2^{mj}} = 0 \tag{2.2}$$

for all $u_1,\cdots,u_n \in \mathcal{A}$. Then, there exists a unique additive function $\Lambda : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying the functional equation *(1.2)* and the inequality

$$\|f(u) - \Lambda(u)\| \leq \frac{1}{2(n-1)} \sum_{i=\frac{1-j}{2}}^{\infty} \frac{\Delta\left(2^{ij}u\right)}{2^{ij}} \tag{2.3}$$

for all $u \in \mathcal{A}$, where $\Delta\left(2^{ij}u\right)$ and $\Lambda(u)$ are defined as

$$\Delta\left(2^{ij}u\right) = \delta\left(\underbrace{2^{ij}u,\cdots,2^{ij}u}_{n-times}\right) \tag{2.4}$$

and

$$\Lambda(u) = \lim_{m\to\infty} \frac{f\left(2^{mj}u\right)}{2^{mj}} \tag{2.5}$$

for all $u \in \mathcal{A}$.

Proof. We bring the proof for the case $j = 1$ completely. Replacing $(u_1,\cdots,u_n)$ by $(u,\cdots,u)$ in (2.1), we get

$$\|(n-1)f(2u) - 2(n-1)f(u)\| \leq \delta(u,\cdots,u) \tag{2.6}$$

for all $u \in \mathcal{A}$. Define $\Delta(u) = \delta(u,\cdots,u)$ and it follows from (2.6), we have

$$\left\|\frac{f(2u)}{2} - f(u)\right\| \leq \frac{\Delta(u)}{2(n-1)} \tag{2.7}$$

for all $u \in \mathcal{A}$. Setting u by $2u$ and multiply by 2 in (2.7), we arrive at

$$\left\|\frac{f\left(2^2u\right)}{2^2} - \frac{f(2u)}{2}\right\| \leq \frac{\Delta(2u)}{4(n-1)} \tag{2.8}$$

for all $u \in \mathcal{A}$. Applying the triangle inequality it follows from (2.7) and (2.8) that

$$\left\|\frac{f\left(2^2u\right)}{2^2} - f(u)\right\| \leq \frac{1}{2(n-1)}\left[\Delta(u) + \frac{\Delta(2u)}{2}\right] \tag{2.9}$$

for all $u \in \mathcal{A}$. Generalizing, for any positive integer m, one can reach

$$\left\|\frac{f(2^mu)}{2^m} - f(u)\right\| \leq \frac{1}{2(n-1)} \sum_{i=0}^{m-1} \frac{\Delta\left(2^iu\right)}{2^i} \tag{2.10}$$

for all $u \in \mathcal{A}$. Thus, the sequence $\{\frac{f(2^m x)}{2^m}\}$ is a Cauchy sequence $\mathcal{B}$. Indeed, replacing u by $2^n u$ and divided by 2^n in (2.10) and using (2.2), we find

$$\begin{aligned}\left\|\frac{f\left(2^{m+n}u\right)}{2^{m+n}} - \frac{f(2^n u)}{2^n}\right\| &= \frac{1}{2^n}\left\|\frac{f\left(2^{m+n}u\right)}{2^m} - f(2^n u)\right\| \\ &\leq \frac{1}{2(n-1)}\sum_{i=0}^{m-1}\frac{\Delta\left(2^{i+n}u\right)}{2^{i+n}} \\ &\rightarrow \quad 0 \quad as \quad n \quad \rightarrow \quad \infty \end{aligned} \tag{2.11}$$

for all $u \in \mathcal{A}$. Since $\mathcal{B}$ is complete, there exists a mapping $\Lambda : \mathcal{A} \longrightarrow \mathcal{B}$ such that

$$\Lambda(u) = \lim_{m\to\infty}\frac{f\left(2^m x\right)}{2^m}$$

for all $u \in \mathcal{A}$. Letting m tends to infinity in (2.10), we arrive (2.3) for all $u \in \mathcal{A}$. Replacing $(u_1, \cdots, u_n)$ by $(2^m u_1, \cdots, 2^m u_n)$ and divided by 2^m in (2.1), using the definition of $\Lambda(u)$ and letting m tends to infinity, we see that Λ satisfies the additive functional equation (1.2) for all $u_1, \cdots, u_n \in \mathcal{A}$. To prove $\Lambda(u)$ is unique, let Λ' be another additive functional equation satisfying (1.2) and (2.3) such that

$$\Lambda(u) = \frac{\Lambda(2^n u)}{2^n} \qquad and \qquad \Lambda'(u) = \frac{\Lambda'(2^n u)}{2^n}$$

for all $u \in \mathcal{A}$. Now, we have

$$\begin{aligned}\|\Lambda(u) - \Lambda'(u)\| &= \frac{1}{2^n}\|\Lambda(2^n u) - \Lambda'(2^n u)\| \\ &\leq \frac{1}{2^n}\{\|\Lambda(2^n u) - f(2^n u)\| + \|f(2^n u) - \Lambda'(2^n u)\|\} \\ &\leq \frac{1}{(n-1)}\sum_{i=0}^{\infty}\frac{\Delta\left(2^{n+i}u\right)}{2^{n+i}} \\ &\rightarrow \quad 0 \quad as \quad n \quad \rightarrow \quad \infty\end{aligned}$$

for all $u \in \mathcal{A}$. Thus $\Lambda(u)$ is unique. Setting u by $\frac{u}{2}$ in (2.7), we get

$$\left\|f(u) - 2f\left(\frac{u}{2}\right)\right\| \leq \frac{\Delta\left(\frac{u}{2}\right)}{(n-1)} \tag{2.12}$$

for all $u \in \mathcal{A}$. Substituting u by $\frac{u}{2}$ and multiply by 2 in (2.12), we obtain

$$\left\|2f\left(\frac{u}{2}\right) - 2^2 f\left(\frac{u}{2^2}\right)\right\| \leq \frac{2\Delta\left(\frac{u}{2^2}\right)}{(n-1)} \tag{2.13}$$

for all $u \in \mathcal{A}$. The relations (2.12) and (2.13) imply that

$$\left\|f(u) - 2^2 f\left(\frac{u}{2^2}\right)\right\| \leq \frac{1}{(n-1)}\left[\Delta\left(\frac{u}{2}\right) + 2\Delta\left(\frac{u}{2^2}\right)\right] \tag{2.14}$$

for all $u \in \mathcal{A}$. The rest of the proof is similar to that of case $j = 1$. This finishes the proof. Q.E.D.

The following corollary is a direct consequence of some stabilities for the functional equation (1.2).

