A matrix application on absolute weighted arithmetic mean summability factors of infinite series

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Abstract

In this present paper, we have generalized a main theorem dealing with $|\bar{N}, p_n|_k$ summability of non-decreasing sequences to $|A, p_n|_k$ summability method by using almost increasing sequences and taking normal matrices in place of weighted mean matrices

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1 Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n^α the nth Cesàro mean of order α, with $\alpha > -1$, of the sequence (s_n), that is (see [5])

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \tag{1.1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad \texttt{for} \quad n > 0. \tag{1.2}$$

A series $\sum a_n$ is said to be summable $\mid C, \alpha \mid_k$, $k \geq 1$, if (see [6])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n^\alpha - u_{n-1}^\alpha \mid^k < \infty. \tag{1.3}$$

If we set α=1, then we have $\mid C, 1 \mid_k$ summability. Let (p_n) be a sequence of positive number such that

$$P_n = \sum_{v=0}^{\infty} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \;\; i \geq 1). \tag{1.4}$$

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The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.5}$$

defines the sequence (w_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [7]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |w_n - w_{n-1}|^k < \infty. \tag{1.6}$$

In the special case when $p_n = 1$ for all values of n (respect. $k = 1$), then $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (respect. $|\bar{N}, p_n|$) summability.
Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{1.7}$$

The series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [11])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} \left| \bar{\Delta} A_n(s) \right|^k < \infty. \tag{1.8}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s). \tag{1.9}$$

Note that in the special case if we take $p_n = 1$ for all n, $|A, p_n|_k$ summability is the same as $|A|_k$ summability (see [12]). Also, if we take $a_{nv} = \frac{p_v}{P_n}$, then $|A, p_n|_k$ summability reduces to $\left|\bar{N}, p_n\right|_k$ summability. Furthermore, if we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then $|A, p_n|_k$ summability is the same as $|C, 1|_k$ summability.

2 The Known Results

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (z_n) and two positive constants A and B such that $Az_n \leq b_n \leq Bz_n$ (see [1]). It is known that every increasing sequences is an almost increasing sequence but the converse need not be true. Quite recently, Bor has proved the following theorems concerning on summability factors of the absolute weighted mean.

Theorem 2.1 [3] Let (X_n) be a positive non-decreasing sequence and suppose that there exists sequences (β_n) and (λ_n) such that

$$|\Delta \lambda_n| \leq \beta_n, \tag{1.10}$$

$$\beta_n \to 0 \quad \text{as} \quad n \to \infty \tag{1.11}$$

$$\sum_{n=1}^{\infty} n |\Delta \beta_n| X_n = O(1), \tag{1.12}$$

$$|\lambda_n| X_n = O(1). \tag{1.13}$$

If

$$\sum_{n=1}^{m} \frac{|s_n|^k}{n} = O(X_m) \quad \text{as} \quad m \to \infty, \tag{1.14}$$

and (p_n) is a sequence that

$$P_n = O(np_n), \tag{1.15}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{1.16}$$

then the series $\sum a_n \frac{P_n \lambda_n}{np_n}$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.
Later on, Bor has recently proved the following theorem using under weaker conditions.
Theorem 2.2 [4] Let (X_n) be a positive non-decreasing sequence. If the sequences (X_n) , (β_n), (λ_n), and (p_n) satisfy the conditions (1.10-1.13), (1.15-1.16), and

$$\sum_{n=1}^{m} \frac{|s_n|^k}{nX_n^{k-1}} = O(X_m) \quad \text{as} \quad m \to \infty, \tag{1.17}$$

then the series $\sum a_n \frac{P_n \lambda_n}{np_n}$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

3 The Main Results

The aim of this paper is to generalize Theorem 2.2 for $|A, p_n|_k$ summability factors using almost increasing sequences in place of positive non-decreasing sequence. So, we have generalized Theorem 2.2 under weaker hypothesis by using normal matrices.
Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \quad \bar{\Delta} a_{nv} = a_{nv} - a_{n-1,v}, \quad a_{-1,0} = 0 \tag{1.18}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{\Delta}\bar{a}_{nv}, \quad n = 1, 2, ... \tag{1.19}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{1.20}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{1.21}$$