Corollary 2.3. Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a mapping fulfilling the inequality

$$\begin{aligned} &\Big\| f(u_1+u_2) + f(u_2+u_3) + \cdots\cdots + f(u_{n-1}+u_n) \\ &\qquad - [f(u_1) + 2f(u_2) + \cdots\cdots + 2f(u_{n-1}) + f(u_n)]\Big\| \\ &\qquad \le \begin{cases} \kappa, \\ \kappa \sum_{i=1}^{n} ||u_i||^{\gamma} \\ \kappa \prod_{i=1}^{n} ||u_i||^{\gamma} \\ \kappa \{\sum_{i=1}^{n} ||u_i||^{n\gamma} + \prod_{i=1}^{n} ||u_i||^{\gamma}\} \\ \kappa \sum_{i=1}^{n} ||u_i||^{\gamma_i} \\ \kappa \prod_{i=1}^{n} ||u_i||^{\gamma_i} \\ \kappa \{\sum_{i=1}^{n} ||u_i||^{\sum_{i=1}^{n}\gamma_i} + \prod_{i=1}^{n} ||u_i||^{\gamma_i}\} \end{cases} \end{aligned} \tag{2.15}$$

where $\kappa, \gamma, \gamma_i (i = 1, \cdots, n) > 0$ and for all $u_1, \cdots, u_n \in \mathcal{A}$. Then there exists a unique additive function $\Lambda : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying the functional equation *(1.2)* and

$$\|f(u) - \Lambda(u)\| \le \begin{cases} \dfrac{\kappa}{|n-1|}, & \\ \dfrac{\kappa n ||u||^{\gamma}}{(n-1)|2-2^{\gamma}|}, & \gamma \neq 1 \\ \dfrac{\kappa ||u||^{n\gamma}}{(n-1)|2-2^{n\gamma}|}, & n\gamma \neq 1 \\ \dfrac{\kappa (n+1) ||u||^{n\gamma}}{(n-1)|2-2^{n\gamma}|}, & n\gamma \neq 1 \\ \displaystyle\sum_{i=1}^{n} \frac{\kappa ||u||^{\gamma_i}}{(n-1)|2-2^{\gamma_i}|}, & \gamma_i (i = 1, \cdots, n) \neq 1 \\ \dfrac{\kappa n ||u||^{\sum_{i=1}^{n}\gamma_i}}{(n-1)|2-2^{\sum_{i=1}^{n}\gamma_i}|}, & \sum_{i=1}^{n}\gamma_i \neq 1 \\ \dfrac{\kappa (n+1) ||u||^{\sum_{i=1}^{n}\gamma_i}}{(n-1)|2-2^{\sum_{i=1}^{n}\gamma_i}|}, & \sum_{i=1}^{n}\gamma_i \neq 1, \end{cases} \tag{2.16}$$

for all $u \in \mathcal{A}$.

Proof. Consider

$$\delta(u_1, \cdots, u_n) = \begin{cases} \kappa, \\ \kappa \sum_{i=1}^{n} ||u_i||^{\gamma} \\ \kappa \prod_{i=1}^{n} ||u_i||^{\gamma} \\ \kappa \{\sum_{i=1}^{n} ||u_i||^{n\gamma} + \prod_{i=1}^{n} ||u_i||^{\gamma}\} \\ \kappa \sum_{i=1}^{n} ||u_i||^{\gamma_i} \\ \kappa \prod_{i=1}^{n} ||u_i||^{\gamma_i} \\ \kappa \{\sum_{i=1}^{n} ||u_i||^{\sum_{i=1}^{n}\gamma_i} + \prod_{i=1}^{n} ||u_i||^{\gamma_i}\} \end{cases}$$

where $\kappa, \gamma, \gamma_i (i = 1, \cdots, n) > 0$ and for all $u_1, \cdots, u_n \in \mathcal{A}$ in Theorem 2.2. Q.E.D.

3 Stability Results: 2-Banach Space and Quasi 2-Banach space

In this section, we examine the generalized Ulam-Hyers stability of the additive functional equation (1.2) in 2-Banach Spaces and quasi 2-Banach spaces. The concept of linear 2-normed spaces has been investigated by S. Gahler [10] in 1964 and has been developed extensively in different subjects by many authors. In the 1960, S. Gahler [9, 10, 11] and A. White [24, 25] introduced the concept of 2-Banach spaces; see also [17]. One of the axioms of the 2-norm is the parallelepiped inequality, which is actually a fundamental one in the theory of 2-normed spaces. C. Park replaced precisely this inequality (analogously as in the normed spaces) with a new condition, which actually means that he gave the following definition for quasi 2-normed space. In order to define completeness, the concepts of Cauchy sequences and convergence are required.

Definition 3.1. Let X be a linear space of dimension greater than 1. Suppose $\|(\bullet,\bullet)\|$ is a real-valued function on $X \times X$ satisfying the following conditions:

(2N1) $\|(x,y)\| = 0$ if and only if x, y are linearly dependent vectors,

(2N2) $\|(x,y)\| = \|(y,x)\|$ for all $x, y \in X$,

(2N3) $\|(\lambda x,y)\| = |\lambda|\|(x,y)\|$ for all $\lambda \in R$ and for all $x, y \in X$,

(2N4) $\|(x+y,z)\| \leq \|(x,z)\| + \|(y,z)\|$ for all $x, y, z \in X$.

Then $\|(\bullet,\bullet)\|$ is called a 2-norm on X and the pair $(X, \|(\bullet,\bullet)\|)$ is called 2-normed linear space.

Definition 3.2. A sequence $\{x_n\}$ in a linear 2-normed space X is called a Cauchy sequence if there are two points $y, z \in X$ such that y and z are linearly independent,

$$\lim_{l,m\to\infty} \|(x_l - x_m, y)\| = 0 \qquad and \qquad \lim_{l,m\to\infty} \|(x_l - x_m, z)\| = 0.$$

A sequence $\{x_n\}$ in a linear 2-normed space X is called a convergent sequence if there is an $x, w \in X$ such that

$$\lim_{n\to\infty} \|(x_n - x, w)\| = 0$$

for all $y \in X$. If $\{x_n\}$ converges to x, write $x_n \to x$ as $n \to \infty$ and call x the limit of $\{x_n\}$. In this case, we also write $\lim_{n\to\infty} x_n = x$. A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-Banach space.

Definition 3.3. Let X be a linear space of dimension greater than or equal to 2 . Suppose $||(\bullet,\bullet)||$ is a real-valued function on $X \times X$ satisfying the following conditions:

(Q2N1) $||(x,y)|| = 0$ if and only if x, y are linearly dependent vectors,

(Q2N2) $||(x,y)|| = ||(y,x)||$ for all $x, y \in X$,

(Q2N3) $||(\lambda x,y)|| = |\lambda|||(x,y)||$ for all $\lambda \in R$ and for all $x, y \in X$,

(Q2N4) It exists a constant $K \geq 1$ such that $||(x+y,z)|| \leq K(||(x,z)|| + ||(y,z)||)$ for all $x, y, z \in X$.

Then $||(\bullet,\bullet)||$ is called a quasi 2-norm on X and the pair $(X, ||(\bullet,\bullet)||)$ is called quasi 2-normed linear space. The smallest possible number K such that it satisfies the condition (Q2N4) is called a modulus of concavity of the quasi 2-norm $||(\bullet,\bullet)||$.

Sometimes the condition (Q2N4) called the triangle inequality. Further, M. Kir and M. Acikgoz [16] gave few examples of trivial quasi 2-normed spaces and consider the question about completing the quasi 2-normed space. A quasi 2-normed space in which every Cauchy sequence is a convergent sequence is called a quasi 2-Banach space.

Theorem 3.4. Let $\mathcal{A}$ be a 2-normed space and $\mathcal{B}$ be a 2-Banach space. Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a mapping satisfying the functional inequality

$$\left\|\left(\mathfrak{F}(u_1, \cdots, u_n), v\right)\right\| \leq \delta_2(u_1, \cdots, u_n) \tag{3.1}$$

where $\delta_2 : \mathcal{A}^n \longrightarrow [0, \infty)$ is a function such that

$$\lim_{m\to\infty} \frac{\delta_2\left(2^{mj}u_1, \cdots, 2^{mj}u_n\right)}{2^{mj}} = 0 \tag{3.2}$$

for all $u_1, \cdots, u_n \in \mathcal{A}$ and all $v \in \mathcal{B}$. Then, there exists a unique additive mapping $\Lambda : \mathcal{A} \to \mathcal{B}$ satisfying the functional equation *(1.2)* and the inequality

$$\|(f(u) - \Lambda(u), v)\| \leq \frac{1}{2(n-1)} \sum_{i=\frac{i-j}{2}}^{\infty} \frac{\Delta_2\left(2^{ij}u\right)}{2^{ij}} \tag{3.3}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$, where $j \in \{-1, +1\}$ and $\Delta_2\left(2^{ij}u\right)$, $\Lambda(u)$ are defined as

$$\Delta_2\left(2^{ij}u\right) = \delta_2\left(\underbrace{2^{ij}u, \cdots, 2^{ij}u}_{n-times}\right) \tag{3.4}$$

and

$$\Lambda(u) = \lim_{m\to\infty} \frac{f\left(2^{mj}u\right)}{2^{mj}} \tag{3.5}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$.

Proof. Replacing $(u_1, \cdots, u_n)$ by $(u, \cdots, u)$ in (3.1), we get

$$\|((n-1)f(2u) - 2(n-1)f(u), v)\| \leq \delta_2(u, \cdots, u) \tag{3.6}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$. Define $\Delta_2(u) = \delta_2(u, \cdots, u)$. It follows from (3.6), we have

$$\left\|\left(\frac{f(2u)}{2} - f(u), v\right)\right\| \leq \frac{\Delta_2(u)}{2(n-1)} \tag{3.7}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$. Setting u by $2u$ and multiply by 2 in (3.7), we arrive

$$\left\|\left(\frac{f\left(2^2u\right)}{2^2} - \frac{f(2u)}{2}, v\right)\right\| \leq \frac{\Delta_2(2u)}{4(n-1)} \tag{3.8}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$. With the use of triangle inequality it follows from (3.7) and (3.8), we obtain

$$\left\| \left(\frac{f\left(2^2 u\right)}{2^2} - f(u), v \right) \right\| \leq \frac{1}{2(n-1)} \left[\Delta_2(u) + \frac{\Delta_2(2u)}{2} \right] \tag{3.9}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$. Generalizing, for any positive integer m, one can reach

$$\left\| \left(\frac{f\left(2^m u\right)}{2^m} - f(u), v \right) \right\| \leq \frac{1}{2(n-1)} \sum_{i=0}^{m-1} \frac{\Delta_2\left(2^i u\right)}{2^i} \tag{3.10}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$. The remaining part of the proof is similar to that of Theorem 2.2. Hence, the proof is complete. Q.E.D.

Corollary 3.5. Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a mapping fulfilling the inequality

$$\begin{aligned} &\left\| (\mathfrak{F}(u_1, \cdots, u_n), v) \right\| \\ &\leq \begin{cases} \kappa, \\ \kappa \sum_{i=1}^n ||(u_i, v)||^\gamma \\ \kappa \prod_{i=1}^n ||(u_i, v)||^\gamma \\ \kappa \left\{ \sum_{i=1}^n ||(u_i, v)||^{n\gamma} + \prod_{i=1}^n ||(u_i, v)||^\gamma \right\} \\ \kappa \sum_{i=1}^n ||u_i||^{\gamma_i} ||v||^{\gamma_i} \\ \kappa \prod_{i=1}^n ||u_i||^{\gamma_i} ||v||^{\gamma_i} \\ \kappa \left\{ \sum_{i=1}^n ||u_i||^{\sum_{i=1}^n \gamma_i} ||v||^{\sum_{i=1}^n \gamma_i} + \prod_{i=1}^n ||u_i||^{\gamma_i} ||v||^{\gamma_i} \right\} \end{cases} \end{aligned} \tag{3.11}$$

where $\kappa, \gamma, \gamma_i (i = 1, \cdots, n) > 0$, for all $u_1, \cdots, u_n \in \mathcal{A}$ and all $v \in \mathcal{B}$. Then, there exists a unique additive function $\Lambda : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying the functional equation *(1.2)* and

$$\|(f(u) - \Lambda(u), v)\| \leq \begin{cases} \dfrac{\kappa}{|n-1|}, & \\ \dfrac{\kappa n ||(u, v)||^\gamma}{(n-1)|2 - 2^\gamma|}, & \gamma \neq 1 \\ \dfrac{\kappa ||(u, v)||^{n\gamma}}{(n-1)|2 - 2^{n\gamma}|}, & n\gamma \neq 1 \\ \dfrac{\kappa (n+1) ||(u, v)||^{n\gamma}}{(n-1)|2 - 2^{n\gamma}|}, & n\gamma \neq 1 \\ \displaystyle\sum_{i=1}^n \frac{\kappa ||u||^{\gamma_i} ||v||^{\gamma_i}}{(n-1)|2 - 2^{\gamma_i}|}, & \gamma_i (i = 1, \cdots, n) \neq 1 \\ \dfrac{\kappa n ||u||^{\sum_{i=1}^n \gamma_i} ||v||^{\sum_{i=1}^n \gamma_i}}{(n-1)|2 - 2^{\sum_{i=1}^n \gamma_i}|}, & \sum_{i=1}^n \gamma_i \neq 1 \\ \dfrac{\kappa (n+1) ||u||^{\sum_{i=1}^n \gamma_i} ||v||^{\sum_{i=1}^n \gamma_i}}{(n-1)|2 - 2^{\sum_{i=1}^n \gamma_i}|}, & \sum_{i=1}^n \gamma_i \neq 1, \end{cases} \tag{3.12}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$.

In analogy with Theorem 3.4, we have the following theorem for the stability of functional equation (1.2) in a quasi 2-normed space. Since, its proof is similar to the proof of Theorem 3.4, we omit it.

Theorem 3.6. Let $j \in \{-1, +1\}$ and $\mathcal{A}$ be quasi 2-normed space and $\mathcal{B}$ be quasi 2-Banach space. Suppose that $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a function satisfying the functional inequality

$$\left\|(\mathfrak{F}(u_1, \cdots, u_n), v)\right\| \leq \delta_3(u_1, \cdots, u_n) \tag{3.13}$$

where $\delta_3 : \mathcal{A}^n \longrightarrow [0, \infty)$ is a function such that

$$\lim_{m\to\infty} \frac{\delta_3\left(2^{mj}u_1, \cdots, 2^{mj}u_n\right)}{2^{mj}} = 0 \tag{3.14}$$

for all $u_1, \cdots, u_n \in \mathcal{A}$ and all $v \in \mathcal{B}$. Then, there exists a unique additive mapping $\Lambda : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying the functional equation *(1.2)* and the inequality

$$\|(f(u) - \Lambda(u), v)\| \leq \frac{K^{m-1}}{2(n-1)} \sum_{i=\frac{i-j}{2}}^{\infty} \frac{\Delta_3\left(2^{ij}u\right)}{2^{ij}} \tag{3.15}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$, where $j = \pm 1$. The function $\Delta_3\left(2^{ij}u\right)$ and $\Lambda(u)$ are defined as

$$\Delta_3\left(2^{ij}u\right) = \delta_3\left(\underbrace{2^{ij}u, \cdots, 2^{ij}u}_{n-times}\right) \tag{3.16}$$

and

$$\Lambda(u) = \lim_{m\to\infty} \frac{f\left(2^{mj}u\right)}{2^{mj}} \tag{3.17}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$.