With this notation we have the following theorem.
Theorem 3.1 Let $A = (a_{nv})$ be a positive normal matrix such that

$$\bar{a}_{n0} = 1, \ n = 0, 1, ..., \tag{1.22}$$

$$a_{n-1,v} \geq a_{nv}, \ \text{for} \ n \geq v + 1, \tag{1.23}$$

$$a_{nn} = O(\frac{p_n}{P_n}), \tag{1.24}$$

$$\sum_{v=1}^{n-1} a_{vv} \hat{a}_{n,v+1} = O(a_{nn}) \tag{1.25}$$

and let (X_n) be an almost increasing sequence. If the sequences (X_n), (β_n), (λ_n), and (p_n) satisfy the conditions of Theorem 2.2, then the series $\sum a_n \frac{P_n \lambda_n}{np_n}$ is summable $|A, p_n|_k$, $k \geq 1$.
We need the following lemmas for the proof of Theorem 3.1.
Lemma 3.1 [8] Under the conditions on (X_n), (β_n), and (λ_n) as expressed in the statement of Theorem 2.2, we have the following:

$$nX_n\beta_n = O(1), \tag{1.26}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{1.27}$$

Lemma 3.2 [10] If the conditions (1.15) and (1.16) of Theorem 2.1 are satisfied, then $\Delta\left(\frac{P_n}{np_n}\right) = O\left(\frac{1}{n}\right)$.
Remark Under the conditions on the sequence (λ_n) of Theorem 2.1, we have that (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$ (see [3]).

4 Proof of Theorem 3.1

Let (V_n) denotes the A-transform of the series $\sum a_n \frac{P_n \lambda_n}{np_n}$. Then, by the definition, we have that

$$\bar{\Delta} V_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v \frac{P_v \lambda_v}{vp_v}.$$

Applying Abel's transformation to this sum, we have that

$$\bar{\Delta} V_n = \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} P_v \lambda_v}{vp_v}\right) \sum_{r=1}^{v} a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{np_n} \sum_{r=1}^{n} a_r$$

$$\bar{\Delta} V_n = \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} P_v \lambda_v}{vp_v}\right) s_v + \frac{\hat{a}_{nn} P_n \lambda_n}{np_n} s_n,$$

by the formula for the difference of products of sequences (see [7]) we have

$$\bar{\Delta} V_n = \frac{a_{nn} P_n \lambda_n}{np_n} s_n + \sum_{v=1}^{n-1} \frac{P_v \lambda_v}{vp_v} \Delta_v(\hat{a}_{nv}) s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_v \Delta\left(\frac{P_v}{vp_v}\right) s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \frac{P_{v+1}}{(v+1)p_{v+1}} \Delta\lambda_v s_v$$

$$\bar{\Delta} V_n = V_{n,1} + V_{n,2} + V_{n,3} + V_{n,4}.$$

To complete the proof of Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} | V_{n,r} |^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4. \tag{1.28}$$

Firstly, by applying Abel's transformation and in view of the hypotheses of Theorem 3.1 we have

$$\begin{aligned}
&\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1} \mid V_{n,1}\mid^k \leq \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^k \left(\frac{P_n}{p_n}\right)^k |\lambda_n|^k \frac{|s_n|^k}{n^k}\\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1} |\lambda_n|^k \frac{|s_n|^k}{n^k} = O(1)\sum_{n=1}^{m}\frac{n^{k-1}}{n^k}|\lambda_n|^{k-1}|\lambda_n||s_n|^k\\
&= O(1)\sum_{n=1}^{m}\frac{1}{n}\frac{1}{X_n^{k-1}}|\lambda_n||s_n|^k = O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}\frac{|s_v|^k}{vX_v^{k-1}} + O(1)|\lambda_m|\sum_{n=1}^{m}\frac{|s_n|^k}{nX_n^{k-1}}\\
&= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n + O(1)|\lambda_m|X_m = O(1)\sum_{n=1}^{m-1}\beta_n X_n + O(1)|\lambda_m|X_m = O(1) \quad \text{as} \quad m\to\infty.
\end{aligned}$$

By applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$ and as in $V_{n,1}$, we have that

$$\begin{aligned}
&\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid V_{n,2}\mid^k = \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\frac{P_v\lambda_v}{vp_v}\Delta_v(\hat{a}_{nv})s_v\right|^k\\
&\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|s_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k\right\}\times\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|s_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v||s_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}a_{vv}\frac{1}{X_v^{k-1}}|s_v|^k|\lambda_v|\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k
\end{aligned}$$

$$\begin{aligned}
&= O(1)\sum_{v=1}^{m}\frac{1}{X_v^{k-1}}|s_v|^k|\lambda_v|\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}\frac{1}{X_v^{k-1}}|s_v|^k|\lambda_v|\frac{1}{v^k}v^{k-1} = O(1)\sum_{v=1}^{m}\frac{1}{X_v^{k-1}}|s_v|^k|\lambda_v|\frac{1}{v} = O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1. Also, since $\Delta\left(\frac{P_v}{vp_v}\right) = O\left(\frac{1}{v}\right)$, by Lemma 3.2, we have

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid V_{n,3}\mid^k = \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\Delta\left(\frac{P_v}{vp_v}\right)\lambda_v s_v\right|^k$$
$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1} a_{vv}^{1-k}\hat{a}_{n,v+1}|\lambda_v|^k|s_v|^k\frac{1}{v^k}\right\}\times\left\{\sum_{v=1}^{n-1}a_{vv}\hat{a}_{n,v+1}\right\}^{k-1}$$
$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^{k-1}\hat{a}_{n,v+1}|\lambda_v|^k|s_v|^k\frac{1}{v^k}$$
$$= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v||s_v|^k\frac{1}{v}\sum_{n=v+1}^{m+1}\hat{a}_{n,v+1}$$
$$= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v||s_v|^k\frac{1}{v}$$
$$= O(1)\sum_{v=1}^{m}\frac{1}{X_v^{k-1}}|\lambda_v||s_v|^k\frac{1}{v} = O(1) \quad \text{as} \quad m\to\infty,$$

by virtue of the hypotheses of Theorem 3.1. Finally, by virtue of the hypotheses of Theorem 3.1, by Lemma 3.1, we have $v\beta_v = O(\frac{1}{X_v})$, then

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid V_{n,4}\mid^k = \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\frac{P_{v+1}}{(v+1)p_{v+1}}\Delta\lambda_v s_v\right|^k$$
$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}a_{vv}^{1-k}\hat{a}_{n,v+1}|\Delta\lambda_v|^k s_v|^k\right\}\times\left\{\sum_{v=1}^{n-1}a_{vv}\hat{a}_{n,v+1}\right\}^{k-1}$$
$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}a_{vv}^{1-k}\hat{a}_{n,v+1}|\Delta\lambda_v|^k|s_v|^k$$
$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1}|s_v|^k|\Delta\lambda_v|^k\sum_{n=v+1}^{m+1}\hat{a}_{n,v+1}$$
$$= O(1)\sum_{v=1}^{m}|s_v|^k(v\beta_v)^{k-1}\beta_v = O(1)\sum_{v=1}^{m}v\beta_v|s_v|^k\frac{1}{vX_v^{k-1}}$$
$$= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\frac{|s_r|^k}{rX_r^{k-1}} + O(1)m\beta_m\sum_{v=1}^{m}\frac{|s_v|^k}{vX_v^{k-1}}$$

$$= O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v + O(1)m\beta_m X_m = O(1)\sum_{v=1}^{m-1}|(v+1)\Delta\beta_v - \beta_v|X_v + O(1)m\beta_m X_m$$
$$= O(1)\sum_{v=1}^{m-1}v|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1}X_v\beta_v + O(1)m\beta_m X_m = O(1) \quad \text{as} \quad m\to\infty,$$

This completes the proof of Theorem 3.1 .

5 Conclusions

1. If we take (X_n) as a positive non-decreasing sequence and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 3.1, then we obtain Theorem 2.2 and if we put $k = 1$ in Theorem 2.2, we have a known result of Mishra and Srivastava dealing with $\left|\bar{N}, p_n\right|$ summability factors of infinite series (see [10]).
2. If we take $p_n = 1$ for all values of n in Theorem 3.1, then we get a new result dealing with the $|A|_k$ summability method.
3. If we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n in Theorem 3.1, then we obtain a known result of Mishra and Srivastava concerning the $\mid C, 1 \mid_k$ summability factors of infinite series (see [9]).

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