Corollary 3.7. Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping fulfilling the inequality

$$\left\|(\mathfrak{F}(u_1, \cdots, u_n), v)\right\| \leq \begin{cases} \kappa, \\ \kappa \sum_{i=1}^{n} ||(u_i, v)||^{\gamma} \\ \kappa \prod_{i=1}^{n} ||(u_i, v)||^{\gamma} \\ \kappa \{\sum_{i=1}^{n} ||(u_i, v)||^{n\gamma} + \prod_{i=1}^{n} ||(u_i, v)||^{\gamma}\} \\ \kappa \sum_{i=1}^{n} ||u_i||^{\gamma_i} ||v||^{\gamma_i} \\ \kappa \prod_{i=1}^{n} ||u_i||^{\gamma_i} ||v||^{\gamma_i} \\ \kappa \left\{\sum_{i=1}^{n} ||u_i||^{\sum_{i=1}^{n}\gamma_i} ||v||^{\sum_{i=1}^{n}\gamma_i} + \prod_{i=1}^{n} ||u_i||^{\gamma_i} ||v||^{\gamma_i}\right\} \end{cases} \tag{3.18}$$

where $\kappa, \gamma, \gamma_i (i = 1, \cdots, n) > 0$, for all $u_1, \cdots, u_n \in \mathcal{A}$ and all $v \in \mathcal{B}$. Then, there exists a mapping

$\Lambda : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying the functional equation *(1.2)* and

$$\|(f(u)-\Lambda(u),v)\| \leq \begin{cases} \dfrac{K^{m-1}\kappa}{|n-1|}, & \\ \dfrac{K^{m-1}\kappa n||u,v||^{\gamma}}{(n-1)|2-2^{\gamma}|}, & \gamma \neq 1 \\ \dfrac{K^{m-1}\kappa||u,v||^{n\gamma}}{(n-1)|2-2^{n\beta\gamma}|}, & n\gamma \neq 1 \\ \dfrac{K^{m-1}\kappa(n+1)||u,v||^{n\gamma}}{(n-1)|2-2^{n\gamma}|}, & n\gamma \neq 1 \\ \displaystyle\sum_{i=1}^{n} \dfrac{K^{m-1}\kappa||u||^{\gamma_i}||v||^{\gamma_i}}{(n-1)|2-2^{\gamma_i}|}, & \gamma_i(i=1,\cdots,n) \neq 1 \\ \dfrac{K^{m-1}\kappa n||u||^{\sum_{i=1}^{n}\gamma_i}||v||^{\sum_{i=1}^{n}\gamma_i}}{(n-1)|2-2^{\sum_{i=1}^{n}\gamma_i}|}, & \sum_{i=1}^{n}\gamma_i \neq 1 \\ \dfrac{K^{m-1}\kappa(n+1)||u||^{\sum_{i=1}^{n}\gamma_i}||v||^{\sum_{i=1}^{n}\gamma_i}}{(n-1)|2-2^{\sum_{i=1}^{n}\gamma_i}|}, & \sum_{i=1}^{n}\gamma_i \neq 1, \end{cases} \tag{3.19}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$.

4 Stability Results : Quasi-β-2-Banach Space

In this section, we investigate the generalized Ulam-Hyers stability of the additive functional equation (1.2) in quasi-β-2-Banach spaces.

Definition 4.1. Let X be a linear space of dimension greater than or equal to 2. Suppose $||(\bullet,\bullet)||$ is a real-valued function on $X \times X$ satisfying the following conditions:

(QB2N1) $||(x,y)|| = 0$ if and only if x, y are linearly dependent vectors,

(QB2N2) $||(x,y)|| = ||(y,x)||$ for all $x, y \in X$,

(QB2N3) $||(\lambda x,y)|| = |\lambda|^{\beta}||(x,y)||$ for all $\lambda \in R$ and for all $x, y \in X$ where β is a a real number with $0 < \beta \leq 1$

(QB2N4) It exists a constant $K \geq 1$ such that $||(x+y,z)|| \leq K(||(x,z)|| + ||(y,z)||)$ for all $x, y, z \in X$.

The pair $(X, ||(\bullet,\bullet)||)$ is called quasi-β-normed space if $||(\bullet,\bullet)||$ is a quasi-β-2-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$.

Definition 4.2. A quasi-β-2-Banach space is a complete quasi-β-normed space.

Throughout this section, we take $\mathcal{A}$ as a quasi-β-2-Banach space and $\mathcal{B}$ as a quasi-β-2-Banach space.

Theorem 4.3. let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a mapping satisfying the functional inequality

$$\|(\mathfrak{F}(u_1,\cdots,u_n),v)\| \leq \delta_4(u_1,\cdots,u_n) \tag{4.1}$$

where $\delta_4 : \mathcal{A}^n \longrightarrow [0, \infty)$ be a function such that

$$\lim_{m\to\infty} \frac{\delta_4\left(2^{mj}u_1, \cdots, 2^{mj}u_n\right)}{2^{mj}} = 0 \tag{4.2}$$

for all $u_1, \cdots, u_n \in \mathcal{A}$ and all $v \in \mathcal{B}$. Then, there exists a unique mapping $\Lambda : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying the functional equation *(1.2)* and the inequality

$$\|(f(u) - \Lambda(u), v)\| \leq \frac{K^{m-1}}{2^\beta (n-1)^\beta} \sum_{i=\frac{i-j}{2}}^{\infty} \frac{\Delta_4\left(2^{ij}u\right)}{2^{ij}} \tag{4.3}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$, where $j \in \{-1, +1\}$ and $\Delta_4\left(2^{ij}u\right)$, $\Lambda(u)$ are defined as

$$\Delta_4\left(2^{ij}u\right) = \delta_4\left(\underbrace{2^{ij}u, \cdots, 2^{ij}u}_{n-times}\right) \tag{4.4}$$

and

$$\Lambda(u) = \lim_{m\to\infty} \frac{f\left(2^{mj}u\right)}{2^{mj}} \tag{4.5}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$.

Proof. Interchanging $(u_1, \cdots, u_n)$ into $(u, \cdots, u)$ in (4.1), we get

$$\|((n-1)f(2u) - 2(n-1)f(u), v)\| \leq \delta_4(u, \cdots, u) \tag{4.6}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$. Consider $\Delta_4(u) = \delta_4(u, \cdots, u)$, and so the relation (4.6) implies that

$$\left\|\left(\frac{f\,(2u)}{2} - f(u), v\right)\right\| \leq \frac{\Delta_4(u)}{2^\beta(n-1)^\beta} \tag{4.7}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$. Substituting u by $2u$ and multiply by 2 in (4.7), we arrive at

$$\left\|\left(\frac{f\left(2^2u\right)}{2^2} - \frac{f(2u)}{2}, v\right)\right\| \leq \frac{\Delta_4(2u)}{2 \cdot 2^\beta(n-1)^\beta} \tag{4.8}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$. With the use of triangle inequality it follows from (4.7) and (4.8), we obtain

$$\left\|\left(\frac{f\left(2^2u\right)}{2^2} - f(u), v\right)\right\| \leq \frac{K}{2^\beta(n-1)^\beta}\left[\Delta_4(u) + \frac{\Delta_4(2u)}{2}\right] \tag{4.9}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$. Repeating this way, for any positive integer m, one can reach

$$\left\|\left(\frac{f\,(2^m u)}{2^m} - f(u), v\right)\right\| \leq \frac{K^{m-1}}{2^\beta(n-1)^\beta} \sum_{i=0}^{m-1} \frac{\Delta_4\left(2^i u\right)}{2^i} \tag{4.10}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$. Similar the rest of the proof Theorem 2.2, we can finish the proof. Q.E.D.

Corollary 4.4. Let $f:\mathcal{A}\longrightarrow\mathcal{B}$ be a mapping fulfilling the inequality

$$\|(\mathfrak{F}(u_1,\cdots,u_n),v)\|\leq\begin{cases}\kappa,\\ \kappa\sum_{i=1}^{n}||(u_i,v)||^{\gamma}\\ \kappa\prod_{i=1}^{n}||(u_i,v)||^{\gamma}\\ \kappa\left\{\sum_{i=1}^{n}||(u_i,v)||^{n\gamma}+\prod_{i=1}^{n}||(u_i,v)||^{\gamma}\right\}\\ \kappa\sum_{i=1}^{n}||u_i||^{\gamma_i}||v||^{\gamma_i}\\ \kappa\prod_{i=1}^{n}||u_i||^{\gamma_i}||v||^{\gamma_i}\\ \kappa\left\{\sum_{i=1}^{n}||u_i||^{\sum_{i=1}^{n}\gamma_i}||v||^{\sum_{i=1}^{n}\gamma_i}+\prod_{i=1}^{n}||u_i||^{\gamma_i}||v||^{\gamma_i}\right\}\end{cases}\tag{4.11}$$

where $\kappa,\gamma,\gamma_i(i=1,\cdots,n)>0$, for all $u_1,\cdots,u_n\in\mathcal{A}$ and all $v\in\mathcal{B}$. Then, there exists a unique mapping $\Lambda:\mathcal{A}\longrightarrow\mathcal{B}$ satisfying the functional equation *(1.2)* and

$$\|(f(u)-\Lambda(u),v)\|\leq\begin{cases}\dfrac{K^{m-1}\kappa}{2^{\beta}(n-1)^{\beta}|n-1|}, & \\ \dfrac{2K^{m-1}\kappa n||u,v||^{\gamma}}{2^{\beta}(n-1)^{\beta}|2-2^{\beta\gamma}|}, & \beta\gamma\neq 1\\ \dfrac{2K^{m-1}\kappa||u,v||^{n\beta\gamma}}{2^{\beta}(n-1)^{\beta}|2-2^{n\gamma}|}, & n\beta\gamma\neq 1\\ \dfrac{2K^{m-1}\kappa(n+1)||u,v||^{n\gamma}}{2^{\beta}(n-1)^{\beta}|2-2^{n\beta\gamma}|}, & n\beta\gamma\neq 1\\ \sum_{i=1}^{n}\dfrac{2K^{m-1}\kappa||u||^{\gamma_i}||v||^{\gamma_i}}{2^{\beta}(n-1)^{\beta}|2-2^{\beta\gamma_i}|}, & \beta\gamma_i(i=1,\cdots,n)\neq 1\\ \dfrac{2K^{m-1}\kappa n||u||^{\sum_{i=1}^{n}\gamma_i}||v||^{\sum_{i=1}^{n}\gamma_i}}{2^{\beta}(n-1)^{\beta}|2-2^{\sum_{i=1}^{n}\beta\gamma_i}|},+ & \sum_{i=1}^{n}\beta\gamma_i\neq 1\\ \dfrac{2K^{m-1}\kappa(n+1)||u||^{\sum_{i=1}^{n}\gamma_i}||v||^{\sum_{i=1}^{n}\gamma_i}}{2^{\beta}(n-1)^{\beta}|2-2^{\sum_{i=1}^{n}\beta\gamma_i}|}, & \sum_{i=1}^{n}\beta\gamma_i\neq 1,\end{cases}\tag{4.12}$$

for all $u\in\mathcal{A}$ and all $v\in\mathcal{B}$.

5 Stability Results : Fuzzy Quasi-β-2-Banach Space

5.1 Definitions And Notations

Definition 5.1. Let X be a linear space of dimension greater than or equal to 2 . A function $N: X\times X\times\mathbb{R}\to[0,1]$ is said to be a fuzzy quasi-β-2-norm on X if for all $x,y,z\in X$ and all $s,t\in\mathbb{R}$,
(2QBFN1) $N(x,z,c)=0$ for $c\leq 0$;
(2QBFN2) $x=0$ if and only if $N(x,z,c)=1$ for all $c>0$;
(2QBFN3) $N(cx,z,t)=N\left(x,z,\frac{t}{|c^{\beta}|}\right)$ if $c\neq 0$ where β is a a real number with $0<\beta\leq 1$
(2QBFN4) $N(x+y,z,s+t)\geq min\{N(x,z,Ks),N(y,z,Kt)\}$; a constant $K\geq 1$
(2QBFN5) $N(x,z,\cdot)$ is a non-decreasing function on $\mathbb{R}$ and $lim_{t\to\infty}N(x,,z,t)=1$;
(2QBFN6) for $x\neq 0, N(x,z,\cdot)$ is (upper semi) continuous on $\mathbb{R}$.
The pair (X,X,N) is called a fuzzy quasi-β-2-Banach space.

Example 5.2. Let X be a linear space. Then

$$N(x,z,t) = \begin{cases} \dfrac{t}{t+\|x\|}, & t>0, \quad x,z \in X, \\ 0, & t \le 0, \quad x,z \in X \end{cases}$$

is a fuzzy quasi-β-2-normed space on X.

Example 5.3. Let X be a linear space. Then

$$N(x,z,t) = \begin{cases} 0, & t \le 0, \\ \dfrac{t}{\|x\|}, & 0 < t \le \|x\|, z \in X \\ 1, & t > \|x\|, z \in X \end{cases}$$

is a fuzzy quasi-β-2-normed space on X.

Definition 5.4. Let X be a fuzzy quasi-β-2-normed space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x, z \in X$ such that $\lim\limits_{n\to\infty} N(x_n - x, z, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim\limits_{n\to\infty} x_n = x$. A sequence x_n in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \ge n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, z, t) > 1 - \varepsilon$. Every convergent sequence in a fuzzy quasi-β-2-normed space is Cauchy. If each Cauchy sequence is convergent, then the 2- norm is said to be complete and the fuzzy quasi-β-2-normed space is called a fuzzy quasi-β-2-Banach space.

5.2 Stability Results

Here, we study the generalized Ulam-Hyers stability of the additive functional equation (1.2) in fuzzy quasi-β-2-Banach space. From now on, we guess $\mathcal{A}$ is a fuzzy quasi-β-2-normed space and $\mathcal{B}$ is a fuzzy quasi-β-2-Banach space.

Theorem 5.5. Let $\eta \in \{-1, +1\}$. Suppose that a mapping $f : \mathcal{A} \longrightarrow \mathcal{B}$ satisfies the inequality

$$N\Big((\mathfrak{F}(u_1, \cdots, u_n), v), r\Big) \ge N'\left(\delta_5(u_1, \cdots, u_n), v, r\right) \tag{5.1}$$

where $\delta_5 : \mathcal{A}^n \longrightarrow [0, \infty)$ is a function such that

$$\lim_{m\to\infty} N'\left(\delta_5\left(2^{\eta m}u_1, \cdots, 2^{\eta m}u_n\right), v, 2^{\eta m} r\right) = 1 \tag{5.2}$$

with the condition

$$N'\left(\delta_5\left(2^{\eta}u_1, \cdots, 2^{\eta}u_n\right), v, r\right) \ge N'\left(\rho^{\eta}\delta_5\left(u_1, \cdots, u_n\right), v, r\right) \tag{5.3}$$

for all $u_1, \cdots, u_n \in \mathcal{A}$ and all $v \in \mathcal{B}$, all $r > 0$ with $0 < \rho^{\eta} < 2$. Then there exists a unique additive function $\Lambda : \mathcal{A} \to \mathcal{B}$ satisfying the functional equation *(1.2)* and the inequality

$$N\left((f(u) - \Lambda(u), v), r\right) \ge N'\left(\Delta_5(u), v, \frac{K\, 2^{\beta}(n-1)^{\beta}|2-\rho| r}{2}\right) \tag{5.4}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$, where $\Delta_5(u)$ and $\Lambda(u)$ are defined as

$$\Delta_5(u) = \delta_5(u, \cdots, u) \tag{5.5}$$

and

$$\Lambda(u) = N - \lim_{m\to\infty} \frac{f(2^{\eta m}u)}{2^{\eta m}} \tag{5.6}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$.

Proof. **Case:1** $\eta = 1$. Replacing $(u_1, \cdots, u_n)$ by $(u, \cdots, u)$ in (5.1), we obtain

$$N\left(((n-1)f(2u) - 2(n-1)f(u), v), r\right) \geq N'\left(\delta_5(u, \cdots, u), v, r\right) \tag{5.7}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$, and all $r > 0$. Setting $\Delta_5(u) = \delta_5(u, \cdots, u)$ in (5.7), we have

$$N\left(((n-1)f(2u) - 2(n-1)f(u), v), r\right) \geq N'\left(\Delta_5(u), v, r\right) \tag{5.8}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$, and all $r > 0$. Using (2QBFN3) in (5.8), we get

$$N\left(\left(\frac{f(2u)}{2} - f(u), v\right), r\right) \geq N'\left(\Delta_5(u), v, 2^\beta(n-1)^\beta r\right) \tag{5.9}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$, and all $r > 0$. Interchanging u into $2^m u$ in (5.9) and applying the propertty (2QBFN3), (5.3), we obtain

$$N\left(\left(\frac{f(2^{m+1}u)}{2^{m+1}} - \frac{f(2^m u)}{2^m}, v\right), \frac{r}{2^{m\beta}}\right) \geq N'\left(\Delta_5(u), v, \frac{2^\beta(n-1)^\beta r}{\rho^{m\beta}}\right) \tag{5.10}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$, all $r > 0$. Replacing r by $\rho^{m\beta}r$ in (5.10), we have

$$N\left(\left(\frac{f(2^{m+1}u)}{2^{m+1}} - \frac{f(2^m u)}{2^m}, v\right), \frac{\rho^{m\beta}r}{2^{m\beta}}\right) \geq N'\left(\Delta_5(u), v, 2^\beta(n-1)^\beta r\right) \tag{5.11}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$, all $r > 0$. It is easy to see that

$$\frac{f(2^m u)}{2^m} - f(u) = \sum_{i=0}^{m-1} \frac{f(2^{i+1}u)}{2^{i+1}} - \frac{f(2^i u)}{2^i} \tag{5.12}$$

for all $u \in \mathcal{A}$. From equations (5.11) and (5.12), we find

$$\begin{aligned} N\left(\left(\frac{f(2^m u)}{2^m} - f(u), v\right)\sum_{i=0}^{m-1}\frac{\rho^{i\beta}\, r}{2^{i\beta}}\right) &\geq \min \bigcup_{i=0}^{m-1}\left\{\left(\frac{f(2^{i+1}u)}{2^{i+1}} - \frac{f(2^i u)}{2^i}, v\right), \frac{K\rho^{i\beta}\, r}{2^{i\beta} i}\right\} \\ &\geq \min \bigcup_{i=0}^{m-1}\left\{N'\left(\Delta_5(u), v, K\, 2^\beta(n-1)^\beta r\right)\right\} \\ &\geq N'\left(\Delta_5(u), v, K\, 2^\beta(n-1)^\beta r\right) \end{aligned} \tag{5.13}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$, all $r > 0$. switching u by $2^l u$ in (5.13) and using (5.3), (2QBFN3), we obtain

$$N\left(\left(\frac{f(2^{m+l}u)}{2^{m+l}} - \frac{f(2^l u)}{2^l}, v, r\right) \geq N'\left(\Delta_5(u), v\right), \frac{K\, 2^{\beta}(n-1)^{\beta} r}{\sum_{i=0}^{m-1} \frac{\rho^{(i+l)\beta}}{2^{(i+l)\beta}}}\right) \tag{5.14}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$, all $r > 0$ and all $m, l \geq 0$. Since $0 < \rho < 2$ and $\sum_{i=0}^{m} \left(\frac{\rho}{2}\right)^i < \infty$, the Cauchy criterion for convergence and (2QBFN5) implies that $\{\frac{f(2^m u)}{2^m}\}$ is a Cauchy sequence in $\mathcal{B}$. Due to the completeness of $\mathcal{B}$, this sequence converges to some point $\Lambda(u) \in \mathcal{B}$. So one can define the mapping $\Lambda(u) : \mathcal{A} \to \mathcal{B}$ by

$$\Lambda(u) = N - \lim_{n\to\infty} \frac{f(2^m u)}{2^m}$$

for all $u \in \mathcal{A}$. Letting $l = 0$ in (5.14), we get

$$N\left(\left(\frac{f(2^m u)}{2^m} - f(u), v\right), r\right) \geq N'\left(\Delta_5(u), v, \frac{K\, 2^{\beta}(n-1)^{\beta} r}{\sum_{i=0}^{m-1} \frac{\rho^{i\beta}}{2^{i\beta}}}\right) \tag{5.15}$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$, all $r > 0$. Letting $m \to \infty$ in (5.15) and using (2QBFN6), we arrive at

$$N\left((f(u) - \Lambda(u), v), r\right) \geq N'\left(\Delta_5(u), v, \frac{K\, 2^{\beta}(n-1)^{\beta}(2-\rho) r}{2}\right)$$

for all $u \in \mathcal{A}$ and all $v \in \mathcal{B}$, all $r > 0$. To prove $\Lambda(u)$ satisfies the (1.2), replacing $(u_1, \cdots, u_n)$ by $(2^m u_1, \cdots, 2^m u_n)$ in (5.1), respectively , we obtain

$$\begin{aligned} N\Bigg(&\frac{f(2^m(u_1+u_2))}{2^m} + \frac{f(2^m(u_2+u_2))}{2^m} + \cdots + \frac{f(2^m(u_{n-1}+u_n))}{2^m} \\ &\quad -\frac{f(2^m u_1)}{2^m} - \frac{2f(2^m u_2)}{2^m} - \cdots - \frac{2f(2^m u_{n-1})}{2^m} - \frac{f(2^m u_n)}{2^m}, v, r\Bigg) \\ &\geq N'\left(\delta_5(2^m u_1, \cdots, 2^m u_n), v, 2^m r\right) \end{aligned} \tag{5.16}$$

for all $u_1, \cdots, u_n \in \mathcal{A}$ and all $v \in \mathcal{B}$, all $r > 0$. Now,

$$\begin{aligned}
&N\left(\left(\Lambda\left(u_1+u_2\right)+\Lambda\left(u_2+u_3\right)+\cdots+\Lambda\left(u_{n-1}+u_n\right)\right.\right.\\
&\qquad\left.\left.-[\Lambda\left(u_1\right)+2\Lambda\left(u_2\right)+\cdots+2\Lambda\left(u_{n-1}\right)+\Lambda\left(u_n\right)], v\right), r\right)\\
&=N\Bigg(\Big(\Lambda\left(u_1+u_2\right)-\frac{f(2^m(u_1+u_2))}{2^m}+\Lambda\left(u_2+u_3\right)-\frac{f(2^m(u_2+u_2))}{2^m}+\cdots+\Lambda\left(u_{n-1}+u_n\right)\\
&\qquad-\frac{f(2^m(u_{n-1}+u_n))}{2^m}-\Lambda\left(u_1\right)+\frac{f(2^m u_1)}{2^m}-2\Lambda\left(u_2\right)+\frac{2f(2^m u_2)}{2^m}\\
&\qquad-\cdots-2\Lambda\left(u_{n-1}\right)+\frac{f(2^m u_{n-1})}{2^m}-\Lambda\left(u_n\right)+\frac{f(2^m u_n)}{2^m}, v\Big), r\Bigg)\\
&\geq \min\left\{N\left(\left(\Lambda\left(u_1+u_2\right)-\frac{f(2^m(u_1+u_2))}{2^m}, v\right), \frac{K\,r}{2n}\right),\ N\left(\left(\Lambda\left(u_2+u_3\right)-\frac{f(2^m(u_2+u_2))}{2^m}, v\right), \frac{K\,r}{2n}\right),\right.\\
&\qquad\cdots N\left(\left(\Lambda\left(u_{n-1}+u_n\right)-\frac{f(2^m(u_{n-1}+u_n))}{2^m}, v\right), \frac{K\,r}{2n}\right), N\left(\left(-\Lambda\left(u_1\right)+\frac{f(2^m u_1)}{2^m}, v\right), \frac{K\,r}{2n}\right),\\
&\qquad N\left(\left(-2\Lambda\left(u_2\right)+\frac{2f(2^m u_2)}{2^m}, v\right), \frac{K\,r}{2n}\right), \cdots\cdots, N\left(\left(-2\Lambda\left(u_{n-1}\right)+\frac{2f(2^m u_{n-1})}{2^m}, v\right), \frac{K\,r}{2n}\right),\\
&\qquad N\left(\left(-\Lambda\left(u_n\right)+\frac{f(2^m u_n)}{2^m}, v\right), \frac{K\,r}{2n}\right), N\left(\frac{f(2^m(u_1+u_2))}{2^m}+\frac{f(2^m(u_2+u_2))}{2^m}+\cdots+\frac{f(2^m(u_{n-1}+u_n))}{2^m}\right.\\
&\qquad\left.-\frac{f(2^m u_1)}{2^m}-\frac{2f(2^m u_2)}{2^m}-\cdots-\frac{2f(2^m u_{n-1})}{2^m}-\frac{f(2^m u_n)}{2^m}, v, \frac{K\,r}{2n}\right) \qquad (5.17)
\end{aligned}$$

for all $u_1, \cdots, u_n \in \mathcal{A}$ and all $v \in \mathcal{B}$, all $r > 0$ Hence, the relations (5.4), (5.16) and (5.17) imply that

$$\begin{aligned}
&N\left(\left(\Lambda\left(u_1+u_2\right)+\Lambda\left(u_2+u_3\right)+\cdots+\Lambda\left(u_{n-1}+u_n\right)\right.\right.\\
&\qquad\left.\left.-[\Lambda\left(u_1\right)+2\Lambda\left(u_2\right)+\cdots+2\Lambda\left(u_{n-1}\right)+\Lambda\left(u_n\right)], v\right), r\right)\\
&\geq \min\left\{N'\left(\Delta_5(u), v, K\,2^\beta(n-1)^\beta r\right), N'\left(\Delta_5(u), v, K\,2^\beta(n-1)^\beta r\right),\ \cdots, N'\left(\Delta_5(u), v, K\,2^\beta(n-1)^\beta r\right),\right.\\
&\qquad N'\left(\Delta_5(u), v, K\,2^\beta(n-1)^\beta r\right), N'\left(\Delta_5(u), v, K\,2^\beta(n-1)^\beta r\right),
\end{aligned} \qquad (5.18)$$

for all $u_1, \cdots, u_n \in \mathcal{A}$ and all $v \in \mathcal{B}$, all $r > 0$. Letting $m \to \infty$ in (5.18) and using (5.2), we see that

$$\begin{aligned}
&N\left(\left(\Lambda\left(u_1+u_2\right)+\Lambda\left(u_2+u_3\right)+\cdots+\Lambda\left(u_{n-1}+u_n\right)\right.\right.\\
&\qquad\left.\left.-[\Lambda\left(u_1\right)+2\Lambda\left(u_2\right)+\cdots+2\Lambda\left(u_{n-1}\right)+\Lambda\left(u_n\right)], v\right), r\right)=1
\end{aligned} \qquad (5.19)$$

for all $u_1, \cdots, u_n \in \mathcal{A}$ and all $v \in \mathcal{B}$, all $r > 0$. Applying (2QBFN2) in (5.19), it gives

$$\begin{aligned}
&\Lambda\left(u_1+u_2\right)+\Lambda\left(u_2+u_3\right)+\cdots+\Lambda\left(u_{n-1}+u_n\right)\\
&\qquad=\Lambda\left(u_1\right)+2\Lambda\left(u_2\right)+\cdots+2\Lambda\left(u_{n-1}\right)+\Lambda\left(u_n\right)
\end{aligned}$$

for all $u_1, \cdots, u_n \in \mathcal{A}$. Therefore, $\mathcal{A}$ satisfies the additive functional equation (1.2). In order to prove $\Lambda(u)$ is unique, let $\Lambda'(u)$ be another additive functional equation satisfying (1.2) and (5.4).

Hence,

$$\begin{aligned} N((\Lambda(u)-\Lambda'(u),v),r) &= N\left(\left(\frac{\Lambda(2^m u)}{2^m}-\frac{\Lambda'(2^m u)}{2^m},v\right),r\right) \\ &\geq \min\left\{N\left(\left(\frac{\Lambda(2^m u)}{2^m}-\frac{f(2^m u)}{2^m},v\right),\frac{K\,r}{2}\right),N\left(\left(\frac{f(2^m u)}{2^m}-\frac{\Lambda'(2^m u)}{2^m},v\right),\frac{K\,r}{2}\right)\right\} \\ &= N'\left(\Delta_5(2^m u),v,\frac{K\,2^\beta(n-1)^\beta r}{2}\right)=N'\left(\Delta_5(u),v,\frac{K\,2^\beta(n-1)^\beta r}{2\rho^m}\right) \end{aligned}$$

for all $u\in\mathcal{A}$ and all $v\in\mathcal{B}$, all $r>0$. Since $\lim_{m\to\infty}\frac{2^\beta(n-1)^\beta r}{2\rho^m}=\infty$, we obtain

$$\lim_{m\to\infty}N'\left(\Delta_5(u),v,\frac{K\,2^\beta(n-1)^\beta r}{2\rho^m}\right)=1.$$

Thus, $N((\Lambda(u)-\Lambda'(u),v),r)=1$ for all $x\in\mathcal{A}$ and all $r>0$, hence $\Lambda(u)=\Lambda'(u)$. Therefore $\Lambda(u)$ is unique.

Case:2 $\eta=-1$. Replacing u by $\frac{u}{2}$ in (5.8) , we reach

$$N\left(\left((n-1)f(u)-2(n-1)f\left(\frac{u}{2}\right),v\right),r\right)\geq N'\left(\Delta_5\left(\frac{u}{2}\right),v,r\right) \tag{5.20}$$

for all $u\in\mathcal{A}$ and all $v\in\mathcal{B}$, and all $r>0$. Using (2QBFN3) in (5.8), we get

$$N\left(\left(f(u)-2f\left(\frac{u}{2}\right),v\right),r\right)\geq N'\left(\Delta_5\left(\frac{u}{2}\right),v,(n-1)^\beta r\right) \tag{5.21}$$

for all $u\in\mathcal{A}$ and all $v\in\mathcal{B}$, and all $r>0$. The rest of the proof is similar tracing to that of above case. This completes the proof. Q.E.D.

The following corollary is the immediate consequence of some stabilities for the functional equation (1.2).

Corollary 5.6. Let $f:\mathcal{A}\to\mathcal{B}$ be a mapping fulfilling the inequality

$$\begin{aligned} N\Big(&f(u_1+u_2)+f(u_2+u_3)+\cdots+f(u_{n-1}+u_n) \\ &-[f(u_1)+2f(u_2)+\cdots+2f(u_{n-1})+f(u_n)],v,r\Big) \\ &\geq\begin{cases} N'(\kappa,v,r) \\ N'\left(\kappa\sum_{i=1}^n\|u_i\|^\gamma,v,r\right) \\ N'\left(\kappa\prod_{i=1}^n\|u_i\|^\gamma,v,r\right) \\ N'\left(\kappa\left\{\sum_{i=1}^n\|u_i\|^{n\gamma}+\prod_{i=1}^n\|u_i\|^\gamma\right\},v,r\right) \\ N'\left(\kappa\sum_{i=1}^n\|u_i\|^{\gamma_i},v,r\right) \\ N'\left(\kappa\prod_{i=1}^n\|u_i\|^{\gamma_i},v,r\right) \\ N'\left(\kappa\left\{\sum_{i=1}^n\|u_i\|^{\sum_{i=1}^n\gamma_i}+\prod_{i=1}^n\|u_i\|^{\gamma_i}\right\},v,r\right) \end{cases} \end{aligned} \tag{5.22}$$

where $\kappa, \gamma, \gamma_i (i = 1, \cdots, n) > 0$ and for all $u_1, \cdots, u_n \in \mathcal{A}$. Then there exists a unique additive function $\Lambda : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying the functional equation *(1.2)* and

$$N\left(f(u) - \Lambda(u), v, r\right) \geq \begin{cases} N'\left(\kappa, v, K\, 2^{\beta}(n-1)^{\beta}|n-1|r\right) & \\ N'\left(2\kappa n||u||^{\gamma}, v, K\, 2^{\beta}(n-1)^{\beta}|2-\rho|r\right), & \beta\gamma \neq 1 \\ N'\left(2\kappa||u||^{n\beta\gamma}, v, K\, 2^{\beta}(n-1)^{\beta}|2-\rho|r\right), & n\beta\gamma \neq 1 \\ N'\left(2\kappa(n+1)||u||^{n\gamma}, v, K\, 2^{\beta}(n-1)^{\beta}|2-\rho|r\right), & n\beta\gamma \neq 1 \\ N'\left(\sum_{i=1}^{n} 2\kappa||u||^{\gamma_i}, v, K\, 2^{\beta}(n-1)^{\beta}|2-\rho|r\right), & \beta\gamma_i \neq 1 \\ N'\left(2\kappa n||u||^{\sum_{i=1}^{n}\gamma_i}, v, K\, 2^{\beta}(n-1)^{\beta}|2-\rho|r\right), & \sum_{i=1}^{n}\beta\gamma_i \neq 1 \\ N'\left(2\kappa(n+1)||u||^{\sum_{i=1}^{n}\gamma_i}, v, K\, 2^{\beta}(n-1)^{\beta}|2-\rho|r\right), & \sum_{i=1}^{n}\beta\gamma_i \neq 1 \end{cases} \tag{5.23}$$

for all $u \in \mathcal{A}$.

Proof. Setting

$$\delta_5(u_1, \cdots, u_n) = \begin{cases} \kappa, \\ \kappa \sum_{i=1}^{n} ||u_i||^{\gamma} \\ \kappa \prod_{i=1}^{n} ||u_i||^{\gamma} \\ \kappa \left\{\sum_{i=1}^{n} ||u_i||^{n\gamma} + \prod_{i=1}^{n} ||u_i||^{\gamma}\right\} \\ \kappa \sum_{i=1}^{n} ||u_i||^{\gamma_i} \\ \kappa \prod_{i=1}^{n} ||u_i||^{\gamma_i} \\ \kappa \left\{\sum_{i=1}^{n} ||u_i||^{\sum_{i=1}^{n}\gamma_i} + \prod_{i=1}^{n} ||u_i||^{\gamma_i}\right\} \end{cases}$$

where $\kappa, \gamma, \gamma_i (i = 1, \cdots, n) > 0$ and for all $u_1, \cdots, u_n \in \mathcal{A}$ in Theorem (5.5) and taking

$$\rho = \begin{cases} 2^{0}, \\ 2^{\beta\gamma}, \\ 2^{n\beta\gamma}, \\ 2^{n\beta\gamma}, \\ 2^{\beta\gamma_i}, \\ 2^{\sum_{i=1}^{n}\beta\gamma_i}, \\ 2^{\sum_{i=1}^{n}\beta\gamma_i}, \end{cases} \tag{5.24}$$

we arrive the desired result. Q.E.D.

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